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On Saint-Venant's principle for micropolar viscoelastic bodies

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Abstract

In this article we establish a spatial decay estimate of Toupin type in the dynamic linear theory of micropolar viscoelastic solids.

1. Introduction

The general theory of micromorphic materials with memory has been developed by Eringen [1]. In Ref. [2], Eringen has established the theory of micropolar viscoelasticity. The propagation conditions and growth equations, which govern the propagation of waves in micropolar viscoelasticity, have been derived and discussed by McCarty and Eringen [3]. Some general theorems in micropolar viscoelasticity have been established in Ref. [4]. In the framework of the nonpolar viscoelasticity, various results concerning Saint-Venant's principle have been established by Sternberg and Al-Khozaie [5], Neapolitan and Edelstein [6], Rionero and Chirità [7] and Chirità [8]. In the present article we generalize the results from Refs. [6,8] to the dynamic linear theory of micropolar viscoelastic bodies. In Section 2 we present the basic equations of the linear theory of micropolar viscoelastic solids. Section 3 is devoted to preliminary results. In Section 4 we establish a spatial decay estimate of Toupin type for micropolar viscoelastic bodies.

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2. Basic equations

We consider a body that at time t_0 occupies the regular region B of Euclidean three-dimensional space and is bounded by the piecewise smooth surface ∂B. The motion of the body is referred to a fixed system of rectangular Cartesian axes $Ox_i (i = 1, 2, 3)$. We designate by $\mathbf{n}$ the outward unit normal of ∂B. Letters in boldface stand for tensors of an order $p \geqslant 1$, and if $\mathbf{v}$ has the order p, we write $v_{ij\ldots s}$ (p subscripts) for the components of $\mathbf{v}$ in the Cartesian coordinate system. We shall employ the usual summation and differentiation conventions: Latin subscripts are understood to range over the integers (1, 2, 3), summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate. In all that follows, we use a superposed dot to denote partial differentiation with respect to the time.

We consider the linear theory of micropolar viscoelastic bodies. In the absence of the body loads, the equations of motion are given by [2]

$$t_{ji,j} = \rho \ddot{u}_i, \tag{1a}$$

$$m_{ji,j} + \epsilon_{irs} t_{rs} = J \ddot{\varphi}_i, \tag{1b}$$

where t_{ij} is the stress tensor, m_{ij} is the couple stress tensor, $\mathbf{u}$ is the displacement vector, $\boldsymbol{\varphi}$ is the micro-rotation vector, ρ is the reference mass density, ϵ_{ijk} is the alternating symbol and J is a coefficient of inertia. The local form of energy balance is [2]

$$\rho \dot{\epsilon} = t_{ij} \dot{e}_{ij} + m_{ij} \dot{\chi}_{ij}, \tag{2}$$

where ϵ is the internal energy density and

$$e_{ij} = u_{j,i} + \epsilon_{jis} \varphi_s, \tag{3a}$$

$$\chi_{ij} = \varphi_{j,i}. \tag{3b}$$

The constitutive equations of the linear theory of homogeneous micropolar viscoelastic bodies are

$$t_{ij}(\mathbf{x},t) = \int_{-\infty}^{t} [A_{ijpr}(t-s) \dot{e}_{pr}(\mathbf{x},s) + B_{ijpr}(t-s) \dot{\chi}_{pr}(\mathbf{x},s)] \, \mathrm{d}s, \tag{4a}$$

$$m_{ij}(\mathbf{x},t) = \int_{-\infty}^{t} [B_{prij}(t-s) \dot{e}_{pr}(\mathbf{x},s) + C_{ijrp}(t-s) \dot{\chi}_{rp}(\mathbf{x},s)] \, \mathrm{d}s, \tag{4b}$$

where the relaxation functions $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are twice continuously differentiable on $[0, \infty)$ and have the properties of symmetry

$$A_{ijrs} = A_{rsij}, \tag{5a}$$

$$C_{ijrs} = C_{rsij}. \tag{5b}$$

Since a viscoelastic material remembers its past history, we must prescribe u_i, φ_i, e_{ij}, χ_{ij}, t_{ij} and m_{ij} up to some instant $t = 0$. The initial data consist of the functions $(u_i^*, \varphi_i^*, e_{Iij}, \chi_{ij}^*, t_{ij}^*, m_{ij}^*) = s^*$, defined on $\bar{B}\times(-\infty, 0)$, which satisfies the field equations. Thus, we have the initial history condition $s^{(i)} = s^*$ where $s^{(i)} = (u_r, \varphi_r, e_{pq}, \chi_{pq}, t_{pq}, m_{pq})$ on $\bar{B}\times(-\infty, 0)$.

In what follows, we assume $s^* = (0, 0, \ldots, 0)$ on $\bar{B}\times(-\infty, 0)$. In this case we have the initial conditions

$$\mathbf{u}(\mathbf{x},0) = \mathbf{0}, \tag{6a}$$

$$\dot{\mathbf{u}}(\mathbf{x},0) = \mathbf{0}, \tag{6b}$$

$$\boldsymbol{\varphi}(\mathbf{x},0) = \mathbf{0}, \tag{6c}$$

$$\dot{\boldsymbol{\varphi}}(\mathbf{x},0) = \mathbf{0}, \quad \mathbf{x} \in \bar{B}, \tag{6d}$$

The constitutive equations can be written in the form

$$t_{ij}(t) = A_{ijrs}(0)e_{rs}(t) + B_{ijrs}(0)\chi_{rs}(t) + \int_0^t [\dot{A}_{ijpq}(t-s)e_{pq}(s) + \dot{B}_{ijpq}(t-s)\chi_{pq}(s)]\,\mathrm{d}s, \tag{7a}$$

$$m_{ij}(t) = B_{rsij}(0)e_{rs}(t) + C_{ijpq}(0)\chi_{pq}(t) + \int_0^t [\dot{B}_{pqij}(t-s)e_{pq}(s) + \dot{C}_{ijpq}(t-s)\chi_{pq}(s)\,\mathrm{d}s, \tag{7b}$$

where, for convenience, we have suppressed the argument $\mathbf{x}$.

Let f be a function of position and time defined on $\bar{B}\times I$, where $I = [0, \infty)$. We say that $f \in C^{M,N}$ if

$$\frac{\partial^m}{\partial x_i \partial x_j \ldots \partial x_p}\left(\frac{\partial^n f}{\partial t^n}\right),$$

exists and is continuous on $\bar{B}\times I$ for $m = 0, 1, \ldots, M$, $n = 0\ 1, \ldots, N$, and $m + n \leqslant \max(M, N)$. We introduce the notion of an admissible process $s = \{u_i, \varphi_i, e_{ij}, \chi_{ij}, t_{ij}, m_{ij}\}$ by which we mean an ordered array of functions u_i, φ_i, e_{ij}, χ_{ij}, t_{ij} and m_{ij} defined on $\bar{B}\times I$ with the following properties

1. u_i, $\dot{u}_i$, $\ddot{u}_i$, φ_i, $\dot{\varphi}_i$, $\ddot{\varphi}_i$, e_{ij}, $\dot{e}_{ij}$, χ_{ij} and $\dot{\chi}_{ij}$ are continuous on $\bar{B}\times I$;
2. t_{ij} and m_{ij} are of class $C^{1,0}$ on $B \times (0, \infty)$;
3. t_{ij}, $t_{ji,j}$, m_{ij} and $m_{ji,j}$ are continuous on $\bar{B}\times I$.

By a viscoelastic process for B corresponding to null body loads we mean an admissible process that satisfies the field Eqs. (1), (3) and (7) on $B \times (0, \infty)$.

The surface traction and surface couple at regular points of ∂B are given by

$$t_i = t_{ji}n_j, \tag{8a}$$

$$m_i = m_{ji} n_j. \tag{8b}$$

In view of initial conditions (6) we have

$$e_{ij}(0) = 0, \tag{9a}$$

$$\chi_{ij}(0) = 0 \tag{9b}$$

on $\bar{B}$.

3. Preliminaries

We introduce the notations

$$\bar{e}_{ij}(t_1,t_2) = e_{ij}(t_1) - e_{ij}(t_2), \tag{10a}$$

$$\bar{\chi}_{ij}(t_1,t_2) = \chi_{ij}(t_1) - \chi_{ij}(t_2) \tag{10b}$$

and

$$W(t_1,t_2,t_3) = \frac{1}{2} A_{ijrs}(t_3)\bar{e}_{ij}(t_1,t_2)\bar{e}_{rs}(t_1,t_2) + B_{ijrs}(t_3)\bar{e}_{ij}(t_1,t_2)\bar{\chi}_{rs}(t_1,t_2) + \frac{1}{2} C_{ijrs}(t_3)\bar{\chi}_{ij}(t_1,t_2)\bar{\chi}_{rs}(t_1,t_2), \tag{11a}$$

$$\Lambda(t_i,t_2,t_3) = \frac{1}{2} \dot{A}_{ijrs}(t_3)\bar{e}_{ij}(t_1,t_2)\bar{e}_{trs}(t_1,t_2) + \dot{B}_{ijrs}(t_3)\bar{e}_{ij}(t_1,t_2)\bar{\chi}_{rs}(t_1,t_2) + \frac{1}{2} \dot{C}_{ijrs}(t_3)\bar{\chi}_{ij}(t_1,t_2)\bar{\chi}_{rs}(t_1,t_2), \tag{11b}$$

$$\Gamma(t_1,t_2,t_3) = \frac{1}{2} \ddot{A}_{ijrs}(t_3)\bar{e}_{ij}(t_1,t_2)\bar{e}_{rs}(t_1,t_2) + \ddot{B}_{ijrs}(t_3)\bar{e}_{ij}(t_1,t_2)\bar{\chi}_{rs}(t_1,t_2) + \frac{1}{2} \ddot{C}_{ijrs}(t_3)\bar{\chi}_{ij}(t_1,t_2)\bar{\chi}_{rs}(t_1,t_2). \tag{11c}$$

Let us note that

$$\bar{e}_{ij}(t,0) = e_{ij}(t),$$

$$\bar{\chi}_{ij}(t,0) = \chi_{ij}(t).$$

Theorem 3.1. *Let* $s = \{u_i, \varphi_i, e_{ij}, \chi_{ij}, t_{ij}, m_{ij}\}$ *be a viscoelastic process for B corresponding to null* *body loads. Then*

$$\int_0^t (t_{ij}\dot{e}_{ij} + m_{ij}\dot{\chi}_{ij})\ \mathrm{d}s = W(t,0,t) - \int_0^t \Lambda(\tau,0,\tau)\ \mathrm{d}\tau - \int_0^t \Lambda(t,\tau,t-\tau)\ \mathrm{d}\tau + \frac{1}{2}\int_0^t \int_0^t \Gamma(r,s,\mid r-s\mid)\ \mathrm{d}r\ \mathrm{d}s. \tag{12}$$

Proof. Clearly, in view of Eqs. (7) and (9), we have

$$\begin{aligned}\int_0^t t_{ij}\dot{e}_{ij}\ \mathrm{d}s &= \int_0^t \left[\frac{\mathrm{d}}{\mathrm{d}s}(t_{ij}e_{ij}) - e_{ij}\dot{t}_{ij}\right] \mathrm{d}s = t_{ij}e_{ij} - t_{ij}(0)e_{ij}(0) - \int_0^t t_{ij}e_{ij}\ \mathrm{d}s = t_{ij}e_{ij} - \int_0^t e_{ij}(s)\{A_{ijmn}(0)\dot{e}_{mn}(s) + B_{ijmn}(0)\dot{\chi}_{mn}(s) + \dot{A}_{ijmn}(0)e_{mn}(s) + \dot{B}_{ijmn}(0)\chi_{mn}(s) \\ &\quad + \int_0^s [\ddot{A}_{ijmn}(s-\tau)e_{mn}(\tau) + \ddot{B}_{ijmn}(s-\tau)\chi_{mn}(\tau)]\ \mathrm{d}\tau\}\ \mathrm{d}s.\end{aligned} \tag{13}$$

If we take into account the identities [9]

$$f_{ij}(t)\int_0^t \dot{M}_{ijrs}(t-\tau)g_{rs}(\tau)\ \mathrm{d}\tau + g_{rs}(t)\int_0^t \dot{M}_{ijrs}(t-\tau)f_{ij}(\tau)\ \mathrm{d}\tau = \int_0^t \dot{M}_{ijrs}(t-\tau)f_{ij}(\tau)g_{rs}(\tau)\ \mathrm{d}\tau - \int_0^t \dot{M}_{ijrs}(t-\tau)[f_{ij}(t) - f_{ij}(\tau)][g_{rs}(t) - g_{rs}(\tau)]\ \mathrm{d}\tau + [M_{ijrs}(t) - M_{ijrs}(0)]f_{ij}(t)g_{rs}(t),$$

$$2\int_0^t \int_0^s M(s-\tau)\ \mathrm{d}s\ \mathrm{d}\tau = \int_0^t \int_0^t M(\mid s-\tau\mid)\ \mathrm{d}s\ \mathrm{d}\tau,$$

$$2\int_0^t \int_0^r \ddot{M}_{ijmn}(r-s)f_{ij}(r)f_{mn}(s)\ \mathrm{d}r\ \mathrm{d}s = \int_0^t \int_0^t \ddot{M}_{ijmn}(\mid r-s\mid)f_{ij}(r)f_{mn}(s)\ \mathrm{d}r\ \mathrm{d}s = \int_0^t \int_0^t \ddot{M}_{ijmn}(\mid r-s\mid)f_{ij}(s)f_{mn}(s)\ \mathrm{d}r\ \mathrm{d}s - \frac{1}{2}\int_0^t \int_0^t \ddot{M}_{ijmn}(\mid r-s\mid)[f_{ij}(r) - f_{ij}(s)][f_{mm}(r) - f_{mn}(s)]\ \mathrm{d}r\ \mathrm{d}s,$$

$$\int_0^t \ddot{M}(|\, s-\tau\,|)\ \mathrm{d}\tau = \int_0^s \ddot{M}(s-\tau)\ \mathrm{d}\tau + \int_s^t \ddot{M}(\tau-s)\ \mathrm{d}\tau = \dot{M}(s) + \dot{M}(t-s) - 2\dot{M}(0),$$

then from Eqs. (5), (7), (9), (11) and (13) we obtain the relation (12).

We introduce the notation

$$T = (t_{ij}, m_{ij}).$$

The magnitude of T is defined by

$$|\, T\,| = (t_{ij}t_{ij} + m_{ij}m_{ij})^{1/2}. \tag{14}$$

Moreover, we introduce the notations

$$\Pi(r,s) = \frac{1}{2}A_{ijpq}(s)t_{ij}(r)t_{pq}(r) + B_{ijpq}(s)t_{ij}(r)m_{pq}(r) + \frac{1}{2}C_{ijpq}(s)m_{ij}(r)m_{pq}(r), \tag{15a}$$

$$\Phi(\tau) = \frac{1}{2}A_{ijrs}(0)z_{ij}(\tau)z_{rs}(\tau) + B_{ijrs}(0)z_{ij}(\tau)y_{rs}(\tau) + \frac{1}{2}C_{ijrs}(0)y_{ij}(\tau)y_{rs}(\tau), \tag{15b}$$

$$\Psi(s,\tau) = \frac{1}{2}\dot{A}_{ijpq}(s-\tau)\alpha_{ij}(s,\tau)\alpha_{pq}(s,\tau) + \dot{B}_{ijrs}(s-\tau)\alpha_{ij}(s,\tau)\beta_{rs}(s,\tau) + \frac{1}{2}\dot{C}_{ijpq}(s-\tau)\beta_{ij}(s,\tau)\beta_{pq}(s,\tau), \tag{15c}$$

where

$$z_{ij} = \frac{1}{\alpha}t_{ij} - \alpha e_{ij}, \tag{16a}$$

$$y_{ij} = \frac{1}{\alpha}m_{ij} - \alpha\chi_{ij}, \tag{16b}$$

$$\alpha_{ij}(s,\tau) = \frac{1}{\alpha}t_{ij}(s) + \alpha e_{ij}(\tau), \tag{16c}$$

$$\beta_{ij}(s,\tau) = \frac{1}{\alpha}m_{ij}(s) + \alpha\chi_{ij}(s), \quad \alpha > 0. \tag{16d}$$

Theorem 3.2. *Let* $s = \{u_i,\ \varphi_i,\ e_{ij},\ \chi_{ij},\ t_{ij},\ m_{ij}\}$ *be a viscoelastic process for* B *corresponding to null body loads. Then*

$$\int_0^t \mid T(\tau) \mid^2 \ \mathrm{d}\tau = \frac{1}{\alpha^2} \int_0^t [2\Pi(\tau,0) - \Pi(\tau,\tau)] \ \mathrm{d}\tau + \alpha^2 \int_0^t [2W(\tau,0,0) - W(\tau,0,t-\tau)] \\ \mathrm{d}\tau - \int_0^t \Phi(\tau) \ \mathrm{d}\tau + \int_0^t \int_0^s \Phi(\tau,s) \ \mathrm{d}\tau \ \mathrm{d}s. \tag{17}$$

Proof. It follows from Eq. (7) that

$$\int_0^t t_{ij} t_{ij} \ \mathrm{d}s = \int_0^t [A_{ijpq}(0) e_{pq}(s) t_{ij}(s) + B_{ijpq}(0) \chi_{pq}(s) t_{ij}(s)] \\ \mathrm{d}s + \int_0^t \int_0^s [\dot{A}_{ijmn}(s-\tau) e_{mn}(\tau) t_{ij}(s) + \dot{B}_{ijmn}(s-\tau) \chi_{mn}(\tau) t_{ij}(s)] \ \mathrm{d}s \ \mathrm{d}\tau. \tag{18}$$

If **M** and **H** have the symmetries

$$M_{ijrs} = M_{rsij},$$

$$H_{ijrs} = H_{rsij},$$

then, for any positive number α, we can write

$$M_{ijmn}(t) f_{mn}(r) g_{ij}(s) = \frac{1}{2\alpha^2} M_{ijpq}(t) f_{pq}(r) f_{ij}(r) + \frac{1}{2}\alpha^2 M_{ijpq}(t) g_{ij}(s) g_{pq}(s) \\ - \frac{1}{2} M_{ijpq}(t) \left(\frac{1}{\alpha} f_{ij}(r) - \alpha g_{ij}(s) \right) \left(\frac{1}{\alpha} f_{pq}(r) - \alpha g_{pq}(s) \right), \tag{19a}$$

$$H_{ijpq}(r,s) f_{pq}(s) g_{ij}(r) = \frac{1}{2} H_{ijpq}(r,s) \left(\frac{1}{\alpha} f_{pq}(s) + \alpha g_{pq}(r) \right) \left(\frac{1}{\alpha} f_{ij}(s) + \alpha g_{ij}(r) \right) \\ - \frac{1}{2\alpha^2} H_{ijpq}(r,s) f_{pq}(s) f_{ij}(s) - \frac{1}{2}\alpha^2 H_{ijpq}(r,s) g_{ij}(r) g_{pq}(r). \tag{19b}$$

Clearly,

$$\int_0^s \dot{F}_{ijpq}(s-\tau) f_{ij}(s) g_{pq}(s) \ \mathrm{d}\tau = (F_{ijpq}(s) - F_{ijpq}(0)) f_{ij}(s) g_{pq}(s), \tag{20a}$$

$$\int_0^t \int_0^s F_{ijpq}(s-\tau) f_{ij}(\tau) g_{pq}(\tau) \, d\tau \, ds = \int_0^t \int_0^t \dot{F}_{ijpq}(s-\tau) f_{ij}(\tau) g_{pq}(\tau) \, ds$$
$$d\tau = \int_0^t (F_{ijpq}(t-\tau) - F_{ijpq}(0)) f_{ij}(\tau) g_{pq}(\tau) \, d\tau. \tag{20b}$$

In view of Eqs. (7), (14)–(16), (18)–(20) we obtain the relation (17).

Now we assume that

1. the quadratic form W is positive definite, that is, there exists a positive constant c_1 so that

$$W(t_1,t_2,t_3) \geq c_1(\bar{e}_{ij}(t_1,t_2)\bar{e}_{ij}(t_1,t_2) + \bar{\chi}_{ij}(t_1,t_2)\bar{\chi}_{ij}(t_1,t_2)) = c_1 \mid F(t_1,t_2) \mid^2 , \tag{21}$$

for any $F(t_1, t_2) = (\bar{e}_{ij}(t_1, t_2), \bar{\chi}_{ij}(t_1, t_2))$,

2. there exists a positive constant c_2 so that

$$W \leq c_2 \mid F \mid^2 ; \tag{22}$$

3. the quadratic forms Λ and Γ have the properties

$$\Lambda \leq 0, \tag{23a}$$

$$\Gamma \geq 0. \tag{23b}$$

These assumptions are consistent with the dissipation inequality for materials with memory. Assumptions 1–3 have been used in classical viscoelasticity to obtain stability results.

Theorem 3.3 *Assume that the hypotheses 1–3 hold. Let $u = \{u_i, \varphi_i, e_{ij}, \chi_{ij}, t_{ij}, m_{ij}\}$ be a viscoelastic process for B corresponding to null body loads. Then*

$$\int_0^t \mid T(\tau) \mid^2 \, d\tau \leq 16 c_2^2 c_1^{-1} \int_0^t \int_0^s [t_{ij}(\tau)\dot{e}_{ij}(\tau) + m_{ij}(\tau)\dot{\chi}_{ij}(\tau)] \, d\tau \, ds. \tag{24}$$

Proof. By Theorem 3.2 and hypotheses 1–3, we obtain

$$\int_0^t \mid T(\tau) \mid^2 \, d\tau \leq \frac{2}{\alpha^2} c_2 \int_0^t \mid T(\tau) \mid^2 \, d\tau + 2\alpha^2 c_2 \int_0^t \mid E(\tau) \mid^2 \, d\tau.$$

If we choose $\alpha = 2\sqrt{c_2}$, then we obtain

$$\int_0^t \mid T(\tau) \mid^2 \, d\tau \leq 16 c_2^2 \int_0^t \mid E(\tau) \mid^2 \, d\tau. \tag{25}$$

From Theorem 3.1 we get

$$\int_0^t [t_{ij}(\tau)\dot{e}_{ij}(\tau) + m_{ij}(\tau)\dot{\chi}_{ij}(\tau)]\ \mathrm{d}\tau \geq c_1 \mid E(t) \mid^2 . \tag{26}$$

Clearly, the inequalities (25) and (26) imply that the relation (24) holds.

4. A decay estimate

In what follows, we assume that region B refers to the interior of a region whose boundary includes a plane portion D_0 (cf. Ref. [10]). We choose the rectangular system of coordinates such that D_0 lies in the x_1Ox_2—plane, and that B lies in the region $x_3 > 0$. We denote by $D(z)$ the intersection of B with the plane $x_3 = z$. We consider the boundary conditions

$$t_{ji}n_j = 0 \tag{27a}$$

$$m_{ji}n_j = 0 \tag{27b}$$

on $(\partial B \backslash D_0) \times (0,\infty)$.

We assume that on D_0 the boundary data are different from zero. We denote by $B(z)$ the region $\{\mathbf{x} \in B: x_3 > z\}$. Let L be the maximum value of x_3 on B.

Theorem 4.1. *Assume that the hypotheses 1–3 hold. Let K denote the set of viscoelastic processes for B that satisfy the initial conditions (6) and the boundary conditions (27), and for each $t \in I$ and $z \in [0, L]$ define the functional $E(\cdot, z, t)$ on K by*

$$E(p,z,t) = \int_0^t \int_{B(z)} \left\{ \frac{1}{2}[\rho \dot{u}_i(s)\dot{u}_i(s) + J\dot{\varphi}_i(s)\dot{\varphi}_i(s)] + \int_0^s [t_{ij}(\tau)\dot{e}_{ij}(\tau) + m_{ij}(\tau)\dot{\chi}_{ij}(\tau)]\ \mathrm{d}\tau \right\} \mathrm{d}v\ \mathrm{d}s \tag{28}$$

for any $p = \{u_i, \varphi_i, e_{ij}, \chi_{ij}, t_{ij}, m_{ij}\} \in K$. Then

$$E(p,z,t) \leq E(p,0,t)\ \exp\left(-\frac{z}{\lambda t}\right), \tag{29}$$

for any $p \in K$, $z \in [0, L]$ and $t \in I$, where

$$\lambda = 2c_2\sqrt{\frac{2(J+\rho)}{\rho c_1 J}}. \tag{30}$$

Proof. It follows from Eqs. (1), (3) and the divergence theorem that

$$\int_{B(z)} (t_{ij}\dot{e}_{ij} + m_{ij}\dot{\chi}_{ij})\ \mathrm{d}v = \int_{D(z)} (t_i\dot{u}_i + m_i\dot{\varphi}_i)\ \mathrm{d}a - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{B(z)} (\rho\dot{u}_i\dot{u}_i + J\dot{\varphi}_i\dot{\varphi}_i)\ \mathrm{d}v. \tag{31}$$

By the initial conditions (6), the boundary conditions (27), the relations (28) and (31), we have

$$E(p,z,t) = \int_0^t \int_0^s \int_{D(z)} (t_i(\tau)\dot{u}_i(\tau) + m_i(\tau)\dot{\varphi}_i(\tau))\ \mathrm{d}a\ \mathrm{d}\tau\ \mathrm{d}s. \tag{32}$$

With the help of the arithmetic–geometric mean inequality we obtain

$$E(p,z,t) \leq \frac{1}{2}t\int_0^t \int_{D(z)} \left(\frac{1}{\beta}\dot{u}_i\dot{u}_i + \frac{1}{\gamma}\dot{\varphi}_i\dot{\varphi}_i + \beta t_{ij}t_{ij} + \gamma m_{ij}m_{ij}\right)\ \mathrm{d}a\ \mathrm{d}\tau, \tag{33}$$

when β and γ are arbitrary positive constants. If we take $\gamma = \beta\rho/J$, then from inequality (33) we obtain

$$E(p,z,t) \leq \frac{1}{2}t\int_0^t \int_{D(z)} \left[\frac{1}{\beta\rho}(\rho\dot{u}_i\dot{u}_i + J\dot{\varphi}_i\dot{\varphi}_i) + \beta\left(1 + \frac{\rho}{J}\right)T^2(\tau)\right]\ \mathrm{d}a\ \mathrm{d}\tau. \tag{34}$$

In view of inequalities (24) and (34) we get

$$\begin{aligned} E(p,z,t) \leq \frac{1}{2}t\int_0^t \int_{D(z)} \Bigg\{ &\frac{1}{\beta\rho}[\rho\dot{u}_i(s)\dot{u}_i(s) + J\dot{\varphi}_i(s)\dot{\varphi}_i(s)] \\ &+ \frac{16\beta c_2^2(J+\rho)}{c_1\rho}\int_0^s [t_{ij}(\tau)\dot{e}_{ij}(\tau) + m_{ij}(\tau)\dot{\chi}_{ij}(\tau)]\ \mathrm{d}\tau \Bigg\}\ \mathrm{d}a\ \mathrm{d}s. \end{aligned} \tag{35}$$

We choose

$$\beta = \frac{1}{2c_2}\sqrt{\frac{c_1 J}{2\rho(J+\rho)}}.$$

Then, relation (35) becomes

$$E(p,z,t) \le \lambda t \int_0^t \int_{D(z)} \left\{ \frac{1}{2}[\rho \dot{u}_i(s)\dot{u}_i(s) + J\dot{\varphi}_i(s)\dot{\varphi}_i(s)] + \int_0^s [t_{ij}(\tau)\dot{e}_{ij}(\tau) + m_{ij}(\tau)\dot{\chi}_{ij}(\tau)]\,\mathrm{d}\tau \right\} \mathrm{d}a \,\mathrm{d}s, \tag{36}$$

where λ is given by the equality (30).

With the help of relation (28) we obtain

$$\frac{\mathrm{d}E(p,z,t)}{\mathrm{d}z} = -\int_0^t \int_{D(z)} \left\{ \frac{1}{2}[\rho \dot{u}_i(\tau)\dot{u}_i(\tau) + J\dot{\varphi}_i(s)\dot{\varphi}_i(s)] + \int_0^s [t_{ij}(\tau)\dot{e}_{ij}(\tau) + m_{ij}(\tau)\dot{\chi}_{ij}(\tau)]\mathrm{d}\tau \right\} \mathrm{d}a \,\mathrm{d}s. \tag{37}$$

It follows from relations (36) and (37) that

$$\lambda t \frac{\mathrm{d}E(p,z,t)}{\mathrm{d}z} + E(p,z,t,) \ge 0.$$

This inequality leads to (29).

For an extensive review of the literature on Saint-Venant's principle the reader is referred to the works of Horgan [11,12].

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