A SURVEY ON PAIRWISE COMPARISON MATRICES OVER ABELIAN LINEARLY ORDERED GROUPS

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ABSTRACT

In this paper, we provide a survey of our results about the pairwise comparison matrices defined over abelian linearly ordered groups.

Keywords: Abelian linearly ordered groups, pairwise comparison matrix, weighting ranking.

1. Introduction

Let $X = \{x_1, x_2,\dots, x_n\}$ be a set of alternatives or criteria. In a multi-criteria evaluation context, a Decision Maker (DM) may state his/her preferences, for the set X, by means of a preference relation

$$A:(x_i,x_j) \to A(x_i,x_j) = a_{ij} \in R, \qquad \textbf{(1)}$$

where a_{ij} represents the preference intensity of x_i over x_j. The preference relation is represented by the *Pairwise Comparison Matrix* (PCM):

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}. \qquad \textbf{(2)}$$

In literature, several kinds of PCMs have been proposed because the entry a_{ij} may assume different meanings: in multiplicative PCMs it represents a *preference ratio*; in additive PCMs it is a *preference difference*; in fuzzy PCMs it encodes a *preference degree* in [0, 1]. A condition of *reciprocity* is assumed for $A = (a_{ij})$ in such way that the preference of x_i over x_j, expressed by a_{ij}, can be exactly read by means of the element a_{ji}. The shape of the reciprocity condition depends on the different kind of PCM:

$$a_{ji} = \frac{1}{a_{ij}}, \qquad a_{ji} = -a_{ij}, \qquad a_{ji} = 1 - a_{ij}, \qquad \textbf{(3)}$$

for multiplicative, additive and fuzzy PCMs, respectively.

The multiplicative PCMs play a basic role in the Analytic Hierarchy Process (AHP), a procedure developed by T.L. Saaty at the end of the 70s (Saaty, 1977, 1980, 1986). In (Basile and D'Apuzzo, 2002, 2006a, 2006b), (D'Apuzzo, Marcarelli and Squillante, 2007), properties of multiplicative PCMs are

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analyzed in order to determine a qualitative ranking on the set *X* and find ordinal and cardinal evaluation vectors representing the ranking. Additive and fuzzy PCMs are investigated for instance in (Barzilai, 1998) and (Chiclana et al., 2009).

In addition to reciprocity, a condition of *consistency* is considered; it takes the following shapes:

$$a_{ik} = a_{ij}\, a_{jk}, \qquad a_{ik} = a_{ij} + a_{jk}, \qquad a_{ik} = a_{ij} + a_{jk} - 0.5, \tag{4}$$

for multiplicative, additive and fuzzy PCMs, respectively. It is satisfied if and only if there exists a vector $\underline{w} = (w_1, ..., w_n)$ that returns the entries of PCM in (2) as follows:

$$\frac{w_i}{w_j} = a_{ij}, \qquad w_i - w_j = a_{ij}, \qquad w_i - w_j + 0.5 = a_{ij}. \tag{5}$$

Vectors satisfying (5) are called *consistent vectors* and the vector of the weights for the alternatives has to be chosen among them. The last equality in (4) is called *additive fuzzy consistency*. Whenever the elements of the PCM belong to]0,1[, for fuzzy PCMs a *multiplicative fuzzy consistency* is also proposed (Chiclana et al., 2009), (Tanino, 1984):

$$a_{ik} = \frac{a_{ij}\, a_{jk}}{a_{ij}\, a_{jk} + (1 - a_{ij})(1 - a_{jk})}. \tag{6}$$

In (Cavallo and D'Apuzzo, 2009a), we prove that (6) is verified if and only if there is a consistent vector such that:

$$\frac{w_i(1 - w_j)}{w_i(1 - w_j) + w_j(1 - w_i)} = a_{ij}. \tag{7}$$

In the case of a multiplicative PCM, Saaty suggests that the comparisons expressed in verbal terms have to be translated into preference ratios a_{ij} taking value in $S = \left\{ \frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1, 2, 3, 4, 5, 6, 7, 8, 9 \right\}$.

Let us stress that, by assuming consistency properties in (4), the assumption of the Saaty scale for multiplicative PCMs, or [0,1] for fuzzy PCMs, restricts the DM's possibility to be consistent: indeed, under the assumption that $a_{ij} \in S$, if the DM expresses the preference ratios $a_{ij} = 5$ and $a_{jk} = 3$, then $a_{ij}\, a_{jk} = 15 > 9$; similarly, under the assumption that $a_{ij} \in [0,1]$, if he/she claims the preference degrees $a_{ij} = 0.9$ and $a_{jk} = 0.8$, then $a_{ij} + a_{jk} - 0.5 = 1.2 > 1$; thus, in these cases, the DM will not be consistent. There is an analogous drawback for the additive PCMs if the elements a_{ij} belong to a closed interval $[-a, a]$, with $a \in]0, +\infty[$.

In order to unify the several approaches to the PCMs and remove some drawbacks, as the ones above described, in [7] the authors introduce PCMs, whose entries belong to a set G structured as an abelian linearly ordered group (alo-group) $(G, \bullet, \leq)$; in this way the reciprocity and consistency conditions are expressed in terms of the group operation $\bullet$, whereas the notion of consistent vector is given by means of the inverse operation $\div$.

In (Cavallo and D'Apuzzo, 2010a, 2010b), (Cavallo, D'Apuzzo and Squillante, 2009, 2010), a study on alo-groups allows us to provide some results for PCMs defined over this structure as: a suitable way to get the weights for the elements of *X*, a measure of PCM's consistency, efficient algorithms to check the consistency and build a consistent PCM starting from a minimum number of pairwise comparisons. In this paper, we provide a survey of these results.

2. The algebraic structure

Let $(G, \bullet, \leq)$ be an alo-group, e its identity element, $a^{(-1)}$ the symmetric of $a \in G$ with respect to $\bullet$, $\div$ the inverse operation of $\bullet$. Starting from the notion of *G-norm* $\|a\| = a \vee a^{(-1)}$, we consider the *G-distance*

$$d_G(a,b) = (a \div b) \vee (b \div a) \in \{g \in G : g \geq e\}. \qquad \textbf{(8)}$$

If $(G,\bullet,\leq)$ is divisible then we consider the power $a^{(q)}$, with q rational number, and define the $\bullet$-mean of n elements as follows:

$$m_\bullet(a_1,...,a_n) = \begin{cases} a_1 & n = 1 \\ \left(\bullet_{i=1}^{n} a_i\right)^{\left(\frac{1}{n}\right)} & n \geq 2 \end{cases} . \qquad \textbf{(9)}$$

$(G,\bullet,\leq)$ is called real alo-group if and only if G is a subset of the real line R and $\leq$ is the total order on G inherited from the usual order on R. Examples of divisible real alo-group are: $(]0,\infty[,\cdot,\leq)$, where $\cdot$ denotes the usual multiplication, and $(]-\infty,\infty[,+,\leq)$; they are called multiplicative and additive alo-groups and (9) provides, for them, the geometric and the arithmetic mean, respectively. In (Cavallo and D'Apuzzo, 2009a), we structure the interval $]0,1[$ as real divisible alo-group by means of a binary operation $\otimes$, defined as follows:

$$a \otimes b = \frac{ab}{ab + (1-a)(1-b)}; \qquad \textbf{(10)}$$

we call $(]0,1[,\otimes,\leq)$ the *fuzzy alo-group*.

2.1 The abelian group $(G^n,\bullet)$

The set $G^n = \{\underline{w} = (w_1,...,w_n) \mid w_i \in G, \forall i = 1,...,n\}$ can be structured as abelian group by setting $\underline{v} \bullet \underline{w} = (v_1 \bullet w_1,...,v_n \bullet w_n)$. Then, $\underline{w} \in G^n$ is a *vector* over G and it is called $\bullet$-*normal vector* if and only if $w_1 \bullet ... \bullet w_n = e$. If $(G,\bullet,\leq)$ is divisible, then $(G^n,\bullet)$ is divisible too (Cavallo and D'Apuzzo, 2010a).

We say that $\underline{w}$ and $\underline{v}$ are $\bullet$-*proportional* if and only if there exists a *constant vector* $\underline{c} = (c,...,c) \in G^n$ such that $\underline{w} = \underline{c} \bullet \underline{v}$ and we show that the proportionality relation $\sim_\bullet$ defined by

$$\underline{w} \sim_\bullet \underline{v} \Leftrightarrow \exists\, c \in G \mid \underline{w} = \underline{c} \bullet \underline{v} \qquad \textbf{(11)}$$

is an equivalence relation (Cavallo and D'Apuzzo, 2010a). Then, $c \bullet \underline{v}$ stays for $\underline{c} \bullet \underline{v}$.

Proposition 1. (Cavallo and D'Apuzzo, 2010b) *Let $N(G^n)$ the set of the $\bullet$-normal vectors. Then $(N(G^n),\bullet)$ is a subgroup of $(G^n,\bullet)$ and, under the assumption of divisibility for $(G,\bullet,\leq)$, $(m_\bullet(\underline{w}))^{(-1)} \bullet \underline{w} \in N(G^n)$, $\forall \underline{w} \in G^n$.*

The function:

$$N : \underline{w} \in G^n \to N(\underline{w}) = (m_\bullet(\underline{w}))^{(-1)} \bullet \underline{w} \in N(G^n) \qquad \textbf{(12)}$$

is called $\bullet$-*normalization function*. If $\underline{w} \sim_\bullet \underline{v}$, then $N(\underline{w}) = N(\underline{v})$.

3. The abelian group of the PCMs over an abelian linearly ordered group $(G,\bullet,\leq)$

Let $G^{n\times n}$ be the set of the n- order matrices $A = (a_{ij})$ with entries in G. By setting $A \bullet B = (a_{ij} \bullet b_{ij})$, for each $A = (a_{ij}),\ B = (b_{ij}) \in G^{n\times n}$, we get that $(G^{n\times n},\bullet)$ is an abelian group. We say that:

- $A = (a_{ij}) \in G^{n\times n}$ is *reciprocal* with respect to $\bullet$ if and only if verifies the *reciprocity* condition:

$$a_{ji} = a_{ij}^{(-1)} \quad \forall i,j. \qquad \textbf{(13)}$$

- $A = (a_{ij}) \in G^{n\times n}$ is a *consistent* PCM with respect to $\bullet$ if and only if verifies the *consistency* condition:

$$a_{ik} = a_{ij} \bullet a_{jk} \quad \forall i,j,k. \qquad \textbf{(14)}$$

Let us denote with *RM* and *CM* the sets of the reciprocal and consistent PCMs, respectively. Then, $(RM,\bullet)$ is a proper subgroup of $(G^{n\times n},\bullet)$ and $(CM,\bullet)$ is a proper subgroup of $(RM,\bullet)$ (Cavallo and D'Apuzzo, 2010b).

Proposition 2. $A = (a_{ij}) \in CM$ *if and only if there exists a vector* $\underline{v} = (v_1,...,v_n) \in G^n$ *such that*

$$a_{ij} = v_i \div v_j \quad \forall i,j. \qquad \textbf{(15)}$$

Then we say that a vector $\underline{v} = (v_1,...,v_n) \in G^n$ verifying (15) is a *consistent vector* for $A = (a_{ij})$.
From now on, $CV(A)$ will denote the set of consistent vectors for $A \in CM$. $CV(A)$ is an equivalence class with respect to the relation of proportionality defined in (11) (Cavallo and D'Apuzzo, 2010a).

Proposition 3. (Cavallo and D'Apuzzo, 2009a) *The following assertions are equivalent:*

1. $A = (a_{ij}) \in CM$;
2. $a_{ik} = a_{ij} \bullet a_{jk} \quad \forall i,j,k: i<j<k$;
3. $d_G(a_{ik}, a_{ij} \bullet a_{jk}) = e \quad \forall i,j,k: i<j<k$;
4. $CV(A) \neq \emptyset$;
5. *for each column* $\underline{a}^k$ of A, $\underline{a}^k \in CV(A)$.

Remark 1. *Whenever* $(G,\bullet,\leq)$ *is one of the real alo-groups* $(]0,\infty[,\cdot,\leq)$, $(]-\infty,\infty[,+,\leq)$ *or* $(]0,1[,\otimes,\leq)$, *consistency in (14) corresponds respectively to multiplicative and additive consistency in (4) or multiplicative fuzzy consistency in (6); an analogous correspondence there exists between consistent vectors defined by (15) and consistent vectors defined by the first two properties in (5) or property in (7).*

In (Cavallo and D'Apuzzo, 2010a) and (Cavallo, D'Apuzzo and Squillante, 2009), we provide the following characterizations of consistency that allow us to provide efficient algorithms to verify the consistency and build a consistent PCM.

Proposition 4. *The following assertions are equivalent:*

1. $A = (a_{ij}) \in CM$;
2. $a_{ik} = a_{i\,i+1} \bullet a_{i+1\,k} \quad \forall i,k: i<k$;
3. $d_G(a_{ik}, a_{i\,i+1} \bullet a_{i+1\,k}) = e \quad \forall i,k: i<k$.

3.1 The $\bullet$-mean vector and the consistency index of a PCM over a divisible alo-group

Let $(G,\bullet,\leq)$ be divisible and $\underline{a}_i$ the *i*-th row of $A = (a_{ij}) \in G^{n\times n}$. Then, the vector $\underline{w}_{m_\bullet}(A) = (m_\bullet(\underline{a}_1)...,m_\bullet(\underline{a}_n))$ is called $\bullet$*-mean vector* associated to A; it is a $\bullet$-normal vector.

Proposition 5. *Let* $(G,\bullet,\leq)$ *be divisible. Then,* $A = (a_{ij}) \in CM$ *if and only if* $\underline{w}_{m_\bullet}(A) \in CV(A)$.

By (8) and Proposition 3, $A=(a_{ij})\notin CM$ if and only if $d_G(a_{ik}, a_{ij}\bullet a_{jk})>e$ for some triple $(i,j,k): i<j<k$; thus, in (Cavallo and D'Apuzzo, 2009a), we provide, for $A\in RM$, the following consistency index:

$$I_G(A)=\left(\underset{T}{\bullet} d_G(a_{ik}, a_{ij}\bullet a_{jk})\right)^{\left(\frac{1}{|T|}\right)}=\left(\underset{T}{\bullet} I_G(A_{ijk})\right)^{\left(\frac{1}{|T|}\right)}, \qquad with \quad T=\{(i,j,k)\,|\, i<j<k\} \qquad \textbf{(16)}$$

that is equal to e if and only if $A=(a_{ij})\in CM$.

Theorem 1. *Let* $\underline{w}_{m_\bullet}(A)=(m_\bullet(\underline{a}_1),...,m_\bullet(\underline{a}_n))$ *be the* $\bullet$*-mean vector associated to* A*. Then:*

$$d_G(a_{ij}, m_\bullet(\underline{a}_i)\div m_\bullet(\underline{a}_j))\begin{cases} = I_G(A)^{\left(\frac{1}{3}\right)} & n=3 \\ \\ \le I_G(A)^{\left(\frac{(n-2)(n-1)}{6}\right)} & n>3 \end{cases} \qquad \forall i,j.$$

Remark 2.
Theorem **1** *justifies* $I_G(A)$ *as a measure of consistency: indeed the more* $I_G(A)$ *is close to* e*, the more* A *is close to be consistent, because the* $\bullet$*-mean vector is close to be a consistent vector.*

Proposition 4 allows us to define a more efficient consistency index $I^*_G(A)$ as the $\bullet$-mean of the distances $d_G(a_{ik}, a_{i\,i+1}\bullet a_{i+1\,k})$ with $i<k$. It results:

$$I^*_G(A)\le I_G(A)^{\left(\frac{n}{3}\right)}. \qquad \textbf{(17)}$$

4. $\bullet$-mean vector as the vector of weights

Let us focus on the problem of deriving weights for the alternatives from a PCM over a divisible alo-group. We stress that:

- the $\bullet$-mean vector $\underline{w}_{m_\bullet}(A)$ is a meaningful vector because each component represents the $\bullet$-mean of the preference intensities of the corresponding alternative over all the others;
- whenever $A=(a_{ij})$ is a consistent PCM, by Proposition 5, $\underline{w}_{m_\bullet}(A)$ is a consistent vector and, among the consistent vectors in $CV(A)$, it is the only normal one. Furthermore, $N(\underline{v})=\underline{w}_{m_\bullet}(A) \quad \forall \underline{v}\in CV(A)$ (see (Cavallo and D'Apuzzo, 2010b));
- whenever $A=(a_{ij})$ is not consistent, by Theorem 1, the more $A=(a_{ij})$ is close to be a consistent PCM the more $\underline{w}_{m_\bullet}(A)$ is close to be a consistent vector;
- $\underline{w}_{m_\bullet}(A)$ verifies the *independence of scale inversion condition*, that is the vectors $\underline{w}_{m_\bullet}(A)$ and $\underline{w}_{m_\bullet}(A^{(-1)})$ provide the same ranking for the alternatives; another vector might not have the same advantage (see (Barzilai, 1998)).

For these reasons, we propose the $\bullet$-mean vector $\underline{w}_{m_\bullet}(A)$ as a suitable vector of weights.

6. Future work

Our future work will be directed to investigate, in the general context of the PCMs over alo-groups, conditions to state the actual ranking and to obtain ordinal and cardinal evaluation vectors able to represent it; a preliminary investigation is already done in (Cavallo and D'Apuzzo, 2009b). Finally, we will deal with the rank reversal problem.

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