On the ring of inertial endomorphisms of an abelian p-group

Ulderico Dardano* - Silvana Rinauro†

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Abstract. An endomorphisms φ of a group G is said inertial if $\forall H \leq G$ $|\varphi(H) : (H \cap \varphi(H))| < \infty$. Here we study the ring of inertial endomorphisms of an abelian torsion group and the group of its units. Also the case of vector spaces is considered. [1]
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1 Introduction and statement of main results

Recently there has been interest for inert subgroups of groups (see [9], [4], [5], for example). A subgroup is said inert if it is commensurable to each conjugate of its. Here we consider *inertial endomorphims*, that is endomorphims mapping setwise subgroups to commensurable ones.

More precisely, if φ is an endomorphism of an abelian group A (from now on always in additive notation) we say:
(RIN) φ is right-inertial iff $\forall H \leq A$ $|\varphi(H) : (H \cap \varphi(H))| < \infty$,
(LIN) φ is left-inertial iff $\forall H \leq A$ $|H : (H \cap \varphi(H))| < \infty$.

In [2] we considered automorphisms of abelian group A and showed that in this case (RIN) and (LIN) are equivalent, when A is periodic. This generalized previous results from [1] and [6]. On the other hand, in [5] authors

*dardano@unina.it
†silvana.rinauro@unibas.it
[1]These results have been featured in invited talks at the Dept's of Mathematics at Universities of Firenze and Padova, I.

consider (RIN) only, which seems to be more adequate for non-invertible maps. Moreover, if A is periodic (LIN) implies (RIN), see Theorem 1. Let us call RIN-endomorphims *inertial*.

Fact *Inertial endomorphisms of any abelian group A fill a subring $\mathcal{I}End(A)$ of the full ring $End(A)$ of endomorphisms of A.*

Clearly $\mathcal{I}End(A)$ contains the ideal $FEnd(A)$ of endomorphisms with finite image and the subring $PEnd(A)$ of power endomorphisms (say *multiplications*) of A.

Here we have a characterization of inertial endomorphisms of torsion abelian groups.

Theorem 1 *Let A be an abelian periodic group and $\varphi \in \mathrm{End}(A)$. Then φ is inertial iff there is a finite index subgroup $B = D \oplus E \oplus L$ of A such that:*
i) $D \oplus E$ and L are coprime,
ii) D is divisible with finite total rank and E has finite exponent,
iii) φ is power on D, E and L .

Thus φ is inertial iff :

$$(FS) \qquad \exists n\ \forall H \leq A\ |H^{(\varphi)}/H_{(\varphi)}| \leq n.$$

Moreover, φ is LIN iff it is inertial and there are subgroups $B,\ D,\ E,\ L$ as above such that φ is non-zero on D and invertible on E and L

Thus the picture of $\mathcal{I}End(A)$, when A is periodic, can be described in Corollary 1. Note that *an endomorphism of an abelian torsion group is inertial iff it is such on all primary components and multiplication on all but finitely many of them.* Notice also that from Theorem 1 it follows that inertial endomorphisms of an abelian p-group are *elementary*, in the sense they act as a multiplication on a finite index subgroup (they are close to be multiplications, see later), unless *the maximum divisible subgroup $D := div(A)$ of A is non-trivial and has finite rank while A/D has infinite rank and finite exponent.* For short, say that such an A is *critical.* To describe the ring $\mathcal{I}End(A)$ we also need consideration of the, say, *essential exponent* $eexp(A)$ of an infinite p-group A (with finite exponent) , that is the smallest power p^e such that p^eA is finite or, equivalently, the maximum p^e such that $A[p^e]/A[p^{e-1}]$ is infinite. Clearly, the above e is the least finite Ulm-Kaplansky invariant of A. Denote by $\mathcal{J}_p$ the ring of p-adics. For terminology and elementary facts see [7] and [8].

Corollary 1 *Let A be an abelian p-group and $D := div(A)$. Then*
1) If A is non-critical:

$$\mathcal{I}End(A) = PEnd(A) + FEnd(A)$$

and, according to $exp(A) = \infty$ or $exp(A) = p^m$ and $p^e = eexp(A)$, we have

$$PEnd(A) \cap FEnd(A) = 0 \text{ or } = p^e PEnd(A) \simeq p^e\mathbb{Z}/p^m\mathbb{Z}$$

$$\frac{\mathcal{I}End(A)}{FEnd(A)} \simeq \quad \mathcal{J}_p \text{ or } \mathbb{Z}/p^e\mathbb{Z}.$$

2) If A is critical, $p^m := exp(A/D)$ and $p^e := eexp(A/D)$:

$$\mathcal{I}End(A) = PEnd(A) \oplus (FEnd(A) + R)$$

where $\quad R \simeq PEnd(A/D) \simeq \mathbb{Z}(p^m)$ *and* $\quad FEnd(A) \cap R = p^e R.$ *Moreover*

$$\frac{\mathcal{I}End(A)}{FEnd(A)} \simeq \mathbb{Z}(p^e) \oplus \mathcal{J}_p.$$

Concerning invertible inertial endomorphisms of a periodic abelian group, note that these fill a group $\mathcal{I}Aut(A)$. Theorem 1 lead us to the consideration of the normal subgroup filled by the so called *finitary* automorphisms, that is $FAut(A) := \{\gamma \in \mathrm{Aut}(A) \mid [A, \gamma] \ is \ finite\}$, and the group $PAut(A)$ of invertible multiplications.

Corollary 2 *Let A be an abelian p-group and $D := div(A)$. Then*
1) If A is non-critical,

$$\mathcal{I}Aut(A) = PAut(A) \cdot FAut(A)$$

where $PAut(A) \cap FAut(A) = 1$ if $exp(A) = \infty$.
Otherwise, if $p^m := exp(A)$ and $p^e := eexp(A)$, we have

$$PAut(A) \cap FAut(A) = \{x \mapsto rx \mid r \equiv 1 \, mod \, p^e\} \simeq \{\bar{r} \in \mathbb{Z}(p^m) \mid r \equiv 1 \, mod \, p^e\}.$$

2) If A is critical, $p^m := exp(A/D)$ and $p^e := eexp(A/D)$,

$$\mathcal{I}Aut(A) = PAut(A) \times (FAutA \cdot \Gamma)$$

with $FAutA \cdot \Gamma = \{\varphi \in \mathcal{I}Aut(A) \,|\, \varphi_{|D} = 1\}$, $\Gamma \simeq \mathcal{U}(\mathbb{Z}(p^m))$ and

$$FAut(A) \cap \Gamma \simeq \{\bar{r} \in \mathbb{Z}(p^m) \mid r \equiv 1 \, mod \, p^e\}.$$

One may ask a similar question about vector spaces and get a similar picture, without critical case. Let V be a K-vector space and denote by $FEnd(V)$ the ring of K-linear maps which are finitary, that is have image with finite dim. Note that these are precisely the linear maps acting as the zero-map on a finite codimension subspace.

Theorem 2 *Let φ be an endomorphism of an infinite dimension K-vector space A. Then $dim(\varphi(H)/(\varphi(H) \cap H)) < \infty$ for each K-subspace H of V iff φ acts as a scalar multiplication on a finite codimension subspace.*

Therefore the above endomorphisms fill the following subring of $End(V)$:

$$\bar{K} \oplus FEnd(V)$$

where $\bar{K}$ is the field of scalar multiplication and $FEnd(V)$ is the ideal of endomorphisms whose image has finite dimension.

On the other hand, $H \cap \varphi(H)$ has finite codimension in H for each K-subspace H of V it iff φ acts as a scalar non-zero multiplication on a finite codimension subspace. Thus such a φ has the above property as well.

2 Proofs

We first prove the above stated Fact. The corresponding statement for vector spaces has a similar proof and we omit it.

Proposition 1 *1) If φ and ψ are LIN-endomorphism (risp. RIN) of any group G. Then $\varphi\psi$ is LIN (risp. RIN).*
2) RIN-endomorphism of an abelian group A fill a subring $\mathcal{I}End(A)$ of $End(A)$, containing the ideal $FEnd(A)$ of endomorphism with finite image.

Proof. 1) If H is any subgroup of G then from $|H/(H \cap \varphi(H))| < \infty$ it follows $|(H \cap \psi(H))/(H \cap \psi(H) \cap \varphi\psi(H))| \leq |\psi(H)/(\psi(H) \cap \varphi\psi(H))| < \infty$.

2) If φ and ψ both have RIN, then $|(H + \varphi(H))/H| < \infty$ and $|(H + \varphi(H) + \psi(H) + \varphi\psi(H))/(H + \varphi(H)| < \infty$. □

Now we prove Theorem 2, which will serve also for the proof of Theorem 1 in the case A of prime exponent. For a subset X of V and $\varphi \in End(V)$, we denote respectively by $\langle X \rangle = KX$ and $X^{(\varphi)} = K[\varphi]X$ the K-subspace and the $K[\varphi]$-submodule of V spanned by X.

Proof. By contradiction, assume φ is multiplication on no quotient space. We claim that: for all finite dimension subspaces $X \leq A$ such that $X \cap \varphi(X) = 0$ there exists a subspace $X' > X$ with finite dim such that

$$X' \cap \varphi(X') = 0 \text{ and } \varphi(X') > \varphi(X).$$

Therefore, starting at $X_0 = 0$, by transfinite recursion we define $X_{i+1} := X'_i$ and $X_\omega := \cup_i X_i$. We get that both X_ω and $\varphi(X_\omega)$ have infinite dimension and $X_\omega \cap \varphi(X_\omega) = 0$, a contradiction.

To prove the claim, we first prove that if $a \in V$, then

$$dim(Ka^{(\varphi)}) < \infty.$$

This is true as we can consider the natural epimorphism

$$F : K[x] \mapsto Ka^{(\varphi)}$$

mapping 1 to a and x to $\varphi(a)$. If F is injective, we can replace V by $K[x]$ and φ by multiplication by x. If $H := K[x^2]$, then both H and $\varphi(H) = xH$ are infinite dim, while $H \cap xH = 0$, a contradiction. Therefore $(Ka)^{(\varphi)} = im(F)$ has finite dim and the same holds for $Z = X^{(\varphi)}$.

Since φ does not act as a scalar multiplication on A/Z, we can choose $a \in V$ such that $\varphi(a) \notin \langle a, Z\rangle$ and define $X' := \langle a\rangle + X$. If now $y \in X' \cap \varphi(X')$, then $\exists n, s \in K$, $\exists x, x_0 \in X$ such that $y = na + x = s\varphi(a) + \varphi(x_0)$. Thus $s\varphi(a) \in \langle a\rangle + Z$ while $\varphi(a) \notin \langle a\rangle + Z$. Therefore $s = 0$ and $na = 0$ as well. It follows $y = x = \varphi(x_0) \in X \cap \varphi(X) = 0$, as claimed.

Finally, we have seen that if for each K-subspace H, $H \cap \varphi(H)$ has finite codimension in H then φ is multiplication on a finite codim subspace B of A. In particular $\varphi(B) \neq 0$. $\square$

Recall that $\varphi \in \mathrm{End}(A)$ is said to be power or multiplication iff
(PW) $\forall H \leq A \;\; \varphi(H) \subseteq H$,
and that PW endomorphisms of an abelian p-group are *locally universal* that is have form $x \mapsto \alpha_p x$ for a p-adic α_p, when A is a p-group. Also, if $C \leq B \leq A$, we say that φ is PW on B/C iff $C \leq H \leq B$ implies $H^\varphi \subseteq H$.

We say that endomorphisms φ_1 and $\varphi_2 \in \mathrm{End}(A)$ are **close** iff the image of $\varphi_1 - \varphi_2$ is finite, that is they act the same way on a finite index subgroup or -equivalently- modulo a finite order subgroup. This is the congruence in $End(A)$ whose kernel is the ideal $FEnd(A)$ of endomorphisms with finite image. An endomorphism which is close to a (RIN) (resp. LIN) one remains such, clearly. We *say that an endomorphisms is PF iff it is close to a multiplication*. Let us sum up basic facts.

Proposition 2 *PF-endomorphisms of an abelian group A fill a subring of $End(A)$,*

$$PEnd(A) + FEnd(A)$$

where the sum is direct, provided $exp(A) = \infty$. Otherwise, if A is a p-group with $p^m = exp(A) < \infty$ and $p^e = eexp(A)$, there is a natural ring isomorphisms

$$PEnd(A) \cap FEnd(A) \simeq p^e\mathbb{Z}/p^n\mathbb{Z}.$$

Moreover, if φ is PF, then

$$(FS) \qquad \exists n\ \forall X \le A\ \ |X^{(\varphi)}/X_{(\varphi)}| \le n. \qquad \square$$

Here by $X^{(\varphi)}$ (resp. $X_{(\varphi)}$) we mean the smallest (resp. largest) φ-(invariant) subgroup of A containing X (resp. contained in X).

Proof. This is quite elementary. If φ acts as $\alpha \in PEnd(A)$ on $B \le A$ with $|A : B| < \infty$, then $\varphi - \alpha \in FEnd(A)$. Moreover if $C := ker\ (\varphi - \alpha)$, we have that for each $X \le A$ it holds $(X \cap B) \le X_{(\varphi)}$ and $X^{(\varphi)} \le (X + C)$. Thus $|X^{(\varphi)}/X_{(\varphi)}| \le |A/B| \cdot |C| \le |A/B|^2$.

If $0 \ne \alpha \in PEnd(A) \cap FEnd(A)$ we have that exists i such that $ker\ \alpha = A[p^i]$ (clearly p^i is the maximal power of p dividing α). If $A[p^i]$ has finite index in A, then $exp(A) < \infty$ and $e \ge i$. Conversely, if p^e divides α it is plain that $\alpha \in FEnd(A)$. $\square$

Let us now have a look at PW-endomorphisms which are LIN too. Recall that an abelian group A with the *minimal condition* (Min) is just a group with shape $A = F \oplus D$, where F is finite and D is divisible with finite total rank.

Proposition 3 *Let φ be a PW endomorphism of an abelian periodic group. Then φ is LIN iff $A = A_\pi \oplus A_{\pi'}$ coprime summands where A_π has Min and $\varphi_{|A'_\pi}$ is invertible.*

Proof. Assume φ is PW and LIN and let π be the set of primes p such that φ is not invertible on A_p. Then π is finite. Now p divides φ_p for any $p \in \pi$, and hence $\varphi(A[p]) = 0$. It follows that A_p has Min and so A_π has Min as well. Conversely, if $A = A_\pi \oplus A_{\pi'}$ coprime summands where A_π has Min and $\varphi_{|A'_\pi}$ is invertible, then for any $H \leq A$ the quotient $H/\varphi(H)$ is finite, as it has finite rank and exponent, and φ is LIN. $\square$

We prove now a Lemma which extends a result due to D.Robinson [9].

Lemma 1 *Let $a \in A$ be an abelian p-group and $\varphi \in \mathrm{End}(A)$.*
(1) If φ either LIN or RIN, then the torsion subgroup of the φ-submodule $\langle a\rangle^{(\varphi)}$ of A generated by a is finite.
(2) If $|X/X_{(\varphi)}| < \infty$ for all $X \leq A$, then $|X^{(\varphi)}/X| < \infty$ for all $X \leq A$.
(3) If $|X/X_{(\varphi)}| \leq p^m$ for all $X \leq A$, then $|X^{(\varphi)}/X| \leq p^{m^2}$ for all $X \leq A$.

Proof. (1) We may assume $A = \langle a\rangle^{(\varphi)}$. Suppose first a has order prime p and regard A as $\mathbb{Z}_p[x]$-module (where x acts as φ) and consider the natural epimorphism mapping 1 to a and x to $\varphi(a)$:

$$F : \mathbb{Z}_p[x] \mapsto A.$$

If F is injective, we can replace A by $\mathbb{Z}_p[x]$ and φ by multiplication by x. If $H := \mathbb{Z}_p[x^2]$, then $\varphi(H) = xH$ is infinite, while $H \cap xH = 0$, a contradiction. If now a has order p^ϵ, then A/pA is finite, by the above. Moreover, pA is finite by induction on ϵ.

(2) This can be proved in a similar way as case (3)

(3) We claim that *if $a \in A$ has order p^ϵ, then $|\langle a\rangle^{(\varphi)}| \leq p^{(m+1)\epsilon}$.*
Assume first $\epsilon = 1$, that is a has order p and $A_0 := \langle a\rangle^{(\varphi)}$ is elementary abelian. Suppose, by contradiction, the above F is injective. As above, let $H := \mathbb{Z}_p[x^2]$. Then $H_{(\varphi)} = (g(x^2))$ for some polynomial g. Since $|H/H_{(\varphi)}| = p^m < \infty$, we have $g \neq 0$. Then $(g(x^2)) \not\subseteq H$, a contradiction. Therefore, for some $f \in \mathbb{Z}_p[x]$ with degree say n, we have

$$\frac{\mathbb{Z}_p[x]}{(f)} \simeq_\gamma \langle a\rangle^{(\varphi)} = A_0$$

Thus the minimal φ-invariant subgroups of A_0 correspond $1-1$ to the irreducible monic factors of f, which are at most n. Consider a $\mathbb{Z}_p$-basis X of A containing an element in each subgroup of them. The the hyperplane H of equation $x_1 + x_2 + \cdots + x_n = 0$ has index p in $\langle a\rangle^{(\varphi)}$ and $H_{(\varphi)} = 0$ as $H \cap X = \emptyset$. Therefore $|\langle a\rangle^{(\varphi)}| \leq p^{m+1}$.

If $\epsilon > 1$, by induction $B := \langle p^{\epsilon-1}a\rangle^{(\varphi)}$ has order at most $p^{(m+1)(\epsilon-1)}$ and $\langle a\rangle^{(\gamma)}/B$ has order at most p^{m+1} by case $\epsilon = 1$. Therefore $|a^{(\varphi)}| \le p^{(m+1)\epsilon}$, as claimed.

In the general case let X be any subgroup of A and $X_{(\varphi)} = 0$. Thus $|X| =: p^\epsilon \le p^m$. Write $X = \langle a_1\rangle \oplus \cdots \oplus \langle a_r\rangle$ with a_i of order p^{ϵ_i} and $\epsilon_1 + \cdots + \epsilon_r = \epsilon$. Since $|\langle a_i\rangle^{(\varphi)}| \le p^{(m+1)\epsilon_i}$ by the above, we have $|X^{(\varphi)}| \le p^{(m+1)\epsilon}$. So that $|X^{(\varphi)}/X| \le p^{(m+1)\epsilon-\epsilon} \le p^{m^2}$. □

Lemma 2 *Let $\varphi \in \mathrm{End}(A)$ and $D \le A$ divisible and primary. If φ is either LIN or RIN, then φ is PW on D, that is there is a p-adic α that is $\varphi(a) = \alpha a \;\; \forall a \in D$.*

Proof. Without loss of generality, let D have rank 1. If φ is LIN, then $D \le \varphi(D)$ and thus $D = \varphi(D)$. Therefore in both cases LIN or RIN, we have $\varphi(D) \le D$. Thus φ is PW on D. □

Proof of Theorem 1 We may assume A is a p-group with $D := div(A)$ and note that *if A is an elementary abelian*, the statement follows from Theorem 2.

We claim that for any RIN or LIN-endomorphism φ of any p-group A:

(fs) $\quad \forall H \le A \;\; |H^{(\varphi)}/H_{(\varphi)}| < \infty$. Therefore (LIN) $\Rightarrow$ (RIN).

To this aim we may suppose $H_{(\varphi)} = 0$. Thus, since φ is PW on the divisible radical D of A, (see Lemma 2), we have $D \cap H = 0$ and H is reduced. Moreover, by the elementary abelian case, φ is PW on a subgroup of finite index of $A[p]$, we get that $H[p]$ is finite. It follows that H is finite. Then (fs) holds by Lemma 1.

Let now A be any residually finite abelian p-group and assume, by contradiction, that φ acts as a multiplication on no quotient with finite kernel. As in the proof argument of Theorem 2, we note that if φ is LIN (resp. RIN), then *there is no sequence of subgroups X_i with the property* that if we denote $Y_i := X_i \cap \varphi(X_i)$ then we have:

(1) $Y_{i+1} \cap X_i = Y_i$

(2) the sequence $|X_i/Y_i|$ (resp. $|\varphi(X_i)/Y_i|$) is *strictly increasing.*

This is true since otherwise there would exists a subgroup $X_\omega := \cup_i X_i$ with the properties that $|X_\omega/X_\omega \cap \varphi(X_\omega)| \ge |X_i/Y_i| \ge i$ (resp. $|\varphi(X_\omega)/X_\omega \cap \varphi(X_\omega)| \ge |\varphi(X_i)/Y_i| \ge i$) for each i. On the other hand, we will construct now a prohibited sequence X_i, a contradiction. Let X be any finite subgroup

of A. As above the subgroup $K := X^{(\varphi)}$ is finite by Lemma 1. By (fs) there is a φ-subgroup B with finite index such that $B \cap K = 0$. Now, as φ is not PW on $(B+K)/K$, there is $a \in B$ such that $\varphi(a) \notin \langle a, K\rangle$. Let $X' := \langle a\rangle + X$ and $Y' := X' \cap \varphi(X')$. Let us check that
(1) $X \cap Y' = Y$;
(2′) $X' > X + Y'$ and $\varphi(X') > \varphi(X) + Y'$.
In fact, on one hand we have $X \cap Y' = Y$, as if $x \in X \cap Y'$ then $x = s\varphi(a) + \varphi(x_0)$ with $s \in \mathbb{Z}$, $x_0 \in X$ and $s\varphi(a) = x - \varphi(x_0) \in B \cap K = 0$, hence $x = \varphi(x_0) \in Y$ and (1) holds. On the other hand $Y' \leq \langle pa\rangle + Y \not\ni a$ and $Y' \leq \langle p\varphi(a)\rangle + Y \not\ni \varphi(a)$. Indeed if $y' \in Y' = X' \cap \varphi(X')$, then $\exists n, s \in \mathbb{Z}$, $\exists x, x_0 \in X$ such that $y' = na + x = s\varphi(a) + \varphi(x_0)$ where $na = s\varphi(a)$ and $x = \varphi(x_0) \in Y := X \cap \varphi(X)$. It follows that p divides s, hence p divides n as well. Then (2′) holds.

Thus we can define by induction a prohibited sequence as above, since from (1) and (2′) it follows $|X'/Y'| > |X/Y|$ and $|\varphi(X')/Y'| > |\varphi(X)/Y|$.

Let now A be any reduced p-group and let R be a basic subgroup. By (fs), $R^{(\varphi)}/R$ is finite and so $H := R^{(\varphi)}$ is residually finite as well. Also, A/H is divisible. By the above there are a p-adic $\alpha \in \mathcal{J}_p$ and a finite φ-invariant subgroup C of H such that $\varphi = \alpha$ on H/C. As the kernel K/C of $(\varphi - \alpha)_{|A/C}$ contains H/C and its image is reduced, while A/H is divisible, it is clear that $K = A$ and φ is close to α, as wished.

Finally, *assume A is any p-group* and φ is not close to any multiplication. As (fs) holds, at the expense of substituting A with a finite index φ-subgroup we have $A = D \oplus E$ with D divisible and E reduced and both φ-invariant. Now φ is multiplication on both D (see Lemma 2) and a finite index subgroup of E (see above). So we may also assume φ is power on E. Say $\varphi_{|D} = \alpha_1$ and $\varphi_{|E} = \alpha_2$, where $\alpha_1 \neq \alpha_2$ are p-adics.

If E has finite exponent, we may substitute it by $E[p^e]$ where $p^e = eexp(E)$. By the reduced case, φ is power on a subgroup A' of finite index of $A[p^e]$. Then if D has infinite rank, $\alpha_1 \equiv \alpha_2 \ mod\ p^e$ and φ it is multiplication on $D \oplus (E \cap A')$ which has finite index in A, a contradiction. Thus *D has finite rank.*

If by contradiction E has infinite exponent (and by our assumption $\alpha_1 \neq \alpha_2$), then there is a quotient E/S of its which is a Prüfer group (and infinite). By (fs) we can assume S to be φ-invariant and consider $\bar{A} := A/S$. This a divisible group on which φ acts as a (universal) multiplication by Lemma 2, contradicting $\alpha_1 \neq \alpha_2$.

Finally let us show that (FS) holds. If φ is a multiplication on some $B \leq A$ take $n = |A/B|^2$. In the other (critical) case, observe that $H_0 := (D \cap H) + (E \cap H)$ is φ-invariant and the group $(H \cap B)/H_0$ has exponent $\leq exp(E) =: p^m$ and finite rank $r <$ rank of D. Thus $|H/H_{(\varphi)}| \leq np^{mr}$. Then apply Lemma 1. Conversely, it is plain that (FS) implies that φ is inertial. □

Proof of Corollary 1. Let $\varphi \in \mathcal{I}End(A)$. If A is non-critical, apply Theorem 1 and Proposition 2.

If $A = D \oplus E$ is critical, there is a φ-invariant finite index subgroup $E_1 \leq E$. Let $E_2 := E_1[p^e]$. By the above φ acts as multiplication by r on a finite index subgroup of E_2. For each r we may consider $\bar{r} \in PEnd(A)$ acting as the zero map on D and multiplication by r on E. Let $\alpha \in \mathcal{J}_p$ represent the action of φ in D (which is power by Lemma 2). Thus $\varphi - \alpha - \overline{r - \alpha} \in FEnd(A)$ and so $\mathcal{I}End(A) = PEnd(A) + (FEnd(A) + R)$, where $R = \{\bar{r} \mid r \in \mathbb{Z}\} \simeq \mathbb{Z}(p^m)$. Further, if $\alpha = \varphi_0 + \bar{r} \in PEnd(A) \cap (FEnd(A) + R)$, then α act as $\bar{r}$ on a subgroup with finite index and therefore on D. Thus $\alpha = 0$. To prove $FEnd(A) \cap R = p^e R$ apply Proposition 2 to E. □

Proof of Corollary 2. Let $\gamma \in \mathcal{I}Aut(A)$. Suppose A non-critical. Then, according to Theorem 1, there exists $\alpha \in PEnd(A)$ and a finite index subgroup $B \leq A$ such that $\gamma_B = \alpha$ and $\gamma^{-1}\alpha$ acts on B as the identity map. This guarantees that p does not divide α, which is therefore invertible. Further if $\alpha \in PAut(A) \cap FAut(A)$ then $\alpha_B = 1$ on a finite index subgroup $B \leq A$. Then $\alpha = 1$, provided $exp(A) = \infty$. Otherwise, if α acts as the identity map on a finite index B subgroup of A, then $B \geq A[p^e]$ and $\alpha \equiv 1 \; mod \; p^e$ (see Proposition 2).

Suppose now A is critical. Fix E with finite exponent such that $A = D \oplus E$. Consider $\Gamma := \{\zeta \in \mathcal{I}Aut(A) \mid \zeta_D = 1 \textit{ and } \zeta = \zeta_r \textit{ is power } r \textit{ on } E\}$. By Theorem 1, there exists an invertible p-adic α such that $\gamma_D = \alpha$ and $r \in \mathbb{Z}$ such that $\gamma\alpha^{-1}$ acts by means of power r on a finite index subgroup of E. Also $\bar{r} \in \mathcal{U}(\mathbb{Z}(p^m))$, as p does not divide r. Thus $\gamma\alpha^{-1}\zeta_r^{-1} \in FAut(A)$, as wished. The final part of the statement follows from the above argument. □

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Ulderico Dardano, Dipartimento di Matematica e Applicazioni "R.Caccioppoli", Università di Napoli "Federico II", Via Cintia - Monte S. Angelo, I-80126 Napoli, Italy. `dardano@unina.it`

Silvana Rinauro, Dipartimento di Matematica, Informatica ed Economia, Università della Basilicata, Via dell'Ateneo Lucano 10 - Contrada Macchia Romana, I-85100 Potenza, Italy. `silvana.rinauro@unibas.it`