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Independence and convergence in non-additive settings

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Abstract Some properties of convergence for archimedean *t*-conorms and *t*-norms are investigated and a definition of independence for events, evaluated by a decomposable measure, is introduced. This definition generalizes the concept of independence provided by Kruse and Qiang for λ-additive fuzzy measures. Finally, we derive the two Borel–Cantelli lemmas in the context of the general framework considered.

Keywords Borel–Cantelli · ⊥-Decomposable measures · Independence · Conditionally distributive semirings

1 Introduction

Alternatives approaches to one offered by the classical probability theory have been introduced in the literature in order to handle partial information and to obviate some difficulties due to the application of the classic model; see, for instance, Dempster–Shafer Belief functions (Shafer 1976), Possibility measures (Dubois and Prade 1990), Qualitative probability (Fine 1973), Plausibility measures (Friedman and Halpern 2001) and ⊥-decomposable measures (Weber 1984).

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The notion of independence plays a critical role in reasoning about uncertainty and the Borel–Cantelli lemmas are instrumental in proving strong laws of large numbers and the law of iterated logarithm. Ever since theoretical formulations different from the classical probability theory have been introduced, several alternative approaches to the concept of independence have been considered (see, for example Dubois et al. 1994; Fine 1973; Goldszmidt and Pearl 1992). Kruse (1987) defines the concept of independence of two events with respect to the λ-additive fuzzy measures introduced by Sugeno (1974); Qiang extends this concept to a finite or infinite class of events and derives two theorems that correspond to the two Borel–Cantelli lemmas of the classical probability theory.

We focus on the class of set functions that are decomposable with respect to a t-conorm $\perp$, $\perp$-*decomposable measures* for short. These measures are fuzzy measures including the λ-additive measures; they have been widely investigated by Weber (1984), who defines a new integration theory extending the one due to Lebesque; other scholars have provided results for convergence and investigated the coherence of assessments (D'Apuzzo et al. 1991; Squillante et al. 1989; Squillante and Ventre 1998).

In this context, we derive a strict t-norm $\bullet$, from a non-strict archimedean t-conorm $\perp$, in such a way that $\big([0,1], \perp, \bullet\big)$ is a conditionally distributive semiring. Then we provide a suitable concept of independence for events, that generalizes the one introduced by Kruse and Qiang and derive more general Borel–Cantelli like lemmas.

The paper is organized as follows: in Sect. 2 we recall definitions and results related to t-conorms, t-norms, $\perp$-decomposable measures and introduce some new concepts and results that will turn useful in the sequel; in Sect. 3 we prove the First Borel–Cantelli like lemma for a σ-$\perp$-decomposable measure; in Sect. 4 we assume that $\perp$ is a non-strict archimedean t-conorm over $[0, 1]$ and we define a strict t-norm $\bullet$ linked to $\perp$; in Sect. 5 we introduce, by means of the t-norm $\bullet$, a concept of $m_\perp$-independence for events, then we prove Second Borel–Cantelli like lemma.

2 Preliminaries

From now on we will denote with:

- J the real interval [0,1];
- $\perp : (a, b) \in J \to \perp(a, b) = a \perp b \in J$ a *triangolar conorm* (*t-conorm* for short);
- $\top : (a, b) \in J \to \top(a, b) = a \top b \in J$ a *triangolar norm* (*t-norm* for short);
- x' the $\perp$-complement of $x \in J$, that is

$$x' = inf\{y : \perp(x, y) = 1\} \tag{1}$$

- $(X, \mathcal{A})$ a measurable space;
- A^c the complement set $X - A$, for $A \in \mathcal{A}$;
- $m : \mathcal{A} \to J$ a set function verifying the conditions: $m(\emptyset) = 0$ and $m(X) = 1$.

Let us recall that $\perp$ and $\top$ are commutative semigroup operations over J that are non-decreasing in each argument and have 0 and 1 as *unit*, respectively. Each one of them is called *strict* if and only if it is continuous and (strictly) increasing in $(0, 1) \times (0, 1)$ (Klement et al. 2000).

The triple $(J, \perp, \top)$ is called a *conditionally distributive semiring* if and only if the $\top$ is conditionally distributive over $\perp$, that is (Klement et al. 2000):

$$a\top(b\perp c) = (a\top b)\perp(a\top c) \quad \text{whenever} \quad b\perp c < 1 \quad (\textit{conditional distributivity}).$$

Let $g : J \to [0, +\infty]$ and $\theta : J \to J$ be continuous, strictly increasing functions, such that $g(0) = 0$ and $\theta(1) = 1$. By $g^{(-1)}$ and $\theta^{(-1)}$ we will denote the *pseudo-inverse* of g and the *pseudo-inverse* of θ defined by

$$g^{(-1)} : y \in [0, +\infty] \to g^{-1}(\min\{y, g(1)\}),$$
$$\theta^{(-1)} : y \in [0, 1] \to \theta^{-1}(\max\{y, \theta(0)\}).$$

Hence

$$g^{(-1)}(y) = \begin{cases} g^{-1}(y) & \text{for } y \in g(J) = [0, g(1)] \\ 1 & \text{for } y \in [g(1), +\infty] \end{cases},$$

and $g^{(-1)} = g^{-1}$ if and only if $g(1) = +\infty$;

$$\theta^{(-1)}(y) = \begin{cases} \theta^{-1}(y) & \text{for } y \in \theta(J) = [\theta(0), 1] \\ 0 & \text{for } y \in [0, \theta(0)] \end{cases},$$

and $\theta^{(-1)} = \theta^{-1}$ if and only if $\theta(0) = 0$.

2.1 About t-conorms and t-norms

By definition of t-conorm and t-norm

$$a = a\perp 0 = 0\perp a, \quad a = a\top 1 = 1\top a. \tag{2}$$

Furthermore the following boundary conditions can be deduced, for each $a \in [0, 1]$:

$$1 = 1\perp a = a\perp 1, \qquad 0 = 0\top a = a\top 0. \tag{3}$$

So 0 and 1 are idempotent elements for both $\perp$ and $\top$: they are called the *trivial idempotent* elements. As a consequence of (2) and the property of monotonicity we get

$$a\top b \le \min\{a, b\} \le \max\{a, b\} \le a\perp b. \tag{4}$$

Because of the associative property, the operations $\perp$ and $\top$ can be extended by induction to n-ary operations by setting

$$\perp_{i=1}^{n} x_i = (\perp_{i=1}^{n-1} x_i)\perp x_n, \quad \top_{i=1}^{n} x_i = (\top_{i=1}^{n-1} x_i)\top x_n. \tag{5}$$

Due to monotonicity, for each sequence $(x_n)_{n\in\mathbf{N}}$ of elements of J, the following limits can be considered:

$$\bot_{i=1}^{+\infty} x_i = lim_n \bot_{i=1}^{n} x_i, \quad \top_{i=1}^{+\infty} x_i = lim_n \top_{i=1}^{n} x_i. \tag{6}$$

If $x_i = x \ \forall i = 1, 2, ..., n$, then we set

$$x_{\bot}^{(n)} = \bot_{i=1}^{n} x_i, \quad x_{\top}^{(n)} = \top_{i=1}^{n} x_i. \tag{7}$$

With regard to the limits (6) for each $n \in \mathbf{N}$ we consider the *remainder* terms:

$$R_n = \bot_{i>n} x_i = \bot_{i=n+1}^{+\infty} x_i, \quad Q_n = \top_{i>n} x_i = \top_{i=n+1}^{+\infty} x_i. \tag{8}$$

Definition 1 Given a sequence $(x_n)_{n\in\mathbf{N}}$ of elements of J, we say that:

1. $\bot_{n=1}^{+\infty} x_n$ is *strongly equal* to 1 and we set

$$\bot_{n=1}^{+\infty} x_n \equiv 1,$$

   if and only if $R_n = 1$ for each n;
2. $\top_{n=1}^{+\infty} x_n$ is *strongly equal* to 0 and we set

$$\top_{n=1}^{+\infty} x_n \equiv 0,$$

   if and only if $Q_n = 0$ for each n.

2.1.1 Archimedean t-conorms and t-norms and representation theorems

Following (Butnariu and Klement 1993) and (Weber 1984) we define the archimedean property of t-conorms and t-norms:

Definition 2 $\bot$ is called *archimedean* if and only if it is continuous and

$$\bot(a, a) = a \bot a > a \quad \forall a \in (0, 1); \tag{9}$$

$\top$ is called *archimedean* if and only if it is continuous and

$$\top(a, a) = a \top a < a \quad \forall a \in (0, 1). \tag{10}$$

Proposition 1 *Let $\bot$ and $\top$ be continuous. Then $\bot$ and $\top$ are archimedean if and only if they satisfy the limit properties*

$$\lim_n x_{\bot}^{(n)} = 1 \quad \forall\, x \in]0, 1]; \tag{11}$$

$$\lim_n x_{\top}^{(n)} = 0 \quad \forall\, x \in [0, 1[. \tag{12}$$

Proof By (Klement et al. 2000) Proposition 2.15. □

Remark 1 In (Klement et al. 2000) (see Definition 2.9, Theorem 2.12 and Remark 2.20) the archimedean property is defined without the assumption of continuity: a t-conorm $\perp$ is archimedean if and only if it satisfies the limit property (11), whereas a t-norm $\top$ is archimedean if and only if it satisfies the limit property (12). By the limit properties, $\perp$ and $\top$ have only trivial idempotent elements, and, by monotonicity property, verify (9) and (10). Proposition 1 states the equivalence between the conditions (9) and (10) and the limit properties, in case of continuous t-conorms and t-norms.

Proposition 2 *Let $\perp$ and $\top$ be archimedean. Then:*

1. $x \in [0,1[$ *and* $x \perp b = x \Rightarrow b = 0$*;*
2. $x \in]0,1]$ *and* $x \top b = x \Rightarrow b = 1$.

Proof Let us prove the implication 1. It is straightforward that $0 \perp b = 0 \Rightarrow b = 0$. Let $x \in]0,1[$ and $x \perp b = x$. By applying the associative property, we get $x \perp b_{\perp}^{(n)} = x \quad \forall n \geq 1$ and as a consequence

$$x \perp \lim_n b_{\perp}^{(n)} = x.$$

So that $\lim_n b_{\perp}^{(n)} < 1$ because of the first boundary condition in (3) and, as a consequence $b < 1$. Then, by the limit property (11), $b = 0$.
The assertion 2 can be proved by an analogous reasoning. □

Proposition 3 *Let $\perp$ and $\top$ be archimedean and $(x_n)_{n \in \mathbf{N}}$ a sequence of elements of J. Then*

1. $\perp_{i=1}^{+\infty} x_i < 1 \Rightarrow \lim_n R_n = 0$*;*
2. $\top_{i=1}^{+\infty} x_i > 0 \Rightarrow \lim_n Q_n = 1$.

Proof To prove the first implication let us set $s = \perp_{i=1}^{+\infty} x_i$ and, for every n, $s_n = \perp_{i=1}^{n} x_i$. Then $s = s_n \perp R_n$ for all $n \geq 1$ and

$$s = \lim_n (s_n \perp R_n) = s \perp \lim_n R_n. \tag{13}$$

As $R_n \leq s < 1$ (see (4)), by (13) and Proposition 2, we get $\lim_n R_n = 0$.
Let us now set $p = \top_{i=1}^{+\infty} x_i$ and $p_n = \top_{i=1}^{n} x_i$. Then $p = p_n \top Q_n$ for each n and item 2 follows by the equality $p = \lim_n (p_n \top Q_n) = p \top (\lim_n Q_n)$ and Proposition 2. □

Representation theorems have been stated for archimedean t-conorms and t-norms (Klement et al. 2000; Ling 1965; Schweizer and Sklar 1963; Weber 1984); we recall the following theorems providing an additive generator for a t-conorm and a multiplicative generator for a t-norm.

Theorem 1 (Weber 1984) *The two following assertions are equivalent:*

(i) $\perp$ *is archimedean;*

(ii) $\perp$ *has a continuous additive generator, i.e. there exists a continuous, strictly increasing function* $g : J \to [0, +\infty]$ *with* $g(0) = 0$ *such that, for all* $(x, y) \in J^2$,

$$x \perp y = g^{(-1)}(g(x) + g(y)).$$

The function g is uniquely determined up to a positive multiplicative constant; moreover $\perp$ *is strict if and only if* $g(1) = +\infty$.

If the archimedean *t*-conorm $\perp$ is non-strict, then there is an additive generator $\bar{g}$, verifying the condition $\bar{g}(1) = 1$. The function $\bar{g}$ is called *normed generator* of $\perp$ and it is obtained by an additive generator g by setting $\bar{g}(x) = g(x)/g(1)$ for each $x \in [0, 1]$.

Theorem 2 (Klement et al. 2000) *The two following assertions are equivalent:*

(i) $\top$ *is archimedean;*
(ii) $\top$ *has a continuous multiplicative generator, i.e. there exists a continuous, strictly increasing function* $\theta : J \to J$ *with* $\theta(1) = 1$ *such that, for all* $(x, y) \in J^2$,

$$x \top y = \theta^{(-1)}(\theta(x) \cdot \theta(y)).$$

The function θ *is uniquely determined up to a positive constant exponent; moreover,* $\top$ *is strict if and only if* $\theta(0) = 0$.

Example 1 $\perp(a, b) = a + b - ab$ is a strict archimedean t-conorm and an additive generator is $g(x) = -\ln(1 - x)$.

Example 2 For $\lambda > -1$, the operation $U_\lambda(a, b) = min(a+b-\lambda ab, 1)$ is a non-strict archimedean t-conorm and an additive generator is $g_\lambda(x) = \frac{1}{\lambda}\ln(1 + \lambda x)$ (Sugeno 1974; Weber 1984). The normed generator is $\overline{g}_\lambda(x) = \log_{1+\lambda}(1 + \lambda x)$.

Example 3 $\top(a, b) = a \cdot b$ is a strict archimedean t-norm and a multiplicative generator is the identity function $\theta(x) = x$.

Example 4 $\top(a, b) = max(a + b - 1, 0)$ is a non-strict archimedean t-norm and a multiplicative generator is $\theta(x) = e^{x-1}$.

By the representation theorems, if $\perp$ and $\top$ are archimedean, then for each $h \in N$ and for each $n \in \mathbf{N} \cup \{+\infty\}$, $n > h$:

$$\perp_{i=h}^{n} x_i = g^{(-1)}\left(\sum_{i=h}^{n} g(x_i)\right), \quad \top_{i=h}^{n} x_i = \theta^{(-1)}\left(\prod_{i=h}^{n} \theta(x_i)\right). \tag{14}$$

Proposition 4 *Let* $\perp$ *and* $\top$*be archimedean. Then*

1. $\perp_{i=1}^{+\infty} x_i < 1 \Leftrightarrow \sum_{i=1}^{+\infty} g(x_i) < g(1)$;
2. $\perp_{i=1}^{+\infty} x_i = 1 \Leftrightarrow \sum_{i=1}^{+\infty} g(x_i) \geq g(1)$;

3. $\bot_{i=1}^{+\infty} x_i \equiv 1 \Rightarrow \sum_{i=1}^{+\infty} g(x_i) = +\infty$;
4. $\top_{i=1}^{+\infty} x_i > 0 \Leftrightarrow \prod_{i=1}^{+\infty} \theta(x_i) > \theta(0)$;
5. $\top_{i=1}^{+\infty} x_i = 0 \Leftrightarrow \prod_{i=1}^{+\infty} \theta(x_i) \leq \theta(0)$;
6. $\top_{i=1}^{+\infty} x_i \equiv 0 \Rightarrow \prod_{i=1}^{+\infty} \theta(x_i) = 0$.

Proof Items 1, 2, 4 and 5 follow by (14) and the definitions of $g^{(-1)}$ and $\theta^{(-1)}$. Item 3 follows by item 2, for which

$$R_n = 1 \; \forall n \Leftrightarrow \sum_{i>n} g(x_i) \geq g(1) \; \forall n,$$

and the implication

$$\sum_{i=1}^{+\infty} g(x_i) < +\infty \Rightarrow \lim_n \sum_{i \geq n} g(x_i) = 0.$$

Item 6 follows by the equivalence

$$Q_n = 0 \; \forall n \Leftrightarrow \prod_{i>n} \theta(x_i) \leq \theta(0) \; \forall n,$$

derived from item 5, and the implication

$$\prod_{i=1}^{+\infty} \theta(x_i) = p > 0 \Rightarrow \lim_n \prod_{i \geq n} \theta(x_i) = 1,$$

that is the implication 2 in Proposition 3 applied to the t-norm in Example 3. □

Corollary 1 *Let $\bot$ and $\top$ be archimedean. If $\bot$ and $\top$ are non-strict, then*

$$\bot_{i=1}^{+\infty} x_i \equiv 1 \Leftrightarrow \sum_{i=1}^{+\infty} g(x_i) = +\infty, \tag{15}$$

$$\top_{i=1}^{+\infty} x_i \equiv 0 \Leftrightarrow \prod_{i=1}^{+\infty} \theta(x_i) = 0. \tag{16}$$

If $\bot$ and $\top$ are strict, then

$$\bot_{i=1}^{+\infty} x_i = 1 \Leftrightarrow \sum_{i=1}^{+\infty} g(x_i) = +\infty, \tag{17}$$

$$\top_{i=1}^{+\infty} x_i = 0 \Leftrightarrow \prod_{i=1}^{+\infty} \theta(x_i) = 0. \tag{18}$$

Proof Under the assumption that $\bot$ is non-strict, it is $g(x) < +\infty$ for each $x \in J$ and, as a consequence, $\sum_{i=1}^{+\infty} g(x_i) = +\infty \Leftrightarrow \sum_{i>n} g(x_i) = +\infty \quad \forall n$; so, by (14), the implication in Item 3 of Proposition 4 becomes the equivalence (15). Under the assumption that $\top$ is non-strict, $\theta(x) > 0 \ \forall x \in J$ and $\prod_{i=1}^{+\infty} \theta(x_i) = 0 \Leftrightarrow \prod_{i>n} \theta(x_i) = 0 \ \forall n$. So (16) follows by (14) and item 6 of Proposition 4.

The assertions related to a strict t-conorm and a strict t-norm follow by the equalities $g(1) = +\infty$, $\theta(0) = 0$ and items 2 and 5 in Proposition 4. □

Corollary 2 *Let $\bot$ and $\top$ be strict and archimedean.*

1. *if $(x_i)_{i \in N}$ is a sequence of elements of $[0, 1[$, then*

$$\bot_{i=1}^{+\infty} x_i = 1 \Leftrightarrow \bot_{i=1}^{+\infty} x_i \equiv 1. \tag{19}$$

2. *if $(x_i)_{i \in N}$ is a sequence of elements of $]0, 1]$, then*

$$\top_{i=1}^{+\infty} x_i = 0 \Leftrightarrow \top_{i=1}^{+\infty} x_i \equiv 0. \tag{20}$$

Proof Let us prove item 1. As $x_i \neq 1 \ \forall i \in N$, we get $g(x_i) \neq +\infty \ \forall i \in N$ and $\sum_{i=1}^{+\infty} g(x_i) = +\infty \Leftrightarrow \sum_{i>n}^{+\infty} g(x_i) = +\infty \ \forall n$. Therefore, by Eq. 17, we get (19). Item 2 is proved by an analogous reasoning. □

Proposition 5 *Let $\bot$ be a non-strict archimedean t-conorm and g one of its additive generators.*

Then $0' = 1$, $1' = 0$, $(x')' = x$, and

$$x' = g^{-1}(g(1) - g(x)) = \overline{g}^{-1}(1 - \overline{g}(x)). \tag{21}$$

As a consequence $x \bot x' = 1$ for each $x \in J$. If $\bot$ is strict, then $x' = 1$ for every $x \in [0, 1[$ and $1' = 0$.

Proof For the statement related to the case $\bot$ non-strict (see Weber 1984). For the statement related to the case $\bot$ strict, it is enough to stress that $g(x) < g(1) = +\infty$ for $x \in [0, 1[$ and $x \bot y = g^{-1}(g(x) + g(y)) = 1$, if and only if $g(x) + g(y) = +\infty$. □

Proposition 6 *Let $\bot$ be a non-strict archimedean t-conorm. Then, for a and b in J it results:*

$$(a \bot b)' \leq \min\{a', b'\} \leq a' \bot b'. \tag{22}$$

Proof By the associative property of $\bot$, by Proposition 5 and (3), $a' \bot (a \bot b) = (a' \bot a) \bot b = 1$, $b' \bot (a \bot b) = a \bot (b \bot b') = 1$. So the assertion follows by (1) and the monotonicity of $\bot$. □

2.2 $\perp$-decomposable measures

Definition 3 The set function m is:

1. a $\perp$-*decomposable* measure if and only if $m(A \cup B) = m(A) \perp m(B)$, for each pair (A, B) of disjoint sets (Weber 1984);
2. a σ-$\perp$-*decomposable* measure if and only if $m(\bigcup_{k \in \mathbf{N}} A_k) = \perp_{k=1}^{+\infty} m(A_k)$, for each sequence $(A_k)_{k \in \mathbf{N}}$ of disjoint sets (Weber 1984);
3. *continuous from below* or *above* resp. if and only if, for a monotone sequence of sets, $\left(A_k\right)_{k \in \mathbf{N}}$, it results $\lim_k m(A_k) = m(A)$ for $A_k \uparrow A \;\; or \;\; A_k \downarrow A \;\; resp.$ (Weber 1984);
4. a $\perp$-*subdecomposable* measure if and only if $m(A \cup B) \leq m(A) \perp m(B)$;
5. a σ-$\perp$-*subdecomposable* measure if and only if $m(\bigcup_{k \in \mathbf{N}} A_k) \leq \perp_{k=1}^{+\infty} m(A_k)$.

Theorem 3 (Weber 1984)

(i) *m is a $\perp$-decomposable measure $\Rightarrow$ m is monotone;*
(ii) *m is a $\perp$-decomposable measure $\Leftrightarrow$ $m(A \cup B) \perp m(A \cap B) = m(A) \perp m(B)$;*
(iii) *m is a σ-$\perp$-decomposable measure $\Leftrightarrow$ m is $\perp$-decomposable and continuous from below.*

Corollary 3
(j) *m is a $\perp$-decomposable measure $\Rightarrow$ m is $\perp$-subdecomposable;*
(jj) *m is a σ-$\perp$-decomposable measure $\Rightarrow$ m is σ-$\perp$-subdecomposable.*

Proof The assertion (j) follows by implication (ii) of Theorem 3 and by monotonicity of $\perp$, for which $m(A \cup B) \perp 0 \leq m(A \cup B) \perp m(A \cap B)$. In order to prove the implication (jj) let us assume that $\left(A_k\right)_{k \in \mathbf{N}}$ is a sequence of measurable subsets and denote, for every k, with B_k the set $A_k - \bigcup_{i=1}^{k-1} A_i$. Then $(B_k)_{k \in \mathbf{N}}$ is a sequence of disjoint sets and $m(\bigcup_{k \in \mathbf{N}} A_k) = m(\bigcup_{k \in \mathbf{N}} B_k) = \perp_{k=1}^{+\infty} m(B_k)$; so the assertion (jj) follows by (iii) and (i) in Theorem 3. □

The following proposition will be useful in reaching the results of the last section.

Proposition 7 *Let m be a $\perp$-decomposable measure and $A \in \mathcal{A}$. Then*

$$m(A^c) = 0 \Rightarrow m(A) = 1. \tag{23}$$

Moreover, if $\perp$ is archimedean, then:

1. *$\perp$ non-strict $\Rightarrow m(A^c) = (m(A))'$;*
2. *$\perp$ strict and $m(A) \neq 1 \Rightarrow m(A^c) = (m(A))' = 1$.*

Proof By the $\perp$-decomposability

$$m(A) \perp m(A^c) = m(A \cup A^c) = 1, \tag{24}$$

so (23) because 0 is the unit of $\perp$. The item 1 follows by (21), and the equality $\overline{g}(m(X - A)) = 1 - \overline{g}(m(A))$. The item 2 follows by Proposition 5, ensuring that $m(A)' = 1$, by (24) and (1), for which $m(A^c) \geq m(A)'$. □

Definition 4 (Sugeno 1974) The set function m is a λ-additive fuzzy measure if and only if it is continuous from below or above and:

$$A, B \in \mathcal{A}, A \cap B = \emptyset \Rightarrow m(A \cap B) = m(A) + m(B) + \lambda m(A)m(B).$$

In the sequel m_λ will denote a λ-additive fuzzy measure.

Remark 2 Let m_λ be a λ-additive fuzzy measure, with $\lambda > -1$. Then m_λ is decomposable respect to the t-conorm U_λ in the Example 2.

3 First Borel–Cantelli lemma

In this section we assume that $\perp$ is an archimedean t-conorm.

Proposition 8 *Let m be a non-decreasing σ-$\perp$-subdecomposable measure, $\left(A_k\right)_{k\in\mathbf{N}}$ a sequence of measurable events and*

$$A = \lim sup A_k = \bigcap_{n\geq 1}\bigcup_{k\geq n} A_k. \tag{25}$$

Then

$$\perp_{k=1}^{+\infty} m(A_k) < 1 \Rightarrow m(A) = 0. \tag{26}$$

Proof By the monotonicity and the σ-$\perp$-subdecomposibilty of m, we get:

$$m(A) \leq m\left(\bigcup_{k\geq n} A_k\right) \leq \perp_{k\geq n} m(A_k) \quad \forall n.$$

As a consequence $m(A) \leq \lim_n \perp_{k\geq n} m(A_k)$ and the claim follows by Proposition 3. □

Theorem 4 (1st Borel–Cantelli like Lemma) *Let m be a σ-$\perp$-decomposable measure, $\left(A_k\right)_{k\in\mathbf{N}}$ a sequence of events and A the event* (25). *Then the implication* (26) *holds.*

Proof By Theorem 3 and Corollary 3 m is monotone and σ- $\perp$- subdecomposable; then, by Proposition 8, the assertion is achieved. □

4 The strict product t-norm associated to a non-strict archimedean t-conorm

From now on we will assume that $\perp$ is a non-strict archimedean t-conorm, g is one of its additive generators and $\overline{g}$ its normed additive generator.

Then, for $(x, y) \in J \times J$

$$g^{-1}\left(\frac{g(x)g(y)}{g(1)}\right) = \bar{g}^{-1}(\bar{g}(x)) \cdot \bar{g}(y)).$$

So the binary operation $\bullet$ over J defined by

$$x \bullet y = \bar{g}^{-1}(\bar{g}(x)) \cdot \bar{g}(y)) \tag{27}$$

can be built starting from each additive generator of $\perp$.

Proposition 9 *The operation $\bullet$ is a strictly archimedean t-norm and $\bar{g}$ is its multiplicative generator.*

Proof By Theorem 2 it is enough to observe that $\bar{g} : [0, 1] \to [0, 1]$ is a continuous, strictly increasing function and $\bar{g}(1) = 1$, $\bar{g}(0) = 0$. □

Definition 5 The operation $\bullet$ in (27) is called the *product norm* associated to $\perp$.

The equalities in (5), (6) and (8) can be written, for $\top = \bullet$, as follows:

$$\bullet_{i=1}^{n} x_i = (\bullet_{i=1}^{n-1} x_i) \bullet x_n, \quad \bullet_{i=1}^{+\infty} x_i = lim_n \bullet_{i=1}^{n} x_i, \quad Q_n = \bullet_{i>n} x_i = \bullet_{i=n+1}^{+\infty} x_i.$$

Example 5 Let $\perp$ coincide with $U_\lambda(a, b) = min(a + b - \lambda ab, 1)$. Then $\bar{g}_\lambda(x) = \log_{1+\lambda}(1 + \lambda x)$ is its normed generator and the product t-norm $\bullet$ associated to U_λ is:

$$a \bullet b = \frac{(1 + \lambda a)^{\log_{1+\lambda}(1+\lambda b)} - 1}{\lambda}.$$

Proposition 10 *The product norm (27) verifies the following conditions:*

1. $a \bullet b = 0$ *if and only if either* $a = 0$ *or* $b = 0$*;*
2. *if* $a \in]0, 1[$ *then* $a \bullet b = a \Leftrightarrow b = 1$*;*
3. $a \bullet (b \perp c) = (a \bullet b) \perp (a \bullet c)$ *whenever* $b \perp c < 1$ *(conditional distributivity).*

Proof The assertion in item 1 follows by the $2nd$ equality of (3) and by the strict monotonicity of $\bullet$. The assertion in item 2 follows by Proposition 2. To prove item 3 let us assume $b \perp c < 1$; then, by Theorem 1 and definition of $\bar{g}$, we get $\bar{g}(b) + \bar{g}(c) < 1$ and

$$\begin{aligned} a \bullet (b \perp c) &= \bar{g}^{-1}\big(\bar{g}(a) \cdot \bar{g}(b \perp c)\big) = \bar{g}^{-1}\big(\bar{g}(a) \cdot (\bar{g}(b) + \bar{g}(c)\big) = \\ &= \bar{g}^{-1}\big(\bar{g}(a) \cdot \bar{g}(b) + \bar{g}(a) \cdot \bar{g}(c)\big) = (a \bullet b) \perp (a \bullet c). \end{aligned}$$

□

By the previous proposition, $(J, \perp, \bullet)$ is a *conditionally distributive semiring*.

Lemma 1 *For* $h, n \in N \cup \{+\infty\}$, $n > h$,

$$(\perp_{i=h}^{n} x_i)' \leq \bullet_{i=h}^{n} x_i' \leq \perp_{i=h}^{n} x_i'. \tag{28}$$

Proof It is enough to prove the first inequality in (28), because the second inequality follows immediately from the inequalities (4). First we prove that

$$x' \bullet y' \geq (x \perp y)' \quad \forall x, y \in J. \tag{29}$$

The inequality (29) is trivial if $x \perp y = 1$ because $(x \perp y)' = 0$; in the case $x \perp y < 1$, by Theorem 1 and Proposition 5, we get $\bar{g}(x) + \bar{g}(y) < 1$ and

$$\bar{g}(x') \cdot \bar{g}(y') = 1 - \bar{g}(x) - \bar{g}(y) + \bar{g}(x) \cdot \bar{g}(y);$$

so

$$(x' \bullet y') \perp (x \perp y) = \bar{g}^{(-1)}\left(\bar{g}(x') \cdot \bar{g}(y') + \bar{g}(x) + \bar{g}(y)\right) = \bar{g}^{(-1)}\left(1 + \bar{g}(x) \cdot \bar{g}(y)\right) = 1.$$

Then, by (1), (29) is proved and, as a consequence, the inequality (28) is achieved for $n = h + 1$.

Assume now that (28) is verified for $k \geq h + 1$. Then by (29) and the assumed inequality $\bullet_{i=h}^{k} x_i' \geq (\perp_{i=h}^{k} x_i)'$ we get

$$\bullet_{i=h}^{k+1} x_i' = (\bullet_{i=h}^{k} x_i') \bullet x_{k+1}' \geq (\perp_{i=h}^{k} x_i)' \bullet x_{k+1}' \geq (\perp_{i=h}^{k+1} x_i)'.$$

Hence the (28) is proved by induction for each $n \geq h + 1$; by the continuity of $\bullet$, (28) holds also for $n = +\infty$. □

Theorem 5 *The following assertions are equivalent:*

1. $\perp_{i=1}^{+\infty} x_i \equiv 1$;
2. $\bullet_{i=1}^{+\infty} x_i' \equiv 0$.

Proof 1. $\Rightarrow$ 2. By the definition of strong equality in item 1 and Corollary 1, $\Sigma_{i>n} \bar{g}(x_i) = +\infty \quad \forall n \geq 1$. Then, by the inequality "$(1 - x) \leq e^{-x}$" holding for each $x \geq 0$, we get, for each $n \geq 1$:

$$0 \leq \bullet_{i>n} x_i' = \overline{g}^{-1}\left(\prod_{i>n}(1 - \overline{g}(x_i))\right) \leq \overline{g}^{-1}\left(\prod_{i>n} e^{-\overline{g}(x_i)}\right) = \overline{g}^{-1}(e^{-\sum_{i>n} \overline{g}(x_i)}) = 0$$

2. $\Leftarrow$ 1. By the first inequality in Lemma 1 and Proposition 5. □

5 $m_\perp$-independent events and second Borel–Cantelli lemma

In this section we assume that m is a σ-$\perp$-decomposable measure with respect to a non-strict archimedean t-conorm $\perp$.

Definition 6 If $A, B \in \mathcal{A}$, then we say that A and B are $m_\perp$-independent if and only if

$$m(A \cap B) = m(A) \bullet m(B).$$

If $T = \{A_1, A_2, \ldots A_n\}$ is a finite collection of measurable sets, then we say that T is $m_\perp$-independent if and only if

$$m(A_1 \cap \cdots \cap A_n) = m(A_1) \bullet \cdots \bullet m(A_n).$$

An infinite collection of measurable sets is $m_{\perp}$-independent if and only if each of its finite subcollections is.

Proposition 11 *$A, B \in \mathcal{A}$ are $m_{\perp}$-independent if and only if they are independent respect to the normed measure $\bar{g} \circ m$, that is*

$$\bar{g}\big(m(A \cap B)\big) = \bar{g}(m(A))\bar{g}(m(B)). \tag{30}$$

Proof By definition of the product norm $\bullet$. □

Kruse (1987) defines the concept of independence of two events with respect to a λ-additive fuzzy measure m_λ, $\lambda > -1$, as follows:

Definition 7 A and B are m_λ-independent if and only if

$$\log_{1+\lambda}(1 + \lambda m_\lambda(A \cap B)) = \log_{1+\lambda}(1 + \lambda m_\lambda(A)) \log_{1+\lambda}(1 + \lambda m_\lambda(B)).$$

This definition has been extended by Qiang (1995) to a finite or infinite collection of events as we have done in Definition 6. By Proposition 11 our concept of $m_{\perp}$-independence generalizes the concept of m_λ-independence given in Kruse (1987) and Qiang (1995), because m_λ is a decomposable measure with respect to the t-conorm $U_\lambda(a, b) = min(a+b-\lambda ab, 1)$ that has $\overline{g}(x) = \log_{1+\lambda}(1+\lambda x)$ as normed generator.

Proposition 12 *Let $(A_k)_{k\in\mathbf{N}}$ be a set of $m_{\perp}$-independent events. Then $(A_k^c)_{k\in\mathbf{N}}$ is a set of $m_{\perp}$- independent events.*

Proof By Proposition 11. □

Theorem 6 (2nd Borel–Cantelli like Lemma) *Let $(A_k)_{k\in\mathbf{N}}$ be a $m_{\perp}$-independent sequence of events and $A = \lim sup A_k$. Then:*

$$\perp_{k=1}^{+\infty} m(A_k) \equiv 1 \Rightarrow m(A) = 1.$$

Proof Because of the Proposition 7, it is enough to show that $m(A^c) = 0$. Let us observe that

$$m(A^c) = m\left(\bigcup_{n\geq 1}\bigcap_{k\geq n} A_k^c\right) = \lim_n m\left(\bigcap_{k\geq n} A_k^c\right).$$

Furthermore, by Propositions 12 and Proposition 7, we get

$$m\left(\bigcap_{k\geq n} A_k^c\right) = \bullet_{k=n}^{+\infty} m(A_k^c) = \bullet_{k\geq n}^{+\infty} m(A_k)'.$$

Hence the assertion follows by Theorem 5. □

6 Conclusions and future works

We have introduced a concept of strict product norm • associated to a non-strict archimedean *t*-conorm ⊥ over the interval [0, 1] and given a notion of independence in the setting of general ⊥-decomposable measures. Our concept of independence generalizes the concept of m_λ-independence introduced by Kruse (1987) and Qiang (1995). In the context of the λ-additive fuzzy measures, Qiang has stated theorems that correspond to those of the classical probability theory such as Borel–Cantelli lemmas; we prove the correspondent theorems for the more general context of the $m_\perp$-decomposable measures.
In order to continue our analysis we plan to handle with the following problems:

– to look for other convergence properties more general than the properties stated in the present paper, see for instance (Petrov 2004);
– to consider the previous problems in a general algebraic structure (not necessarily a real interval) in order to generally characterize the operations suitable for the above properties.

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