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An efficient representation for kernels in the 2d dynamic displacement discontinuity method for cracked elastic materials

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1 Introduction

The displacement discontinuity method is a rather standard approach to study cracks in elastic materials [1–3]. This is in fact a certain technique to construct the system of Boundary Integral Equations (BIE), or equivalently, Boundary Element Methods (BEM). In the static case this typically results in explicit expressions for the kernels of respective BIE, both in 2d and 3d problems. Unfortunately, in dynamic problems the structure of respective kernels is expressed, as a rule, in terms of some quadratures of very complex form. In the present work we give efficient representations for such kernels in explicit form. They contain Hankel functions admitting efficient rational approximations.

It should be noted that fundamental properties of integral equations in problems for punches and cracks are established in the works of I. I. Vorovich with co-authors [4, 5].

2 Basic relations for a single elementary crack

Let us consider a crack located in a linear isotropic homogeneous elastic medium. The dependence upon time is $e^{-i\omega t}$, where ω is the circular frequency. Equations of motion in the 2d case are

$$\begin{cases} \dfrac{\partial^2 u_x}{\partial x^2} + c^2 \dfrac{\partial^2 u_x}{\partial y^2} + (1-c^2)\dfrac{\partial^2 u_y}{\partial x \partial y} + k_p^2 u_x = 0\,, \\ \dfrac{\partial^2 u_y}{\partial y^2} + c^2 \dfrac{\partial^2 u_y}{\partial x^2} + (1-c^2)\dfrac{\partial^2 u_x}{\partial x \partial y} + k_p^2 u_y = 0\,, \end{cases} \tag{2.1}$$

where

$$k_p = \frac{\omega}{c_p}\,,\ k_s = \frac{\omega}{c_s}\,, \quad c_p^2 = \frac{\lambda + 2\mu}{\rho}\,,\ c_s^2 = \frac{\mu}{\rho}\,, \quad c^2 = \frac{c_s^2}{c_p^2} = \frac{k_p^2}{k_s^2} < 1\,, \tag{2.2}$$

c_p is the longitudinal wave speed, c_s is the transverse wave speed, k_p and k_s are respective wave numbers, $\{u_x(x,y), u_y(x,y)\}$ denote the components of the displacement vector $\mathbf{u}$. The components of the stress tensor are given as follows

$$\begin{aligned} &\frac{\sigma_{xx}}{\lambda + 2\mu} = \frac{\partial u_x}{\partial x} + (1-2c^2)\frac{\partial u_y}{\partial y}\,, \quad \frac{\sigma_{xy}}{\mu} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\,, \\ &\frac{\sigma_{yy}}{\lambda + 2\mu} = \frac{\partial u_y}{\partial y} + (1-2c^2)\frac{\partial u_x}{\partial x}. \end{aligned} \tag{2.3}$$

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Then we study a small isolated (elementary) linear crack of length ε, so that the left tip of the crack is $(-\varepsilon/2, 0)$ and the right one is $(\varepsilon/2, 0)$. Let us assume that the crack faces are subjected to some (uniform) normal and tangential stresses, T_n and T_τ, and the relative displacement of the crack faces caused by this applied load, along x and y, is respectively, $g_x(x)$ and $g_y(x)$, ($|x| \le \varepsilon/2$). Then analytical solution to the problem under consideration can be constructed by using the Fourier transform along the x-axis. This reduces system (2.1) to a system of ordinary differential equations with constant coefficients

$$\begin{cases} c^2 U_x'' + \left(k_p^2 - s^2\right) U_x - is(1-c^2)\, U_y' = 0\,, \\ -is(1-c^2)\, U_x' + U_y'' + \left(k_p^2 - s^2 c^2\right) U_y = 0\,, \end{cases} \tag{2.4}$$

where all derivatives are related to variable y and all Fourier transforms are denoted by respective capital letters.

Solution to this elementary crack problem constructed separately on the upper ($y > 0$) and the lower ($y < 0$) half-planes as a general solution of system (2.4), with the use of the radiation condition, can be represented in the following form:

$$\begin{bmatrix} U_x^\pm \\ U_y^\pm \end{bmatrix} = A^\pm \begin{bmatrix} 1 \\ \mp i\beta/s \end{bmatrix} \mathrm{e}^{\mp\beta y} + B^\pm \begin{bmatrix} 1 \\ \mp is/\gamma \end{bmatrix} \mathrm{e}^{\mp\gamma y},\ \beta(s) = \sqrt{s^2 - k_p^2}\,,\ \gamma(s) = \sqrt{s^2 - k_s^2}\,, \tag{2.5}$$

where $A^\pm = A^\pm(s)$ and $B^\pm = B^\pm(s)$ are some unknown quantities. These four unknown constants should be defined from respective boundary conditions. All upper signs are related to the upper half-plane $y \ge 0$ and the lower ones - to the lower half-plane $y \le 0$.

The boundary conditions over the line $y = 0$ are evident:

$$\begin{aligned} \sigma_{yy}^+(x,0) = \sigma_{yy}^-(x,0)\,, \quad \sigma_{xy}^+(x,0) = \sigma_{xy}^-(x,0)\,, \quad |x| < \infty\,, \\ (u_x^+ - u_x^-)(x,0) = g_x(x)\,, \quad (u_y^+ - u_y^-)(x,0) = g_y(x)\,, \quad |x| \le \infty\,, \end{aligned} \tag{2.6}$$

and can be reduced to the following 4×4 system of linear algebraic equations for constants $A^+,\ B^+,\ A^-,\ B^-$:

$$\begin{cases} \dfrac{ic^2}{s}(2s^2 - k_s^2)A^+ + 2isc^2 B^+ - \dfrac{ic^2}{s}(2s^2 - k_s^2)A^- - 2isc^2 B^- = 0, \\ 2\beta A^+ + \dfrac{2s^2 - k_s^2}{\gamma}\, B^+ + 2\beta A^- + \dfrac{2s^2 - k_s^2}{\gamma}\, B^- = 0\,, \\ A^+ + B^+ - A^- - B^- = G_x\,, \\ \dfrac{i\beta}{s} A^+ + \dfrac{is}{\gamma} B^+ + \dfrac{i\beta}{s} A^- + \dfrac{is}{\gamma} B^- = -\,G_y\,. \end{cases} \tag{2.7}$$

The solution to this system is

$$A^\pm = \pm \frac{s^2}{k_s^2}\, G_x - \frac{is(2s^2 - k_s^2)}{2\beta k_s^2}\, G_y\,, \quad B^\pm = \mp\, \frac{2s^2 - k_s^2}{2k_s^2}\, G_x + \frac{is\gamma}{k_s^2}\, G_y\,. \tag{2.8}$$

Taking into account that $g_x(x) = 0,\ g_y(x) = 0,\ |x| > \varepsilon/2$, one obtains:

$$\begin{aligned} u_x(x,y) = {} & \frac{\mathrm{sgn}(y)}{2\pi k_s^2} \int_{-\varepsilon/2}^{\varepsilon/2} g_x(\xi) d\xi \int_{-\infty}^{\infty} \left[s^2 \mathrm{e}^{i(\xi - x)s - \beta|y|} + (k_s^2/2 - s^2) \mathrm{e}^{i(\xi - x)s - \gamma|y|} \right] ds \\ & + \frac{1}{2\pi k_s^2} \int_{-\varepsilon/2}^{\varepsilon/2} g_y(\xi) d\xi \int_{-\infty}^{\infty} is \left[\frac{k_s^2/2 - s^2}{\beta}\, \mathrm{e}^{i(\xi - x)s - \beta|y|} + \gamma \mathrm{e}^{i(\xi - x)s - \gamma|y|} \right] ds \\ = {} & \frac{i}{2k_s^2} \int_{-\varepsilon/2}^{\varepsilon/2} K_{xx}(\xi - x, y)\, g_x(\xi) d\xi + \frac{i}{2k_s^2} \int_{-\varepsilon/2}^{\varepsilon/2} K_{xy}(\xi - x, y)\, g_y(\xi) d\xi\,, \end{aligned} \tag{2.9a}$$

$$u_y(x,y) = -\frac{1}{2\pi k_s^2}\int\limits_{-\varepsilon/2}^{\varepsilon/2} g_x(\xi)d\xi \int\limits_{-\infty}^{\infty} is\left[\beta \mathrm{e}^{i(\xi-x)s-\beta|y|} + \frac{k_s^2/2-s^2}{\gamma}\mathrm{e}^{i(\xi-x)s-\gamma|y|}\right]ds$$

$$+\frac{\mathrm{sgn}(y)}{2\pi k_s^2}\int\limits_{-\varepsilon/2}^{\varepsilon/2} g_y(\xi)d\xi \int\limits_{-\infty}^{\infty}\left[(k_s^2/2-s^2)\mathrm{e}^{i(\xi-x)s-\beta|y|} + s^2\mathrm{e}^{i(\xi-x)s-\gamma|y|}\right]ds$$

$$= \frac{i}{2k_s^2}\int\limits_{-\varepsilon/2}^{\varepsilon/2} K_{yx}(\xi-x,y)\,g_x(\xi)d\xi + \frac{i}{2k_s^2}\int\limits_{-\varepsilon/2}^{\varepsilon/2} K_{yy}(\xi-x,y)\,g_y(\xi)d\xi\,. \tag{2.9b}$$

The last tabulated integrals are expressed in terms of Hankel functions which possess a precise approximation by rational functions [6]:

$$K_{xx}(x,y) = yN_{xx}(x,y), \quad K_{xy}(x,y) = \frac{\partial}{\partial x}N_{xy}(x,y),$$
$$K_{yx}(x,y) = -\frac{\partial}{\partial x}N_{yx}(x,y), \quad K_{yy}(x,y) = yN_{yy}(x,y), \tag{2.10a}$$

$$N_{xx}(x,y) = \frac{k_s^3}{2}\,\frac{H_1^{(1)}\left(k_s\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}} - M = -\frac{k_s^2}{\pi(x^2+y^2)} + Q_s(x,y) - M,$$

$$Q_s(x,y) = \frac{k_s^2}{2}\left[k_s\frac{H_1^{(1)}\left(k_s\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}} + \frac{2}{\pi(x^2+y^2)}\right],$$

$$N_{yy}(x,y) = \frac{k_p k_s^2}{2}\,\frac{H_1^{(1)}\left(k_p\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}} + M = -\frac{k_s^2}{\pi(x^2+y^2)} + Q_p(x,y) + M, \tag{2.10b}$$

$$Q_p(x,y) = \frac{k_s^2}{2}\left[k_p\frac{H_1^{(1)}\left(k_p\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}} + \frac{2}{\pi(x^2+y^2)}\right],$$

$$M = M(x,y) = k_p\frac{\partial^2}{\partial x^2}\left[\frac{H_1^{(1)}\left(k_p\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}}\right] - k_s\frac{\partial^2}{\partial x^2}\left[\frac{H_1^{(1)}\left(k_s\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}}\right],$$

$$N_{xy}(x,y) = \left(\frac{k_s^2}{2} + \frac{\partial^2}{\partial x^2}\right)H_0^{(1)}\left(k_p\sqrt{x^2+y^2}\right) + \frac{\partial^2}{\partial y^2}H_0^{(1)}\left(k_s\sqrt{x^2+y^2}\right), \tag{2.10c}$$

$$N_{yx}(x,y) = \frac{\partial^2}{\partial y^2}H_0^{(1)}\left(k_p\sqrt{x^2+y^2}\right) + \left(\frac{k_s^2}{2} + \frac{\partial^2}{\partial x^2}\right)H_0^{(1)}\left(k_s\sqrt{x^2+y^2}\right).$$

Note that for kernels N_{xx}, N_{yy} the leading asymptotic term for small argument is extracted, so Q_s, Q_p may be accepted constant on the short interval $(-\varepsilon/2, \varepsilon/2)$.

The components of the stress tensor can be found by analogy:

$$\frac{\sigma_{xx}}{\lambda+2\mu} = \frac{i}{2k_s^2}\int_{-\varepsilon/2}^{\varepsilon/2} L_x^{xx}(\xi-x,y)\,g_x(\xi)d\xi + \frac{i}{2k_s^2}\int_{-\varepsilon/2}^{\varepsilon/2} L_y^{xx}(\xi-x,y)\,g_y(\xi)d\xi\,,$$

$$\frac{\sigma_{xy}}{\mu} = \frac{i}{2k_s^2}\int_{-\varepsilon/2}^{\varepsilon/2} L_x^{xy}(\xi-x,y)\,g_x(\xi)d\xi + \frac{i}{2k_s^2}\int_{-\varepsilon/2}^{\varepsilon/2} L_y^{xy}(\xi-x,y)\,g_y(\xi)d\xi\,, \tag{2.11}$$

$$\frac{\sigma_{yy}}{\lambda+2\mu} = \frac{i}{2k_s^2}\int_{-\varepsilon/2}^{\varepsilon/2} L_x^{yy}(\xi-x,y)\,g_x(\xi)d\xi + \frac{i}{2k_s^2}\int_{-\varepsilon/2}^{\varepsilon/2} L_y^{yy}(\xi-x,y)\,g_y(\xi)d\xi\,,$$

where

$$L_x^{xx}(\xi-x,y) = \frac{\partial}{\partial x}K_{xx}(\xi-x,y) + (1-2c^2)\frac{\partial}{\partial y}K_{yx}(\xi-x,y)\,,$$
$$L_y^{xx}(\xi-x,y) = \frac{\partial}{\partial x}K_{xy}(\xi-x,y) + (1-2c^2)\frac{\partial}{\partial y}K_{yy}(\xi-x,y)\,, \tag{2.12a}$$

$$L_x^{xy}(\xi-x,y) = \frac{\partial}{\partial y}K_{xx}(\xi-x,y) + \frac{\partial}{\partial x}K_{yx}(\xi-x,y)\,,$$
$$L_y^{xy}(\xi-x,y) = \frac{\partial}{\partial y}K_{xy}(\xi-x,y) + \frac{\partial}{\partial x}K_{yy}(\xi-x,y)\,, \tag{2.12b}$$

$$L_x^{yy}(\xi-x,y) = \frac{\partial}{\partial y}K_{yx}(\xi-x,y) + (1-2c^2)\frac{\partial}{\partial x}K_{xx}(\xi-x,y)\,,$$
$$L_y^{yy}(\xi-x,y) = \frac{\partial}{\partial y}K_{yy}(\xi-x,y) + (1-2c^2)\frac{\partial}{\partial x}K_{xy}(\xi-x,y)\,. \tag{2.12c}$$

3 Boundary Element Method for arbitrary crack

Let us consider an arbitrary crack or system of cracks located in the elastic medium. This can be considered as a superposition of N elementary linear cracks of small length ε, which for the sake of brevity we put equal for all cracks. For j-th crack, ($j = 1, \ldots, N$), the contribution of elementary quantities $g_j^x,\ g_j^y$ to the displacements and stresses at any point can be calculated as above. Then full contribution of all elementary cracks is a superposition of these calculated elementary contributions.

By calculating integrals in (2.11), we may consider the quantities $g_j^x,\ g_j^y$ to be approximately constant over the small interval $(-\varepsilon/2, \varepsilon/2)$. Since $\partial/\partial x = -\partial/\partial\xi$ in relations (2.12), it is clear that

$$\int_{-\varepsilon/2}^{\varepsilon/2} L_x^{xx}(\xi-x,y)\,d\xi = R_x^{xx}\left(\frac{\varepsilon}{2}-x,y\right) - R_x^{xx}\left(-\frac{\varepsilon}{2}-x,y\right), \tag{3.1a}$$

$$\int_{-\varepsilon/2}^{\varepsilon/2} L_y^{xx}(\xi-x,y)d\xi = \frac{\varepsilon(1-2c^2)}{2}\frac{\partial}{\partial y}\left\{y\left[Q_p\left(\frac{\varepsilon}{2}-x,y\right)+Q_p\left(-\frac{\varepsilon}{2}-x,y\right)\right]\right\}$$
$$+R_y^{xx}\left(\frac{\varepsilon}{2}-x,y\right) - R_y^{xx}\left(-\frac{\varepsilon}{2}-x,y\right), \tag{3.1b}$$

$$\int_{-\varepsilon/2}^{\varepsilon/2} L_x^{xy}(\xi - x, y)\, d\xi = \frac{\varepsilon}{2}\frac{\partial}{\partial y}\left\{y\left[Q_s\left(\frac{\varepsilon}{2} - x, y\right) + Q_s\left(-\frac{\varepsilon}{2} - x, y\right)\right]\right\}$$
$$+R_x^{xy}\left(\frac{\varepsilon}{2} - x, y\right) - R_x^{xy}\left(-\frac{\varepsilon}{2} - x, y\right), \quad (3.1c)$$

$$\int_{-\varepsilon/2}^{\varepsilon/2} L_y^{xy}(\xi - x, y)\, d\xi = R_y^{xy}\left(\frac{\varepsilon}{2} - x, y\right) - R_y^{xy}\left(-\frac{\varepsilon}{2} - x, y\right), \quad (3.1d)$$

$$\int_{-\varepsilon/2}^{\varepsilon/2} L_x^{yy}(\xi - x, y)\, d\xi = R_x^{yy}\left(\frac{\varepsilon}{2} - x, y\right) - R_x^{yy}\left(-\frac{\varepsilon}{2} - x, y\right), \quad (3.1e)$$

$$\int_{-\varepsilon/2}^{\varepsilon/2} L_y^{yy}(\xi - x, y) d\xi = \frac{\varepsilon}{2}\frac{\partial}{\partial y}\left\{y\left[Q_p\left(\frac{\varepsilon}{2} - x, y\right) + Q_p\left(-\frac{\varepsilon}{2} - x, y\right)\right]\right\}$$
$$+R_y^{yy}\left(\frac{\varepsilon}{2} - x, y\right) - R_y^{yy}\left(-\frac{\varepsilon}{2} - x, y\right), \quad (3.1f)$$

where

$$R_x^{xx}(x, y) = -K_{xx}(x, y) - (1 - 2c^2)\frac{\partial}{\partial y}N_{yx}(x, y), \quad (3.2a)$$

$$R_y^{xx}(x, y) = -K_{xy}(x, y) + (1 - 2c^2)\left\{\frac{k_s^2\, x}{\pi(x^2 + y^2)}\right.$$
$$\left. +k_p\frac{\partial^2}{\partial x \partial y}\left[\frac{yH_1^{(1)}\left(k_p\sqrt{x^2 + y^2}\right)}{\sqrt{x^2 + y^2}}\right] - k_s\frac{\partial^2}{\partial x \partial y}\left[\frac{yH_1^{(1)}\left(k_s\sqrt{x^2 + y^2}\right)}{\sqrt{x^2 + y^2}}\right]\right\}, \quad (3.2b)$$

$$R_x^{xy}(x, y) = -K_{yx}(x, y) + \left\{\frac{k_s^2\, x}{\pi(x^2 + y^2)}\right.$$
$$\left. +k_s\frac{\partial^2}{\partial x \partial y}\left[\frac{yH_1^{(1)}\left(k_s\sqrt{x^2 + y^2}\right)}{\sqrt{x^2 + y^2}}\right] - k_p\frac{\partial^2}{\partial x \partial y}\left[\frac{yH_1^{(1)}\left(k_p\sqrt{x^2 + y^2}\right)}{\sqrt{x^2 + y^2}}\right]\right\}, \quad (3.2c)$$

$$R_y^{xy}(x, y) = -K_{yy}(x, y) + \frac{\partial}{\partial y}N_{xy}(x, y), \quad (3.2d)$$

$$R_x^{yy}(x, y) = -(1 - 2c^2)K_{xx}(x, y) - \frac{\partial}{\partial y}N_{yx}(x, y), \quad (3.2e)$$

$$R_y^{yy}(x, y) = -(1 - 2c^2)K_{xy}(x, y) + \left\{\frac{k_s^2\, x}{\pi(x^2 + y^2)}\right.$$
$$\left. +k_p\frac{\partial^2}{\partial x \partial y}\left[\frac{yH_1^{(1)}\left(k_p\sqrt{x^2 + y^2}\right)}{\sqrt{x^2 + y^2}}\right] - k_s\frac{\partial^2}{\partial x \partial y}\left[\frac{yH_1^{(1)}\left(k_s\sqrt{x^2 + y^2}\right)}{\sqrt{x^2 + y^2}}\right]\right\}. \quad (3.2f)$$

In development of these formulas we took into account that integration of the rational function in (2.10) can be performed explicitly: $\int(\partial/\partial y)[y/(x^2 + y^2)]dx = -x/(x^2 + y^2)$. Besides, in integration of approximately constant functions Q_p and

Q_s over small interval of length ε, instead of to take the value of respective integrand at the central point $\xi = 0$ multiplied by ε, we apply an arithmetic average of the integrand at the end-points. Such a treatment can guarantee that the arising expressions take no singular values which may occur in the case $x = 0,\ y = 0$.

Now, by collecting together all developed formulas, one can deduce that

$$\frac{\sigma_{xx}}{\lambda+2\mu} = \frac{ig_j^x}{2k_s^2}\left[R_x^{xx}\left(\frac{\varepsilon}{2}-x,y\right) - R_x^{xx}\left(-\frac{\varepsilon}{2}-x,y\right)\right] + \frac{ig_j^y}{2k_s^2}\left\langle\frac{\varepsilon(1-2c^2)}{2}\frac{\partial}{\partial y}\right.$$
$$\left.\times\left\{y\left[Q_p\left(\frac{\varepsilon}{2}-x,y\right)+Q_p\left(-\frac{\varepsilon}{2}-x,y\right)\right]\right\} + R_y^{xx}\left(\frac{\varepsilon}{2}-x,y\right) - R_y^{xx}\left(-\frac{\varepsilon}{2}-x,y\right)\right\rangle, \quad (3.3a)$$

$$\frac{\sigma_{xy}}{\mu} = \frac{ig_j^x}{2k_s^2}\left\langle\frac{\varepsilon}{2}\frac{\partial}{\partial y}\left\{y\left[Q_s\left(\frac{\varepsilon}{2}-x,y\right)+Q_s\left(-\frac{\varepsilon}{2}-x,y\right)\right]\right\}\right.$$
$$\left.+R_x^{xy}\left(\frac{\varepsilon}{2}-x,y\right) - R_x^{xy}\left(-\frac{\varepsilon}{2}-x,y\right)\right\rangle + \frac{ig_j^y}{2k_s^2}\left[R_y^{xy}\left(\frac{\varepsilon}{2}-x,y\right) - R_y^{xy}\left(-\frac{\varepsilon}{2}-x,y\right)\right], \quad (3.3b)$$

$$\frac{\sigma_{yy}}{\lambda+2\mu} = \frac{ig_j^x}{2k_s^2}\left[R_x^{yy}\left(\frac{\varepsilon}{2}-x,y\right) - R_x^{yy}\left(-\frac{\varepsilon}{2}-x,y\right)\right] + \frac{ig_j^y}{2k_s^2}\left\langle\frac{\varepsilon}{2}\frac{\partial}{\partial y}\right.$$
$$\left.\times\left\{y\left[Q_p\left(\frac{\varepsilon}{2}-x,y\right)+Q_p\left(-\frac{\varepsilon}{2}-x,y\right)\right]\right\} + R_y^{yy}\left(\frac{\varepsilon}{2}-x,y\right) - R_y^{yy}\left(-\frac{\varepsilon}{2}-x,y\right)\right\rangle, \quad (3.3c)$$

representations valid for arbitrary observation point (x, y).

In the problem for full cracks we have $2N$ unknown quantities $g_j^x,\ g_j^y,\ j = 1,\dots,N$. In order to construct the basic system for BEM, let us recall that for any elementary area with the normal $\bar{n}$ and tangent $\bar{\tau}$ unit vectors the normal and tangential stress over this area is

$$T_n = \sigma_{xx}n_x^2 + 2\sigma_{xy}n_xn_y + \sigma_{yy}n_y^2\,, \quad T_\tau = \sigma_{xx}n_x\tau_x + \sigma_{xy}(n_x\tau_y + n_y\tau_x) + \sigma_{yy}n_y\tau_y\,. \quad (3.4)$$

Passing from the local to a global coordinate system, one may treat the unknown quantities $g_j^x,\ g_j^y$ as tangential and normal ones, respectively. Then the substitution of expressions (3.3) into (3.4) allows one to write out the complete contribution of all elementary cracks to $T_k^n(x_k, y_k),\ T_k^\tau(x_k, y_k)$ as a superposition of elementary expressions (3.4). This can be written symbolically as

$$T_k^n(x_k,y_k) = \sum_{j=1}^N a_{kj}^{nn}\,g_j^n + \sum_{j=1}^N a_{kj}^{n\tau}\,g_j^\tau\,, \quad T_k^\tau(x_k,y_k) = \sum_{j=1}^N a_{kj}^{\tau n}\,g_j^n + \sum_{j=1}^N a_{kj}^{\tau\tau}\,g_j^\tau\,, \quad (3.5)$$

for all elements $k = 1,\dots,N$, where N is the total number of elementary cracks. Now, by using the boundary conditions for the known normal and tangential stresses over the faces of the crack, one comes to a linear algebraic system, in frames of the Displacement Discontinuity Method, like in the standard static case [1]. Note that all kernels are expressed explicitly, and no additional integration is required.

4 Application to wave diffraction in the US echo-scanning

Let us assume that an Ultrasonic sensor is placed in a far zone to register longitudinal far-field wave signals scattered by the considered crack. Typically, the recording of the longitudinal waves by the US transducer is equivalent to a registration of radial component of the displacement vector. In frames of very standard "echo-method" the far-field pressure is measured just in the same direction α from which the incident wave arrives.

Relations (3.5) contain $2N$ unknown quantities $g_j^n,\ g_j^\tau,\ j = 1,\dots,N$. In the considered problem of US incident wave the boundary condition leads to the closed-form $2N \times 2N$ linear algebraic system regarding these unknown quantities:

$$\begin{cases} \displaystyle\sum_{j=1}^N a_{kj}^{nn}g_j^n + \sum_{j=1}^N a_{kj}^{n\tau}g_j^\tau = -[\sigma_{xx}^{inc}n_x^2 + 2\sigma_{xy}^{inc}n_xn_y + \sigma_{yy}^{inc}n_y^2]_k, \quad k = 1,\dots N, \\ \displaystyle\sum_{j=1}^N a_{kj}^{\tau n}g_j^n + \sum_{j=1}^N a_{kj}^{\tau\tau}g_j^\tau = -[\sigma_{xx}^{inc}n_x\tau_x + \sigma_{xy}^{inc}(n_x\tau_y + n_y\tau_x) + \sigma_{yy}^{inc}n_y\tau_y]_k,\ k = 1,\dots N. \end{cases} \quad (4.1)$$

If j-th elementary crack (x_j, y_j) is again defined by its unit normal and tangential vectors, $\bar{n}_j = \{n_{jx}, n_{jy}\}$, $\bar{\tau}_j = \{\tau_{jx}, \tau_{jy}\}$, then the far-field ($R \to \infty$) scattering pattern, which is registered by a sensor of longitudinal waves, can be written in the following discrete form as a superposition of elementary contributions:

$$A(\alpha) = \sqrt{R}\,|u_R(\alpha)| \sim \Bigg| \sum_{j=1}^{N} \varepsilon_j \Big\{ - \big[(\tau_{jx}\cos\alpha + \tau_{jy}\sin\alpha)(n_{jx}\cos\alpha + n_{jy}\sin\alpha)\big]\, g_j^{\tau} + \left[\big(\tau_{jx}\cos\alpha + \tau_{jy}\sin\alpha\big)^2 - \frac{1}{2c^2} \right] g_j^{n} \Big\} \mathrm{e}^{-ik_p(x_j\cos\alpha + y_j\sin\alpha)} \Bigg| , \tag{4.2}$$

where the following far-field representation has been used:

$$\mathrm{e}^{ik_p r} = \exp\left[ik_p \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \right] = \exp\left[ik_p \sqrt{R^2 - 2R\,(x_j\cos\alpha + y_j\sin\alpha) + x_j^2 + y_j^2} \right] \sim \mathrm{e}^{ik_p R}\, \mathrm{e}^{-ik_p(x_j\cos\alpha + y_j\sin\alpha)} . \tag{4.3}$$

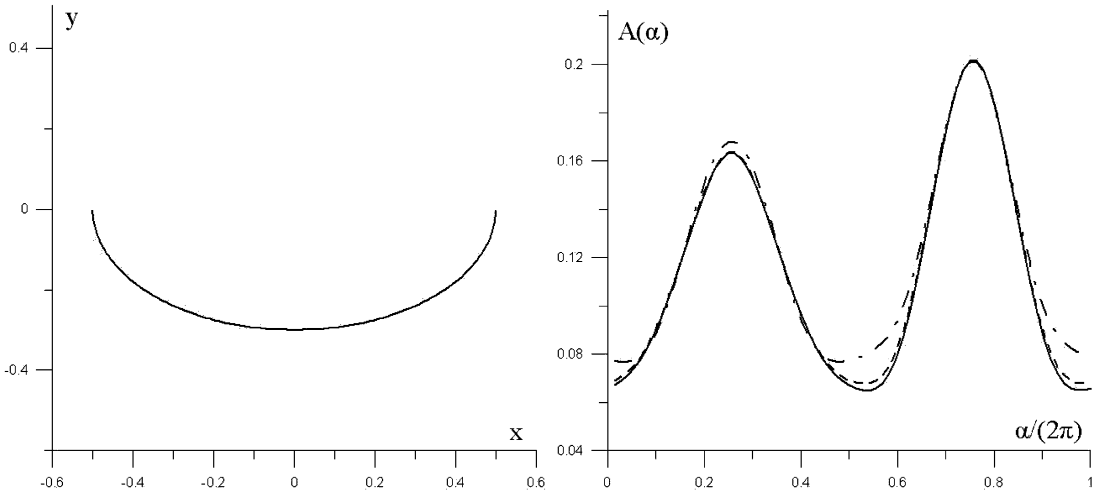

Fig. 1 Geometry of the crack (left) and far-field back-scattered amplitude (right): – · – · – $N = 10$; - - - - $N = 40$; —— $N = 120$.

As an example we consider a crack whose geometry is shown in Fig. 1, left. The right half of the same figure demonstrates the far-field back-scattered amplitude versus incident angle. A good convergence of the proposed method can clearly be observed from this figure. It should be noted that the algorithm, applied to solve the arising system of integral equations, from the numerical point of view is similar to a classical *collocation* technique [7].

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