Transitive matrices, strict preference and intensity operators

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Abstract. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of alternatives and a_{ij} a positive number expressing how much the alternative x_i is preferred to the alternative x_j. Under suitable hypothesis of no indifference and transitivity over the *pairwise comparison* matrix $A = (a_{ij})$, the alternatives can be ordered as a chain "$x_{i_1} \succ x_{i_2} \succ \ldots \succ x_{i_n}$". Then a *coherent priority* vector is a vector giving a weighted ranking agreeing with the chain. An *intensity* vector is a coherent priority vector encoding information about the intensities of the preferences. In this paper we deal with intensity vectors, and we look for operators that transform $A = (a_{ij})$ into an intensity vector, when they act on the row vectors of the matrix.

Keywords. Pairwise comparison matrix, preference relations, coherent priority vector, intensity vector.

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1 Introduction

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of alternatives, and, for every i and j, a_{ij} a positive number expressing how much the alternative x_i is preferred to the alternative x_j: obviously $a_{ij} > 1$ implies that x_i is strictly preferred to x_j, $a_{ij} < 1$ expresses the opposite preference and $a_{ij} = 1$ means that x_i and x_j are indifferent. The preference ratios a_{ij} generate the *pairwise comparison* matrix $A = (a_{ij})$ that is assumed to be reciprocal:

$$\textbf{r)} \quad a_{ji} = \frac{1}{a_{ij}} \quad \forall\, i, j = 1, 2, \ldots, n \quad (\textit{reciprocity}).$$

As a consequence $a_{ii} = 1\ \forall i = 1, 2, \ldots, n$.

Given $A = (a_{ij})$, there exists a vector $\underline{w} = (w_1, w_2, \ldots, w_n)$ verifying

$$\frac{w_i}{w_j} = a_{ij} \quad \forall i, j = 1, 2, \ldots, n \tag{1}$$

if and only if A is *consistent*, that is

$$\textbf{c)} \quad a_{ij} \cdot a_{jk} = a_{ik} \quad \forall\, i,j,k = 1,2,\ldots,n \quad (\textit{consistency}).$$

We call *consistent evaluation vector* each positive vector $\underline{w}$ verifying (1).

$A = (a_{ij})$ is consistent if and only if n is its unique positive eigenvalue and consistent evaluation vectors are [14]: the columns of the matrix, the positive right eigenvectors associated to n, the vectors built by applying the arithmetic or the geometric mean to each row of $A = (a_{ij})$.

If $A = (a_{ij})$ is not consistent, the following problem arises: how to determine a preference order on X and evaluations for the alternatives.

In the classical approach to the above problem (see [9], [11], [13], [14], [15]) *priority* vectors are provided by :

- the *arithmetic* and the *geometric mean operators* that transform the matrix A into a vector, when applied to each row of the given matrix;
- the *right eigenvectors* associated to the greatest eigenvalue of A;
- the *logarithmic least squares method.*

A priority vector $\underline{w} = (w_1, w_2, \ldots, w_n)$ ranks the alternatives by means of the weak order $\succeq_{\underline{w}}$ defined by

$$x_i \succeq_{\underline{w}} x_j \Leftrightarrow w_i \geq w_j,$$

and the *dominance priority* vector $\underline{w}^* = \underline{w}/\sum_i w_i$ supplies the weights of the alternatives.

Hence, in the classical approach, a priority vector provides both the ranking on the set X and the weights for the alternatives. Nevertheless, if the matrix A is not consistent, then the relation $\succeq_{\underline{w}}$ may not respect the effective dominance ratios expressed by the entries of the matrix [4], [5], [16]. In other words it may happen that $w_i > w_j$ whereas $a_{ij} < 1$ means that the alternative x_j is preferred to the alternative x_i. Moreover, even though the order $\succeq_{\underline{\mathbf{w}}}$ agrees with the preference ratios a_{ij}, it may not enclude any information about the intensity of the preferences: indeed it may happen that $\frac{w_i}{w_j} > \frac{w_r}{w_s}$ and $a_{ij} < a_{rs}$.

Therefore the need of a new procedure for getting a ranking on X and evaluations for the alternatives. In order, we have the following tasks:

1^{st} to state the *actual ranking* on X, that is a *qualitative* ranking agreeing with the preference ratios a_{ij};

2^{nd} to look for *ordinal evaluation vectors* $\underline{w} = (w_1, w_2, \ldots, w_n)$ representing the actual ranking;

3^{rd} to find *cardinal evaluation vectors* encoding information about the intensities of the preferences.

In [4], [5], [7] we tackled the first and the second problem under the assumption that $A = (a_{ij})$ is **Rts**, that is it is reciprocal and verifies the conditions:

$$\textbf{t)} \quad a_{ij} > 1 \text{ and } a_{jk} > 1 \Rightarrow a_{ik} > 1 \quad (\textit{transitivity}),$$

s) $a_{ij} \neq 1$ for $i \neq j$ (*no indifference*).

The conditions **s** and **t** ensure that the relation $\succ$ defined by

$$x_i \succ x_j \;\Leftrightarrow\; a_{ij} > 1 \tag{2}$$

is a strict simple order on X [4] (i.e. it is transitive, asymmetric and complete [12]). Hence a **Rts** matrix induces a relation of strict preference on X and a permutation $\alpha = (i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots, n)$ is available ranking the alternatives as follows

$$x_{i_1} \succ x_{i_2} \succ \ldots \succ x_{i_n}. \tag{3}$$

We say that the decreasing chain (3) represents the *actual ranking* and a vector $\underline{w} = (w_1, w_2, \ldots, w_n) \in]0, +\infty[^n$ is a *coherent priority vector* for $A = (a_{ij})$ if and only if:

$$x_i \succ x_j \;\Leftrightarrow\; w_i > w_j; \tag{4}$$

hence, in accordance with (3), we get $w_{i_1} > w_{i_2} > \ldots > w_{i_n}$.

$A = (a_{ij})$ is called *priority matrix* if each column is a coherent priority vector [6], [7].

If $A = (a_{ij})$ is **Rts**, then it is a priority matrix if and only the following condition lying between **t** and **c** is verified [7]:

wc) $a_{ij} > 1$ and $a_{jk} > 1 \Rightarrow a_{ik} > a_{ij} \vee a_{jk}$ (*weak consistency*).

Remark 1. Under the assumption **s** for which $a_{ij} = 1 \Leftrightarrow i = j$, the condition **wc** implies the following other

$$a_{ij} \geq 1 \text{ and } a_{jk} \geq 1 \Rightarrow a_{ik} \geq a_{ij} \vee a_{jk} \tag{5}$$

that is the condition of *strong stochastic transitivity* or (restricted) *max-max transitivity* (see [8], [10], [12], [17]), reformuled for a context in which the comparisons among alternatives are expressed by preference ratios a_{ij} verifying **r**. We stress that (5) does not imply **wc**: indeed, if in the matrix of Example 5 below we change $a_{13} = 7$ in $a_{13} = 5 = a_{12} \vee a_{23}$, then we get a $\max - \max$ transitive matrix, that is not weakly consistent.

A way to find coherent priority vectors taking into account the effective values of the $a_{is}'s$ consists in to aggregate the entries of each row $\underline{a}_i$ of the matrix $A = (a_{ij})$ using an operator

$$F : \underline{u} = (u_1, u_2, \ldots, u_n) \in]0, +\infty[^n \rightarrow F(\underline{u}) \in]0, +\infty[. \tag{6}$$

F transforms $A = (a_{ij})$ into the vector $\underline{w}_F = (F(\underline{a}_1), F(\underline{a}_2), \ldots, F(\underline{a}_n))$. Then we say that F is *ordinal evaluation operator* for the matrix $A = (a_{ij})$ if and only if $\underline{w}_F$ is a coherent priority vector [7]. The class of the ordinal evaluation operators associated to a priority matrix is very large and can be characterized by means of a relation of strict partial order $\triangleright$ emboding the set $]0,+\infty[^n$ (see [7] and Sect. 2.1 below).

The aim of this paper is to tackle the $3rd$ problem under the assumption that $A=(a_{ij})$ is **Rts**. We look for vectors $\underline{w}=(w_1,w_2,\ldots,w_n)$ such that

$$\frac{w_i}{w_j} > \frac{w_h}{w_k} \Leftrightarrow a_{ij} > a_{hk} \quad \text{and} \quad \frac{w_i}{w_j} = \frac{w_h}{w_k} \Leftrightarrow a_{ij} = a_{hk}, \tag{7}$$

and for conditions over A linked to the existence of this type of vector. We call:

- *intensity vector* or a *value ratio vector* for $A=(a_{ij})$ each positive vector verifying (7) (and *dominance vector* an intensity vector normalized to 1);
- *intensity operator* or *value ratio operator* for $A=(a_{ij})$ each operator (6) such that the vector $\underline{w}_F=(F(\underline{a}_1),F(\underline{a}_2),\ldots,F(\underline{a}_n))$ is an intensity vector.

We say that $A=(a_{ij})$ is an *intensity matrix* if and only if its columns are intensity vectors.

The paper is organized as follows. In Sect. 2 we recall a characterization of the **Rts** matrices given in [7] and show which are the coherent priority vectors linked to this type of matrix. In Sect. 2.1 we recall caracterizations of the class of the priority matrices and the related class of the ordinal evaluation operators together with related results that come in useful for reaching our aim. Sect.3 is devoted to the intensity vectors. An intensity vector is a coherent priority vector encoding information about the strengths of preference and the existence of intensity vector implies the following property for $A=(a_{ij})$:

$\delta)$ $a_{ij}>a_{rs} \Leftrightarrow a_{ir}>a_{js}$ and $a_{ij}=a_{rs} \Leftrightarrow a_{ir}=a_{js}$ (*index exchangeability*).

In Sect. 4 the intensity matrices are analyzed. They are characterized by means a partial order $\trianglerighteq$ emboding the set $]0,+\infty[^n$. A further characterization is given for the matrices of the 3th order. In Sect.5 we provide a characterization of the class of the intensity operators linked to an intensity matrix.

2 Preliminaries

Let $X=\{x_1,x_2,\ldots,x_n\}$ be a set of alternatives and $A=(a_{ij})$ the related pairwise comparison matrix. From now on we assume that: $A=(a_{ij})$ is **Rts** and (3) represents the *actual ranking* on X.

We denote by:

- $\tilde{\mathbf{A}}=\{\underline{a}_1,\underline{a}_2,\ldots,\underline{a}_n\}$ the set of the rows of A;
- $\check{\mathbf{A}}=\{\underline{a}^1,\underline{a}^2,\ldots,\underline{a}^n\}$ the set of the columns of A;
- $n(\underline{a}_i)$ the number of the components of $\underline{a}_i\in\tilde{\mathbf{A}}$ greater than 1;
- $n_0(\underline{a}_i)$ the number of the components of $\underline{a}_i$ greater than or equal to 1;
- $\triangleright$ the strict partial order on $R^n_+=]0,+\infty[^n$ defined by

$$\underline{u}=(u_1,\ldots,u_n) \triangleright \underline{v}=(v_1,\ldots,v_n) \Leftrightarrow u_j>v_j \ \forall j=1,2,\ldots,n;$$

- $\trianglerighteq$ the partial order on R_+^n defined by:

$$\underline{u} \trianglerighteq \underline{v} \Leftrightarrow (\underline{u} \ \triangleright \ \underline{v} \quad or \quad \underline{u} \ = \ \underline{v});$$

- $\underline{1}$ the vector $\underline{1} = (1, 1, \ldots, 1)$;
- $\frac{\underline{u}}{\underline{v}}$ the vector $\left(\frac{u_1}{v_1}, \ldots, \frac{u_n}{v_n}\right)$, for $\underline{u}, \underline{v} \in R_+^n$;
- $\overline{A}_{\div}$ the set $\left\{\frac{\underline{a}_i}{\underline{a}_j} : \underline{a}_i, \underline{a}_j \in \tilde{A}\right\}$;
- $A_{\div}$ the set $\left\{\frac{\underline{a}_i}{\underline{a}_j} \in \overline{A}_{\div} : \frac{\underline{a}_i}{\underline{a}_j} \triangleright \underline{1}\right\}$;
- $(u_1^*, u_2^*, \ldots, u_n^*)$ the decreasing rearrangement of the vector $\underline{u} = (u_1, u_2, \ldots, u_n)$.

Since $A = (a_{ij})$ verifies **s**:

$$n_0(\underline{a}_i) = n(\underline{a}_i) + 1. \tag{8}$$

Proposition 1. (Lemma 3.1 and Theorem 3.1 in [7]) *The assertion "$A = (a_{ij})$ is **Rts** and (3) represents the actual ranking " is equivalent to each one of the following others:*

1. $n(\underline{a}_{i_1}) = n - 1 > n(\underline{a}_{i_2}) = n - 2 > \ldots > n(\underline{a}_{i_h}) = n - h > \ldots > n(\underline{a}_{i_n}) = 0$;

2. $n_0(\underline{a}_{i_1}) = n > n_0(\underline{a}_{i_2}) = n-1 > \ldots > n_0(\underline{a}_{i_h}) = n-h+1 > \ldots > n_0(\underline{a}_{i_n}) = 1$.

So, by Proposition 1, the vector $\underline{n}_0(A) = (n_0(\underline{a}_1), n_0(\underline{a}_2), \ldots n_0(\underline{a}_n))$ is a coherent priority vector (see Sect. 1). It is straightforward that a positive vector $\underline{w}$ is a coherent priority vector if and only if $\underline{w} = (\phi(n_0(\underline{a}_1)), \phi(n_0(\underline{a}_2)), \ldots \phi(n_0(\underline{a}_n))$, with ϕ real positive and strictly monotonic function on $]0, +\infty[$.

Example 1. Let A be the matrix

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 1 & 7 & 3 & 6 & 2 \\ 1/7 & 1 & 1/5 & 5 & 2 \\ 1/3 & 5 & 1 & 7 & 2 \\ 1/6 & 1/5 & 1/7 & 1 & 1/3 \\ 1/2 & 1/2 & 1/2 & 3 & 1 \end{pmatrix} \end{array}.$$

As $n(\underline{a}_1) = 4 > n(\underline{a}_3) = 3 > n(\underline{a}_2) = 2 > n(\underline{a}_5) = 1 > n(\underline{a}_4) = 0$, A is **Rts**, the actual ranking is $x_1 \succ x_3 \succ x_2 \succ x_5 \succ x_4$ and $\underline{n}_0(A) = (5, 3, 4, 1, 2)$ is a coherent priority vector.

Finally we stress that,under the assumption that $A = (a_{ij})$ is **Rts**, the property of consistency can be expressed as the following proposition shows.

Proposition 2. *The condition of consistency **c** is equivalent to*

$$a_{ik} > 1 \Rightarrow a_{ik} = a_{ij} a_{jk} \quad \forall\, j = 1, 2, \ldots, n. \tag{9}$$

Proof. It is enough to show that, if (9) holds, then $a_{ik} = a_{ij} a_{jk}$ also in the case $a_{ik} \leq 1$. If $a_{ik} = 1$ then, by **s**, $i = k$ and, by **r**, $a_{ik} = a_{ii} = a_{ij} a_{ji}$. If $a_{ik} < 1$, then, by **r**, $a_{ki} > 1$ and by (9), $a_{ki} = a_{kj} a_{ji} \quad \forall j = 1, 2, \ldots, n$; the last inequalities, by **r**, are equivalent to the following others: $a_{ik} = a_{ij} a_{jk} \quad \forall j = 1, 2, \ldots, n$. $\square$

2.1 Priority matrices and ordinal evaluation operators

By definition, $A = (a_{ij})$ is a *priority* matrix if and only if each column $\underline{a}^h$ is a coherent priority vector, that is:

$$x_i \succ x_j \Leftrightarrow (a_{ih} > a_{jh} \; \forall h = 1, 2, \ldots, n).$$

Proposition 3. [7] *The following conditions are equivalent*

1. $A = (a_{ij})$ *is a priority matrix;*
2. $x_i \succ x_j \Leftrightarrow \underline{a}_i \triangleright \underline{a}_j \Leftrightarrow \underline{a}^j \triangleright \underline{a}^i$*;*
3. $\tilde{A}$ *is completely ordered by the relation* $\triangleright$*;*
4. $A = (a_{ij})$ *is weakly consistent.*

Hence, by Proposition 3, A is a priority matrix if and only if, in accordance with (3), $\underline{a}_{i_1} \triangleright \underline{a}_{i_2} \triangleright \ldots \triangleright \underline{a}_{i_n}$.

Corollary 1. *Let* $A = (a_{ij})$ *be a priority matrix and* $\alpha = (i_1, i_2, \ldots, i_n)$ *the permutation of* $(1, 2, \ldots, n)$ *providing the actual ranking (3). Then*

$$\begin{array}{l} \underline{a}^{i_1} = \underline{a}_\wedge = (\wedge_h a_{1h}, \wedge_h a_{2h}, \ldots, \wedge_h a_{nh}), \\ \underline{a}^{i_n} = \underline{a}_\vee = (\vee_h a_{1h}, \vee_h a_{2h}, \ldots, \vee_h a_{nh}). \end{array} \tag{10}$$

Corollary 2. *The following conditions are equivalent:*

1. $A = (a_{ij})$ *is a priority matrix;*
2. $(a_{ih} > a_{jh}$ *for some* $h) \Leftrightarrow \underline{a}_i \triangleright \underline{a}_j$*;*
3. $(a_{ij} > a_{is}$ *for some* $i) \Leftrightarrow \underline{a}^j \triangleright \underline{a}^s$*.*

Theorem 1. *An operator (6) is an ordinal evaluation operator for the class of the priority matrices if and only if it is strictly increasing as to* $\triangleright$*, that is*

$$\underline{u} \triangleright \underline{v} \Rightarrow F(\underline{u}) > F(\underline{v}).$$

Proof. By Proposition 3. □

Corollary 3. [7] *Let* ϕ *denote a strict monotonic function on* R^n_+ *and* $\underline{p} = (p_1, p_2, \ldots, p_n)$ *a non negative weighting vector verifying the condition* $\sum p_i = 1$. *The class of the ordinal evaluation operators for priority matrices includes*

- *the quasilinear mean operators* ([1], [2])

$$F_{\phi\underline{p}}(u_1, u_2, \ldots, u_n) = \phi^{-1}\left(\sum_{i=1}^{n} p_i \phi(u_i)\right), \tag{11}$$

- *the ordered quasilinear means*

$$O_{\phi\underline{p}}(u_1, u_2, \ldots, u_n) = F_{\phi\underline{p}}(u_1^*, u_2^*, \ldots, u_n^*) = \phi^{-1}\left(\sum_{i=1}^{n} p_i \phi(u_i^*)\right) \tag{12}$$

(see also [3]).

By Corollary 3 we get that the class of the ordinal evaluation operators for priority matrices includes the *weighted geometric mean operators*

$$F_{\ln \underline{p}}(u_1, u_2, \ldots, u_n) = u_1^{p_1} u_2^{p_2} \ldots u_n^{p_n}, \tag{13}$$

the OWA (*ordered weighted averaging*) *operators* [18]

$$O_{\underline{p}}(u_1, u_2, \ldots, u_n) = F_{\underline{p}}(u_1^*, u_2^*, \ldots, u_n^*) = \sum_{i=1}^{n} p_i u_i^*,$$

and, as particulare cases of OWA operators, the *min* operator and the *max* operator

$$F_\wedge(u_1, \ldots, u_n) = \min\{u_1, \ldots, u_n\}, \quad F_\vee(u_1, \ldots, u_n) = \max\{u_1, \ldots, u_n\}.$$

The above results can be applied to show that every right positive eigenvector associated to the maximum eigenvalue of a priority matrix is a coherent priority vector [7].

3 Intensity vectors

An *intensity vector* for $A = (a_{ij})$ is a positive vector $\underline{w} = (w_1, w_2, \ldots, w_n)$ that represents the intensities of the preference ratios in accordance with the equivalences (7).

Proposition 4. *A positive vector $\underline{w} = (w_1, w_2, \ldots, w_n)$ is an intensity vector for $A = (a_{ij})$ if and only if*

$$\frac{w_i}{w_j} > \frac{w_r}{w_s} \geq 1 \Leftrightarrow a_{ij} > a_{rs} \geq 1 \quad \textit{and} \quad \frac{w_i}{w_j} = \frac{w_r}{w_s} \geq 1 \Leftrightarrow a_{ij} = a_{rs} \geq 1. \tag{14}$$

Proof. It is enough to show that the condition (14) implies (7). Applying the first equivalence in (14) we get

$$1 \geq \frac{w_i}{w_j} > \frac{w_r}{w_s} \Leftrightarrow \frac{w_s}{w_r} > \frac{w_j}{w_i} \geq 1 \Leftrightarrow a_{sr} > a_{ji} \geq 1$$

and, by the condition of reciprocity,

$$1 \geq \frac{w_i}{w_j} > \frac{w_r}{w_s} \Leftrightarrow 1 \geq a_{ij} > a_{rs}. \tag{15}$$

By (15) and the first equivalence in (14) we get

$$\frac{w_i}{w_j} > 1 > \frac{w_r}{w_s} \Leftrightarrow a_{ij} > a_{jj} = a_{rr} > a_{rs}. \tag{16}$$

So the first equivalence in (14) implies the first equivalence in (7). The second equivalence in (7) follows from the first one. □

Corollary 4. *The following conditions are equivalent*

1. $\underline{w} = (w_1, w_2, \ldots, w_n)$ *is an intensity vector;*
2. $\underline{w} = (w_1, w_2, \ldots, w_n)$ *is a coherent priority vector verifying the conditions:*

$$\frac{w_i}{w_j} > \frac{w_r}{w_s} > 1 \Leftrightarrow a_{ij} > a_{rs} > 1 \quad \textit{and} \quad \frac{w_i}{w_j} = \frac{w_r}{w_s} > 1 \Leftrightarrow a_{ij} = a_{rs} > 1. \tag{17}$$

Proof. 1. $\Rightarrow$ 2. Let $\underline{w} = (w_1, w_2, \ldots, w_n)$ be an intensity vector. By choosing $r = s$ in (14) we get

$$\frac{w_i}{w_j} > 1 \Leftrightarrow a_{ij} > 1$$

that is $w_i > w_j \Leftrightarrow x_i \succ x_j$. Then $\underline{w}$ is a coherent priority vector. The equivalences (17) follow from Proposition 4 and from the condition **s**.

2. $\Rightarrow$ 1. By definition of coherent priority vector and condition **s** we get

$$\frac{w_i}{w_j} = \frac{w_r}{w_s} = 1 \Leftrightarrow (x_i = x_j \ \text{ and } \ x_r = x_s) \Leftrightarrow a_{ij} = a_{rs} = 1$$

$$\frac{w_i}{w_j} > 1 = \frac{w_r}{w_s} \Leftrightarrow (x_i \succ x_j \ \text{ and } \ x_r = x_s) \Leftrightarrow a_{ij} > a_{rs} = 1.$$

Then, by (17) and Proposition 4, the implication is proved. □

Proposition 5. *The existence of an intensity vector* $\underline{w} = (w_1, w_2, \ldots, w_n)$ *implies that* $A = (a_{ij})$ *verifies the condition of index exchangeability* δ.

Proof. The claim follows from (7) and the following chain of equivalences

$$\frac{w_i}{w_j} > \frac{w_r}{w_s} \Leftrightarrow \frac{w_i}{w_r} > \frac{w_j}{w_s}, \qquad \frac{w_i}{w_j} = \frac{w_r}{w_s} \Leftrightarrow \frac{w_i}{w_r} = \frac{w_j}{w_s}. \ \square$$

Example 2. The **Rts** matrix

$$\begin{pmatrix} 1 & 4 & 7 & 8 \\ 1/4 & 1 & 5 & 6 \\ 1/7 & 1/5 & 1 & 4 \\ 1/8 & 1/6 & 1/4 & 1 \end{pmatrix}$$

induces the ranking $x_1 \succ x_2 \succ x_3 \succ x_4$ on the set $X = \{x_1, x_2, x_3, x_4\}$ and it is a priority matrix because, in accordance with the actual ranking, we get $\underline{a}_1 \triangleright \underline{a}_2 \triangleright \underline{a}_3 \triangleright \underline{a}_4$ (see Proposition 3). The condition δ is not verified: indeed $a_{13} > a_{24}$, but $a_{12} = a_{34}$. Since δ is not verified there do not exist any intensity vector.

4 Intensity matrices

By definition, $A = (a_{ij})$ is an *intensity matrix* if and only if each column $\underline{a}^h$ is an intensity vector, that is

$$\iota) \begin{cases} a_{ij} > a_{rs} \Leftrightarrow \left(\dfrac{a_{ih}}{a_{jh}} > \dfrac{a_{rh}}{a_{sh}} \quad \forall h \in \{1, 2, \ldots, n\} \right) \\ a_{ij} = a_{rs} \Leftrightarrow \left(\dfrac{a_{ih}}{a_{jh}} = \dfrac{a_{rh}}{a_{sh}} \quad \forall h \in \{1, 2, \ldots, n\} \right) \end{cases}.$$

Proposition 6. *$A = (a_{ij})$ is an intensity matrix if and only if*

$$a_{ij} > a_{rs} \Leftrightarrow \frac{\underline{a}_i}{\underline{a}_j} \triangleright \frac{\underline{a}_r}{\underline{a}_s}, \qquad a_{ij} = a_{rs} \Leftrightarrow \frac{\underline{a}_i}{\underline{a}_j} = \frac{\underline{a}_r}{\underline{a}_s}.$$

Proof. By ι and definition of $\triangleright$. □

The following propositions make easier to verify the intensity of the matrix $A = (a_{ij})$.

Proposition 7. *$A = (a_{ij})$ is an intensity matrix if and only if A is a priority matrix and the following equivalences hold*

$$a_{ij} > a_{rs} > 1 \Leftrightarrow \frac{\underline{a}_i}{\underline{a}_j} \triangleright \frac{\underline{a}_r}{\underline{a}_s} \triangleright \underline{1}, \qquad a_{ij} = a_{rs} > 1 \Leftrightarrow \frac{\underline{a}_i}{\underline{a}_j} = \frac{\underline{a}_r}{\underline{a}_s} \triangleright \underline{1}.$$

Proof. By Corollary 4, $\underline{a}^h$ is an intensity vector if and only if it is a coherent priority vector and:

$$a_{ij} > a_{rs} > 1 \Leftrightarrow \frac{a_{ih}}{a_{jh}} > \frac{a_{rh}}{a_{sh}} > 1, \qquad a_{ij} = a_{rs} > 1 \Leftrightarrow \frac{a_{ih}}{a_{jh}} = \frac{a_{rh}}{a_{sh}} > 1.$$

Hence A is an intensity matrix if and only if A is a priority matrix and

$$\begin{aligned} a_{ij} > a_{rs} > 1 &\Leftrightarrow \left(\frac{a_{ih}}{a_{jh}} > \frac{a_{rh}}{a_{sh}} > 1 \quad \forall h \in \{1, 2, \ldots, n\} \right), \\ a_{ij} = a_{rs} > 1 &\Leftrightarrow \left(\frac{a_{ih}}{a_{jh}} > \frac{a_{rh}}{a_{sh}} > 1 \quad \forall h \in \{1, 2, \ldots, n\} \right). \end{aligned}$$

By definition of $\triangleright$ the assertion is proved. □

By Proposition 7 an intensity matrix is a priority matrix, but a priority matrix might not be an intensity matrix as Example 2 shows.

Proposition 8. *$A = (a_{ij})$ is an intensity matrix if and only if A is a priority matrix and*

$$\iota^*) \quad (i \neq r \text{ and } j \neq s) \Rightarrow \begin{pmatrix} a_{ij} > a_{rs} > 1 \Leftrightarrow \dfrac{\underline{a}_i}{\underline{a}_j} \triangleright \dfrac{\underline{a}_r}{\underline{a}_s} \triangleright \underline{1} \\ \\ a_{ij} = a_{rs} > 1 \Leftrightarrow \dfrac{\underline{a}_i}{\underline{a}_j} = \dfrac{\underline{a}_r}{\underline{a}_s} \triangleright \underline{1} \end{pmatrix}.$$

Proof. By Proposition 7 it is enough to show that, if A is a priority matrix and the condition ι^* holds, then the equivalences in the right side of ι^* are also verified for $i = r$ and $j = s$. By Corollary 2 and the property **r** we get

$$a_{ij} > a_{is} \Leftrightarrow \underline{a}^j \rhd \underline{a}^s \Leftrightarrow \underline{a}_s \rhd \underline{a}_j \Leftrightarrow \frac{\underline{a}_i}{\underline{a}_j} \rhd \frac{\underline{a}_i}{\underline{a}_s},$$

$$a_{ij} > a_{rj} \Leftrightarrow \underline{a}_i \rhd \underline{a}_r \Leftrightarrow \frac{\underline{a}_i}{\underline{a}_j} \rhd \frac{\underline{a}_r}{\underline{a}_j}.$$

To complete the proof it is enough to stress that, because of Proposition 3, $a_{ij} = a_{is} \Leftrightarrow j = s$ and $a_{ij} = a_{rj} \Leftrightarrow i = r$, and, by Corollary 2,

$$a_{ij} > 1 \Leftrightarrow \frac{\underline{a}_i}{\underline{a}_j} \rhd \underline{1}. \ \square$$

By Proposition 8 the intensity of a priority matrix can be verified just checking the pairs (a_{ij}, a_{rs}) of entries that are greater than 1 and do not lie in the same row or in the same column.

Example 3. Let us consider the **Rts** matrix

$$\begin{pmatrix} 1 & 9/7 & 3 & 7 \\ 7/9 & 1 & 2 & 4 \\ 1/3 & 1/2 & 1 & 2 \\ 1/7 & 1/4 & 1/2 & 1 \end{pmatrix}.$$

The actual ranking is $x_1 \succ x_2 \succ x_3 \succ x_4$ and the matrix is a priority matrix. Limiting ourselves to the comparisons between entries that are greater than 1 and don't belong to the same line, we just consider the inequalities $7 = a_{14} > 2 = a_{23}$, $4 = a_{24} > 3 = a_{13}$, $3 = a_{13} > 2 = a_{34}$, $2 = a_{23} > 9/7 = a_{12}$, $2 = a_{34} > 9/7 = a_{12}$, and the equality $2 = a_{23} = a_{34}$.

It is easy to verify that

$$\frac{\underline{a}_1}{\underline{a}_4} \rhd \frac{\underline{a}_2}{\underline{a}_3}, \quad \frac{\underline{a}_2}{\underline{a}_4} \rhd \frac{\underline{a}_1}{\underline{a}_3}, \quad \frac{\underline{a}_1}{\underline{a}_3} \rhd \frac{\underline{a}_3}{\underline{a}_4}, \quad \frac{\underline{a}_2}{\underline{a}_3} \rhd \frac{\underline{a}_1}{\underline{a}_2}, \quad \frac{\underline{a}_3}{\underline{a}_4} \rhd \frac{\underline{a}_1}{\underline{a}_2}, \quad \frac{\underline{a}_2}{\underline{a}_3} = \frac{\underline{a}_3}{\underline{a}_4}.$$

Then the matrix is an intensity matrix.

Theorem 2. $A = (a_{ij})$ *is an intensity matrix if and only if it verifies the condition of* index exchangeability δ *and :*

α) $\quad A_{\div} = \{\frac{\underline{a}_i}{\underline{a}_j} \in \overline{A}_{\div} : \frac{\underline{a}_i}{\underline{a}_j} \rhd \underline{1}\}$ *is completely ordered by* $\unrhd$.

Proof. Let A be an intensity matrix. Then the assertion α follows from Proposition 7 and by the comparability of the relation $\geq$ on the set of the real numbers. The condition δ follows from Proposition 5.

Assume now that α and δ are verified. From δ we have

$$\frac{a_{ij}}{a_{rs}} > 1 \Leftrightarrow \frac{a_{ir}}{a_{js}} > 1 \quad \text{and} \quad \frac{a_{ij}}{a_{rs}} = 1 \Leftrightarrow \frac{a_{ir}}{a_{js}} = 1; \tag{18}$$

and, by α, one of the following relation is true

$$\frac{\underline{a}_i}{\underline{a}_j} \rhd \frac{\underline{a}_r}{\underline{a}_s}, \quad \frac{\underline{a}_r}{\underline{a}_s} \rhd \frac{\underline{a}_i}{\underline{a}_j}, \quad \frac{\underline{a}_i}{\underline{a}_j} = \frac{\underline{a}_r}{\underline{a}_s}.$$

Assume that $\frac{\underline{a}_i}{\underline{a}_j} \rhd \frac{\underline{a}_r}{\underline{a}_s}$, that is

$$\frac{a_{ih}}{a_{jh}} > \frac{a_{rh}}{a_{sh}} \quad \forall h = 1, 2, \ldots, n.$$

Then, by choosing $h = i, j, r, s$ we get

$$a_{ij} > \frac{a_{ri}}{a_{si}}, \quad a_{ij} > \frac{a_{rj}}{a_{sj}}, \quad \frac{a_{ir}}{a_{jr}} > a_{rs}, \quad \frac{a_{is}}{a_{js}} > a_{rs},$$

respectively, and as a consequence,

$$(a_{ij})^2 \frac{a_{ir}}{a_{js}} \frac{a_{is}}{a_{jr}} > (a_{rs})^2 \frac{a_{ri}}{a_{sj}} \frac{a_{rj}}{a_{si}}.$$

As $\frac{a_{rj}}{a_{si}} = \frac{a_{is}}{a_{jr}}$, the above inequality becomes $\left(\frac{a_{ij}}{a_{rs}}\right)^2 > \left(\frac{a_{ri}}{a_{sj}}\right)^2$ that is

$$\frac{a_{ij}}{a_{rs}} > \frac{a_{ri}}{a_{sj}}. \tag{19}$$

If we suppose $\frac{a_{ij}}{a_{rs}} < 1$, then by (19) we get $\frac{a_{ri}}{a_{sj}} < 1$, that is $\frac{a_{ir}}{a_{js}} > 1 > \frac{a_{ij}}{a_{rs}}$. But, the above inequalities contradicts (18). Suppose $\frac{a_{ij}}{a_{rs}} = 1$, then by (19) we get $\frac{a_{ri}}{a_{sj}} < 1$, that is $\frac{a_{ir}}{a_{js}} > 1 = \frac{a_{ij}}{a_{rs}}$ and this is absurd by (18). So necessarily $\frac{a_{ij}}{a_{rs}} > 1$. We proved that

$$\frac{\underline{a}_i}{\underline{a}_j} \rhd \frac{\underline{a}_r}{\underline{a}_s} \Rightarrow a_{ij} > a_{rs}.$$

In analogous way we can prove that

$$\frac{\underline{a}_r}{\underline{a}_s} \rhd \frac{\underline{a}_i}{\underline{a}_j} \Rightarrow a_{rs} > a_{ij} \quad \text{and} \quad \frac{\underline{a}_i}{\underline{a}_j} = \frac{\underline{a}_r}{\underline{a}_s} \Rightarrow a_{ij} = a_{rs}.$$

Obviously the above implications are actually equivalences and so the assertion is proved. $\square$

By Theorem 2, if there exist two elements $\frac{\underline{a}_i}{\underline{a}_j}$ and $\frac{\underline{a}_r}{\underline{a}_s}$ of $A_{\div}$ not comparable by means of $\trianglerighteq$, then $A = (a_{ij})$ is not an intensity matrix. For example, we can claim that the matrix in Example 2 is not an intensity matrix because $\frac{\underline{a}_1}{\underline{a}_3} = (7, 20, 7, 2)$ and $\frac{\underline{a}_2}{\underline{a}_4} = (2, 6, 20, 6)$ are not comparable via $\trianglerighteq$.

Next Proposition indicates a condition that an intensity matrix has to verify and that allow us to characterize the intensity matrices of the third order.

Proposition 9. *Let $A = (a_{ij})$ be an intensity matrix. Then*

$$\gamma)\begin{cases} a_{ij} \neq a_{jk} \Leftrightarrow (a_{ij} \wedge a_{jk})^2 < a_{ik} < (a_{ij} \vee a_{jk})^2 \\ a_{ij} = a_{jk} \Leftrightarrow a_{ik} = (a_{jk})^2 = a_{ij}a_{jk} \end{cases}.$$

Proof. We limit ourselves to prove the direct implications in γ because the inverse implications are evident.

Suppose $a_{ij} = a_{jk}$. Then, by applying the condition ι, we get $\frac{a_{ik}}{a_{jk}} = \frac{a_{jk}}{a_{kk}}$. That is $a_{ik} = a_{jk}^2$.

Suppose now $a_{ij} \neq a_{jk}$. We have to prove that

1. $a_{ij} > a_{jk} \Rightarrow (a_{ij})^2 > a_{ik} > (a_{jk})^2$;
2. $a_{ij} < a_{jk} \Rightarrow (a_{ij})^2 < a_{ik} < (a_{jk})^2$.

Let us assume $a_{ij} > a_{jk}$. Then by choosing $h = i$ in the second side of the first equivalence in ι and by $\mathbf{r}$, we get

$$a_{ij} > a_{jk} \Rightarrow \frac{a_{ii}}{a_{ji}} > \frac{a_{ji}}{a_{ki}} \Leftrightarrow (a_{ji})^2 < a_{ki} \Leftrightarrow (a_{ij})^2 > a_{ik};$$

whereas, for $h = k$,

$$a_{ij} > a_{jk} \Rightarrow \frac{a_{ik}}{a_{jk}} > \frac{a_{jk}}{a_{kk}} \Leftrightarrow a_{ik} > (a_{jk})^2.$$

Hence $(a_{jk})^2 < a_{ik} < (a_{ij})^2$. The implication 1. is indeed proved. The proof of the implication 2. is analogous. □

By Theorem 2 and Proposition 9 an intensity matrix verifies the conditions γ and δ. But γ and δ don't imply that the matrix is an intensity matrix.

Example 4. The matrix

$$\begin{pmatrix} 1 & 9/8 & 3 & 8 \\ 8/9 & 1 & 2 & 4 \\ 1/3 & 1/2 & 1 & 2 \\ 1/8 & 1/4 & 1/2 & 1 \end{pmatrix}$$

is a priority matrix verifying the conditions γ and δ. Nevertheless we stress that $4 = a_{24} > 3 = a_{13}$, whereas $\frac{a_{24}}{a_{44}} = \frac{a_{14}}{a_{34}} = 4$. Then $\underline{a}^4$ is not an intensity vector.

We recall that in a consistent matrix each column is consistent evaluation vector, that is a positive vector werifying (1). Of course a consistent evaluation vector is an intensity vector. As an obvious consequence we get

Proposition 10. *Let $A = (a_{ij})$ be consistent. Then A is an intensity matrix.*

The reverse implication does not hold.

Example 5. In the matrix

$$\begin{pmatrix} 1 & 5 & 7 \\ 1/5 & 1 & 2 \\ 1/7 & 1/2 & 1 \end{pmatrix}$$

a_{12} and a_{23} are the only elements greater than 1 that do not lie in the same line and we get together $a_{12} > a_{23}$ and $\frac{\underline{a}_1}{\underline{a}_2} \triangleright \frac{\underline{a}_2}{\underline{a}_3}$. Then, by Proposition 8, the above matrix is an intensity matrix; but it is not a consistent matrix because $a_{12}a_{23} = 10 \neq a_{13} = 7$.

4.1 The case $n = 3$

From Proposition 9 we derive the following characterization of an intensity matrix of the third order.

Proposition 11. *Let $A = (a_{ij})$ be a matrix of the 3rd order and $x_{i_1} \succ x_{i_2} \succ x_{i_3}$ be the actual ranking. Then A is an intensity matrix if and only if it is a priority matrix and one of the following conditions is verified:*

$$a_{i_1 i_2} = a_{i_2 i_3} \textit{ and } a_{i_1 i_3} = (a_{i_2 i_3})^2; \tag{20}$$

$$(a_{i_1 i_2} \wedge a_{i_2 i_3})^2 < a_{i_1 i_3} < (a_{i_1 i_2} \vee a_{i_2 i_3})^2. \tag{21}$$

Proof. Assume A is an intensity matrix. Then A is a priority matrix and, by Proposition 9, almost one of the condition (20) or (21) is true.

Viceversa, assume that A is a priority matrix and either (20) or (21) is verified. Let (20) be verified. Then $a_{i_1 i_3} = a_{i_1 i_2} a_{i_2 i_3}$ and, as $a_{i_1 i_2}, a_{i_2 i_3}, a_{i_1 i_3}$ are the only entries of A greater than 1, by **r** and Proposition 2 we get that A is consistent. By Proposition 10 A is an intensity matrix.

Suppose now that (21) is verified. By Proposition 8, to prove that A is an intensity matrix it is enough to prove the equivalences:

1. $a_{i_1 i_2} > a_{i_2 i_3} \Leftrightarrow \frac{a_{i_1 h}}{a_{i_2 h}} > \frac{a_{i_2 h}}{a_{i_3 h}} \quad \forall h \in \{i_1, i_2, i_3\}$;
2. $a_{i_1 i_2} < a_{i_2 i_3} \Leftrightarrow \frac{a_{i_1 h}}{a_{i_2 h}} < \frac{a_{i_2 h}}{a_{i_3 h}} \quad \forall h \in \{i_1, i_2, i_3\}$;

The inverse implications in 1. and 2. are proved by choosing $h = i_2$ in the right side and applying the condition **r**.

Let us prove the direct implications. Assume $a_{i_1 i_2} > a_{i_2 i_3}$. Then, for $h = i_3$, the inequality

$$\frac{a_{i_1 h}}{a_{i_2 h}} > \frac{a_{i_2 h}}{a_{i_3 h}} \tag{22}$$

becomes $a_{i_1 i_3} > (a_{i_2 i_3})^2 = (a_{i_1 i_2} \wedge a_{i_2 i_3})^2$, that is true by hypothesis. For $h = i_2$, (22) becomes $a_{i_1 i_2} > a_{i_2 i_3}$ and it is true for assumption. Finally, for $h = i_1$ and by applying **r**, (22) becomes $(a_{i_1 i_2})^2 > a_{i_1 i_3}$ and it is true by hypothesis.

Assume now $a_{i_1 i_2} < a_{i_2 i_3}$. For $h = i_3$, the inequality

$$\frac{a_{i_1 h}}{a_{i_2 h}} < \frac{a_{i_2 h}}{a_{i_3 h}} \tag{23}$$

becomes $a_{i_1 i_3} < (a_{i_2 i_3})^2$ and it is true by hypothesis. For $h = i_2$, (23) becomes $a_{i_1 i_2} < a_{i_2 i_3}$ and it is true for assumption. Finally, for $h = i_1$ and because of **r**, (23) becomes $(a_{i_1 i_2})^2 < a_{i_1 i_3}$ and it is true by hypothesis. □

By Proposition 11 the matrix in the Example 5 is an intensity matrix. Indeed $x_1 \succ x_2 \succ x_3$ and $a_{23}^2 < a_{13} < a_{12}^2$.

As we have stressed in the proof of Proposition 11, under the assumption $x_{i_1} \succ x_{i_2} \succ x_{i_3}$, the condition (20) implies the condition of consistency for the matrix A. So by Propositions 10 and 11 we get the following

Corollary 5. *Let $A = (a_{ij})$ be a matrix of the 3th order and $x_{i_1} \succ x_{i_2} \succ x_{i_3}$ be the actual ranking. Then A is an intensity matrix if and only if either A is consistent or A is a priority matrix and (21) is verified.*

5 Intensity operators linked to an intensity matrix

An operator (6) is an *intensity operator* for the matrix A if and only if the vector $\underline{w}_F = (F(\underline{a}_1), F(\underline{a}_2), \dots, F(\underline{a}_n))$ is an intensity vector.

Proposition 12. *Let $A = (a_{ij})$ be an intensity matrix. Then the operator (6) is an intensity operator for A if and only if it is an ordinal evaluation operator and*

$$\frac{F(\underline{a}_i)}{F(\underline{a}_j)} > \frac{F(\underline{a}_r)}{F(\underline{a}_s)} > 1 \Leftrightarrow \frac{\underline{a}_i}{\underline{a}_j} \triangleright \frac{\underline{a}_r}{\underline{a}_s} \triangleright \underline{1}, \qquad \frac{F(\underline{a}_i)}{F(\underline{a}_j)} = \frac{F(\underline{a}_r)}{F(\underline{a}_s)} > 1 \Leftrightarrow \frac{\underline{a}_i}{\underline{a}_j} = \frac{\underline{a}_r}{\underline{a}_s} \triangleright \underline{1}$$

Proof. The assertion follows by the definition of ordinal evaluation operator, Corollary 4 and Proposition 7. □

Proposition 13. *The weighted geometric mean operator (13) is an intensity operator for the class of the intensity matrices.*

Proof. Assume that A is an intensity matrix and F is the operator (13). Then

$$\frac{F(\underline{a}_i)}{F(\underline{a}_j)} = \left(\frac{a_{i1}}{a_{j1}}\right)^{p_1} \left(\frac{a_{i2}}{a_{j2}}\right)^{p_2} \cdots \left(\frac{a_{in}}{a_{jn}}\right)^{p_n}, \tag{24}$$

$$\frac{F(\underline{a}_r)}{F(\underline{a}_s)} = \left(\frac{a_{r1}}{a_{s1}}\right)^{p_1} \left(\frac{a_{r2}}{a_{s2}}\right)^{p_2} \cdots \left(\frac{a_{rn}}{a_{sn}}\right)^{p_n}, \tag{25}$$

and, by Corollary 3, F is an ordinal evaluation operator.

Suppose that $\underline{a}_i \triangleright \underline{a}_j$ and $\underline{a}_r \triangleright \underline{a}_s$. Since A is an intensity matrix, by Theorem 2, condition α), one of the following situations happens

$$\frac{\underline{a}_i}{\underline{a}_j} \triangleright \frac{\underline{a}_r}{\underline{a}_s}, \quad \frac{\underline{a}_r}{\underline{a}_s} \triangleright \frac{\underline{a}_i}{\underline{a}_j}, \quad \frac{\underline{a}_i}{\underline{a}_j} = \frac{\underline{a}_r}{\underline{a}_s}, \tag{26}$$

and, by (24) and (25),

$$\frac{\underline{a}_i}{\underline{a}_j} \triangleright \frac{\underline{a}_r}{\underline{a}_s} \Rightarrow \frac{F(\underline{a}_i)}{F(\underline{a}_j)} > \frac{F(\underline{a}_r)}{F(\underline{a}_r)}, \quad \frac{\underline{a}_i}{\underline{a}_j} = \frac{\underline{a}_r}{\underline{a}_s} \Rightarrow \frac{F(\underline{a}_i)}{F(\underline{a}_j)} = \frac{F(\underline{a}_r)}{F(\underline{a}_r)}. \tag{27}$$

The above implications are actually equivalences. Indeed only one of the relations (26) is true and, by the implications (27), if $\frac{F(\underline{a}_i)}{F(\underline{a}_j)} > \frac{F(\underline{a}_r)}{F(\underline{a}_s)}$, then $\frac{\underline{a}_i}{\underline{a}_j} \triangleright \frac{\underline{a}_r}{\underline{a}_s}$ and, if $\frac{F(\underline{a}_i)}{F(\underline{a}_j)} = \frac{F(\underline{a}_r)}{F(\underline{a}_s)}$, then $\frac{\underline{a}_i}{\underline{a}_j} = \frac{\underline{a}_r}{\underline{a}_s}$. □

Proposition 14. *The min operator $F_\wedge$ and the max-operator $F_\vee$ are intensity operators for the class of the intensity matrices.*

Proof. It is enough to observe that, an intensity matrix is a priority matrix, and so, by Corollary 1, the vectors $\underline{w}_{F_\wedge} = \underline{a}_\wedge$ and $\underline{w}_{F_\vee} = \underline{a}_\vee$ are colums of the matrix A. □

Proposition 15. *Let $A = (a_{ij})$ be consistent and F an operator (6). If F is homogeneus of order $p > 0$, then F is an intensity operator and the components of the vector $\underline{w}_F$ verify the condition*

$$\frac{w_h}{w_k} = (a_{hk})^p \tag{28}$$

Proof. It is enough to show that, if F is homogeneous of order $p > 0$, then $\underline{w}_F$ verifies (28). By the consistency, for every choice of h and k, we get the following chain of equalities

$$a_{h1} = a_{hk} \cdot a_{k1}, \quad a_{h2} = a_{hk} \cdot a_{k2}, \quad \ldots \quad a_{hn} = a_{hk} \cdot a_{kn},$$

and, by homogeneity of F,

$$\frac{w_h}{w_k} = \frac{F(a_{h1}, a_{h2}, ..., a_{hn})}{F(a_{k1}, a_{k2}, ..., a_{kn})} = \frac{F(a_{hk}a_{k1}, a_{hk}a_{k2}, ..., a_{hk}a_{kn})}{F(a_{k1}, a_{k2}, ..., a_{kn})} =$$

$$= \frac{a_{hk}^p F(a_{k1}, a_{k2}, ..., a_{kn})}{F(a_{k1}, a_{k2}, ..., a_{kn})} = (a_{hk})^p.$$

This proves the assertion. □

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