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Stress concentration effects in microstretch elastic bodies

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Abstract

This paper is concerned with the plane strain problem of the equilibrium theory of microstretch elastic bodies. First, we study the problem of stress concentration in the neighbourhood of a circular hole located in a plane subjected to the action of constant loads at a great distance from the hole. Then, the problem of a rigid inclusion in an infinite body is studied.

1. Introduction

There has been very much written in recent years on the subject of the theory of continua in which the deformation is described not only by the usual displacement vector field, but by other vector or tensor fields as well. The theory of microstretch continua was introduced by Eringen [1–3] in order to study micromorphic materials whose microelements can undergo expansions and contractions. The material points of the microstretch bodies can stretch and contract independently of their translations and rotations. A microstretch body can model composite materials and various porous bodies (cf. [2]). The linear theory of microstretch elastic bodies was introduced in [1,2]. The theory of microstretch elastic solids is a generalization of the micropolar theory [4].

In this paper we study the problem of stress concentration in microstretch elastic bodies. This problem is of great practical and technological importance and in the context of classical elastostatics the problem has been a subject of intensive study (see e.g. [5,6]). In the framework of the theory of micropolar elasticity the problem of stress concentration around holes was studied in various papers (see e.g. [4,7–9]). In Section 2 we present the basic equations of the equilibrium theory of microstretch elastic bodies and derive the equations of the plane strain problem for

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homogeneous and isotropic bodies. Section 3 is concerned with the problem of a cylindrical cavity in an infinite solid subjected to the action of constant loads at a great distance from the hole. The representation used in solving of the problem refers to displacement vector, microrotation vector and microstretch function. The solution is presented in a closed form. In Section 4 we study the problem of a cylindrical rigid inclusion in an infinite body which is uniformly stretched along one axis.

2. Basic equations

We consider the theory of microstretch elastic solids established by Eringen [1,2]. We assume that the body occupies at some instant the regular region B of three-dimensional Euclidean space. We let $\overline{B}$ denote the closure of B, call ∂B the boundary of B, and designate by $\boldsymbol{n}$ the outward unit normal of ∂B. Letters in boldface stand for tensors of an order $p \geqslant 1$, and if $\boldsymbol{v}$ has the order p, we write $v_{ij\cdots s}$ (p subscripts) for the components of $\boldsymbol{v}$ in the Cartesian coordinate system Ox_i $(i = 1, 2, 3)$. We shall employ the usual summation and differentiation conventions: Greek subscripts are understood to range over the integers $(1, 2)$, whereas Latin subscripts—unless otherwise specified—to the range $(1, 2, 3)$; summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate.

We confine our attention to the equilibrium theory of linearly microstretch elastic materials. The basic equations of the equilibrium theory of homogeneous and isotropic microstretch elastic solids, in the absence of the body loads, consist of the equations of equilibrium

$$t_{ji,j} = 0, \quad m_{ji,j} + \varepsilon_{irs} t_{rs} = 0, \quad h_{i,i} - s = 0, \tag{2.1}$$

the constitutive equations

$$\begin{aligned} t_{ij} &= \lambda e_{kk} \delta_{ij} + (\mu + \kappa) e_{ij} + \mu e_{ji} + \sigma \psi \delta_{ij}, \\ m_{ij} &= \alpha \kappa_{kk} \delta_{ij} + \beta \kappa_{ji} + \gamma \kappa_{ij} + b_0 \varepsilon_{sji} \psi_{,s}, \\ h_i &= \xi \varphi_{,i} + b_0 \varepsilon_{ijs} \kappa_{sj}, \\ s &= \sigma e_{rr} + b \psi, \end{aligned} \tag{2.2}$$

and the geometrical equations

$$e_{ij} = u_{j,i} + \varepsilon_{jik} \varphi_k, \quad \kappa_{ij} = \varphi_{j,i}. \tag{2.3}$$

Here, t_{ij} is the stress tensor, m_{ij} is the couple stress tensor, h_k is the microstress vector, s is the net pressure involved in dilatation, e_{ij} and κ_{ij} are strain measures, u_i is the displacement vector, φ_i is the microrotation vector, ψ is the microstretch function, ε_{irs} is the alternating symbol, δ_{ij} is the Kronecker's delta and λ, μ, κ, α, β, γ, σ, ξ, b and b_0 are constitutive constants.

Throughout this paper we assume that the internal energy density is a positive definite quadratic form. Thus, the constitutive coefficients satisfy the conditions [2]

$$b(3\lambda + 2\mu + \kappa) - 3\sigma^2 > 0, \quad 2\mu + \kappa > 0, \quad \kappa > 0, \quad \xi > 0, \quad b > 0,$$
$$3\alpha + \beta + \gamma > 0, \quad \gamma + \beta > 0, \quad \gamma - \beta > 0. \tag{2.4}$$

The components of surface traction $\boldsymbol{t}$, the component of the surface moment $\boldsymbol{m}$ and the microtraction h at regular points of ∂B are defined by

$$t_i = t_{ji}n_j, \quad m_i = m_{ji}n_j, \quad h = h_j n_j, \tag{2.5}$$

respectively.

We assume that the region B refers to a right cylinder with the open cross section Σ and the smooth lateral boundary Π. The rectangular Cartesian coordinate frame is supposed to be chosen in such a way that the x_3-axis is parallel to the generators of B. We denote by L the boundary of Σ. The state of plane strain of the cylinder B, parallel to the plane x_1Ox_2, is characterized by

$$u_\alpha = u_\alpha(x_1, x_2), \quad u_3 = 0, \quad \varphi_\alpha = 0, \quad \varphi_3 = \varphi(x_1, x_2), \quad \psi = \psi(x_1, x_2), \qquad (x_1, x_2) \in \Sigma. \tag{2.6}$$

The above restrictions, in conjunction with the geometrical equations (2.3) and the constitutive equations (2.2), imply that e_{ij}, κ_{ij}, t_{ij}, m_{ij}, h_i and s are all independent of x_3. It follows from (2.3) and (2.6) that the non-zero strain measures are given by

$$e_{\alpha\beta} = u_{\beta,\alpha} + \varepsilon_{\beta\alpha 3}\varphi, \quad \kappa_{\beta 3} = \varphi_{\beta,\alpha}, \quad \kappa_{\alpha 3} = \varphi_{,\alpha}. \tag{2.7}$$

The constitutive equations show that the non-zero components of the stress tensor, couple stress tensor and microstress vector are $t_{\alpha\beta}$, $m_{\alpha 3}$, t_{33}, $m_{3\alpha}$ and h_α. Further,

$$\begin{aligned} t_{\alpha\beta} &= \lambda e_{\rho\rho}\delta_{\alpha\beta} + (\mu + \kappa)e_{\alpha\beta} + \mu e_{\beta\alpha} + \sigma\psi\delta_{\alpha\beta}, \\ m_{13} &= \gamma\kappa_{13} + b_0\psi_{,2}, \quad m_{23} = \gamma\kappa_{23} - b_0\psi_{,1}, \\ h_1 &= \xi\psi_{,1} - b_0\kappa_{23}, \quad h_2 = \xi\psi_{,2} + b_0\kappa_{13}, \quad s = \sigma e_{\rho\rho} + b\psi. \end{aligned} \tag{2.8}$$

The equations of equilibrium (2.1) reduce to

$$\begin{aligned} &t_{\beta\alpha,\beta} = 0, \\ &m_{\rho 3,\rho} + \varepsilon_{3\alpha\beta}t_{\alpha\beta} = 0, \\ &h_{\alpha,\alpha} - s = 0 \end{aligned} \tag{2.9}$$

on Σ. We assume that on the boundary of the body there are prescribed the surface loads. Given the surface traction $\tilde{\boldsymbol{t}}$, the surface moment $\widetilde{\boldsymbol{m}}$ and the surface microtraction $\tilde{h}$ on Π, with $\tilde{\boldsymbol{t}}$, $\widetilde{\boldsymbol{m}}$ and $\tilde{h}$ independent of x_3 and $\tilde{t}_3 = 0$, $\widetilde{m}_\alpha = 0$, the boundary conditions on the lateral surface become

$$t_{\beta\alpha}n_\beta = \tilde{t}_\alpha, \quad m_{\alpha 3}n_\alpha = \widetilde{m}_3, \quad h_\alpha n_\alpha = \tilde{h} \text{ on } L, \tag{2.10}$$

where $\tilde{t}_\alpha$, $\widetilde{m}_3$ and $\tilde{h}$ are prescribed functions.

In what follows we are interested in a plane strain problem with the displacement vector, the microrotation vector and the microstretch function being specified in cylindrical coordinates (r, θ, z) as follows:

$$
\begin{aligned}
&u_r = u(r,\theta), \quad u_\theta = v(r,\theta), \quad u_z = 0, \\
&\varphi_r = 0, \quad \varphi_\theta = 0, \quad \varphi_z = \varphi(r,\theta), \quad \psi = \psi(r,\theta), \quad (r,\theta) \in \Theta.
\end{aligned}
\tag{2.11}
$$

The axis Oz of the cylindrical coordinate system is taken along the axis of the cylinder. The geometrical equations (2.7) become

$$
\begin{aligned}
&e_{rr} = \frac{\partial u}{\partial r}, \quad e_{\theta\theta} = \frac{1}{r}\left(\frac{\partial v}{\partial \theta} + u\right), \quad e_{r\theta} = \frac{\partial v}{\partial r} - \varphi, \\
&e_{\theta r} = \frac{1}{r}\left(\frac{\partial u}{\partial \theta} - v\right) + \varphi, \quad \kappa_{rz} = \frac{\partial \varphi}{\partial r}, \quad \kappa_{\theta z} = \frac{1}{r}\frac{\partial \varphi}{\partial \theta}.
\end{aligned}
\tag{2.12}
$$

The equilibrium equations (2.9) take the form

$$
\begin{aligned}
&\frac{\partial t_{rr}}{\partial r} + \frac{1}{r}\frac{\partial t_{\theta r}}{\partial \theta} + \frac{1}{r}(t_{rr} - t_{\theta\theta}) = 0, \\
&\frac{\partial t_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial t_{\theta\theta}}{\partial \theta} + \frac{1}{r}(t_{r\theta} + t_{\theta r}) = 0, \\
&\frac{\partial m_{rz}}{\partial r} + \frac{1}{r}\frac{\partial m_{\theta z}}{\partial \theta} + \frac{1}{r}m_{rz} + t_{r\theta} - t_{\theta r} = 0, \\
&\frac{1}{r}\frac{\partial}{\partial r}(r h_r) + \frac{1}{r}\frac{\partial h_\theta}{\partial \theta} - s = 0.
\end{aligned}
\tag{2.13}
$$

The constitutive equations can be written in the form

$$
\begin{aligned}
&t_{rr} = (\lambda + 2\mu + \kappa)e_{rr} + \lambda e_{\theta\theta} + \sigma\psi, \\
&t_{\theta\theta} = \lambda e_{rr} + (\lambda + 2\mu + \kappa)e_{\theta\theta} + \sigma\psi, \\
&t_{r\theta} = (\mu + \kappa)e_{r\theta} + \mu e_{\theta r}, \quad t_{\theta r} = (\mu + \kappa)e_{\theta r} + \mu e_{r\theta}, \\
&m_{rz} = \gamma\kappa_{rz} + b_0\frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad m_{\theta z} = \gamma\kappa_{\theta z} - b_0\frac{\partial \psi}{\partial r}, \\
&h_r = \xi\frac{\partial \psi}{\partial r} - b_0\kappa_{\theta z}, \quad h_\theta = \frac{1}{r}\xi\frac{\partial \psi}{\partial \theta} + b_0\kappa_{rz}, \\
&s = \sigma\left[\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta}\right] + b\psi.
\end{aligned}
\tag{2.14}
$$

The plane strain problem consists in the finding of the functions u, v, φ and ψ on Σ which satisfy the Eqs. (2.12)–(2.14) on Σ and the boundary conditions.

3. Stress concentration around a circular hole

In this section we study the problem of a cylindrical cavity in an infinite solid. We assume that the region B is defined by $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 > a^2\}$, where $a > 0$ is a given constant. The body is in equilibrium in the absence of body loads. We suppose that the surface of the cavity is free of surface loads and that the body is subject to a field of simple tension at infinity. Let P be the constant tension field at a plane $x_1 =$ constant at infinity. We assume that the surface moment and the surface microtraction vanish at infinity. The body is in a state of plane strain, parallel to the plane x_1Ox_2. In this case the domain Σ is given by $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 > a^2, x_3 = 0\}$.

The boundary conditions on the surface of the cavity can be expressed as

$$t_{rr} = 0, \quad t_{r\theta} = 0, \quad m_{rz} = 0, \quad h_r = 0 \quad \text{for } r = a. \tag{3.1}$$

The conditions at infinity require that the stress distribution must reduce to that of a body without cavity. Thus we have the following conditions at infinity:

$$\begin{aligned} &t_{rr} = \tfrac{1}{2}P(1 + \cos 2\theta), \quad t_{\theta\theta} = \tfrac{1}{2}P(1 - \cos 2\theta), \\ &t_{r\theta} = t_{\theta r} = -\tfrac{1}{2}P \sin 2\theta, \quad m_{rz} = m_{\theta z} = 0, \\ &h_r = 0, \quad h_\theta = 0, \end{aligned} \tag{3.2}$$

where P is a given constant.

The problem consists in the finding of the functions u, v, φ and ψ on Σ which satisfy the Eqs. (2.12)–(2.14) on Σ and the conditions (3.1) and (3.2). We seek the solution of the problem in the form

$$\begin{aligned} &u = F(r) + U(r)\cos 2\theta, \quad v = V(r)\sin 2\theta, \\ &\varphi = W(r)\sin 2\theta, \quad \psi = G(r) + \Phi(r)\cos 2\theta, \end{aligned} \tag{3.3}$$

where F, G, U, V, W and Φ are functions only on r. It follows from (2.12), (3.3) and (2.14) that

$$\begin{aligned} t_{rr} &= (\lambda + 2\mu + \kappa)\frac{\mathrm{d}F}{\mathrm{d}r} + \frac{1}{r}\lambda F + \sigma G \\ &\quad + \left[(\lambda + 2\mu + \kappa)\frac{\mathrm{d}U}{\mathrm{d}r} + \frac{1}{r}\lambda(U + 2V) + \sigma\Phi\right]\cos 2\theta, \\ t_{\theta\theta} &= \lambda\frac{\mathrm{d}F}{\mathrm{d}r} + \frac{1}{r}(\lambda + 2\mu + \kappa)F + \sigma G \\ &\quad + \left[\lambda\frac{\mathrm{d}U}{\mathrm{d}r} + \frac{1}{r}(\lambda + 2\mu + \kappa)(U + 2V) + \sigma\Phi\right]\cos 2\theta, \\ t_{r\theta} &= \left[(\mu + \kappa)\frac{\mathrm{d}V}{\mathrm{d}r} - \kappa W - \frac{1}{r}\mu(2U + V)\right]\sin 2\theta, \\ t_{\theta r} &= \left[\mu\frac{\mathrm{d}V}{\mathrm{d}r} + \kappa W - \frac{1}{r}(\mu + \kappa)(2U + V)\right]\sin 2\theta, \end{aligned}$$

$$
\begin{aligned}
m_{rz} &= \left(\gamma \frac{\mathrm{d}W}{\mathrm{d}r} - \frac{2}{r} b_0 \Phi\right) \sin 2\theta, \\
m_{\theta z} &= \left(\frac{2}{r}\gamma W - b_0 \frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) \cos 2\theta - b_0 \frac{\mathrm{d}G}{\mathrm{d}r}, \\
h_r &= \xi\left(\frac{\mathrm{d}G}{\mathrm{d}r} + \frac{\mathrm{d}\Phi}{\mathrm{d}r} \cos 2\theta\right) - \frac{2}{r} b_0 W \cos 2\theta, \\
h_\theta &= \left(-\frac{2}{r}\xi\Phi + b_0 \frac{\mathrm{d}W}{\mathrm{d}r}\right) \sin 2\theta, \\
s &= \sigma\left(\frac{\mathrm{d}F}{\mathrm{d}r} + \frac{1}{r}F\right) + bG \\
&\quad + \left\{\sigma\left[\frac{\mathrm{d}U}{\mathrm{d}r} + \frac{1}{r}(U + 2V)\right] + b\Phi\right\} \cos 2\theta.
\end{aligned}
\tag{3.4}
$$

If we substitute (3.4) into the equilibrium equations (2.13), then we obtain the following equations:

$$
\begin{aligned}
&(\lambda + 2\mu + \kappa)\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rF)\right] + \sigma\frac{\mathrm{d}G}{\mathrm{d}r} = 0, \\
&\xi\left(\frac{\mathrm{d}^2G}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}G}{\mathrm{d}r} - \frac{b}{\xi}G\right) - \frac{1}{r}\sigma\frac{\mathrm{d}}{\mathrm{d}r}(rF) = 0, \\
&(\lambda + 2\mu + \kappa)\left(r^2\frac{\mathrm{d}^2U}{\mathrm{d}r^2} + r\frac{\mathrm{d}U}{\mathrm{d}r}\right) + 2(\lambda + \mu)r\frac{\mathrm{d}V}{\mathrm{d}r} + \sigma r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r} \\
&\quad - (\lambda + 6\mu + 5\kappa)U - 2(\lambda + 3\mu + 2\kappa)V + 2\kappa rW = 0, \\
&(\mu + \kappa)\left(r^2\frac{\mathrm{d}^2V}{\mathrm{d}r^2} + r\frac{\mathrm{d}V}{\mathrm{d}r}\right) - 2(\lambda + \mu)r\frac{\mathrm{d}U}{\mathrm{d}r} - \kappa r^2\frac{\mathrm{d}W}{\mathrm{d}r} \\
&\quad - 2(\lambda + 3\mu + 2\kappa)U - (4\lambda + 9\mu + 5\kappa)V - 2\sigma r\Phi = 0, \\
&\gamma\left(r^2\frac{\mathrm{d}^2W}{\mathrm{d}r^2} + r\frac{\mathrm{d}W}{\mathrm{d}r} - 4W\right) + \kappa r^2\frac{\mathrm{d}V}{\mathrm{d}r} + \kappa r(2U + V) - 2\kappa r^2 W = 0, \\
&\xi\left(r^2\frac{\mathrm{d}^2\Phi}{\mathrm{d}r^2} + r\frac{\mathrm{d}\Phi}{\mathrm{d}r} - 4\Phi - \frac{b}{\xi}r^2\Phi\right) - \sigma r^2\frac{\mathrm{d}U}{\mathrm{d}r} - \sigma r(U + 2V) = 0.
\end{aligned}
\tag{3.5}
$$

The first equation of (3.5) implies that

$$
\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rF) + \frac{\sigma}{\lambda + 2\mu + \kappa}G = C_1, \tag{3.6}
$$

where C_1 is an arbitrary constant. In view of (3.6), the second equation of (3.5) can be written in the form

$$\frac{d^2G}{dr^2} + \frac{1}{r}\frac{dG}{dr} - \zeta^2 G = \frac{\sigma}{\xi}C_1, \tag{3.7}$$

where

$$\zeta^2 = \frac{1}{\xi}\left(b - \frac{\sigma^2}{\lambda + 2\mu + \kappa}\right). \tag{3.8}$$

It follows from (2.4) that $\zeta^2 > 0$. The solution of Eq. (3.7) is

$$G = A_1 K_0(\zeta r) + A_1^* I_0(\zeta r) - \frac{\sigma}{\xi\zeta^2}C_1,$$

where I_n and K_n are the modified Bessel functions of order n, and A_1 and A_1^* are arbitrary constants. Since the function G must be finite at infinity we have $A_1^* = 0$. Thus, we get

$$G = A_1 K_0(\zeta r) - \frac{\sigma}{\xi\zeta^2}C_1. \tag{3.9}$$

It follows from (3.6) and (3.9) that

$$F = \frac{b}{2\xi\zeta^2}C_1 r + \frac{1}{r}C_2 + \frac{\sigma A_1}{\zeta(\lambda + 2\mu + \kappa)}K_1(\zeta r), \tag{3.10}$$

where C_2 is an arbitrary constant.

Now we introduce the independent variable t through the relation

$$t = \ln r, \tag{3.11}$$

and denote

$$\mathrm{D} = \frac{d}{dt}.$$

Then, Eq. (3.5, parts 3 and 4) can be written in the form

$$\begin{aligned} &[\mathrm{D}^2 - (1 + 4c_1)]U + 2[(1 - c_1)\mathrm{D} - (1 + c_1)]V = -e^t(c_2\mathrm{D}\Phi + 2c_3 W), \\ &[(1 - c_1)\mathrm{D} + (1 + c_1)]U + [c_1\mathrm{D}^2 - (4 + c_1)]V = e^t(2c_2\Phi + c_3\mathrm{D}W), \end{aligned} \tag{3.12}$$

where

$$c_1 = \frac{\mu + \kappa}{\lambda + 2\mu + \kappa}, \quad c_2 = \frac{\sigma}{\lambda + 2\mu + \kappa}, \quad c_3 = \frac{\kappa}{\lambda + 2\mu + \kappa}, \quad c_4 = \frac{\kappa}{\gamma}. \tag{3.13}$$

The general solution of the homogeneous system (3.12) which corresponds to a finite stress field at infinity is given by

$$\begin{aligned} U_0 &= b_1 \mathrm{e}^{-t} + B_2 \mathrm{e}^{-3t} + B_3 \mathrm{e}^{t}, \\ V_0 &= -c_1 B_1 \mathrm{e}^{-t} + B_2 \mathrm{e}^{-3t} - B_3 \mathrm{e}^{t}, \end{aligned} \tag{3.14}$$

where B_1, B_2 and B_3 are arbitrary constants. Particular solution of the system (3.12) can be seen to be

$$\begin{aligned} U^* &= -\frac{1}{2} c_2 (\mathrm{e}^t S_1 + \mathrm{e}^{-3t} S_2) - \frac{1}{2c_1} c_3 (\mathrm{e}^t R_1 - \mathrm{e}^{-3t} R_2), \\ V^* &= \frac{1}{2} c_2 (\mathrm{e}^t S_1 - \mathrm{e}^{-3t} S_2) + \frac{1}{2c_1} c_3 (\mathrm{e}^t R_1 + \mathrm{e}^{-3t} R_2), \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} S_1(t) &= \int^t \Phi(s)\,\mathrm{d}s, \quad S_2(t) = \int^t \mathrm{e}^{4s} \Phi(s)\,\mathrm{d}s, \\ T_1(t) &= \int^t W(s)\,\mathrm{d}s, \quad R_2(t) = \int^t \mathrm{e}^{4s} W(s)\,\mathrm{d}s. \end{aligned} \tag{3.16}$$

With the help of (3.11), (3.14) and (3.15) we obtain

$$\begin{aligned} U = {} & B_1 r^{-1} + B_2 r^{-3} + B_3 r \\ & - \frac{1}{2} c_2 \left[r \int^r x^{-1} \Phi(x)\,\mathrm{d}x + r^{-3} \int^r x^3 \Phi(x)\,\mathrm{d}x \right] \\ & - \frac{1}{2c_1} c_3 \left[r \int^r x^{-1} W(x)\,\mathrm{d}x - r^{-3} \int^r x^3 W(x)\,\mathrm{d}x \right], \\ V = {} & - c_1 B_1 r^{-1} + B_2 r^{-3} - B_3 r \\ & + \frac{1}{2} c_2 \left[r \int^r x^{-1} \Phi(x)\,\mathrm{d}x - r^{-3} \int^r x^3 \Phi(x)\,\mathrm{d}x \right] \\ & + \frac{1}{2c_1} c_3 \left[r \int^r x^{-1} W(x)\,\mathrm{d}x + r^{-3} \int^r x^3 W(x)\,\mathrm{d}x \right]. \end{aligned} \tag{3.17}$$

If we substitute U and V from (3.17) into (3.5, parts 5 and 6) we obtain the equations

$$\begin{aligned} & r^2 \frac{\mathrm{d}^2 W}{\mathrm{d}r^2} + r \frac{\mathrm{d}W}{\mathrm{d}r} - (4 + \delta^2 r^2) W = -2c_4 B_1, \\ & r^2 \frac{\mathrm{d}^2 \Phi}{\mathrm{d}r^2} + r \frac{\mathrm{d}\Phi}{\mathrm{d}r} - (4 + \zeta^2 r^2) \Phi = -\frac{2c_1 \sigma}{\xi} B_1, \end{aligned} \tag{3.18}$$

where

$$\delta^2 = \frac{\kappa(2\mu+\kappa)}{\gamma(\mu+\kappa)}. \tag{3.19}$$

The solutions of Eqs. (3.18) which generate finite stresses for $r \to \infty$ are given by

$$\begin{aligned} W &= A_2K_2(\delta r) + \frac{2}{\delta^2}c_4B_1r^{-2}, \\ \Phi &= A_3K_2(\zeta r) + \frac{2}{\xi\zeta^2}c_1\sigma B_1 r^{-2}, \end{aligned} \tag{3.20}$$

where A_2 and A_3 are arbitrary constants. If we substitute (3.20) into relations (3.17) we obtain

$$\begin{aligned} U = {} & \frac{1}{r}d_1B_1 + \frac{1}{r^3}B_2 + B_3r - \frac{1}{2\delta c_1}c_3A_2[K_3(\delta r) - K_1(\delta r)] \\ & + \frac{1}{2\zeta}c_2A_3[K_3(\zeta r) + K_1(\zeta r)], \\ V = {} & -\frac{1}{r}c_2d_2B_1 + \frac{1}{r^3}B_2 - B_3r - \frac{1}{2\delta c_1}c_3A_2[K_3(\delta r) + K_1(\delta r)] \\ & + \frac{1}{2\zeta}c_2A_3[K_3(\zeta r) - K_1(\zeta r)], \end{aligned} \tag{3.21}$$

where

$$d_1 = 1 + \frac{c_3c_4}{c_1\delta^2}, \quad d_2 = 1 + \frac{c_2\sigma}{\xi\zeta^2}. \tag{3.22}$$

We introduce the notations

$$\begin{aligned} & q_1 = d_1 - 2c_1d_2, \quad q_2 = 2d_1 - c_1d_2, \\ & Q_1 = \frac{1}{2\mu+\kappa}\left[-(2\mu+\kappa)d_1 - 2\lambda c_1d_2 + \frac{2c_1\sigma^2}{\xi\zeta^2}\right], \\ & Q_2 = \frac{1}{2\mu+\kappa}\left[(\lambda+2\mu+\kappa)q_1 - \lambda d_1 + \frac{2c_1\sigma^2}{\xi\zeta^2}\right], \\ & Q_3 = \frac{1}{2\mu+\kappa}\left[(\mu+\kappa)c_1d_2 - q_2\mu - \frac{2\kappa c_4}{\delta^2}\right], \\ & Q_4 = \frac{1}{2\mu+\kappa}\left[\mu c_1d_2 - (\mu+\kappa)q_2 + \frac{2\kappa c_4}{\delta^2}\right], \\ & k = (2\lambda+2\mu+\kappa)b - \sigma^2. \end{aligned} \tag{3.23}$$

It follows from (3.4), (3.9), (3.10) and (3.20)–(3.23) that

$$
\begin{aligned}
t_{rr} &= \frac{k}{2\xi\zeta^2}C_1 - (2\mu+\kappa)r^{-2}C_2 - \frac{(2\mu+\kappa)}{\zeta}c_4A_1r^{-1}K_1(\zeta r)\\
&\quad + (2\mu+\kappa)\left\{Q_1r^{-2}B_1 - 3r^{-4}B_2 + B_3 + \frac{1}{2\delta c_1}c_3A_2r^{-1}[K_1(\delta r)+3K_3(\delta r)]\right.\\
&\quad \left. - \frac{1}{4\zeta}c_2A_3[6r^{-1}K_3(\zeta r) - \zeta K_2(\zeta r) + \zeta K_0(\zeta r)]\right\}\cos 2\theta,\\
t_{\theta\theta} &= \frac{k}{2\xi\zeta^2}C_1 + (2\mu+\kappa)r^{-2}C_2 + (2\mu+\kappa)c_2A_1\left[K_0(\zeta r) + \frac{1}{\zeta r}K_1(\zeta r)\right]\\
&\quad + (2\mu+\kappa)\left\{Q_2r^{-2}B_1 + 3r^{-4}B_2 - B_3 - \frac{1}{2\delta c_1}c_3A_2r^{-1}[K_1(\delta r)+3K_3(\delta r)]\right.\\
&\quad \left. + \frac{1}{4\zeta}c_2A_3[3\zeta K_2(\zeta r) + \zeta K_0(\zeta r) - 6r^{-1}K_3(\zeta r)]\right\}\cos 2\theta,\\
t_{r\theta} &= (2\mu+\kappa)\left\{Q_3B_1r^{-2} - 3r^{-4}B_2 - B_3 + \frac{1}{4\delta c_1}c_3A_2[6r^{-1}K_3(\delta r) + \delta K_0(\delta r) - \delta K_2(\delta r)]\right.\\
&\quad \left. - \frac{1}{2\zeta}c_2A_3r^{-1}[3K_3(\zeta r) + K_1(\zeta r)]\right\}\sin 2\theta,\\
t_{\theta r} &= (2\mu+\kappa)\left\{Q_4B_1r^{-2} - 3r^{-4}B_2 - B_3 + \frac{1}{4\delta c_1}c_3A_2[6r^{-1}K_3(\delta r) + 3\delta K_2(\delta r) + \delta K_0(\delta r)]\right.\\
&\quad \left. - \frac{1}{2\zeta}c_2A_3r^{-1}[3K_3(\zeta r) + K_1(\zeta r)]\right\}\sin 2\theta,\\
m_{rz} &= -\left\{\gamma A_2[\delta K_1(\delta r) + 2r^{-1}K_2(\delta r)] + 2b_0r^{-1}A_3K_2(\zeta r) + 4B_1r^{-3}\left(\frac{\gamma c_4}{\delta^2} + \frac{b_0c_1\sigma}{\xi\zeta^2}\right)\right\}\sin 2\theta,\\
m_{\theta z} &= \left\{2\gamma A_2r^{-1}K_2(\delta r) + b_0A_3[\zeta K_1(\zeta r) + 2r^{-1}K_2(\zeta r)] + 4B_1r^{-3}\left(\frac{\gamma c_4}{\delta^2} + \frac{b_0c_1\sigma}{\xi\zeta^2}\right)\right\}\cos 2\theta\\
&\quad + b_0\zeta A_1K_1(\zeta r),\\
h_r &= -\xi\zeta A_1K_1(\zeta r) - \left\{\xi A_3[\zeta K_1(\zeta r) + 2r^{-1}K_2(\zeta r)]\right.\\
&\quad \left. + 2b_0A_2r^{-1}K_2(\delta r) + 4B_1r^{-3}\left(\frac{c_1\sigma}{\zeta^2} + \frac{b_0c_4}{\delta^2}\right)\right\}\cos 2\theta,\\
h_\theta &= -\left\{2\xi r^{-1}A_3K_2(\zeta r) + b_0A_2[\delta K_1(\delta r) + 2r^{-1}K_2(\delta r)] + 4B_1r^{-3}\left(\frac{c_1\sigma}{\zeta^2} + \frac{c_4b_0}{\delta^2}\right)\right\}\sin 2\theta.
\end{aligned}
\tag{3.24}
$$

On the basis of (3.24), the conditions at infinity (3.2) reduce to

$$B_3 = \frac{1}{2(2\mu+\kappa)}P, \quad C_1 = \frac{\xi\zeta^2}{k}P. \tag{3.25}$$

We note that the restrictions (2.4) imply that $k > 0$.

We introduce the notations

$$\begin{aligned} &\Lambda(z;p) = 6z^{-1}K_3(pz) - pK_2(pz) + pK_0(pz),\\ &\Gamma(z;p) = pK_1(pz) + \frac{2}{p}K_2(pz), \quad \mathscr{E}(z) = K_1(z) + 3K_3(z),\\ &\ell_1 = 4a^{-3}(\gamma c_4\sigma^{-2} + c_1\sigma b_0\xi^{-1}\zeta^{-2}),\\ &\ell_2 = 4a^{-3}(c_1\sigma\zeta^{-2} + b_0c_4\delta^{-2}),\\ &J = \gamma\xi\Gamma(a;\delta)\Gamma(a;\zeta) - 4b_0^2a^{-2}K_2(\zeta a)K_2(\delta a),\\ &T_1 = \xi\ell_1\Gamma(a;\zeta) - 2b_0\ell_2a^{-1}K_2(\zeta a),\\ &T_2 = \gamma\ell_2\Gamma(a;\delta) - 2\ell_1b_0a^{-1}K_2(\delta a). \end{aligned} \tag{3.26}$$

The boundary conditions (3.1) reduce to

$$\begin{aligned} &C_2 = \frac{Pa^2}{2(2\mu+\kappa)}, \quad A_1 = 0, \quad A_2 = -\frac{1}{J}T_1B_1, \quad A_3 = -\frac{1}{J}T_2B_1,\\ &H_1B_1 - 3a^{-4}B_2 = -\frac{P}{2(2\mu+\kappa)}, \quad H_2B_1 - 3a^{-4}B_2 = \frac{P}{2(2\mu+\kappa)}, \end{aligned} \tag{3.27}$$

where

$$\begin{aligned} &H_1 = Q_1a^{-2} - \frac{1}{2Ja\delta c_1}c_3T_1\mathscr{E}(\delta a) + \frac{1}{4\zeta J}c_2T_2\Lambda(a;\zeta),\\ &H_2 = Q_3a^{-2} - \frac{1}{4\delta c_1J}c_3T_1\Lambda(a;\delta) + \frac{1}{2\zeta aJ}c_2T_2\mathscr{E}(\zeta a). \end{aligned} \tag{3.28}$$

From (3.27) we find that

$$B_1 = \frac{P}{(2\mu+\kappa)(H_2-H_1)}, \quad B_2 = \frac{P(H_1+H_2)a^4}{6(2\mu+\kappa)(H_2-H_1)}, \tag{3.29}$$

so that all constants C_α, A_i and B_i are determined. Substituting (3.29), (3.27) and (3.25) into (3.9), (3.10), (3.20), (3.21) and (3.3) we obtain u_r, u_θ, φ_z and ψ. The stresses are determined from (3.24). The value of $t_{\theta\theta}$ at the periphery of the cavity is given by

$$t_{\theta\theta} = P\left(1 + \frac{2}{1+f}\cos 2\theta\right), \tag{3.30}$$

where

$$f = N^{-1}[8J\delta c_1 a\zeta(H_2 - H_1) - N],$$
$$N = 4J\delta c_1\zeta a^{-1}Q_2 + 4J\delta c_1 a\zeta H_1 + 2c_3T_1\zeta\mathscr{E}(\delta a) - c_1c_2a\delta T_2[3\zeta K_2(\zeta a) + \zeta K_0(\zeta a) - 6a^{-1}K_3(\zeta a)]. \quad (3.31)$$

4. The problem of a rigid inclusion

In this section we study the problem of a rigid cylindrical inclusion in an infinite body which is uniformly stretched along the axis Ox_1. We assume that the elastic body occupies the region $B = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 > a^2\}$, where a is a positive constant. We assume that the region $\{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 < a^2\}$ is occupied by a rigid body. We consider the following boundary conditions:

$$u_r = 0, \quad u_\theta = 0, \quad \varphi_z = 0, \quad \psi = 0 \text{ on } r = a, \quad (4.1)$$

and the conditions (3.2) at infinity. The body B is in a state of plane strain parallel to the plane x_1Ox_2 in the absence of body loads. We seek the solution in the form (3.3). It follows from (3.24) that the conditions at infinity (3.2) reduce to (3.25). With the help of (3.9), (3.10), (3.20) and (3.21) we find that the conditions (4.1) can be written in the form

$$A_1 = \frac{\sigma}{k}P[K_0(\zeta a)]^{-1}, \quad C_2 = -\frac{ba^2}{2k}P - \frac{c_2 a}{\zeta}A_1K_1(\zeta a),$$
$$A_2 = -\frac{2}{\delta^2a^2}c_4B_1[K_2(\delta a)]^{-1},$$
$$A_3 = -\frac{2}{\xi\zeta^2a^2}c_1\sigma B_1[K_2(\zeta a)]^{-1}, \quad (4.2)$$
$$\left[d_1a^2 + \frac{c_3c_4a}{c_1\delta^3}L(\delta a) - \frac{c_1c_2\sigma a}{\xi\zeta^3}L(\zeta a)\right]B_1 + B_2 = -\frac{Pa^4}{2(2\mu+\kappa)},$$
$$\left[-c_2d_2a^2 + \frac{c_3c_4a}{c_1\delta^3}L(\delta a) - \frac{c_1c_2\sigma a}{\xi\zeta^3}L(\zeta a)\right]B_1 + B_2 = \frac{Pa^4}{2(2\mu+\kappa)},$$

where

$$L(z) = [K_3(z) + K_1(z)][K_2(z)]^{-1}. \quad (4.3)$$

From (4.2) we obtain

$$B_1 = -\frac{Pa^2}{(2\mu+\kappa)(d_1+c_2d_2)},$$
$$B_2 = \frac{Pa^2}{2(2\mu+\kappa)(d_1+c_2d_2)}\left[(d_1 - c_2d_2)a^2 + \frac{2c_3c_4a}{c_1\delta^3}L(\delta a) - \frac{2c_1c_2\sigma a}{\xi\zeta^3}L(\zeta a)\right]. \quad (4.4)$$

The solution of the problem has the form (3.3) where the constants A_i, B_i and C_α are given by (3.25), (4.2) and (4.4). The stress tensor, the couple stress tensor and the microstress vector can be determined from the relations (3.24). In particular, the values of t_{rr} and $t_{r\theta}$ on the boundary of the inclusion have the form

$$t_{rr} = \frac{1}{2}P + \frac{1}{2k}(2\mu+\kappa)bP\left[1 + \frac{2\sigma}{\zeta ab}(c_2 - c_4)\frac{K_1(\zeta a)}{K_0(\zeta a)}\right] - \frac{Pa^2}{d_1 + c_2 d_2}\left\{(Q_1 + d_1 - 2c_2 d_2)a^{-2} + \frac{2c_3 c_4}{c_1 \delta^3 a^3}\frac{K_1(\delta a)}{K_2(\delta a)} - \frac{4c_1 c_2 \sigma}{\xi \zeta^3 a^3}\frac{K_1(\zeta a)}{K_2(\zeta a)}\right\}\cos 2\theta,$$

$$t_{r\theta} = -\frac{Pa^2}{d_1 + c_2 d_2}\left\{\left(Q_3 + \frac{1}{2}d_1 - \frac{5}{2}c_2 d_2\right)a^{-2} - \frac{2c_1 c_2 \sigma}{\xi \zeta^2 a^3}\frac{K_1(\zeta a)}{K_2(\zeta a)} + \frac{4c_3 c_4}{c_1 a^3 \delta^3}\frac{K_1(\delta a)}{K_2(\delta a)}\right\}\sin 2\theta.$$

As in [10], we can study the behaviour of an infinite microstretch elastic body with a spherical cavity.

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