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NONLOCAL FIELD THEORY DRIVEN BY A DEFORMED PRODUCT. GENERALIZATION OF KALB–RAMOND DUALITY**

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A modification of the standard product used in local field theory by means of an associative deformed product is proposed. We present a class of deformed products, one for every spin $S = 0, 1/2, 1$, that induces a nonlocal theory, displaying different form for different fields. This type of deformed product is naturally supersymmetric and it has an intriguing duality.

Keywords: Deformed product; duality; Kalb–Ramond field; Moyal product; nonlocal field theory; supersymmetry.

1. Introduction

The main problem of any local quantum field theory is the presence of ultra-violet divergences. In fact the S-matrix is expressed in terms of the products of causal functions of the field operators. Since the causal functions have fairly strong singularities on the light cone, the products of such functions are not mathematically defined. This problem arises from the ill-defined nature of the product of two local field operators at the same space-time point. This generates one of the main

After acceptance of this paper, the authors have become aware of another important work by P. Morgan, *J. Math. Phys.* **48 (2007) 122302.

problems of quantum field theory — the so-called problem of ultraviolet divergences. There are different regularization procedures to deal with such divergences, making the S-matrix elements mathematically meaningful. These are, for example, the subtraction procedure interpreted in various manners in a local field theory, the summation of asymptotic series for the Green functions and super-propagators, a nonlocal generalization of the theory. In particular, the nonlocal quantum field theory which replaces local quantum field theory is very old, dating from 1950's, starting with Pais and Uhlenbleck (1950), Efimov and coworkers [1] (1970-onwards), Moffat, Woodard and coworkers (1990) [2, 3]. The basic idea to try to avoid "infinities" was to assume a nonlocal interaction and thus to provide a natural cut-off. One has to build a nonlocal quantum field theory which is a self-consistent scheme satisfying all principles of conventional quantum field theory (unitarity, causality, relativistic invariance, etc.) and providing the basis for correct description of nonlocality effects. There are some problems with the gauge invariance because of nonlocality in gauge field interactions. Nevertheless, there are different nonlocal approaches which give a good possibility of building a completely ultraviolet finite theory of fundamental interactions.

(i) One way is to introduce nonlocality in the interaction term [4] writing down the Lagrangian of scalar fields in the form

$$L = \phi(x)^\dagger(\partial^2 + m^2)\phi(x) + \lambda\Phi(x)^\dagger\Phi(x)\,, \tag{1}$$

where the nonlocal field $\Phi(x)$ is obtained from the local one $\phi(x)$ by "smearing" over the nonlocality domain with the characteristic scale l_0. Without specifying the nature of this nonlocality, and introducing the phenomenological form factor K, the nonlocal field $\Phi(x)$ is defined as

$$\Phi(x) \equiv \int dyK(x-y)\phi(y) = K(l_0^2\partial^2)\phi(x)\,, \tag{2}$$

where the nonlocal operator $K(l_0^2\partial^2)$ can be written in the form

$$K(l_0^2\partial^2) = \sum_{n=0}^{\infty}\frac{c_n}{(2n)!}(l_0^2\partial^2)^n\,, \tag{3}$$

K being an entire function without any zeros. Then the generalized function $K(x-y) = K(l_0^2\partial^2)\delta(x-y)$ belongs to one of the spaces of nonlocal generalized functions which was introduced and explored in the works of Efimov [1]. One rewrites the Lagrangian (1) in terms of the nonlocal fields $\Phi(x)$ as

$$L = \Phi(x)^\dagger(\partial^2 + m^2)\Phi(x) + \lambda\Phi(x)^\dagger\Phi(x)\,, \tag{4}$$

in this way the $\phi(x)$-propagator (smeared propagator) is obtained by taking the Fourier transform of

$$\frac{\exp\left[\frac{p^2-m^2}{l_0^2}\right]}{p^2-m^2+i\epsilon}\,. \tag{5}$$

This suggests interpreting the nonlocal quantum field theory as an effective theory valid up to an energy scale l_0; and for energy scales beyond l_0, one has to replace the nonlocal quantum field theory by a more fundamental theory of constituents having its own larger mass scale and coupling constant. In a sense, this formulation can be viewed either as a regularization, or as a physical theory with a finite mass parameter l_0. Such a theory preserves causality at tree-level in the S-matrix (it is the same as the local one), but it suffers from quantum causality violations, which are a serious limitation.

(ii) Another way to have a finite quantum field theory was proposed in [5] on the basis of infinite-component fields, which results in the introduction of a special form of nonlocality.

(iii) Another way to introduce nonlocality in the theory is to consider a special class of field theories with higher derivatives. In the canonical formulation, one usually considers Lagrangians with only first derivatives. However, higher derivative theories, including nonlocal theories, also have many physical applications. For example, when one integrates out high energy degrees of freedom in a local field theory, the low-energy effective action is generically nonlocal [6]. Higher derivative theories were also considered in order to find a finite quantum field theory [7], before the advent of renormalization. Moreover, theories with infinitely many derivatives are unavoidable in string theory [8, 9]. There are other examples, such as higher derivative gravity [10], meson-nucleon interactions [11], and spacetime noncommutative field theory [12, 13], and so on. In most cases, higher derivative terms appear as higher-order corrections in the effective Lagrangian, hence a perturbative approximation scheme would already be very useful.

(iv) Another example of nonlocality is in quantum mechanics where it has long been an intriguing topic in the past decades, and so far there has been no experiment contradicting nonlocality. It refers to the correlation between two particles separated in space, e.g. the entanglement derived from the Bell theory [14] and well confirmed in many experiments [15]. All these experiments used massless photons as carriers of the states, and the nonlocality is of the Bell type.

Finally we would like to mention our approach, in which nonlocality is introduced through the deformation of the product, and in a perturbative approximation it is reduced to a field theory with higher derivatives.

We shall now introduce the plan of the paper. In Sec. 2 we review the deformed products, i.e. deformed field theories and their nonlocal properties. In Sec. 3 we propose a class of deformed products which are associative, with different expressions for spin $S = 0, 1/2, 1$, respectively. In Sec. 4 we exhibit their properties. In Sec. 5 we study the deformed interaction term. In Secs. 6 and 7 we show that the scalar field theory with our deformed product is the same as found by Moffat [16]. Concluding remarks are presented in Sec. 8.

2. Review of Deformed Field Theory

Deformation quantization was born as an attempt to interpret the quantization of a classical system as an associative deformation (i.e. via star-products) of the algebra of classical observables. This idea was behind the mind of many mathematical physicists and physicists [17, 18] as illustrated by the historical developments which led to deformation quantization. By deformed it is meant that the standard point-wise multiplication of functions has been replaced by a new product which may or may not be commutative. Recently, algebras of functions with a deformed product have been studied intensively [19]. These are deformed (star-) products which remain associative but not commutative. There is a class of K-deformed products which generate deformed associative products, see for example [20]. Before talking about deformed field theory we first summarize the concept of deformed coordinate spaces as quantum spaces. Deformed coordinate spaces are defined in terms of coordinates x_μ and their commutation relations. The θ, k, q-deformations are the best known examples [21]. They are: the canonical relations $[x_\mu, x_\nu] = \theta_{\mu\nu}$, which for constant $\theta_{\mu\nu}$ leads to the so-called $\theta_{\mu\nu}$-deformed coordinate space; the Lie-type relations where the coordinates form a Lie algebra $[x_\mu, x_\nu] = C^\lambda_{\mu\nu} x_\lambda$ and $C^\lambda_{\mu\nu}$ are the structure constants, this framework leads to the k-deformed coordinate space; and then the quantum group relations, $x^\mu x^\nu = R^{\mu\nu}_{\rho\sigma} x^\rho x^\sigma \theta_{\mu\nu}$ where the R-matrix defines a quantum group, this leads to the q-deformed spaces.

Deformations of mathematical structures have been used at different moments in physics. When Galilean transformations between inertial systems were seen not to describe adequately the physical world, a deformation of the group law arose as the solution to this paradox. The Lorentz group is a deformation of the Galilei group in terms of the parameter c . In this deformation scheme, the old structure is seen as a limit or contraction when the parameter takes a preferred value. Hence a deformation, an inverse of contraction (in the sense of Segal–Wigner–Inonu contraction), is one of the methods of generalization of a physical theory [22]. The undeformed theory can be recovered from the deformed one when taking a limit of deformation parameter to some value, e.g. nonrelativistic, classical physics, the undeformed theory, is recovered from relativistic physics when taking the velocity of light $c \to \infty$, and, from the point of view of quantum physics, when taking the Planck constant $h \to 0$. The mathematical structure of quantum mechanics has also an ingredient of deformation with respect to classical mechanics. This naive concept has been applied to field theories on noncommutative spaces considered as deformations of flat Euclidean or Minkowski spaces. Since noncommutative geometry generalizes standard geometry in using a noncommutative algebra of functions, it is naturally related to the simpler context of deformation theory. The star product is a product in the space of formal power series in $\hbar$ whose coefficients are functions on the phase space. Thus, a product of fields on NC spaces can be expressed as a deformed product or star-product [23, 24] of fields on commutative spaces [25, 26]. The star product can be seen as a higher-order f-dependent differential operator acting on the function g. The noncommutativity is governed by a parameter such that the commutative case appears in the limit where this parameter approaches zero. The simplest and most well known example of $\star$-product is the Moyal–Weyl product, and it first appeared in quantum mechanics [27]. It was first introduced by

H. Weyl for his quantization procedure and later by Moyal [22] to relate functions on phase space to quantum mechanical operators in Hilbert space. For this reason the $\star$-product is called in the literature as Moyal–Weyl product. In the Moyal–Weyl $\star$-product representation the noncommutative coordinates $\hat{x}_\mu$ (and their functions) are mapped to commutative coordinates x_μ with commutative pointwise product replaced by deformed (nonlocal) $\star$-product defined as

$$\begin{aligned} f \star g(x) &\equiv \exp\left(i\theta^{\mu\nu}\partial_\mu^y \partial_\nu^z f(y) g(z)\right)|_{y=z=x} \\ &= \sum_{n=1}^{\infty} \left(\frac{i}{2}\right)^n \frac{1}{n!} \theta^{\rho_1\sigma_1} \cdot \theta^{\rho_n\sigma_n} (\partial_{\rho_1} \cdot \partial_{\rho_n} f(x))(\partial_{\sigma_1} \cdot \partial_{\sigma_n} g(x)) \,. \end{aligned} \tag{6}$$

This implies the presence of (infinitely many) derivatives in the action, hence the theory becomes nonlocal, and the noncommutative quantum field theories are a special case of a nonlocal quantum field theory. Other $\star$-products will occur for different orderings. The way of looking at noncommutative geometry in terms of deformed products can give different insights. In fact a deformed gauge theory leads to a theory with a larger symmetry structure, i.e. the enveloping algebra structure, and it exhibits its nonlocal nature. Nevertheless, the commutative field theories can be recovered from their noncommutative counterparts when the noncommutativity tensor approaches zero: $\theta_{\mu\nu} \to 0$. A property of noncommutative field theories is the presence of nonlocal interaction terms, which explicitly breaks Lorentz invariance. In fact under the integration, the star-product of fields does not affect the quadratic parts of the Lagrangian, whereas it gives rise to a nonlocal interaction part.

Hence, Feynman rules in momentum space are modified with respect to the commutative ones, in fact the vertices are modified by a phase factor. The deformed vertices differ from the nondeformed ones by a factor of type $\cos(1/2 p_\mu \theta^{\mu\nu} p_\nu)$. When $\theta_{\mu\nu} \to 0$, the deformed vertex reduces to the nondeformed one.

One can give a star-product quantization scheme following [28, 29], and see that there is a class of star products (K-star products) which are obtained via a specific deformation procedure [29]. Much more recently, noncommutative geometry has entered physics in different contexts.

One context is string theory. In their pioneering paper, Connes, Douglas and Schwarz [30] introduced noncommutative spaces (tori) as possible compactification manifolds of space-time. Noncommutative geometry arises as a possible scenario for short-distance behaviour of physical theories. In the framework of open string theory [31], Seiberg and Witten in [12] identified limits in which the whole string dynamics, in presence of a B-field, is described by a deformed gauge theory in terms of a Moyal–Weyl star product on space-time. The field theory associated to string theory, in the low-energy limit, is nonlocal, because the fields in the action are multiplied by a (deformed) star-product. The deformed theories enjoy renormalization properties as well as UV/IR connection reminiscent of string theory. Other approaches connecting deformation theory to theories of gravity have also appeared in the literature. Among others, there is the deformation quantization of M-theory [32], quantum anti-de Sitter spacetime [33], q-gravity [34] and gauge theories of quantum groups [35]. Another area covered by noncommutativity is supersymmetric theory. The deformation aspects of supersymmetric field theories were investigated in [36–39]. Analogous to noncommutative field theories on bosonic spacetime, noncommutative superfield theories can be formulated in ordinary superspace by multiplying functions given on it via a $\star$-product which is generated by some bi-differential operator or Poisson structure P. It defines a deformed superspace and leads to deformed products for general superfields. There will be symmetries of the undeformed (local) field theory which are explicitly broken in the deformed (nonlocal) case. In this case only free actions preserve all supersymmetries while interactions get deformed and are not invariant under all standard supersymmetry transformations, because the integral of the star product of two superfields is not deformed, while in the case of three or more superfields the integral is deformed.

In [37] the authors present a variety of deformations, both for $N = 1$ and extended ($N = 2$) supesymmetry in $D = 4$, which vary according to the differential operators chosen to construct the Poisson bracket that afterwards becomes quantized with a star product of Moyal–Weyl type. For example in [37], the first deformation has the advantage of being manifestly supersymmetric, while the second, although it explicitly breaks half of the supersymemtry, allows the definition of chiral and antichiral superfields, which form subalgebras of the star product. Another way to construct a deformed field theory is with derivatives which are an essential input for the construction of deformed field equations such as the deformed Klein–Gordon or Dirac equations [40].

At the end we would like to emphasize some properties of deformed field theory as a nonlocal theory. The quantum deformation modifies the behavior of relativistic theories at distances comparable to and smaller than the length l corresponding to the deforming parameter. It appears that, by virtue of the deformation of local product of fields in the interaction, the vertex will be replaced by a deformed nonlocal product, with the nonlocality extending to distances of order l.

Such a quantized space-time geometry can provide additional convergence factors or even a finite quantum field theory. Indeed, if one introduces a masslike deformation parameter, it occurs also as a regularizing parameter.

There are also attempts to remove the ultraviolet divergences by introducing nonlocality into the interaction Lagrangian. Hence the advantage of the nonlocal character of the deformed product is the following: first, one has succeeded in introducing into the interaction Lagrangian all the ambiguity in the choice of the shape and the value of the "elementary" length; second, the amplitudes of the physical processes have no additional singularities in the finite region of change of the invariant momentum variables as compared to the local theory.

Nonlocal quantum field theory faces, however, many difficulties. One of the main difficulties in constructing the non-local quantum field theory appears to be the formulation of macro-causality of the S-matrix. Then it seems that a reasonable macro-causality condition imposed on the S-matrix would be a generalization of the micro-causality condition [1]. However, as one can see in Efimov's paper [1], there is indeed a causality violation but, from the physical point of view, the problem may be formulated in such a way that the amount of causality violation would satisfy the usual requirements imposed on nonlocal theories. Indeed, using the Lagrangian of the quantized field system, Efimov expands the S-matrix in the small coupling constant. Therefore, in the case of the small coupling constant interaction, the violation of causality at large distances is rather small. Another property is the unitarity. The postulate of unitarity of the S-matrix in quantum field theory is one of the principal requirements for the theory to be regarded as self-consistent and physically acceptable. Efimov for example, in Ref. [41], proves the unitarity of the S-matrix in the n-th order of perturbation theory in a nonlocal quantum field theory.

3. A Class of Deformed Products

In this section we propose a class of associative deformed products, with different expressions for spin $S = 0, 1/2, 1$, respectively. In [29] a deformed operator product was introduced (K-product) in the form $\hat{f}_K \hat{g} = \hat{f}\hat{K}\hat{g}$ where $\hat{K}$ is a generic operator. It satisfies the associativity condition

$$(\hat{f} \triangle_K \hat{g}) \triangle_K \hat{h} = \hat{f} \triangle_K (\hat{g} \triangle_K \hat{h}) . \tag{7}$$

As emphasized in [20] the K-deformed products are a way to generate new associative products. To construct our new deformed product we take into consideration this one and that the star-product in Quantum Mechanics, because of its nonlocal nature, can be described through an integral kernel [42]. This integral kernel plays the role of the structure function for the product and the star-product reduces to the more familiar asymptotic expansion with a particular choice of this kernel. Inspired by all these properties, and bearing in mind that the star product is a particular associative deformed product, we import this formalism used in Quantum Mechanics to Field Theory to define a new deformed product $(A\diamondsuit_\theta B)(x)$ according to

$$(A\diamondsuit_\theta B)(x) \equiv \int_{\mathbb{R}^4}\int_{\mathbb{R}^4} A(y)L(x,y,z)\, B(z)\, dy\, dz\,, \tag{8}$$

where the integral kernel L has different shape for each spin. The associativity condition for operator symbols implies that the kernel $L(x, y, z)$ satisfies the nonlinear equation

$$\int L(x_1,x_2,y)L(y,x_3,x_4)dy = \int L(x_1,y,x_4)L(x_2,x_3,y)dy\,. \tag{9}$$

Our kernel is of the type $\delta(x, z)[\exp \theta f(\partial)]\delta(x, y)$, hence it fulfils the associativity condition (9) thanks to the properties of a Dirac δ-functional.

For every spin we have a different choice of integral kernel. In the case of $S = 0$, i.e. a scalar field, we start with a theory with a static term

$$\int d^4x \frac{1}{2} m_2^2 . \phi \diamondsuit \phi \,. \tag{10}$$

With a particular choice of kernel, we can write

$$\delta L_{\rm scal} = \delta(x,z)[-1 + \theta\Box]\delta(x,y)\,, \tag{11}$$

and we obtain a dynamical theory. It can be seen as a leading order term of expansion, in the parameter θ, of the following general definition:

$$L_{\rm scal} := \delta(x,z)[\exp \theta\Box]\delta(x,y)\,. \tag{12}$$

In this form it actually displays a "static nature", as we will show later.

In the case of $S = 1$, i.e. a vector field, we start with a theory with a static term

$$\int d^4x \frac{1}{2} m_1^2 A_\mu \diamondsuit A^\mu \,. \tag{13}$$

The particular choice of kernel is now such that

$$\begin{aligned} \delta L_{\rm vect}(x,y,z) &= \delta(x,z)[D_\theta]_\mu^\nu \delta(x,y) = \delta(x,z)[\delta_\mu^\nu + \theta\Delta_\mu^\nu]\delta(x,y) \\ &= \delta(x,z)\left[\delta_\mu^\nu + \theta\left(\Box_y \delta_\mu^\nu - \left(1 - \frac{1}{\alpha}\right)\nabla^\nu\nabla_\mu\right)\right]\delta(x,y)\,, \end{aligned} \tag{14}$$

which leads to a dynamical theory. It can be seen as a leading-order term in the expansion, in the parameter θ, of the following general definition:

$$L_{\rm vect} := \delta(x,z)[\exp \theta\Delta_\nu^\mu]\delta(x,y)\,. \tag{15}$$

In the case of spin $S = 1/2$, i.e. a spinor field, we start with a theory with a static term

$$\int d^4x \frac{1}{2} m_3 \psi \diamondsuit \overline{\psi} \,. \tag{16}$$

Our particular choice of kernel is such that

$$\delta L_{\rm matter}(x,y,z) = \delta(x,z)[-1 + i\sqrt{\theta}\gamma_\mu\partial^\mu]\delta(x,y)\,. \tag{17}$$

It can be seen as a leading-order term in the expansion, in the parameter θ, of the following general definition:

$$L_{\rm matter} := \delta(x,z)[\exp -i\sqrt{\theta}\gamma_\mu\partial^\mu]\delta(x,y)\,. \tag{18}$$

4. Properties of the Deformed Product

In this section we analyze some peculiar properties of our deformed product, outlining the possible implications.

4.1. *Supersymmetric nature of the deformed product and its apparent dynamical nature*

To achieve correspondence to lowest-order theory we must impose the condition: $m_i\sqrt{\theta} = 1$, which then implies $m_i = m = \frac{1}{\sqrt{\theta}}$. This requires, of course, supersymmetry, i.e. that the fields ϕ, ψ, A_μ belong to a massive vector $N = 1$ superfield. Hence, to lowest order, the "static" massless theory in the corresponding deformed product for scalar, vector, fermionic fields is "equivalent" to a dynamical massive supermultiplet, i.e. if we restrict ourselves to the leading term, linear in the deformation parameter, we get naturally a supersymmetric formulation, as in a dual deformation.

In principle one can think of having a class of deformed products and, corresponding to different choices of kernel, to build a deformed supersymmetryc Wess–Zumino model, as in a supersymmetric U(1) theory, introducing the dynamics to lowest order in the kernel and not in the supersymmetric formulation, as in [43]. In our case, with suitable choice of kernel in the scalar, spinor and vector Lagrangian, we can reproduce in a natural way a supersymmetric action which suffers from nonlocality at subsequent orders in θ. The Lagrangian for a globally supersymmetric matter multiplet is [44]

$$L^{N=1}_{\text{matter}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\bar{\psi}\gamma_\mu\partial^\mu\psi - \frac{1}{2}m\bar{\psi}\psi + \frac{1}{2}m^2A_\mu A^\mu - \frac{1}{2}m^2\phi^2\,, \tag{19}$$

where A^μ is a vector field, ψ is a Majorana spinor field, and ϕ is a pseudoscalar field.

All fields must have the same mass. The above Lagrangian in superspace formalism is

$$L^{N=1}_{\text{matter}} = [\Phi\dagger\Phi]_D + m[\Phi\Phi]_F + m[V^2_{WZ}]_D + \frac{1}{32}[W^\alpha W_\alpha]_F\,. \tag{20}$$

With our prescription, it becomes

$$L^{N=1}_{\text{matter}} = \frac{1}{2}m^2A_\mu\diamondsuit_{S=1}A^\mu + \frac{1}{2}m\psi\diamondsuit_{S=1/2}\overline{\psi} + \frac{1}{2}m^2\phi\diamondsuit_{S=0}\phi\,. \tag{21}$$

What seems to happen is that the dynamics, which is put at the level of superfields in a supersymmetric Lagrangian, is found in a nonlocal theory to the first nonvanishing order in the θ parameter.

Hence one can think of using the deformed product, to lowest order in the θ parameter, to obtain the dynamics, i.e. one can encode dynamics in a product and the other way around. Changing deformed product means having a different dynamics. We can have infinitely many derivatives in order to reproduce, at different orders in θ, the dynamics of the system. This is a first step towards obtaining a more elaborate model and the dynamics in the deformed product, where the θ parameter is essential, and its smallness is important to make sure that higher-order (derivative) terms are of no importance. Thus, *the dynamics can be seen as a "perturbative" effect which disappears in the global "nonperturbative" static expression.*

4.2. *Nonlocal nature of the deformed product*

Hence, starting with the free static action and using the (bi)-differential operator to lowest order in the parameter, one obtains a known massive free-field theory, that is a lower order approximation of a more complicated nonlocal theory. In this way we have a nonlocal field theory from a local theory in which we deformed the product on a nonlocality domain with a characteristic scale $\sqrt{\theta}$. While in [45, 3, 46, 47] one replaces a local field by a "smearing" field, in our model we put the nonlocality in the product, as in the star product. Thus, the deformed quantum field theory is a particular case of a nonlocal quantum field theory.

4.3. *Duality of the Kalb–Ramond field in this deformed product*

In this section we analyze an intriguing property of a type of deformed product. We show how the duality property of a Kalb–Ramond is changed by using a particular deformed product at lower order in θ. A well-known result is that a massless field $H_{\nu\rho\sigma} = \partial_{[\nu} B_{\rho\sigma]}$ in an undeformed product is equivalent to a massless scalar field, i.e. the degrees of freedom of the antisymmetric tensor field $B_{\rho\sigma}$ are only one:

$$\partial^\mu \phi = \frac{1}{6} \epsilon^{\mu\nu\rho\sigma} H_{\nu\rho\sigma} \, , \tag{22}$$

while the massive field is equivalent to a massive vector field, i.e. the degrees of freedom of $B_{\rho\sigma}$ are three:

$$H^\mu = \frac{1}{6} \epsilon^{\mu\nu\rho\sigma} H_{\nu\rho\sigma} \, . \tag{23}$$

In view of these properties, we want to generalize the duality of Kalb–Ramond in the deformed case. We show that a massless deformed Kalb–Ramond theory is dual to a U(1)-breaking theory, i.e. it is equivalent to a massive vector field theory, with the above choice of duality and choosing a particular kernel for the deformed product. The presence of deformation leaves nontrivial transverse and longitudinal modes, unlike the classical undeformed massless case. For this purpose, we are interested in a model ruled by the deformed action

$$\begin{aligned} S_H &= \frac{1}{3!} \int d^4 x H \star_\theta H(x) \\ &:= \int d^4 x \left[\int_{\mathbb{R}^4} \int_{\mathbb{R}^4} H_{\alpha\beta\gamma}(y) L^{\alpha\beta\gamma}_{\rho\sigma\tau}(x, y, z) \, H^{\rho\sigma\tau}(z) \, dy \, dz \right] , \end{aligned} \tag{24}$$

while the action $S_B = \frac{1}{3!} \int d^4 x H_{\mu\nu\rho}(B) H^{\mu\nu\rho}(B)$ is deformed through the integral kernel L chosen to recover the U(1) gauge theory, and is given by

$$L^{\alpha\beta\gamma}_{\rho\sigma\tau}(x, y, z) := \frac{1}{6} \epsilon^{\mu\alpha\beta\gamma} \epsilon_{\nu\rho\sigma\tau} [D_\theta]^\nu_\mu \, , \tag{25}$$

and $[D_\theta]_\mu^\nu$ reads as

$$[D_\theta]_\mu^\nu := \delta(x,z)\left[\delta_\mu^\nu + \frac{\theta^2}{m^2}\Delta_\mu^\nu\right]\delta(x,y)$$
$$= \delta(x,z)\left[\delta_\mu^\nu + \frac{\theta^2}{m^2}\left(\Box_y\delta_\mu^\nu - \left(1-\frac{1}{\alpha}\right)\nabla^\nu\nabla_\mu\right)\right]\delta(x,y)\,. \tag{26}$$

The presence of α in (26) reflects what we know about QED in its formulation with functional integrals in the Lorenz gauge to obtain an invertible operator on the potential A_μ. Actually, its inclusion for a massive vector field model is not compelling, it is a redundant term, i.e. we are summing a vanishing term ($\alpha = \infty$). With this choice of L and with the following duality transformation:

$$H_{\mu\nu\rho} = \frac{1}{\sqrt{\theta}}\epsilon_{\mu\nu\rho\sigma}A^\sigma, \tag{27}$$

to first order in θ, the fundamental (at high energy) deformed action (24) is dual to the action of a massive spin-1 field with mass $m_\theta = \frac{1}{\sqrt{\theta}} \sim M_{pl}$ (effective theory at low energy), i.e.

$$S_{\rm dual} = S_H = \int d^4x\left[\frac{1}{2}m_\theta^2 A_\mu A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2\alpha}(\partial_\mu A^\mu)^2\right], \tag{28}$$

where $m = \frac{1}{\sqrt{\theta}}$ and $\partial_\mu A^\mu$ can be shown to vanish.

In general we can give a formal expression of L to every order in θ:

$$[D_{\rm global}]_\mu^\nu := \delta(x,z)\left[\exp\frac{\theta^2}{m^2}\Delta_\mu^\nu\right]\delta(x,y)\,. \tag{29}$$

Its expansion in powers of θ is in terms of derivatives of increasing order.

To zeroth order in θ (or $\theta = 0$), we recover the duality of the massless Kalb–Ramond to a massless scalar field ϕ, putting $A^\sigma = \partial^\sigma\phi$. To first order in $\frac{\theta^2}{m^2} = \tilde{\theta}$, a deformed massless Kalb–Ramond (gauge invariant) is dual to a massive spin-1 field.

The operator Δ_μ^ν in Eq. (29) is a hyperbolic operator. To have a meaningful expression, we have to make a Wick rotation, so that it becomes an elliptic operator. In this way the action of the massive vector field can be seen as the lowest order (effective theory) of a (broader) nonlocal theory, which is reduced at zeroth order to a free scalar theory. A massive spin-1 theory may be regarded as the low-energy limit of a fundamental deformed theory, where the low-energy limit is set by the massive term $\frac{1}{\sqrt{\theta}}$.

On using different representations of deformed product it is possible to find a "dual" deformed product, i.e. we can use duality in the reverse order. Then we start with $\int d^4x\frac{1}{2}m_1^2 A_\mu\diamondsuit A^\mu$, and after the duality transformation we have $\int d^4x\frac{1}{2}H\diamondsuit H$, hence we are able to write a dual deformed product to first perturbative order. In this way the duality and the deformation are connected, and when $\theta \to 0$, the deformed product and the duality disappear.

5. Deformed Interaction Term

In this section we analyze the deformed interaction term between matter and a vector field, and in its dual version.

5.1. *Deformed interaction term between matter and vector field*

We have a U(1) theory of an unknown particle at high energy which can decay (because we do not observe it) and in principle it can produce an observable physics. Thus, we can imagine a coupling with matter and we study the decay. We construct a model in which the deformed (high energy) action of matter and its interaction term with the vector field corresponds to a low-energy action of massive fermions plus the vector-matter interaction plus correction terms involving derivatives. The general Lagrangian density for a vector field $L(V_\mu, \psi)$ describing all their interactions is given by

$$\begin{aligned} L = &-\frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}M^2 V_\mu V^\mu + \beta\partial_\nu V_\mu V^\mu V^\nu \\ &+ \gamma V_\mu V_\nu V^\mu V^\nu + i\overline{\psi}\gamma_\mu\partial^\mu\psi - \overline{\psi}m\psi + V_\mu\overline{\psi}\gamma^\mu\psi \,. \end{aligned} \tag{30}$$

To obtain the correspondence with a vector theory we can consider different combinations, i.e. $(\bar{\psi}\diamondsuit\gamma^\mu\psi)A_\mu$, or $(\bar{\psi}\gamma^\mu\diamondsuit\psi)A_\mu$. We can take account of two possibilities in the form

$$\begin{aligned} S_{\psi\bar{\psi}A} &= \frac{1}{2}\int d^4x[\bar{\psi}\diamondsuit(m - \gamma^\mu A_\mu)\psi + \bar{\psi}(m - \gamma^\mu A_\mu)\diamondsuit\psi] \\ &:= \int d^4x \left\{ \int_{\mathbb{R}^4}\int_{\mathbb{R}^4} [\bar{\psi}(y)L_{\text{matter}}(x,y,z)(m - \gamma^\mu A_\mu)\,\psi(z) \right. \\ &\qquad \left. + \;\bar{\psi}(y)(m - \gamma^\mu A_\mu)L_{\text{matter}}(x,y,z)\,\psi(z)\,dy\,dz] \right\}, \end{aligned} \tag{31}$$

hence with the integral kernel L_{matter} chosen to recover the vector-fermion action

$$L_{\text{matter}}(x,y,z) := \delta(x,z)(-1 + \sqrt{\theta}\gamma_\mu\partial^\mu)\delta(x,y)\,, \tag{32}$$

to first order in θ, the fundamental deformed action (31) is an action of a massive spin-1 in interaction with matter plus a correction term, i.e.

$$\begin{aligned} S^{\text{total}} &= S_A + S_{\psi\bar{\psi}A} \\ &= \int d^4x \left[\frac{1}{2}m_\theta^2 A_\mu A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\,m\,\sqrt{\theta}\overline{\psi}\gamma_\mu\partial^\mu\psi - \overline{\psi}m\psi \right. \\ &\qquad \left. - \frac{1}{\sqrt{\theta}\,M}A_\mu\overline{\psi}\gamma^\mu\psi - \frac{i}{M}\overline{\psi}\partial^\rho(A_\rho\psi)\right]. \end{aligned} \tag{33}$$

To have a correct correspondence we have the condition $m\sqrt{\theta} = 1$, i.e. at high energy the only possible mass is "driven" by the θ term, and the charge of coupling is $Q = \frac{1}{M\sqrt{\theta}}$.

In general we can give a formal expression of L to every order in θ as in (18):

$$[L_{\text{matter}-\text{global}}] := \delta(x,z)[\exp -i\sqrt{\theta}\gamma_\mu\partial^\mu]\delta(x,y). \tag{34}$$

We thus find that the vector field combines with matter to become, at low energy, a massive vector field, and it interacts through the standard coupling described by the previous Eq. (33). The presence of deformation "seems" to introduce a *dynamics* in a *static* system to zeroth order.

5.2. *Deformed interaction term between matter and Kalb–Ramond field*

We construct a model in which we show that the deformed (high energy) action of matter and its interaction term with Kalb–Ramond corresponds, after a duality transformation, to a low-energy action of massive fermions plus the vector-matter interaction plus correction terms involving derivatives. Such a tensor field, which appears, for example, in the massless sector of a heterotic string theory, is assumed to coexist with gravity in the bulk, in a five-dimensional Randall–Sundrum scenario [48]. It has a well-known geometric interpretation as the spacetime torsion. We consider the most general gauge-invariant action of a second-rank antisymmetric Kalb–Ramond tensor gauge theory, including the coupling with matter modes [48]:

$$\mathcal{L}_{\psi\bar{\psi}H} = -\frac{1}{M_{Pl}}\bar{\psi}[i\gamma^\mu\sigma^{\nu\lambda}H_{\mu\nu\lambda}]\psi\,. \tag{35}$$

The general Lagrangian density for a vector field $L(V_\mu, \psi)$ describing all their interactions is given by

$$\begin{aligned} L = {} & -\frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{1}{2}M^2V_\mu V^\mu + \beta\partial_\nu V_\mu V^\mu V^\nu \\ & + \gamma V_\mu V_\nu V^\mu V^\nu + i\overline{\psi}\gamma_\mu\partial^\mu\psi - \overline{\psi}m\psi + V_\mu\overline{\psi}\gamma^\mu\psi\,. \end{aligned} \tag{36}$$

To obtain the correspondence with a vector theory we can have different combinations, i.e. $\frac{1}{M_{Pl}}(\bar{\psi}\diamondsuit\gamma^\mu\sigma^{\nu\lambda}\gamma_5\psi)H_{\mu\nu\lambda}$, or $\frac{1}{M_{Pl}}(\bar{\psi}\gamma^\mu\sigma^{\nu\lambda}\gamma_5\diamondsuit\psi)H_{\mu\nu\lambda}$. We can take account of two possibilities by writing

$$\begin{aligned} S^{\text{dual}}_{\psi\bar{\psi}H} = {} & \frac{1}{2}\int d^4x\left[\bar{\psi}\diamondsuit\left(m - \frac{1}{M_{Pl}}\gamma^\mu\sigma^{\nu\lambda}\gamma_5H_{\mu\nu\lambda}\right)\psi\right. \\ & \left. + \;\bar{\psi}\left(m - \frac{1}{M_{Pl}}\gamma^\mu\sigma^{\nu\lambda}H_{\mu\nu\lambda}\right)\diamondsuit\psi\right] \\ := {} & \int d^4x\left\{\int_{\mathbb{R}^4}\int_{\mathbb{R}^4}\left[\bar{\psi}(y)L_{\text{matter}}(x,y,z)\left(m - \frac{1}{M_{Pl}}\gamma^\mu\sigma^{\nu\lambda}\gamma_5H_{\mu\nu\lambda}\right)\psi(z)\right.\right. \\ & \left.\left. + \bar{\psi}(y)\left(m - \frac{1}{M_{Pl}}\gamma^\mu\sigma^{\nu\lambda}H_{\mu\nu\lambda}\right)L_{\text{matter}}(x,y,z)\,\psi(z)\,dy\,dz\right]\right\}, \end{aligned} \tag{37}$$

hence with the integral kernel $L_{\rm matter}$ chosen to recover the vector-fermion action

$$L_{\rm matter}(x,y,z) := \delta(x,z)(-1+\sqrt{\theta}\gamma_\mu\partial^\mu)\delta(x,y)\,, \tag{38}$$

and with the duality transformation (27) to first order in θ, the fundamental non-commutative action (37) is dual to the action of a massive spin-1 pseudo-vector field in interaction with matter plus a correction term, i.e.

$$\begin{aligned} S_{\rm dual}^{\rm total} &= S_H + S_{\psi\bar{\psi}H}^{\rm dual} \\ &= \int d^4x\left[\frac{1}{2}m_\theta^2 A_\mu A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\,m\sqrt{\theta}\overline{\psi}\gamma_\mu\partial^\mu\psi - \overline{\psi}m\psi \right. \\ &\quad \left. - \frac{1}{\sqrt{\theta}\,M_{Pl}}A_\mu\overline{\psi}\gamma^\mu\psi - \frac{i}{M_{Pl}}\overline{\psi}\partial^\rho(A_\rho\psi)\right], \end{aligned} \tag{39}$$

where we have used the identity $\gamma_\lambda\Sigma_{\mu\nu} = i[g_{\lambda\mu}\gamma_\nu - g_{\lambda\nu}\gamma_\mu + i\epsilon_{\lambda\mu\nu\rho}\gamma_5\gamma_\rho]$. To have a correct correspondence we have the condition $m\sqrt{\theta}=1$, i.e. at high energy the only possible mass is "driven" by the θ term, and the charge of coupling is $Q = \frac{1}{M_{Pl}\sqrt{\theta}}$.

In general we can give a formal expression of L to every order in θ:

$$[L_{\rm matter-global}] := \delta(x,z)[\exp -i\sqrt{\theta}\gamma_\mu\partial^\mu]\delta(x,y)\,. \tag{40}$$

We thus find that the Kalb–Ramond field combines with matter to become at low energy a massive vector filed, and it interacts through the standard coupling described by the previous Eq. (39).

6. Application of the Deformed Product to a Free Scalar Field

In this section we evaluate the Green function and dispersion relation for the free scalar action with Lagrangian density (12), showing its static nature. Then we analyze the dynamical scalar field theory with our deformed product, showing that, in a sense, it is equivalent to the one found by Moffat [16].

6.1. *Green function and dispersion relation of the nonlocal model. A fictitious dynamical theory*

Given any differential operator D on R^4 one can define a map $\sigma(D)$ called the *symbol* of D: $\sigma : D \to \sigma(D) \equiv e^{-\alpha k_\mu x^\mu} D e^{\alpha k_\mu x^\mu}$. The associated equation of motion is $D\psi = 0$, to which there corresponds the dispersion relation $\sigma(D;k,\omega)=0$, e.g. $\omega=\omega(k)$. For $D=\Box$ one obtains

$$\sigma(\Box + m^2) = \alpha^2(\vec{k}\cdot\vec{k} - \omega^2) + m^2. \tag{41}$$

By setting $\sigma(D;k,\omega)=0$ one obtains the dispersion relation

$$E^2 = \vec{k}\cdot\vec{k} + m^2\,, \tag{42}$$

which leads to the following wave equation:

$$(\Box + m^2)\phi = 0\,. \tag{43}$$

The Green function is defined by

$$G(x,x') = \int d^4k \frac{e^{ik_\mu(x^\mu - x'^\mu)}}{\sigma(\Box + m^2)}\,, \tag{44}$$

and it satisfies the equation

$$(\Box + m^2)G(x,x') = -\delta(x-x') = -\int d^4k e^{ik_\mu(x^\mu - x'^\mu)}\,. \tag{45}$$

In our nonlocal model for a free scalar field, the symbol is $\sigma(m^2 \exp\theta\Box; k, \omega) = m^2 \exp(-k_\mu k^\mu \theta) = m^2 \sum_{n=0}^{\infty}(-k_\mu k^\mu \theta)^n$. If we instead consider a finite approximation, at small θ we have correction terms. Taking into account that the symbol maps $\exp\theta\Box \to \exp -k_\mu k^\mu \theta$, the Green function is

$$\begin{aligned}
G(x,x') &= m^2 \int d^4k \frac{1}{(2\pi)^4} e^{ik_\mu(x^\mu - x'^\mu)} \exp(-k_\mu k^\mu \theta) \\
&= m^2 \int d^4k \frac{1}{(2\pi)^4} e^{\delta^{\mu\nu}(\sqrt{\theta}k_\mu - \frac{i}{2\sqrt{\theta}}(x_\mu - x'_\mu))(\sqrt{\theta}k_\nu - \frac{i}{2\sqrt{\theta}}(x_\nu - x'_\nu))} \\
&\quad \times \exp\left[-\frac{1}{4\theta}\delta^{\mu\nu}(x_\mu - x'_\mu)(x_\nu - x'_\nu)\right] \\
&= m^2 \prod_{r=1}^{4} \int dk_r \frac{1}{(2\pi)^4} e^{\delta^{jl}(\sqrt{\theta}k_j - \frac{i}{2\sqrt{\theta}}(x_j - x'_j))(\sqrt{\theta}k_l - \frac{i}{2\sqrt{\theta}}(x_l - x'_l))} \\
&\quad \times \exp\left[-\frac{1}{4\theta}\delta^{\mu\nu}(x_\mu - x'_\mu)(x_\nu - x'_\nu)\right] \\
&= m^2 \left(\frac{1}{64\pi\theta}\right)^2 \exp\left[-\frac{1}{4\theta}\delta^{\mu\nu}(x_\mu - x'_\mu)(x_\nu - x'_\nu)\right],
\end{aligned} \tag{46}$$

where we have used the Gaussian integral, and $G(k) = (m^2 \exp -k_\mu k^\mu \theta)^{-1} = \frac{1}{-m^2 + k^2 + \sum_{n=2}^{\infty}(-k_\mu k^\mu \theta)^n}$, ($\theta = m^{-2}$), while in [16] the modified Feynman propagator in momentum space is $i\Delta_F(k) = \frac{i \exp 1/2k_\mu \tau^{\mu\nu} k_\nu \theta}{k^2 - m^2 + i\epsilon}$. We note that the exact expression of the Green function does not show any pole, corresponding to the static nature of the global theory, while dynamics can be recovered from a perturbative expansion.

7. Application of the Deformed Product to a Free Scalar Field Theory. Analogy with the Moffat Model

In this section we show that the free scalar field theory with our deformed product, in a sense, is the same as that found by Moffat [16]. As we saw in the above subsection the product does not induce a dynamics, because the dispersion relation obtained by setting to zero the symbol does not have a solution. This means that we have to start with an ordinary dynamical action rewritten through this deformed product.

Our deformed product can be seen as an application of the product proposed by Moffat [16] in which we find supersymmetry in the different shape of the deformed product, but not in the superspace formalism:

$$(\hat{\phi}_1 \diamondsuit \hat{\phi}_2)(\rho) = \left[\exp\left(-\frac{1}{2}\tau^{\mu\nu}\frac{\partial}{\partial\rho^\mu}\frac{\partial}{\partial\eta^\nu}\right)\phi_1(\rho)\phi_2(\eta)\right]_{\rho=\eta}$$
$$= \phi_1(\rho)\phi_2(\rho) - \frac{1}{2}\tau^{\mu\nu}\frac{\partial}{\partial\rho^\mu}\phi_1(\rho)\frac{\partial}{\partial\rho^\nu}\phi_2(\rho) + O(\tau^2)\,. \qquad (47)$$

Here, by comparison with the Moffat product in [16], $\tau^{\mu\nu} = \delta^{\mu\nu}\theta$ and $\rho = \eta$. Hence our product is a particular application of it and it is associative and commutative. In our case the modified Feynman propagator $\bar{\Delta}_F$ is defined by the vacuum expectation value of the time-ordered $\star$-product

$$i\bar{\Delta}_F(x-y) \equiv \langle 0|T(\phi(x)\diamondsuit\phi(y))|0\rangle$$
$$= \frac{i}{(2\pi)^4}\int\frac{d^4k\,\exp[-ik(x-y)]\exp[\frac{1}{2}(k^2\theta)]}{k^2-m^2+i\epsilon}\,. \qquad (48)$$

In momentum space this gives

$$i\bar{\Delta}_F(k) = \frac{i\exp\left[\frac{1}{2}(k^2\theta)\right]}{k^2-m^2+i\epsilon}\,, \qquad (49)$$

which reduces to the standard commutative field theory form for the Feynman propagator

$$i\Delta_F(k) = \frac{i}{k^2-m^2+i\epsilon}, \qquad (50)$$

in the limit $|\theta^{\mu\nu}| \to 0$. The free-field ϕ^2 theory is nonlocal, unlike the corresponding one in ordinary local field theory, resulting in a modified Feynman propagator $\bar{\Delta}_F(k)$ and modified dispersion relation. It turns out to be a particular case of that treated by Moffat in [16].

8. Concluding Remarks

In our paper, a modification of the standard product used in local field theory by means of an associative deformed product has been proposed. We have built a class of deformed products, one for every spin $S = 0, 1/2, 1$, that induces a nonlocal theory, displaying different form for different fields. This type of deformed product is naturally supersymmetric and it has an intriguing duality.

It now remains to be seen whether a suitable variant of our construction can lead to a product different from the one used by Moffat [16].

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