Spreads of $PG(3,q)$ and ovoids of polar spaces

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Abstract

To any spread $\mathcal{S}$ of $PG(3,q)$ corresponds a family of locally hermitian ovoids of the Hermitian surface $H(3,q^2)$, and conversely; if in addition $\mathcal{S}$ is a semifield spread, then each associated ovoid is a translation ovoid, and conversely.

In this paper we calculate the translation group of the locally hermitian ovoids of $H(3,q^2)$ arising from a given semifield spread, and we characterize the p-semiclassical ovoid constructed in [4] as the only translation ovoid of $H(3,q^2)$ whose translation group is abelian.

If $\mathcal{S}$ is a spread of $PG(3,q)$ and $\mathcal{O}(\mathcal{S})$ is one of the associated ovoids of $H(3,q^2)$, then using the duality between $H(3,q^2)$ and $Q^-(5,q)$, another spread of $PG(3,q)$, say $\mathcal{S}_2$, can be constructed. On the other hand, using the Barlotti-Cofman representation of $H(3,q^2)$, one more spread of a 3-dimensional projective space, say $\mathcal{S}_1$, arises from the ovoid $\mathcal{O}(\mathcal{S})$. In [8] some questions are posed on the relations among $\mathcal{S}$, $\mathcal{S}_1$ and $\mathcal{S}_2$; here we prove that $\mathcal{S}$ and $\mathcal{S}_2$ are isomorphic and the ovoids $\mathcal{O}(\mathcal{S})$ and $\mathcal{O}(\mathcal{S}_1)$, corresponding to $\mathcal{S}$ and $\mathcal{S}_1$ respectively, under the Plücker map, are isomorphic.

1 Introduction

A spread $\mathcal{S}$ of $\Sigma = PG(3,q)$ is a set of q^2+1 mutually skew lines partitioning the point-set of Σ. Let $\mathcal{S}$ be a spread of Σ and choose homo-

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geneous projective coordinates (x_0, x_1, x_2, x_3) in such a way that the lines $l_\infty = \{(0,0,c,d)\colon c,d \in F_q\}$ and $l_0 = \{(a,b,0,0)\colon a,b \in F_q\}$ belong to $\mathcal{S}$. Then for each line M of $\mathcal{S}$ different from l_∞, there is a unique 2×2 matrix J_M over F_q such that $M = \{(a,b,c,d)\colon (c,d) = (a,b)J_M,\ a,b \in F_q\}$. The set $\mathcal{C}_\mathcal{S} = \{J_M\colon M \in \mathcal{S}\}$ has the following properties: (i) $\mathcal{C}_\mathcal{S}$ contains q^2 elements, (ii) the zero matrix belongs to $\mathcal{C}_\mathcal{S}$, (iii) $X - Y$ is non-singular for all X, $Y \in \mathcal{C}_\mathcal{S}$, $X \neq Y$. The set $\mathcal{C}_\mathcal{S}$ is called the *spread set* associated with $\mathcal{S}$ with respect to l_∞ and l_0. Conversely, starting from a set $\mathcal{C}$ of 2×2 matrices over F_q satisfying (i), (ii) and (iii), the set of lines $\mathcal{S} = \{l_M\colon M \in \mathcal{C}\} \cup \{l_\infty\}$ where $l_M = \{(a,b,c,d)\colon (c,d) = (a,b)M,\ \ a,b \in F_q\}$ is a spread of Σ and $\mathcal{C}_\mathcal{S} = \mathcal{C}$. A spread $\mathcal{S}$ is a *semifield spread* if there exists a collineation group of $\mathcal{S}$ fixing a line l pointwise and acting regularly on the set of the q^2 lines of $\mathcal{S}$ different from l. Equivalently, $\mathcal{S}$ is a semifield spread with respect to the line l_∞ if and only if $\mathcal{C}_\mathcal{S}$ is closed under the sum.

Regard Σ as a canonical subgeometry of $\Sigma^* = PG(3,q^2)$ and let σ be the involutory collineation of Σ^* pointwise fixing Σ. Let $\mathcal{S}$ be any spread of Σ and l be a fixed line of $\mathcal{S}$. A plane π of Σ^* is an *indicator plane* of $\mathcal{S}$ through the line l if $\pi \cap \Sigma = l$. The set $I_\pi(\mathcal{S}) = \{m^* \cap \pi |\, m \in \mathcal{S} \setminus \{l\}\}$ (where m^* is the extension of the line m in Σ^*) is called the *indicator set* of $\mathcal{S}$ in π. Such a set consists of q^2 points and any secant line of it meets l^* in a point not on l. Conversely, any set of points I of $\pi \setminus l^*$ satisfying the above properties defines a spread $\mathcal{S} = \{< P, P^\sigma > \cap \Sigma | P \in I\} \cup \{l\}$ of Σ containing l such that $I_\pi(\mathcal{S}) = I$ ([2] and [6]). The spread $\mathcal{S}$ is regular if and only if $I_\pi(\mathcal{S})$ is either an affine line (*classical indicator set*) or an affine Baer subplane (*semiclassical indicator set*). Following [5], we say that two indicator sets I_1 and I_2 in Σ^* lying on the indicator planes π_1 and π_2, respectively, passing through the line l^*, are *isomorphic* if the associated spreads of Σ are; the indicator sets I_1 and I_2 are *equivalent* if there exists a collineation ψ fixing the Baer subline l such that $\psi(I_1) = I_2$. Note that isomorphic indicator sets may be not equivalent, while in [5, Prop 3.1] it is proven that equivalent indicator sets are isomorphic.

A Hermitian surface $\mathcal{H} = H(3,q^2)$ of $PG(3,q^2)$ is the set of all isotropic points of a non-degenerate unitary polarity. A line of $PG(3,q^2)$ meets $\mathcal{H}$ in 1, $q+1$ or q^2+1 points. The former are the *tangents* and the latter are the *generators* of $\mathcal{H}$. The intersections of order $q+1$ are Baer sublines and are often called *chords*, whereas the lines meeting $\mathcal{H}$ in a Baer subline are called *hyperbolic lines*.

An ovoid $\mathcal{O}$ of $\mathcal{H}$ is a set of q^3+1 points which has exactly one common point with every generator of $\mathcal{H}$. The ovoid $\mathcal{O}$ is called *locally hermitian* with respect to one of its points, say P, if it is the union of q^2 chords of $\mathcal{H}$ through P; the ovoid $\mathcal{O}$ is called *translation* ovoid with respect to its point P if there is a collineation group of $\mathcal{H}$ fixing P, leaving invariant all the generators through P, and acting regularly on the points of $\mathcal{O} \setminus \{P\}$. Note that any translation ovoid is locally hermitian ([3]).

In [10] by using the so-called *Shult embedding* the author proves that any indicator set $I = I_\pi(\mathcal{S})$ in $\pi \simeq PG(2, q^2)$ gives rise to a locally hermitian ovoid of $\mathcal{H}$, and conversely. Let $\mathcal{O}_\pi(\mathcal{S})$ be the locally hermitian ovoid of $\mathcal{H}$ arising from the spread $\mathcal{S}$ via the indicator set $I_\pi(\mathcal{S})$. In [8] it has been proved that if the spread $\mathcal{S}$ is a semifield spread then, for any choice of π, the ovoid $\mathcal{O}_\pi(\mathcal{S})$ is a translation ovoid, and conversely.

In [4], starting from semiclassical indicator sets, the authors construct some translation ovoids of $\mathcal{H}$ and their translation groups; among them, only one, namely the p-*semiclassical ovoid* (permutable semiclassical ovoid), has an elementary abelian p-group ($q = p^r$). In Section 2 we determine the translation group of all translation ovoids arising from a semifield spread $\mathcal{S}$ and we characterize the p-semiclassical ovoid as the only example whose translation group is abelian.

In [8], by using the Barlotti-Cofman representation of the Hermitian surface $\mathcal{H}$ it is shown that any locally hermitian ovoid $\mathcal{O}_\pi(\mathcal{S})$ of $\mathcal{H}$ defines an ovoid, say $\mathbb{O}$, of the hyperbolic quadric $Q^+(5, q)$, and conversely; if $\mathcal{O}_\pi(\mathcal{S})$ is a translation ovoid, then also $\mathbb{O}$ is. In Section 3 we answer a question posed in [8], proving that the ovoids $O(\mathcal{S})$ (which is the image of $\mathcal{S}$ under the Plücker map) and $\mathbb{O}$ are isomorphic for any choice of the indicator plane π.

By duality any locally hermitian ovoid $\mathcal{O} = \mathcal{O}_\pi(\mathcal{S})$ of $\mathcal{H}$ with respect to a point P gives rise to a locally hermitian spread of $Q^-(5, q)$ with respect to the line L (dual of P); such a spread defines a spread $\mathcal{S}_2$ of $L^\perp \simeq PG(3, q)$, where $\perp$ is the polarity induced by $Q^-(5, q)$, see [11]. If the spread $\mathcal{S}$ is a semifield spread then the spread $\mathcal{S}_2$ also is. In Section 4 we prove that $\mathcal{S}$ and $\mathcal{S}_2$ are isomorphic for any choice of the indicator plane π. In the case $\mathcal{S}$ is a semifield spread, the question on the relation between $\mathcal{S}$ and $\mathcal{S}_2$ was posed in [8, Sect. 4.3].

2 Translation ovoids of $H(3, q^2)$

Let $\mathcal{O}$ be a translation ovoid of $\mathcal{H} = H(3, q^2)$ with respect to a point P. Then $\mathcal{O}$ is a locally hermitian ovoid of $\mathcal{H}$ with respect to P and it arises from a semifield spread $\mathcal{S}$ of a 3-dimensional projective space over F_q via the Shult embedding ([8]). Choose the homogeneous projective coordinates (x_0, x_1, x_2, x_3) in $PG(3, q^2)$ in such a way that $\mathcal{H} : x_0 x_3^q - x_3 x_0^q + x_2 x_1^q - x_1 x_2^q = 0$ and $P = (0, 0, 0, 1)$; then in [5] it has been proved that

$$\begin{aligned}\mathcal{O} = \mathcal{O}_\pi(\mathcal{S}) = \mathcal{O}(\lambda, h, k) \quad = \quad & \{(1, -v - \lambda^q u, h(u, v) + \lambda^q k(u, v), \\ & \mu + \lambda(vk(u, v) - uh(u, v)))| u, v, \mu \in F_q\} \cup \{P\}\end{aligned}$$

where $\pi : x_1 = \lambda x_0$ $(\lambda \in F_{q^2} \setminus F_q)$ and $h, k : F_q \times F_q \to F_q$. The spread set $\mathcal{C}_\mathcal{S}$ associated with $\mathcal{S}$ with respect to l_∞ and l_0 consists of the matrices:

$$X_{uv} = \begin{pmatrix} v & h(u, v) \\ u & k(u, v) \end{pmatrix},$$

with $u, v \in F_q$. Since $\mathcal{S}$ is a semifield spread, then $\mathcal{C}_\mathcal{S}$ is closed under the sum and hence h and k are additive functions.

Let $\mathbb{U} = PGU(4, q^2)$ denote the group of the linear collineations of $PG(3, q^2)$, leaving $\mathcal{H}$ invariant. The subgroup E of $\mathbb{U}$ fixing P and leaving invariant all the generators through P has size q^5 ([9]) and direct computations show that E consists of the matrices

$$\begin{pmatrix} 1 & \alpha & \beta & c - \alpha\beta^q \\ 0 & 1 & 0 & -\beta^q \\ 0 & 0 & 1 & \alpha^q \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \alpha, \beta \in F_{q^2}, c \in F_q.$$

The following theorem holds.

Theorem 1 *The translation group of $\mathcal{O} = \mathcal{O}(\lambda, h, k)$ with respect to P is*

$$G = \left\{ \begin{pmatrix} 1 & -v - \lambda^q u & h(u, v) + \lambda^q k(u, v) & c + (v + \lambda^q u)(h(u, v) + \lambda k(u, v)) \\ 0 & 1 & 0 & -h(u, v) - \lambda k(u, v) \\ 0 & 0 & 1 & -v - \lambda u \\ 0 & 0 & 0 & 1 \end{pmatrix}; \right.$$

$$\left. u, v, c \in F_q \right\}.$$

Proof: Direct calculations show that G is a subgroup of E and that the point $P_0 = (1,0,0,0)$ of $\mathcal{O}(\lambda,h,k)$ is mapped to the point $(1, -v-\lambda^q u, h(u,v) + \lambda^q k(u,v), \mu + \lambda(vk(u,v) - uh(u,v))$ of $\mathcal{O}(\lambda,h,k)$ by the element of G with $c = \mu - vh(u,v) - \lambda^{q+1}uk(u,v) - uh(u,v)(\lambda+\lambda^q)$. This implies that G acts regularly on the points of $\mathcal{O}(\lambda,h,k) \setminus \{P\}$. □

From now on, let Tr and N denote the trace and the norm functions of F_{q^2} over F_q, respectively.

The set $\tilde{E} = F_{q^2} \times F_q \times F_{q^2}$ equipped with the product $(\alpha,c,\beta)\circ(\alpha',c',\beta') = (\alpha+\alpha', c+c'+Tr(\alpha'\beta^q), \beta+\beta')$ is a group, whose center is $Z = \{(0,c,0) |\ c \in F_q\}$.

The Hermitian surface $\mathcal{H}$ is an elation generalized quadrangle of order (q^2,q) with respect all of its points, which can be also described as a coset geometry, and in this model the elation group of $\mathcal{H}$ is isomorphic to $\tilde{E}$. For more details, see e.g. [9].

The map $\theta : (\alpha,c,\beta) \in \tilde{E} \mapsto \begin{pmatrix} 1 & \alpha & \beta & c-\alpha\beta^q \\ 0 & 1 & 0 & -\beta^q \\ 0 & 0 & 1 & \alpha^q \\ 0 & 0 & 0 & 1 \end{pmatrix} \in E$ is an isomorphism and a straightforward calculation shows that $\tilde{G} = \theta^{-1}(G) = \{(-v-\lambda^q u, c, h(u,v)+\lambda^q k(u,v)) | u,v,c \in F_q\}$.

In [4] the authors, starting from indicator sets associated with the desarguesian spread, construct some translation ovoids of $H(3,q^2)$ and their translation groups; among them, only one, namely the p-semiclassical ovoid, has an elementary abelian p-group ($q = p^r$, p odd). We conclude this section characterizing translation ovoids of $H(3,q^2)$, $q = p^r$, whose translation group is abelian.

Theorem 2 *The translation group G of $\mathcal{O}(\lambda,h,k)$ is abelian if and only if $\mathcal{O}(\lambda,h,k)$ is* p-*semiclassical.*

Proof: Since G and $\tilde{G}$ are isomorphic, we work on $\tilde{G}$. The group $\tilde{G}$ is abelian if and only if for any $u,u',v,v' \in F_q$,

$$Tr((-v'-\lambda^q u')(h(u,v)+\lambda k(u,v)) + (v+\lambda^q u)(h(u',v')+\lambda k(u',v'))) = 0$$

i.e., as $\lambda^q = Tr(\lambda) - \lambda$,

$$\begin{aligned} 2N(\lambda)(uk(u',v') - u'k(u,v)) + Tr(\lambda)[uh(u',v') - u'h(u,v) + vk(u',v') \\ -v'k(u,v)] + 2(vh(u',v') - v'h(u,v)) = 0. \quad (1) \end{aligned}$$

Note that, since $h, k : F_q \times F_q \mapsto F_q$ are additive maps, they are linear over F_p and hence we can write

$$h(u,v) = a_0 u + \sum_{i=1}^{r-1} a_i u^{p^i} + b_0 v + \sum_{i=1}^{r-1} b_i v^{p^i} = a_0 u + h_1(u) + b_0 v + h_2(v) \quad (2)$$

and

$$k(u,v) = c_0 u + \sum_{i=1}^{r-1} c_i u^{p^i} + d_0 v + \sum_{i=1}^{r-1} d_i v^{p^i} = c_0 u + k_1(u) + d_0 v + k_2(v) \quad (3)$$

with $a_i, b_i, c_i, d_i \in F_q$.
Now suppose $p = 2$. Since $Tr(\lambda) \neq 0$, from (1) it follows

$$uh(u', v') - u'h(u, v) + vk(u', v') - v'k(u, v) = 0 \quad (4)$$

for all $u, u', v, v' \in F_q$. Putting $u = u' = 0$ in (4) we get $vk(0, v') = v'k(0, v)$ for all $v, v' \in F_q$ and hence $k(0, v) = d_0 v$. Similarly, putting $v = v' = 0$ in (4) we obtain $h(u, 0) = a_0 u$. Substituting in (4) with $u = u'$, we have $uh(0, v' - v) + (v - v')k(u, 0) = 0$ for all $u, v, v' \in F_q$, from which it follows $b_0 = c_0$ and $b_i = c_i = 0$ for all $i = 1, \dots, r-1$. Summing up, in even characteristic, if $\tilde{G}$ is abelian, then

$$h(u, v) = a_0 u + b_0 v \quad \text{and} \quad k(u, v) = b_0 u + d_0 v.$$

This is impossible, as such maps do not define any spread of Σ.

On the other hand, let p be odd. From (1) with $u = u' = 0$ and taking (2) and (3) into account, it follows

$$\sum_{i=0}^{r-1} (Tr(\lambda) d_i + 2b_i) v v'^{p^i} = \sum_{i=0}^{r-1} (Tr(\lambda) d_i + 2b_i) v' v^{p^i}$$

for all $v, v' \in F_q$. Then $b_i = \frac{-Tr(\lambda)}{2} d_i$ for all $i = 1, \dots, r-1$. Similarly, from (1) with $v = v' = 0$, it follows

$$\sum_{i=0}^{r-1} (2N(\lambda) c_i + Tr(\lambda) a_i) u u'^{p^i} = \sum_{i=0}^{r-1} ((2N(\lambda) c_i + Tr(\lambda) a_i) u' u^{p^i}$$

for all $u, u' \in F_q$. Therefore, $c_i = \frac{-Tr(\lambda)}{2N(\lambda)} a_i$ for $i = 1, \dots, r-1$. Substituting in (2) and in (3), we get

$$h_2(v) = -\frac{Tr(\lambda)}{2} k_2(v) \tag{5}$$

$$k_1(u) = -\frac{Tr(\lambda)}{2N(\lambda)} h_1(u). \tag{6}$$

Moreover, with $u = u'$ and $v = 0$, from (1) it follows

$$2N(\lambda) uk(0, v') + Tr(\lambda)(uh(0, v') - v'k(u, 0)) - 2v'h(u, 0) = 0$$

for all $u, v' \in F_q$. The above condition, taking (5) and (6) into account, yields

$$\frac{4N(\lambda) - Tr(\lambda)^2}{2} \Big(uk_2(v') - \frac{v'h_1(u)}{N(\lambda)} \Big) = 0.$$

Since $\lambda \in F_{q^2} \setminus F_q$, we have $4N(\lambda) - Tr(\lambda)^2 \neq 0$; hence $h_1(u) = k_2(v') = 0$ for all $u, v' \in F_q$. From (5) and (6) it turns out that if $\tilde{G}$ is abelian, then the spread $\mathcal{S}$ of $PG(3, q)$ is regular. As q is odd, without loss of generality, we can consider the regular spread $\mathcal{S}_m$ arising for $h(u, v) = u$ and $k(u, v) = mv$ for m a fixed nonsquare in F_q. With this choice, from (1) it follows that $(m\lambda^{q+1} - 1)(uv' - u'v) = 0$ for all $u, u', v, v' \in F_q$ and hence $\lambda^{q+1} = 1/m$. Since such an equation admits $q+1$ solutions in $F_{q^2} \setminus F_q$, there exist exactly $q+1$ translation ovoids arising from S_m with an abelian translation group. In [5] it has been shown that there are $q+1$ p-semiclassical translation ovoids arising from a fixed regular spread $\mathcal{S}$ of $PG(3, q)$, and each has an abelian group. Hence, the translation ovoids $\mathcal{O}(\lambda, u, mv)$ of $H(3, q^2)$, with $\lambda^{q+1} = m$ are p-semiclassical translation ovoids. $\square$

Remark 1 In [5] it is also proved that the $q+1$ p-semiclassical translation ovoids arising from a given regular spread are all isomorphic. Hence, there exists (up to isomorphisms) a unique translation ovoid of $H(3, q^2)$ with an abelian translation group.

3 Translation ovoids of $H(3,q^2)$ and ovoids of $Q^+(5,q)$

In the setting of the previous section, let $\mathcal{S}$ be the spread of $\Sigma = PG(3,q)$ containing the lines l_∞ and l_0 and defined by the functions h and k. Then

$$\begin{aligned}\mathcal{O}(\lambda,h,k) &= \{(1,-v-\lambda^q u, h(u,v)+\lambda^q k(u,v), \mu+\lambda(vk(u,v)-uh(u,v))) : \\ &\quad u,v,\mu \in F_q\} \cup \{(0,0,0,1)\},\end{aligned}$$

with $\lambda \in F_{q^2} \setminus F_q$, are the locally hermitian ovoids of the Hermitian surface $\mathcal{H} : x_0x_3^q - x_0^q x_3 + x_2x_1^q - x_2^q x_1 = 0$ of $\Gamma = PG(3,q^2)$ arising from $\mathcal{S}$.

An element $x \in F_{q^2}$ can be uniquely written as $x = x_0 + \lambda x_1$, where $x_0, x_1 \in F_q$. To any point $R = (a,b,c,d) \in \Gamma$, with $a = a_0 + \lambda a_1$, $b = b_0 + \lambda b_1$, $c = c_0 + \lambda c_1$, $d = d_0 + \lambda d_1$, there corresponds the line l_R of a 7-dimensional projective space over F_q, say Λ, passing through the point $(a_0,a_1,b_0,b_1,c_0,c_1,d_0,d_1)$ defined as $l_R = \{(y_0,y_1,\dots,y_7) | \exists \mu \in F_{q^2} : \mu a = y_0 + \lambda y_1, \mu b = y_2 + \lambda y_3, \mu c = y_4 + \lambda y_5, \mu d = y_6 + \lambda y_7\}$. The set $\mathcal{R}_\lambda = \{l_R : R \in \Gamma\}$ turns out to be a normal spread of Λ (for more details see [7]). We say that the pair $(\Lambda, \mathcal{R}_\lambda)$ is the *F_q-linear representation* of Γ with respect to the basis $\{1,\lambda\}$.

The F_q-linear representation of $\mathcal{H}$ with respect to the basis $\{1,\lambda\}$ is the hyperbolic quadric $Q^+(7,q)$ with equation $y_0y_7 - y_1y_6 + y_4y_3 - y_5y_2 = 0$. Let $P = (0,0,0,1) \in \mathcal{H}$. Then, the F_q-linear representation with respect to the basis $\{1,\lambda\}$ of the polar plane $P^\rho : x_0 = 0$ (where ρ is the unitary polarity induced by $\mathcal{H}$) is the 5-dimensional projective space $\Omega : y_0 = y_1 = 0$ of Λ equipped with the normal spread $\mathcal{N}$ induced by $\mathcal{R}_\lambda$ on it. Note that $l_P : y_0 = \cdots = y_5 = 0$ is a line of $\mathcal{N}$. Let $\Omega' = PG(6,q) : y_1 = 0$. Define an incidence structure $\pi(\Omega',\Omega,\mathcal{N})$ as follows. The points are either the points of $\Omega' \setminus \Omega$ or the elements of $\mathcal{N}$. The lines are either the planes of Ω' which intersect Ω in a line of $\mathcal{N}$ or the regular spreads of the 3-dimensional projective spaces $\langle A, B\rangle$, where A and B are distinct lines of $\mathcal{N}$; the incidence is the natural one. As $\mathcal{N}$ is normal, $\pi = \pi(\Omega',\Omega,\mathcal{N})$ is isomorphic to Γ (for more details see [1]). Let $\Phi : \Gamma \mapsto \pi$ be the isomorphism defined by $\Phi(R) = l_R$ if $R \in P^\rho$ and $\Phi(R) = l_R \cap \Omega'$ if $R \notin P^\rho$. Note that, if $R = (1, b_0 + \lambda b_1, c_0 + \lambda c_1, d_0 + \lambda d_1)$ with $b_0, b_1, c_0, c_1, d_0, d_1 \in F_q$, then $\Phi(R) = (1,0,b_0,b_1,c_0,c_1,d_0,d_1)$.

Moreover, $\Omega' \cap Q^+(7,q)$ is a quadratic cone $\mathcal{K}$ with vertex the point $V = (0,0,0,0,0,0,1,0) \in l_P$. A base of $\mathcal{K}$ is the hyperbolic quadric $\mathcal{Q} = Q^+(5,q)$

obtained by intersecting $\mathcal{K}$ with the 5-dimensional projective space $\Delta : y_1 = y_6 = 0$.

The set

$$\begin{aligned}\Phi(\mathcal{O}(\lambda,h,k)\setminus P)\cap\Delta \;=\; & \{(1,0,-v-Tr(\lambda)u,u,h(u,v)+Tr(\lambda)k(u,v),\\ & -k(u,v),0,vk(u,v)-uh(u,v)) : u,v\in F_q\}\end{aligned}$$

union the point $Q = l_P \cap \Delta = (0,0,0,0,0,0,0,1)$ is an ovoid $\mathbb{O}_\lambda$ of $\mathcal{Q}$ (see [8, Thm. 6]). Let $\mathcal{S}_1(\lambda)$ denote the spread of a projective space $PG(3,q)$ which is the image of $\mathbb{O}_\lambda$ under the inverse of the Plücker map. Let $\varphi_\lambda : \Lambda \mapsto \Lambda$ be the collineation with equations

$$y_0' = y_0, \quad y_1' = y_1, \quad y_2' = y_3, \quad y_3' = -y_5,$$

$$y_4' = -y_2 - Tr(\lambda)y_3, \quad y_5' = y_4 + Tr(\lambda)y_5, \quad y_6' = y_6, \quad y_7' = -y_7.$$

Then φ_λ fixes both Δ and the Klein quadric $\mathcal{Q}$ and maps the ovoid $\mathbb{O}_\lambda$ to the ovoid

$$\begin{aligned}\mathbb{O} \;=\; & \{(1,0,u,k(u,v),v,h(u,v),0,uh(u,v)-vk(u,v)) : u,v\in F_q\}\\ & \cup\{(0,0,0,0,0,0,0,1)\}.\end{aligned}$$

It can be easily seen that such an ovoid is the image of the spread $\mathcal{S}$, under the Plücker map between the lines of Σ and the points of $\mathcal{Q}$. We have proved the following

Theorem 3 *The ovoids $\mathbb{O}$ and $\mathbb{O}_\lambda$ are isomorphic for any choice of λ.* □

Corollary 1 *The ovoid $\mathcal{O}(\lambda,h,k)$ is a translation ovoid of $\mathcal{H}$ with respect to the point $P = (0,0,0,1)$ if and only if $\mathbb{O}_\lambda$ is a translation ovoid of $\mathcal{Q}$ with respect to the point $Q = (0,0,0,0,0,0,0,1)$.*

Proof: Since $\mathcal{O}(\lambda,h,k)$ is a translation ovoid if and only if $\mathcal{S}$ is a semifield spread (see [8, Thm. 5 and Cor. 2]), the result follows from Theorem 3. □

4 A construction of a spread in $PG(3,q)$

Let $\mathcal{S}$ be a spread of $\Sigma = PG(3,q)$. Embed Σ in $\Sigma^* = PG(3,q^2)$ in such a way that $\Sigma = Fix(\sigma)$, where σ is an involutory collineation of Σ^*. Let π be an indicator plane of $\mathcal{S}$ in $PG(3,q^2)$. Denote by l the line of $\mathcal{S}$ such that l is in π and by $I_\pi(\mathcal{S})$ the indicator set of $\mathcal{S}$ in the plane π. Consider the point-line dual plane of π: this is a plane $\tilde{\pi}$, in which l^* (the extension of l in Σ^*) is represented by a point P, the Baer subline l by a Baer subpencil $\tilde{l}$ through P and $I_\pi(\mathcal{S})$ by a set $\mathcal{F}$ of q^2 lines not containing P, any two of them intersecting at a point of $\tilde{\pi}\setminus\tilde{l}$. The set of lines $\mathcal{F}$ is called a *Shult set*, following [5] and [8]. Fix a Hermitian surface $\mathcal{H} = H(3,q^2)$ in such a way that $P \in \mathcal{H}$ and $\tilde{\pi} \cap \mathcal{H} = \tilde{l}$. Let u be the polarity defined by $\mathcal{H}$. The elements of $\mathcal{F}^u$ are hyperbolic lines of $\mathcal{H}$ through P, hence the set $\mathcal{O} = \bigcup_{m\in\mathcal{F}}(m^u \cap \mathcal{H})$ is a locally hermitian ovoid of $\mathcal{H}$ (see [10]). The ovoid $\mathcal{O}$ corresponds via a Plücker map ρ to a locally hermitian spread $\mathbb{S}$ of $Q^-(5,q)$ with respect to the line $L = P^\rho$. Let $\Lambda = L^\perp$, where $\perp$ is the orthogonal polarity induced by $Q^-(5,q)$. If M is a line of $\mathbb{S}$ different from L then the line $m_{L,M} = \langle L, M\rangle^\perp$ is a line of Λ disjoint from $\langle L, M\rangle$. Moreover the set of lines $\mathcal{S}_2 = \{m_{L,M} \colon M \in \mathbb{S}, M \neq L\} \cup \{L\}$ turns out to be a spread of Λ ([11]). With this notation, the following holds.

Theorem 4 *The spreads $\mathcal{S}$ and $\mathcal{S}_2$ are isomorphic, for any choice of l and π, and for any embedding of the indicator plane of the spread $\mathcal{S}$ as a tangent plane to a Hermitian surface at the point-line dual of the line l.*

Proof: In order to prove the isomorphism between $\mathcal{S}$ and $\mathcal{S}_2$, we review the above construction embedding the involved spreads in the same 3-dimensional projective space over F_{q^2}.

Let $\mathcal{S}$ be a spread of Σ, fix a line l of $\mathcal{S}$ and fix an indicator plane, say π, of $\mathcal{S}$ in Σ^*, such that $l^* \subset \pi$. Let $I = I_\pi(\mathcal{S})$ be the indicator set of $\mathcal{S}$ in π. Choose a hyperbolic quadric $Q^+(5,q^2)$ of a $PG(5,q^2)$ containing Σ^* such that $Q^+(5,q^2) \cap \Sigma^* = \pi \cup \pi^\sigma$. Let ϕ be the inverse of a Plücker map from $Q^+(5,q^2)$ to the lineset of $\Gamma = PG(3,q^2)$, such that π is a latin plane, i.e. π^ϕ is a ruled plane of Γ. Note that the map ϕ restricted to π is a point-line duality between π and π^ϕ. Let $X = (l^*)^{\phi_{|\pi}}$ and let $\mathcal{B}$ denote the Baer cone which is the image of the points of l under $\phi_{|\pi}$. Also, the indicator set I corresponds, under $\phi_{|\pi}$, to a Shult set $\mathcal{F}$ of π^ϕ with respect to $\mathcal{B}$.

Let $\mathcal{H} = H(3,q^2)$ be any Hermitian surface of Γ such that $\mathcal{H} \cap \pi^\phi = \mathcal{B}$ and let $\mathcal{O}$ be the locally hermitian ovoid of $\mathcal{H}$ arising from $\mathcal{F}$. The ovoid $\mathcal{O}$ is locally hermitian with respect to X and applying the Pluüker map ϕ^{-1} to $\mathcal{H}$, we get an elliptic quadric $Q^-(5,q)$ containing l embedded in $Q^+(5,q^2)$ and the locally hermitian spread $\mathcal{O}^{\phi^{-1}}$ of $Q^-(5,q)$ with respect to l. Moreover, $\mathcal{O}^{\phi^{-1}}$ defines the spread $\mathcal{S}_2 = \{m_{l,M} \colon M \in \mathcal{O}^{\phi^{-1}}, M \neq l\} \cup \{l\}$ of $l^\perp = \Sigma_1 \simeq PG(3,q)$, where $\perp$ is the polarity defined by $Q^-(5,q)$. Note that $l \subset l^*$ implies that $\Sigma_1 \subset \Sigma^*$, since Σ^* is the polar space of l^* with respect to $Q^+(5,q^2)$. By construction, π is an indicator plane of $\mathcal{S}_2$. Also, the extension of any line $m_{l,M}$ of $\mathcal{S}_2$ intersects the plane π in a point, say x, which is orthogonal to every point of the regulus of the hyperbolic quadric $Q^+(3,q) = \langle l, M\rangle \cap Q^-(5,q)$ containing the lines l and M. Hence x^ϕ is a line of π^ϕ whose polar line (with respect to $\mathcal{H}$) is one of the hyperbolic lines of $\mathcal{O}$, i.e. x^ϕ is a line of the Shult set $\mathcal{F}$. So the indicator set of $\mathcal{S}_2$ in π is I as well. Let ψ be a collineation of Σ^* mapping Σ to Σ_1. Hence $\mathcal{S}^\psi$ is a spread of Σ_1 containing l and its indicator set in π^ψ is I^ψ. Since both $\mathcal{S}_2$ and $\mathcal{S}^\psi$ are spreads of Σ_1 with equivalent indicator sets I and I^ψ, respectively, by [5, Prop. 3.1] such spreads are isomorphic. $\square$

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