Some results on Spreads and Ovoids

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Abstract

We survey some results on ovoids and spreads of finite polar spaces, focusing on the ovoids of $H(3, q^2)$ arising from spreads of $PG(3, q)$ via indicator sets and Shult embedding, and on some related constructions. We conclude with a remark on symplectic spreads of $PG(2n-1, q)$.

1 Introduction

Let q be any prime power and let $PG(2n-1, q)$ be the projective space of dimension $2n-1$ over the Galois field $GF(q)$. A $(n-1)$-spread $\mathcal{S}$ of $PG(2n-1, q)$ is a set of q^n+1 mutually skew $(n-1)$-dimensional subspaces; hence the elements of $\mathcal{S}$ partition the pointset of $PG(2n-1, q)$. Spreads of $PG(2n-1, q)$ define translation planes of order q^n , with kernel containing $GF(q)$, embedding $PG(2n-1, q)$ as a hyperplane in a $PG(2n, q)$ and using the well known André-Bruck/Bose construction, and conversely. This relationship is probably the main motivation for the study of spreads, and the most studied case is $n = 2$.

Bruck in [8] introduced indicator sets in finite desarguesian projective planes of square order, and their links with line spreads of projective 3-spaces have been studied in the next years by Bruck himself in [9] and by Bruen in [10]; a few years later, Lunardon in [15] further studied that relationship, mainly from the synthetic geometric point of view: with any spread of $PG(3, q)$ a family of indicator sets is associated. Indicator sets have been somehow aside for many years, until Shult in [19] proved that a suitable set of lines, presently called aShult set, defines a locally Hermitian ovoid of the Hermitian variety via the so-called Shult embedding, and conversely.

As a Shult set is the point-line dual of an indicator set, there immediately followed a link between spreads of $PG(3, q)$ and families of locally Hermitian ovoids of $H(3, q^2)$, which was first studied by Cossidente, Ebert, Marino and Siciliano in [11] focusing on those associated with the regular spread, the so called classical and semiclassial ovoids of the Hermitian variety. In the subsquent paper [12] Cossidente, Lunardon, Marino and Polverino classified the ovoids arising from the regular spread and from a (proper) semifield spread via the above construction, while in [2] Bader, Marino, Polverino and Trombetti further studied the collineation group of the translation ovoids constructed via a Shult embedding and pointed out that two constructions which could be performed (a family of ovoids of the Klein quadric from the given family of locally Hermitian ovoids of the Hermitian variety via a construction of Lunardon [17] and a family

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of line spreads from the given family of Shult sets via a construction of Thas [21]) do not produce any new example.

Here we deal with these results and we conclude the paper with a remark linking symplectic spreads of $PG(2n-1,q)$ and Thas maximal arcs in projective planes of order q^n and kernel containing $GF(q)$.

2 Spreads of $PG(3,q)$, ovoids of $H(3,q^2)$ and some related constructions

2.1 Spreads, indicator sets, Shult sets

View $\Sigma = PG(3,q)$ as a canonical subgeometry of a $\Sigma^* = PG(3,q^2)$; let σ be the collineation of Σ^* fixing Σ pointwise (hence $\sigma^2 = id$) and let $\mathcal{S}$ be any spread of Σ. Fix a line l in $\mathcal{S}$. A plane $\pi \cong PG(2,q^2)$ of Σ^* is an *indicator plane* of $\mathcal{S}$ if $\pi \cap \Sigma = l$; the *indicator set* of $\mathcal{S}$ in π is $I_\pi(\mathcal{S}) = \{m^* \cap \pi | m \in \mathcal{S}\}$, where m^* denotes the unique line of Σ^* containing m. The set $I_\pi(\mathcal{S})$ has size q^2 and none of its secants contains points of l; conversely, any set I' of points of π satisfying the previous two properties canonically defines a spread, namely $\mathcal{S}' = \{< Q, Q^\sigma > \cap \Sigma \mid Q \in I'\} \cup \{l\}$ and $I_\pi(\mathcal{S}') = I'$. Hence, with any spread $\mathcal{S}$ a family is associated of indicator sets $I_\pi(\mathcal{S})$. Furthermore, the spread $\mathcal{S}$ is regular if and only if any $I_\pi(\mathcal{S})$ is either an affine line (*classical indicator set*) or an affine Baer subplane (*semiclassical indicator set*). For more details, see e.g. [8], [9], [10] and [15].

Let $\Sigma = PG(3,q), \Sigma^* = PG(3,q^2)$, the plane π and the line l be as above, and denote by l^* the line of Σ^* containing l. Let $\hat{\pi}$ be the dual plane of π and let P denote the point of $\hat{\pi}$ corresponding to the line l^*. The points of l are mapped to the lines of a cone $\hat{l}$ of $\hat{\pi}$ having vertex P. Let $\mathcal{F}$ be the set of lines of $\hat{\pi}$ corresponding to the points of the indicator set I. Then: (i) $\hat{\pi}$ is a projective plane with a distinguished degenerate Hermitian variety (the Baer subpencil $\hat{l}$ with vertex P); (ii) $\mathcal{F}$ is a set of q^2 lines of $\hat{\pi}$, none of which contains P; (iii) any two distinct lines of $\mathcal{F}$ intersect in a point not on the Baer subpencil. Any set of lines satisfying the above three properties is called a *Shult set*. Conversely, a Shult set defines, by any polarity of its plane, an indicator set. In conclusion, with any line spread a family of indicator sets or, equivalently, a family of Shult sets is associated.

2.2 Shult embedding

A Hermitian surface $\mathcal{H} = H(3,q^2)$ of $PG(3,q^2)$ is the set of all isotropic points of a non-degenerate unitary polarity. A line of $PG(3,q^2)$ meets $\mathcal{H}$ in 1 (*tangent*) or $q+1$ (*hyperbolic line*) or q^2+1 (*generator*) points. The hyperbolic lines intersect $\mathcal{H}$ in Baer sublines which are called *chords*.

An ovoid $\mathcal{O}$ of $\mathcal{H}$ is a set of q^3+1 points such that any generator of $\mathcal{H}$ contains exactly one point of $\mathcal{O}$. The Hermitian curve $H(2,q^2)$, intersection of $\mathcal{H}$ with any of its secant planes, is the *classical* ovoid. An ovoid is called *locally Hermitian* with respect to a point P if it is the union of q^2 chords of $\mathcal{H}$ through P and is called *translation* with respect to a point P if there is a collineation group of $\mathcal{H}$ fixing P, all the generators through P, and acting regularly on the points of $\mathcal{O} \setminus \{P\}$. Note that any translation ovoid is locally Hermitian ([7]) but not conversely, and a classical ovoid of $\mathcal{H}$ is a translation ovoid with respect to each of its points.

Start off with a spread $\mathcal{S}$ of $PG(3,q)$, fix a line l in $\mathcal{S}$, an indicator plane π through l as above, construct the indicator set and polarize to a Shult set $\mathcal{F}$ with respect to the subpencil $\hat{l}$ in the plane $\hat{\pi} = PG(2,q^2)$; embed the plane $\hat{\pi}$ in a $PG(3,q^2)$ containing a Hermitian surface $\mathcal{H}$ such that $\hat{\pi}$ is the tangent plane to $\mathcal{H}$ at P and $\hat{l} = \mathcal{H} \cap \ \hat{\pi}$; denote by ρ be the polarity defined by $\mathcal{H}$. Then Shult has proved in [19] that $\mathcal{O}_\pi(\mathcal{S}) = \bigcup\{L^\rho | L \in \mathcal{F}\}$ is an ovoid of $\mathcal{H}$, which is, by construction, locally Hermitian with respect to its point P. The above construction is presently called a *Shult embedding* following [11].

We explicitly note that on the other hand, via the so-called *Hermitian embedding* defined by Cossidente, Ebert, Marino and Siciliano in [11], symplectic spreads of $PG(3,q)$ are characterised as those corresponding to indicator sets embedded in a Hermitian variety $\mathcal{H}$, and conversely. Namely, let δ be a symplectic polarity commuting with the unitary polarity ρ associated with $\mathcal{H} = H(3,q^2)$. The map $\sigma = \delta \circ \rho = \rho \circ \delta$ is a (non-linear) collineation, fixing $q^3 + q^2 + q + 1$ points on $\mathcal{H}$ but no point off $\mathcal{H}$, and leaving invariant q^3+q^2+q+1 generators of $\mathcal{H}$. Also, noting that any fixed point (invariant generator resp.) is incident with $q+1$ invariant generators (fixed points resp.), yields a symmetric configuration which extends in a suitable way to a symplectic polar space $\mathcal{W} = W(3,q)$ embedded in a subgeometry $\Sigma = PG(3,q)$ of the starting Σ^* containing $\mathcal{H}$. In this context, the totally isotropic lines of Σ with respect to δ are exactly the lines of $\mathcal{W}$. Let $\mathcal{S}$ be a spread of Σ whose lines are isotropic with respect to δ. Let l be a line of $\mathcal{S}$ and denote by l^* the line of Σ^* containing l, which is a generator of $\mathcal{H}$. Fix a point $P \in l^* \setminus l$, hence $P^\rho \cap \mathcal{W} = l$. The indicator set is contained in $\mathcal{H}$ and consists of the points in which all the extended lines of $\mathcal{S}$ meet P^ρ. The construction above can be reversed. Unfortunately, the Hermitian embedding does not produce any locally Hermitian ovoid (whereas the Shult embedding does) because the dual lines of the starting Hermitian indicator set not necessarily are hyperbolic lines of $\mathcal{H}$.

2.3 Semiclassical ovoids of $H(3,q^2)$

In [11] the Shult embedding is used to construct the classical ovoid and two semiclassical ovoids of $\mathcal{H}(3,q^2)$ arising from classical and semiclassical indicator sets, respectively. Also, the groups of those ovoids are computed, proving that if $q > 3$ there exist at least two (non isomorphic) semiclassical ovoids of $\mathcal{H}(3,q^2)$, depending on the elliptic quadric $\mathcal{Q} = Q^-(3,q)$, image of the points of the indicator set being permutable or not, i.e. the polarity defined by the $Q^+(5,q)$ containing $\mathcal{Q}$ commutes with the unitary polarity associated with the Hermitian variety. The first one, called the p-*semiclassical ovoid* (permutable semiclassical ovoid), has an elementary abelian p-group ($q = p^r$).

The notion of commuting polarities was introduced by Tits in 1955, and Segre in 1965 studied Hermitian geometry over finite fields, also investigating the polarities commuting with a unitary one. Starting with Segre's results, recently Cossidente, de Resmini and Marino in [13] have studied various geometrical and combinatorial properties of permutable polarities, with special regard to unitary polarities commuting with orthogonal ones, focusing on the relationship between (regular) symplectic spreads of $PG(3,q)$ and some remarkable subsets of the Hermitian curve $H(2, q^2)$, the so-called C_F-sets after Donati and Durante. FutHermore, they discuss symplectic polarities commuting with unitary polarities.

In order to compute the number of non isomorphic ovoids of $\mathcal{H}(3, q^2)$ arising via the Shult embedding, the following definition has been introduced by Cossidente, Lunardon, Marino and Polverino in [12]: two indicator sets I_1 and I_2 in the same Σ^* lying on the indicator planes π_1 and π_2, respectively, passing through the line l^*, are said *isomorphic* if the associated spreads of Σ are, whereas they are said *equivalent* if there is a collineation of Σ^* mapping I_1 to I_2 and fixing the Baer subline l. Note that equivalent indicator sets are isomorphic, whereas it is worth noting that isomorphic indicator sets may be non equivalent. With this approach, they can prove that two locally Hermitian ovoids of $\mathcal{H}(3, q^2)$ are isomorphic if and only if the corresponding indicator sets are equivalent, and consequently show that the number θ of non isomorphic semiclassical ovoids of $\mathcal{H}(3, q^2)$ is $\frac{q-3}{2} + 1$ if q is a prime with $q \geq 3$, whereas the following bounds hold for any q : $2 \leq \theta \leq \frac{q-2}{2}$ if q is even and $q > 2$, and $2 \leq \theta \leq \frac{q-3}{2} + 1$ if q is odd and $q > 3$. For further details, see [12].

2.4 Translation ovoids and their group

To obtain futher information on the collineation group of the ovoids arising from the Shult embedding, we specialize to a distinguished class of spreads, namely semifield spreads. A spread $\mathcal{S}$ is a *semifield spread* with respect to its line ℓ_∞ if there exists a group fixing the line ℓ_∞ pointwise and acting regularly on the set of the q^2 lines of $\mathcal{S}$ different from ℓ_∞. Moreover, if $\mathcal{S}$ is a semifield spread with respect to the line ℓ_∞ then, for any choice of the indicator plane π such that $\ell_\infty \subset \pi$, the ovoid $\mathcal{O}_\pi(\mathcal{S})$ is a translation ovoid with respect to the point P, and conversely.

Choose homogeneous projective coordinates (x_0, x_1, x_2, x_3) in such a way that $\mathcal{S} = \{\ell_\infty, \ell_{u,v} | u, v \in GF(q)\}$ with $\ell_\infty : x_0 = x_1 = 0$, and $\ell_{u,v} = \{(a, b, c, d) : (c, d) = (a, b)X_{u,v},\ a, b \in GF(q)\}$ where $X_{u,v} = \begin{pmatrix} v & h(u,v) \\ u & k(u,v) \end{pmatrix}$ with $h, k : GF(q) \times GF(q) \to GF(q),\quad h(0, 0) = k(0, 0) = 0$. Since $\mathcal{S}$ is a semifield spread, then $\{X_{u,v} | u, v \in GF(q)\}$ is closed under addition hence h and k are additive functions.

If $\pi_\lambda : x_1 = \lambda x_0$ is any indicator plane through ℓ_∞, where $\lambda \in GF(q^2) \setminus GF(q)$, then $I_{\pi_\lambda}(\mathcal{S}) = I_\lambda(\mathcal{S}) = \{(1, \lambda, v + \lambda u, h(u, v) + \lambda k(u, v)) : u, v \in GF(q)\}$ and

$$\mathcal{O}_\lambda(\mathcal{S}) = \{(1, -v - \lambda^q u, h(u, v) + \lambda^q k(u, v), \alpha + \lambda(vk(u, v) - uh(u, v))) : \\ u, v, \alpha \in GF(q)\} \cup \{P = (0, 0, 0, 1)\}$$

is the locally Hermitian ovoid (with respect to P) of $\mathcal{H}(3, q^2) : y_0 y_3^q - y_3 y_0^q + y_2 y_1^q - y_1 y_2^q = 0$ arising via the Shult embedding (for more details, see [12]).

Let $PGU(4, q^2)$ be the group of the linear collineations of $PG(3, q^2)$ leaving $\mathcal{H}$ invariant. The subgroup E of $PGU(4, q^2)$ fixing P and leaving invariant all the generators through P has size q^5 ([18]) and direct computations show that E consists of the matrices

$$\begin{pmatrix} 1 & \alpha & \beta & c-\alpha\beta^q \\ 0 & 1 & 0 & -\beta^q \\ 0 & 0 & 1 & \alpha^q \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \alpha,\beta \in GF(q^2), c \in GF(q).$$

The subgroup of E acting as translation group on the ovoid $\mathcal{O}_\lambda(\mathcal{S})$ is explicitly computed in [2] as

$$G = \left\{ \begin{pmatrix} 1 & -v-\lambda^q u & h(u,v)+\lambda^q k(u,v) & c+(v+\lambda^q u)(h(u,v)+\lambda k(u,v)) \\ 0 & 1 & 0 & -h(u,v)-\lambda k(u,v) \\ 0 & 0 & 1 & -v-\lambda u \\ 0 & 0 & 0 & 1 \end{pmatrix}; \right.$$

$$\left. u,v,c \in F_q \right\}.$$

As $H(3,q^2)$ can also be viewed as an elation generalised quadrangle, which can be represented as a coset geometry with elation group $(\tilde{E},\circ)$ where $\tilde{E} = GF(q^2)\times GF(q)\times GF(q^2)$ and $(\alpha,c,\beta)\circ(\alpha',c',\beta') = (\alpha+\alpha', c+c'+Tr(\alpha'\beta^q), \beta+\beta')$ with $\alpha,\beta \in GF(q^2)$ and $c \in GF(q)$ (see e.g. [3]), the map

$$\psi : (\alpha,c,\beta) \in \tilde{E} \to \begin{pmatrix} 1 & \alpha & \beta & c-\alpha\beta^q \\ 0 & 1 & 0 & -\beta^q \\ 0 & 0 & 1 & \alpha^q \\ 0 & 0 & 0 & 1 \end{pmatrix} \in E$$

is an isomorphism and the translation group of any translation ovoid $\mathcal{O}_\lambda(\mathcal{S})$ arising from a semifield spread $\mathcal{S}$ via the Shult embedding is isomorphic to the preimage of G

$$\tilde{G} = \psi^{-1}(G) = \{(-v-\lambda^q u, \alpha, h(u,v)+\lambda^q k(u,v)) : \ u,v,\alpha \in GF(q)\}$$

which turns out to be abelian if and only if $\mathcal{O}_\lambda(\mathcal{S})$ is p-semiclassical (see [2]).

Recall that the permutable semiclassical ovoid was the only translation ovoid constructed in [11] admitting an elementary abelian p-group , $q = p^r$, and in [12] it is proved that the $q+1$ p-semiclassical translation ovoids arising from a given regular spread are all isomorphic. Hence there exists (up to isomorphism) a unique translation ovoid of $H(3,q^2)$ with an abelian translation group, namely the p-semiclassical. For more details, see [2].

2.5 Ovoids of $Q^+(5,q)$ from indicator sets

Let $\mathcal{S}$ be any spread of $\Sigma = PG(3,q)$ containing the lines ℓ_∞ and ℓ_0 and defined by the functions h and k as in Section 2.4. Here, as $\mathcal{S}$ may not be a semifield spread, hence h and k may not be additive.

Then $\mathcal{O}_\lambda(\mathcal{S})$ are the locally Hermitian ovoids of the Hermitian surface $\mathcal{H} : x_0x_3^q - x_0^q x_3 + x_2 x_1^q - x_2^q x_1 = 0$ of $\Gamma = PG(3,q^2)$ arising from $\mathcal{S}$, as λ varies in $GF(q^2)\setminus GF(q)$.

The projective plane $\pi = PG(V,q^2)$ is the lattice of the $GF(q^2)-$subspaces of the 3-dimensional vector space V (over $GF(q^2)$); as V can also be viewed as a 6-dimensional vector space over $GF(q)$, a 5-dimensional projective space $= PG(V,q) = PG(5,q)$ arises. A point (line resp.) of π is defined by a

$GF(q^2)-$subspace of dimension 1 (2 resp.), which can be considered as a $GF(q)-$subspace of dimension 2 (4 resp.); hence the pointset of π is mapped to a lineset of Ω, which is a normal spread $\mathcal{R}_\lambda$, and any line of π is mapped to a 3-space with a regular spread consisting of the images of the points of the line itself . The pair $(\Omega, \mathcal{R}_\lambda)$ is the *F_q-linear representation* of π with respect to the basis $\{1, \lambda\}$ (for more details see [16]).

Embed the above $(\Omega, \mathcal{R}_\lambda)$ in $\Omega' = PG(6, q)$ as a hyperplane and define the point-line geometry $\pi(\Omega', \Omega, \mathcal{R}_\lambda)$ as follows. The points are either the points of $\Omega' \setminus \Omega$ or the elements of $\mathcal{R}_\lambda$. The lines are either the planes of Ω' which intersect Ω in a line of $\mathcal{R}_\lambda$ or the regular spreads of the 3-dimensional projective spaces $\langle A, B\rangle$, where A and B are distinct lines of $\mathcal{R}_\lambda$; the incidence is the natural one. As $\mathcal{R}_\lambda$ is normal, $\pi(\Omega', \Omega, \mathcal{R}_\lambda)$ is isomorphic to a $PG(3, q^2)$ containing π, and the isomorphism extends the linear representation. This is the *Barlotti-Cofman representation* of $PG(3, q^2)$ (for more details see [5]).

Lunardon in [17] has shown that the image of a Hermitian variety having π as a tangent plane, in the Barlotti-Cofman representation, is a cone having vertex in Ω and basis a suitable $Q^+(5, q)$ of Ω', and that any locally Hermitian ovoid $\mathcal{O}_\pi(\mathcal{S})$ with respect to P of $\mathcal{H}$ is mapped to an ovoid, say $\mathbb{O}_\lambda$, of the hyperbolic quadric $Q^+(5, q)$, and conversely; if $\mathcal{O}_\pi(\mathcal{S})$ is a translation ovoid, then $\mathbb{O}_\lambda$ is too.

On the other hand, to the line spread $\mathcal{S}$ there corresponds, via the Klein map, an ovoid $\mathcal{O}(\mathcal{S})$ of the Klein quadric. Answering a question posed in [17], in [2] it is shown that the ovoid $\mathcal{O}(\mathcal{S})$ is isomorphic to any $\mathbb{O}_\lambda$, for any choice of the indicator plane π, therefore no new ovoids of $Q^+(5, q)$ can be constructed in this way.

2.6 Spreads from indicator sets via locally Hermitian spreads of $Q^-(5, q)$

Let $\mathcal{S}$ be any spread of $\Sigma = PG(3, q)$. Embed Σ in $\Sigma^* = PG(3, q^2)$ in such a way that $\Sigma = Fix(\sigma)$, where σ is an involutory collineation of Σ^*. Let π be an indicator plane of $\mathcal{S}$ in $PG(3, q^2)$. Denote by l the line of $\mathcal{S}$ such that l is in π and by $I_\pi(\mathcal{S})$ the indicator set of $\mathcal{S}$ in the plane π. Consider the point-line dual plane of π: this is a plane $\tilde{\pi}$, in which l^* (the extension of l in Σ^*) is represented by a point P, the Baer subline l by a Baer subpencil $\tilde{l}$ through P and $I_\pi(\mathcal{S})$ by a set $\mathcal{F}$ of q^2 lines not containing P, any two of which them intersect at a point of $\tilde{\pi} \setminus \tilde{l}$. (The set of lines $\mathcal{F}$ is the associated Shult set.) Fix a Hermitian surface $\mathcal{H} = H(3, q^2)$ in such a way that $P \in \mathcal{H}$ and $\tilde{\pi} \cap \mathcal{H} = \tilde{l}$. Let ρ be the polarity defined by $\mathcal{H}$. The elements of $\mathcal{F}^\rho$ are hyperbolic lines of $\mathcal{H}$ through P, hence the set $\mathcal{O}_\pi = \bigcup_{m \in \mathcal{F}}(m^\rho \cap \mathcal{H})$ is a locally Hermitian ovoid of $\mathcal{H}$. (Note that the ovoid depends on the choice of the indicator plane π.) The ovoid $\mathcal{O}$ corresponds, via the Klein map κ, to a locally Hermitian spread $\mathbb{S}_\pi$ of $Q^-(5, q)$ with respect to the line $L = P^\kappa$. Let $\Lambda = L^\perp$, where $\perp$ is the orthogonal polarity induced by $Q^-(5, q)$. If M is a line of $\mathbb{S}_\pi$ different from L then $m_{L,M} = \langle L, M\rangle^\perp$ is a line of Λ disjoint from $\langle L, M\rangle$. Moreover the set of lines $\mathcal{S}'_\pi = \{m_{L,M} \colon M \in \mathbb{S}, M \neq L\} \cup \{L\}$ turns out to be a spread of Λ as proved by Thas in [21]. If the spread $\mathcal{S}$ is a semifield spread then the spread $\mathcal{S}'_\pi$ also is. In [2] it is proved that $\mathcal{S}$ and $\mathcal{S}'_\pi$ are isomorphic for any choice of the indicator plane π, the proof being obtained by reviewing the above construction embedding the involved spreads in the same 3-dimensional projective space over

$GF(q^2)$. In the case $\mathcal{S}$ is a semifield spread, the question on the relation between $\mathcal{S}$ and $\mathcal{S}'_\pi$ was posed in [17, Sect. 4.3].

3 Symplectic spreads and Thas arcs

Let $PG(2n-1,q)$ be the projective $(2n-1)-$dimensional space over F_q. A *spread* of $PG(2n-1,q)$ is a set of q^n+1 pairwise disjoint $(n-1)-$dimensional subspaces which partition the pointset of $PG(2n-1,q)$. A spread is *symplectic* if all of its elements are totally isotropic with respect to some polarity of the space, defined by a nonsingular alternating bilinear form of the underlying vector space. For more details, the reader is referred e.g. to [14].

In [20] Thas gave the following construction of a maximal arc, which is called *Thas arc*: let $Q^- = Q^-(2n-1,q)$ be an elliptic quadric of $PG(2n-1,q)$, $n \geq 2$, and let $\mathcal{S}^-$ be a spread of $Q^-(2n-1,q)$. Fix an $(n-1)-$spread $\mathcal{S}$ of $H = PG(2n-1,q)$ intersecting $Q^-(2n-1,q)$ in $\mathcal{S}^-$. Embed $PG(2n-1,q)$ as a hyperplane in $PG(2n,q)$ and fix a point $x \in PG(2n,q) \setminus PG(2n-1,q)$. The set $\{< x, y > | y \in Q^-\} \setminus Q^-$ is a maximal $(q^{2n-1} - q^n + q^{n-1}; q^{n-1})-$arc of the projective plane of order q^n defined by $\mathcal{S}$ via the usual André-Bruck/Bose construction. We recall that, following Barlotti in [4], a $\{k; m\}-$arc in a finite projective plane of order s is a set of k points such that m is the greatest number of collinear points in the set, and an arc is *maximal* if k attains its maximal value, i.e. $k = sm - s + m$. In [6] Blockhuis, Hamilton and Wilbrink proved that no Thas arcs exist for q odd, as conjectured in [20].

Recently, using some intersection properties of symplectic spreads and non-singular quadrics, it has been proved in [1] that a translation plane of order q^n , q even, with kernel containing $GF(q)$, is defined by a symplectic spread if and only if it contains a Thas arc.

In the following Bibliography a huge number of actually relevant papers and books are missing, for obvious reasons of space. We have just listed some items we explicitly refer to, and we apologize to the Authors of the many missing ones.

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