A JOURNEY FROM THE OCTONIONIC $\mathbb{P}^2$ TO A FAKE $\mathbb{P}^2$

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ABSTRACT. We discover a family of surfaces of general type with $K^2 = 3$ and $p_g = q = 0$ as free C_{13} quotients of special linear cuts of the octonionic projective plane $\mathbb{OP}^2$. A special member of the family has 3 singularities of type A_2, and is a quotient of a fake projective plane. We use the techniques of [BF20] to define this fake projective plane by explicit equations in its bicanonical embedding.

1. Introduction

Fake projective planes are complex projective surfaces of general type with Hodge numbers equal to those of the usual projective plane $\mathbb{CP}^2$. There are exactly 50 complex conjugate pairs, constructed as ball quotients in [CS11] and they are fascinating gemstones in the vast mine of algebraic surfaces of general type. The first explicit equations of a pair of fake projective planes were constructed in [BK19], and additional six pairs were given explicitly in [BF20]. We refer the reader to [BK19] for more background and history.

Many fake projective planes $\mathbb{P}^2_{fake}$ admit an action of the cyclic group C_3. The quotient $\mathbb{P}^2_{fake}/C_3$ is then a singular surface with $K^2 = 3$ and three singular points of type A_2, see [Ke08]. It can be deformed to construct interesting smooth surfaces with $K^2 = 3$, genus $p_g = 0$, and irregularity $q = 0$. In [BF20] the process was reversed, namely a fake projective plane was constructed as a Galois triple cover of a special singular member of a family of surfaces with $K^2 = 3$. Since the current paper is in many ways analogous, we describe [BF20] in some detail in the next paragraph.

The paper [BF20] first builds a family of special complete intersections of seven Plücker hyperplanes in the Grassmannian $\mathrm{Gr}(3, \mathbb{C}^6)$ which admit a free action of the cyclic group C_{14}. This gives a family of surfaces W which has $K_W^2 = 3,\ p_g = q = 0$. Then the authors find an element of this family such that the quotient by C_{14} has an additional C_3 symmetry and three A_2 singularities. Its Galois cover (that was not at all easy to construct) is a fake projective plane with the automorphism group $(C_3)^2$, labeled by $(\mathrm{C2}, p = 2, \emptyset, d_3D_3)$ in Cartwright-Steger classification [CS11+].

We have tried to see whether the approach of [BF20] can be used to construct other fake projective planes. In the table [BCP11, Table 1], the group C_{14} is the largest possible fundamental group of a $K^2 = 3$ surface coming from a deformation of C_3 quotient of a fake projective plane. The second largest group is C_{13}, which is likely just a coincidence. Still, we hoped to follow the recipe of [BF20] for this case and the current paper is the result of our efforts. We tried to realize the fundamental covers of these $K^2 = 3$ surfaces as complete intersections in some homogeneous space. We were not quite able to do it, instead we constructed them as *almost* complete intersections of the 16 dimensional octonionic projective plane $\mathbb{OP}^2$ in $\mathbb{P}^{26}$ by certain 15 linear equations, equivariant with respect to an order 13 element in the Cartan subgroup of the E_6 group of automorphisms of $\mathbb{OP}^2$.

Afterwards, the process was rather similar to that of [BF20], although there were some technical complications due to 13 being odd. This meant that we could not construct an unramified double covers of our degenerate $K^2 = 3$ surface. As a result, we had more difficulty controlling the size of the coefficients and had to work with $60K$ decimal digit numbers at some intermediate steps.

The main result of out paper can be stated as follows.

Theorem 1. *Let d_1 and d_2 be the roots of*

$$0 = 2187 + 7290d + 23433d^2 + 21640d^3 + 66393d^4 - 21640d^5 + 23433d^6 - 7290d^7 + 2187d^8$$

approximately given by $(d_1, d_2) \approx (1.93 + 2.30\,\mathrm{i}, 0.0125 - 0.515\,\mathrm{i})$. *Consider the surface* X_0 *given by* 26 *quadratic equations* (3) *in* $\mathbb{P}^{11}$ *with a natural* C_{13} *action* $t_i \mapsto \zeta_{13}^i t_i$. *Then this group* C_{13} *acts freely on* X_0 *and the quotient* X_0/C_{13} *has three* A_2 *singularities. This quotient admits a Galois triple cover, which is a fake projective plane denoted by* $(\mathrm{C18}, p = 3, \emptyset, d_3D_3)$ *in* [CS11+].

The paper is organized as follows. In Section 2 we describe our motivation for using the octonionic projective plane and the special linear cuts that achieve our goal of constructing surfaces with $K^2 = 3,\ p_g = q = 0$. We also describe how we found a special element of this family with three A_2 singularities. In Section 3 we briefly explain the construction of the fake projective plane $\mathbb{P}^2_{fake}$, labeled by $(\mathrm{C18}, p = 3, \emptyset, d_3D_3)$ in [CS11+], and state some open problems. An interested reader can access the details of the computations in [BBF20+], mostly in the form of Mathematica files.

Acknowledgements. Our computations relied heavily on Mathematica software package [Math] and to some extent on Julia [Ju], Macaulay2 [Mac] and Magma [Mag]. We thank John Cremona for helping us access the number theory server at the University of Warwick. We thank the anonymous referees for suggestions to improve the exposition. Anders Buch was partially supported by the NSF grant DMS-1503662.

2. Special cuts of the octonionic projective plane

2.1. **Motivation.** As was mentioned in the Introduction, we set out to find a family of surfaces of general type with $K^2 = 3,\ p_g = q = 0$ and fundamental group C_{13}. Here is how this search led us to consider cuts of the octonionic projective plane $\mathbb{OP}^2$.

Let X be the universal cover of a surface in question, with a free action of an order 13 automorphism g. Then $K_X^2 = 39$ and $\chi(K_X) = 13$. It is reasonable to expect that $h^1(X, K_X) = 0$ and $h^0(X, K_X) = 12$. Then the pluricanonical ring $\bigoplus_{n\geq 0} H^0(X, nK_X)$ of X must have the graded dimension

$$\sum_{n\geq 0} \dim H^0(X, nK_X)\, t^n = 1 + 12t + 52t^2 + 130t^3 + \cdots = \frac{1 + 9t + 19t^2 + 9t^3 + t^4}{(1-t)^3}.$$

It is also reasonable to assume that the pluricanonical ring is generated in degree one, so X is embedded into $\mathbb{P}^{11}$. It is also plausible that its image is cut out by $12(12+1)/2 - 52 = 26$ quadrics. By the Holomorphic Lefschetz formula, as in [H11, Theorem 2.1], the trace of the action of g on $H^0(X, K_X)$ is (-1), which means that $H^0(X, K_X)$ has a basis of eigenvectors of g with eigenvalues ζ_{13}^i for $i = 1, \ldots, 12$. Similarly, the action of g on the space of quadrics splits it into 13 two-dimensional eigenspaces.

Inspired by [BF20], we undertook a rather exhaustive computer search for homogeneous varieties of degree 39 and other relevant invariants, but were not successful. However, the octonionic projective plane $\mathbb{OP}^2$ has degree 78, and we observed the following remarkable coincidence: the homogeneous coordinate ring of $\mathbb{OP}^2$ has graded dimension

$$\frac{(1 + 9t + 19t^2 + 9t^3 + t^4)(1+t)}{(1-t)^{17}}.$$

More specifically, $\mathbb{OP}^2$ is the dim 16 singular locus of the E_6-invariant cubic in $\mathbb{P}^{26}$ cut out by the 27 quadratic equations which are the partial derivatives of the cubic. So our idea was to take a linear cut of $\mathbb{OP}^2$ by 15 equations (so that we are in $\mathbb{P}^{11}$) which only drop the dimension by 14. We also want one of the quadratic equations to reduce to zero on the linear cut. The best analogy would be cutting a quadric $xy = zw$ with Hilbert series $\frac{1+t}{(1-t)^3}$ by two linear equations $x = 0$ and $z = 0$ to get a line with the Hilbert series $\frac{1}{(1-t)^2}$ but, ultimately, it was a lucky guess.

2.2. **Octonionic projective plane.** There are several incarnations of the E_6-invariant cubic found in the literature. We used the one in Jacob Lurie's undergraduate thesis [L11], namely

$$\begin{aligned}
&-P_{10}P_{13}P_{16} - P_{11}P_{14}P_{17} - P_{12}P_{15}P_{18} - P_{16}P_{17}P_{18} + P_1P_{10}P_{19} - P_1P_{18}P_2 + P_{11}P_2P_{20} - P_1P_{14}P_{21}\\
&-P_{16}P_{20}P_{21} - P_{18}P_{22}P_{23} - P_{17}P_{19}P_{24} - P_{13}P_2P_{24} - P_{14}P_{15}P_{25} - P_{19}P_{22}P_{25} - P_{12}P_{13}P_{26} - P_{20}P_{23}P_{26}\\
&-P_{10}P_{11}P_{27} - P_{21}P_{24}P_{27} - P_{25}P_{26}P_{27} + P_{12}P_{21}P_3 - P_{10}P_{23}P_3 - P_2P_{25}P_3 - P_{15}P_{20}P_4 + P_{13}P_{22}P_4\\
&-P_{17}P_3P_4 - P_{12}P_{19}P_5 + P_{14}P_{23}P_5 - P_{27}P_4P_5 - P_{11}P_{22}P_6 + P_{15}P_{24}P_6 - P_1P_{26}P_6 - P_{16}P_5P_6\\
&-P_{11}P_{12}P_7 - P_{23}P_{24}P_7 + P_{16}P_{25}P_7 - P_1P_4P_7 - P_{10}P_{15}P_8 - P_{21}P_{22}P_8 + P_{17}P_{26}P_8 - P_2P_5P_8\\
&-P_{13}P_{14}P_9 - P_{19}P_{20}P_9 + P_{18}P_{27}P_9 - P_3P_6P_9 - P_7P_8P_9
\end{aligned}$$

The 27 variables $P_1, \dots, P_{27}$ are indexed by the lines on the Fermat cubic surface in $\mathbb{CP}^3$ and the terms correspond to triples of coplanar lines. The sign prescription is more intricate, given in terms of the C_3 action on the cubic, see [L11]. The octonionic projective plane $\mathbb{OP}^2$ is cut out by the 27 partial derivatives of the above cubic.

There is a Cartan subgroup $(\mathbb{C}^*)^6$ of E_6 that acts diagonally on the variables P_i. We picked an element g of order 13 of it which acts by $P_i \mapsto \zeta_{13}^{a_i} P_i$ with the weights a_i given by

$$(6, 7, 7, 10, 3, 6, 10, 3, 0, 5, 8, 8, 4, 9, 5, 4, 9, 0, 2, 11, 11, 12, 1, 2, 12, 1, 0).$$

As the reader can see, the action of g on the variables has a three-dimensional eigenspace of weight zero and 12 two-dimensional eigenspaces of other weights. For the three invariant variables P_9, P_{18}, P_{27}, the corresponding partial derivatives of the cubic

$$\begin{aligned}
&-P_{13}P_{14} - P_{19}P_{20} + P_{18}P_{27} - P_3P_6 - P_7P_8,\\
&-P_{12}P_{15} - P_{16}P_{17} - P_1P_2 - P_{22}P_{23} + P_{27}P_9,\\
&-P_{10}P_{11} - P_{21}P_{24} - P_{25}P_{26} - P_4P_5 + P_{18}P_9
\end{aligned} \tag{1}$$

involve all of the variables P_i.

At this point, our expectations of the g-action on $H^0(X, K_X)$ indicate that we need to take a linear cut by

$$\begin{aligned}
(&P_9, P_{18}, P_{27}, P_{23} + d_1P_{26}, P_{19} + d_2P_{24}, P_5 + d_3P_8, P_{13} + d_4P_{16}, P_{10} + d_5P_{15}, P_1 + d_6P_6,\\
&P_2 + d_7P_3, P_{11} + d_8P_{12}, P_{14} + d_9P_{17}, P_4 + d_{10}P_7, P_{20} + d_{11}P_{21}, P_{22} + d_{12}P_{25})
\end{aligned}$$

for some constants $d_1, \dots, d_{12}$. Moreover, we want a linear combination of the g-invariant quadrics (1) to vanish on the linear subspace of the cut. In view of the Cartan subgroup symmetry, it is reasonable to pick this linear combination to be the sum of the above quadrics. This gives 6 simple equations on d_i, namely $d_i d_{13-i} = -1$. We have been able to verify by computer at a specific point that the resulting scheme is a smooth surface of degree 13 and is thus a good candidate for our X.

Specifically, the 26 quadrics that cut out X are given by

$$\begin{aligned}
&-t_{10}^2 + d_2t_2t_5 - t_1t_6 - d_2t_{11}t_9,\ t_3^2 + t_2t_4 - d_1t_{12}t_7 + t_{11}t_8,\ -d_1t_1t_5 - d_1t_{12}t_7 + d_2t_{11}t_8 - t_{10}t_9,\\
&-t_{12}t_4 - t_{11}t_5 + t_{10}t_6 - t_7t_9,\ -t_4t_6 - t_3t_7 + d_2t_2t_8 - d_1t_1t_9,\ t_3t_4 + t_2t_5 + t_1t_6 - t_{12}t_8,\\
&d_1t_1t_2 + d_1t_{12}t_4 + t_{10}t_6 - t_8^2,\ -d_2t_{11}t_{12} + t_5^2 + t_3t_7 + t_1t_9,\ d_2t_{11}t_2 - t_{10}t_3 - t_6t_7 + t_4t_9,\\
&t_4^2 - t_3t_5 + d_2t_2t_6 + d_1t_1t_7,\ -t_{12}t_6 + t_{11}t_7 - t_{10}t_8 - t_9^2,\ -d_2t_2t_3 + t_1t_4 + d_2t_{11}t_7 - t_{10}t_8,\\
&t_{10}t_{12} + t_4t_5 - t_2t_7 - t_1t_8,\ d_1t_1t_3 - d_1t_{12}t_5 + d_2t_{11}t_6 - t_8t_9,\ -t_{10}t_{11} + t_3t_5 + t_2t_6 - d_1t_{12}t_9,\\
&-d_2t_{11}^2 + d_1t_{10}t_{12} - t_4t_5 + t_3t_6,\ d_2t_2^2 + t_1t_3 - t_{10}t_7 - t_8t_9,\ d_1t_1t_{12} + t_6t_7 - t_5t_8 - t_4t_9,\\
&-d_1t_{12}^2 + t_5t_6 + t_3t_8 - t_2t_9,\ d_1t_1^2 - d_2t_{11}t_4 - t_{10}t_5 + t_7t_8,\ -t_{12}t_3 - t_{11}t_4 + t_7t_8 + t_6t_9,\\
&d_1d_2t_{12}t_2 - d_2t_{11}t_3 - t_{10}t_4 - t_6t_8,\ -t_1t_{11} - t_{10}t_2 + t_5t_7 - t_3t_9,\ d_1t_1t_{10} + t_5t_6 + t_4t_7 + d_2t_2t_9,\\
&d_2t_{12}t_2 + t_{10}t_4 - t_7^2 - t_5t_9,\ d_1t_1t_{11} + t_6^2 + t_4t_8 + t_3t_9
\end{aligned} \tag{2}$$

in the homogeneous coordinates $(t_1 : \ldots : t_{12})$ of $\mathbb{P}^{11}$. The action of g is $t_i \mapsto \zeta_{13}^i t_i$.

Remark 2.1. *The action of the Cartan subgroup of E_6 reduces the dimension of the space of parameters d from six to two (taking into account the need to preserve the invariant quadric that has to vanish on the cut reduces E_6 to F_4). We expect the total family to have dimension four, but it is not clear how one can build it. What makes the elements above special is that these surfaces X admit*

an additional C_3 symmetry that extends the C_{13} action to the semidirect product of these two groups. Namely, by scaling the variables (but still calling them t_i) we could rewrite the equations (2) *as*
(3)

$$\begin{aligned}
&-t_{10}^2-d_1d_2^2(t_2t_5+t_1t_6-t_{11}t_9),\ d_1d_2^2t_3^2+t_2t_4+t_{12}t_7-t_{11}t_8,\ d_1d_2^2t_1t_5+t_{12}t_7-d_2(t_{11}t_8+t_{10}t_9),\\
&-t_{12}t_4+d_1d_2(-t_{11}t_5+t_{10}t_6+d_2t_7t_9),\ t_4t_6+d_2(-t_3t_7-t_2t_8+d_1d_2t_1t_9),\ d_1d_2(t_3t_4-t_2t_5+d_2t_1t_6)-t_{12}t_8,\\
&d_1d_2^2t_1t_2+t_{12}t_4+d_2t_{10}t_6-d_2t_8^2,\ t_{11}t_{12}+d_1d_2(t_5^2-t_3t_7+d_2t_1t_9),\ -t_{11}t_2-t_{10}t_3+t_6t_7+t_4t_9,\\
&t_4^2+d_1d_2^2(t_3t_5+t_2t_6-t_1t_7),\ -t_{12}t_6+t_{11}t_7-t_{10}t_8-d_1d_2^2t_9^2,\ -d_1d_2^2t_2t_3+d_2t_1t_4+d_2t_{11}t_7-t_{10}t_8,\\
&t_{10}t_{12}-d_1d_2(t_4t_5-t_2t_7+d_2t_1t_8),\ d_1d_2^2t_1t_3+t_{12}t_5-d_2(t_{11}t_6+t_8t_9),\ t_{10}t_{11}-d_1d_2(d_2t_3t_5-t_2t_6+t_{12}t_9),\\
&-d_2t_{11}^2+t_{10}t_{12}+d_2t_4t_5+d_1d_2^2t_3t_6,\ t_{10}t_7+d_1d_2(t_2^2+d_2t_1t_3-t_8t_9),\ t_1t_{12}-t_6t_7+t_5t_8-t_4t_9,\\
&-t_{12}^2-d_1d_2^2(t_5t_6-t_3t_8+t_2t_9),\ d_1d_2^2t_1^2+t_{11}t_4+t_{10}t_5-t_7t_8,\ t_{11}t_4-d_2(t_{12}t_3+t_7t_8)+d_1d_2^2t_6t_9,\\
&-t_{10}t_4+d_1d_2(t_{12}t_2+d_2t_{11}t_3-t_6t_8),\ d_2t_1t_{11}-t_{10}t_2+d_2t_5t_7-d_1d_2^2t_3t_9,\ -t_4t_7+d_1d_2(t_1t_{10}-t_5t_6+d_2t_2t_9),\\
&d_2t_{12}t_2+t_{10}t_4-d_2t_7^2+d_1d_2^2t_5t_9,\ t_4t_8+d_1d_2(-t_1t_{11}+t_6^2+d_2t_3t_9)
\end{aligned}$$

with the additional symmetry $t_i\mapsto t_{3i \bmod 13}$. The details are in [BBF20+, Section2.nb].

2.3. **Constructing a cut with A_2 singularities.** Our method of constructing a fake projective plane largely followed the blueprint of [BF20].

We set $d_3=d_4=d_5=d_6=1$ and tried to find out which (d_1,d_2) give singular cuts. In order to achieve this, we worked on an affine coordinate chart of $\mathbb{OP}^2$ which can be obtained by solving the equations of $\mathbb{OP}^2$ for eleven of the variables as follows.

$$\begin{aligned}
&P_4=P_{10}P_{16}+P_2P_{24}+P_{12}P_{26}+P_{14}P_9, P_6=-P_{14}P_{17}+P_2P_{20}-P_{10}P_{27}-P_{12}P_7,\\
&P_8=-P_1P_{14}-P_{16}P_{20}-P_{24}P_{27}+P_{12}P_3, P_{11}=P_{15}P_{24}-P_1P_{26}-P_{16}P_5-P_3P_9,\\
&P_{13}=P_{15}P_{20}+P_{17}P_3+P_{27}P_5+P_1P_7, P_{18}=-P_{20}P_{26}-P_{10}P_3+P_{14}P_5-P_{24}P_7,\\
&P_{19}=-P_{14}P_{15}-P_{26}P_{27}-P_2P_3+P_{16}P_7, P_{21}=-P_{10}P_{15}+P_{17}P_{26}-P_2P_5-P_7P_9,\\
&P_{22}=1, P_{23}=-P_{12}P_{15}-P_{16}P_{17}-P_1P_2+P_{27}P_9, P_{25}=P_1P_{10}-P_{17}P_{24}-P_{12}P_5-P_{20}P_9
\end{aligned}$$

We obtained this chart by connecting the formulas for the Cartan cubic from [GE96] and [L11]. We then further solved for five of the variables to reduce their number while still keeping the equations relatively short. Then we looked for tangent vectors for the surfaces with $d_1=1$ that lie in a codimension three subspace, by a multivariable Newton method starting at random points. The idea is that some of these would happen at values of d_2 where the surface $X=X_{1,d_2}$ acquires a node. After some trial and error we saw that solutions to

$$-27-34d_2-397d_2^2-172d_2^3-821d_2^4+190d_2^5-83d_2^6+16d_2^7=0$$

give singular surfaces. As in [BF20], we then perturbed d_1 slightly to $1+10^{-20}$ to find a nearby point on the locus of singular surfaces. This lead us to conjecture that generic points (d_1,d_2) on the curve

$$\begin{aligned}
0=&-4d_1^3+8d_1^4-4d_1^5-12d_1^2d_2-16d_1^3d_2+28d_1^4d_2-39d_1^5d_2+12d_1^6d_2-12d_1d_2^2-28d_1^2d_2^2-54d_1^3d_2^2\\
&+78d_1^4d_2^2-34d_1^5d_2^2+28d_1^6d_2^2-12d_1^7d_2^2-4d_2^3-39d_1d_2^3-34d_1^2d_2^3-277d_1^3d_2^3+192d_1^4d_2^3-277d_1^5d_2^3\\
&+54d_1^6d_2^3-16d_1^7d_2^3+4d_1^8d_2^3-8d_2^4-28d_1d_2^4-78d_1^2d_2^4-192d_1^3d_2^4+192d_1^5d_2^4-78d_1^6d_2^4+28d_1^7d_2^4\\
&-8d_1^8d_2^4-4d_2^5-16d_1d_2^5-54d_1^2d_2^5-277d_1^3d_2^5-192d_1^4d_2^5-277d_1^5d_2^5+34d_1^6d_2^5-39d_1^7d_2^5+4d_1^8d_2^5+12d_1d_2^6\\
&+28d_1^2d_2^6+34d_1^3d_2^6+78d_1^4d_2^6+54d_1^5d_2^6-28d_1^6d_2^6+12d_1^7d_2^6-12d_1^2d_2^7-39d_1^3d_2^7-28d_1^4d_2^7-16d_1^5d_2^7\\
&+12d_1^6d_2^7+4d_1^3d_2^8+8d_1^4d_2^8+4d_1^5d_2^8
\end{aligned}$$

give nodal X_{d_1,d_2}.

We then looked for singular points of this curve. There were several such points, one of which was a cusp of the curve. We focused our attention on it and discovered a surface X_{d_1,d_2} with 39 A_2 singularities. Specifically, both d_1 and d_2 can be given as roots of

$$0=2187+7290d+23433d^2+21640d^3+66393d^4-21640d^5+23433d^6-7290d^7+2187d^8$$

approximately given by $(d_1,d_2)\approx(1.93+2.30\,\mathrm{i}, 0.0125-0.515\,\mathrm{i})$. Of course, the same is true for all of the Galois conjugates of this pair. From now on we will call this surface X_0.

We observed that four of the Galois conjugate pairs of (d_1, d_2) give isomorphic surfaces. To see that, we noticed that scheme X_{d_1,d_2} cut out by (3) is isomorphic to $X_{-1/d_2,d_1}$ under the coordinate change

(4) $\quad (t_1, \ldots, t_{12}) \mapsto (d_2t_5, t_{10}, d_2t_2, -d_1t_7, t_{12}, t_4, -d_1d_2t_9, -d_1d_2t_1, d_2t_6, -d_1t_{11}, -d_1d_2t_3, -d_1t_8)$

The idea behind it was to use $i \to 5i \bmod 13$, and we heavily relied on Mathematica computations, see [BBF20+, Section2.nb].

We then used the symmetry (4) to average the C_{13}-invariants of the coordinate ring of the surface X_0 suspected to have A_2 singularities, to get X_0/C_{13} defined in the 13-dimensional weighted projective space $W\mathbb{P}(2^4, 3^{10})$ by 9 equations of degree five and 29 equations of degree six. Due to the above symmetrization, the coefficients were in the field $\mathbb{Q}(\sqrt{-2})$.

2.4. **Finding singular points.** It was not entirely trivial to find the singular points of X_0/C_{13}. We did it by calculating a degree 12 equation in the first four variables which gives a (non-normal) image of X_0/C_{13} in $\mathbb{P}^3$. Then we looked for its curves of singularities by finding multiple singular points on random hyperplane cuts. Then we have looked for singular points outside of the curve of singularities, and indeed hit upon A_2 singularities. We were then able to verify that these were the only singularities by computing the degree of the singular locus over a finite field. As in the case of [BF20], the A_2 singularities were not defined over the quadratic extension of $\mathbb{Q}$, but a coordinate change gave us a model of $X_0/C_{13} \subset W\mathbb{P}^{13}$ still defined over $\mathbb{Q}(\sqrt{-2})$ and with three singular points defined over $\mathbb{Q}$.

3. Constructing the fake projective plane

3.1. **Constructing the triple cover.** By the work of Keum [Ke12], the surface X_0/C_{13} admits a Galois triple cover which is a fake projective plane. In this, it is very similar to the situation in [BF20] and we employed the same general method. It was useful that in both cases there was an additional order three automorphism σ because the FPP had a $C_3 \times C_3$ group of automorphisms.

In order to construct a smooth Galois triple cover of X_0/C_{13} we need to find Weyl divisors on X_0/C_{13} which are not Cartier. The smallest effective Weyl divisors are images of elements of the linear system $|4H|$ on the fake projective plane which are eigenvectors for the covering action, where H is an ample generator of the Picard of $\mathbb{P}^2_{fake}$ such that $3H = K$. These images D_1, D_2 and D_3 are further permuted by the order three automorphism σ of X_0/C_{13}. The key idea is that $3D_1$, $3D_2$, $3D_3$ and $D_1 + D_2 + D_3$ are Cartier divisors on X_0/C_{13}, in fact they are zeroes of sections of $4K_{X_0/C_{13}}$, which we denote by f, $\sigma(f)$, $\sigma^2(f)$ and d respectively. After scaling, these sections satisfy

$$f\,\sigma(f)\sigma^2(f) = d^3$$

and one can try to find them by viewing the above as a system of cubic equations on the coefficients of f and d. Moreover f and d should have certain behavior on the exceptional lines at the blowup of A_2 singularities, namely the proper preimage of $f = 0$ should intersect one of the exceptional lines of the blowup at a triple point, while the proper preimage of $d = 0$ intersects it once at the same point. We refer the reader to [BF20] for details.

The nature of X_0/C_{13} made the computations more challenging. In particular, at some point we had to work with random points on the surface computed with 6×10^4 digits of accuracy. The equations for f and d had coefficients in $\mathbb{Q}(\sqrt{-2})$ which were about 1.5×10^4 digits long. As in [BF20], we solved it over a finite field of 19 elements, but now we used a p-adic version of the Newton's method to quickly gain the needed accuracy.

After finding f and d, we added $(\sigma(f)/f)^{\frac{1}{3}}$ to the field of rational functions of X_0/C_{13} and computed the sections of the bicanonical linear system in terms of appropriate linear systems on X_0/C_{13}. Once the triple cover $\mathbb{P}^2_{fake}$ was constructed, we used the fixed points of the automorphisms of $\mathbb{P}^2_{fake}$ to get a basis with nicer equations, only about 100 digits long coefficients, see [BBF20+, Section3.nb] for details.

We checked that the Galois triple cover of X_0/C_{13} is a fake projective plane by essentially the same methods as in [BK19], with minor innovations. As before, to verify smoothness of the surface cut out by these equations, we looked at the reduction modulo a prime. Then in theory one wants to argue that size 7 minors of the 10×84 Jacobian matrix have no common zeroes on the surface. Of course, it is impossible to even store all of these minors, so the idea was to take some sufficiently general linear combinations thereof. Namely, we multiplied the Jacobian matrix on the left and on the right by random invertible matrices modulo the prime in question and took the leading minor of the matrix. We repeated it three times and saw that these three elements of the Jacobian ideal, together with the original ideal have trivial Hilbert polynomial. This computation was done in Magma, see [BBF20+, CheckSmoothness] for details.

To pay homage to the theorem-proof style of exposition we will present the proof of Theorem 1 from the Introduction. Note that it is based heavily on computer calculations.

Proof. (of Theorem 1) This $\mathbb{P}^2_{fake}$ is labeled by $(\mathrm{C18}, p = 3, \emptyset, d_3D_3)$ in the classification of [CS11+], since it is the only one with an automorphism group that contains $(C_3)^2$ and Picard group that contains C_{13}. □

3.2. **Open questions.** Let us now discuss open problems related to this construction.

The first question is how to verify that the special cuts X_{d_1,d_2} of $\mathbb{OP}^2$ are simply connected, which we strongly suspect. This would imply that $\pi_1(X_{d_1,d_2}/C_{13}) = C_{13}$, as opposed to only having π_1 with a C_{13} quotient. It might perhaps follow from our construction and [CS11+], but a more direct argument is desirable. Unfortunately, since these are not complete intersections, the Lefschetz Hyperplane theorem can not be applied, so other methods are needed.

A related question is how to construct non-C_3-invariant deformations of X_{d_1,d_2}. It looks like they will no longer be cuts of $\mathbb{OP}^2$ but perhaps one can get them by carefully examining the equations (2). This has no bearing on the topic of fake projective planes, but these surfaces would be of independent interest.

The quotient of the fake projective plane $(\mathrm{C18}, p = 3, \emptyset, d_3D_3)$ by $(C_3)^2$ is also covered by $(\mathrm{C18}, p = 3, \{2I\})$. This fake projective plane in turn covers a surface $(\mathrm{C18}, p = 3, \{2\})$, which is covered by three other fake projective planes. The method of [BF20] is, unfortunately, not quite applicable here, so how do we find (the equations of) these other surfaces?

It is known from [CS11+] that $(\mathrm{C18}, p = 3, \emptyset, d_3D_3)$ has Picard group $C_2 \times C_2 \times C_{13}$. While the C_{13} part can be inferred from our construction (even though it may not be entirely trivial to follow), the other two factors are mysterious. It would be interesting to see them explicitly, and they may be useful in answering both the previous and the next questions.

A perennial question is how one can reduce the size of the coefficients in the equations of a fake projective plane. There are currently only ad hoc tools that are not very successful, except in [BK19] case.

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