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Dissipative probability vector fields and generation of evolution semigroups in Wasserstein spaces

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Abstract
We introduce and investigate a notion of multivalued λ-dissipative probability vector field (MPVF) in the Wasserstein space $\mathcal{P}_2(\mathsf{X})$ of Borel probability measures on a Hilbert space X. Taking inspiration from the theories of dissipative operators in Hilbert spaces and of Wasserstein gradient flows for geodesically convex functionals, we study local and global well posedness of evolution equations driven by dissipative MPVFs. Our approach is based on a measure-theoretic version of the Explicit Euler scheme, for which we prove novel convergence results with optimal error estimates under an abstract stability condition, which do not rely on compactness arguments and also hold when X has infinite dimension. We characterize the limit solutions by a suitable Evolution Variational Inequality (EVI), inspired by the Bénilan notion of integral solutions to dissipative evolutions in Banach spaces. Existence, uniqueness and stability of EVI solutions are then obtained under quite general assumptions, leading to the generation of a semigroup of nonlinear contractions.

Keywords Measure differential equations/inclusions in Wasserstein spaces · Probability vector fields · Dissipative operators · Evolution variational inequality · Explicit Euler scheme

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Contents

1 Introduction

The aim of this paper is to study the local and global well posedness of evolution equations for Borel probability measures driven by a suitable notion of probability vector fields in an Eulerian framework.

For the sake of simplicity, let us consider here a finite dimensional Euclidean space X with scalar product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$ (our analysis however will not be confined to finite dimension and will be carried out in a separable Hilbert space) and the space $\mathcal{P}(\mathsf{X})$ (resp. $\mathcal{P}_b(\mathsf{X})$) of Borel probability measures in X (resp. with bounded support).

A Cauchy-Lipschitz approach, via vector fields

A first notion of vector field can be described by maps $\boldsymbol{b} : \mathcal{P}_b(\mathsf{X}) \to \mathrm{C}(\mathsf{X}; \mathsf{X})$, typically taking values in some subset of continuous vector fields in X (as the locally Lipschitz ones of $\mathrm{Lip}_{loc}(\mathsf{X}; \mathsf{X})$), and satisfying suitable growth-continuity conditions. In this respect, the evolution driven by $\boldsymbol{b}$ can be described by a continuous curve $t \mapsto \mu_t \in \mathcal{P}_b(\mathsf{X})$, $t \in [0, T]$, starting from an initial measure $\mu_0 \in \mathcal{P}_b(\mathsf{X})$ and satisfying the continuity equation

$$\begin{cases} \partial_t \mu_t + \nabla \cdot (\boldsymbol{v}_t \mu_t) = 0, & \text{in } (0, T) \times \mathsf{X}, \qquad (1.1a) \\ \boldsymbol{v}_t = \boldsymbol{b}[\mu_t], & \mu_t\text{-a.e. for every } t \in (0, T), \qquad (1.1b) \end{cases}$$

in the distributional sense, i.e.

$$\int_0^T \int_{\mathsf{X}} \Big(\partial_t \zeta + \langle \nabla \zeta, \boldsymbol{v}_t \rangle\Big) \,\mathrm{d}\mu_t \,\mathrm{d}t = 0, \quad \boldsymbol{v}_t = \boldsymbol{b}[\mu_t], \quad \text{for every } \zeta \in \mathrm{C}^1_c((0, T) \times \mathsf{X}). \tag{1.2}$$

If $\boldsymbol{b}$ is sufficiently smooth, solutions to (1.1c,d) can be obtained by many techniques. Recent contributions in this direction are given by the papers [5, 10, 26, 27], we also mention [28, 29] for the analysis in presence of sources. In particular, in [5] the aim of the authors is to develop a suitable Cauchy-Lipschitz theory in Wasserstein spaces for differential inclusions which generalizes (1.1b) to multivalued maps $\boldsymbol{b} : \mathcal{P}_b(\mathsf{X}) \rightrightarrows \mathrm{Lip}_{loc}(\mathsf{X}; \mathsf{X})$ and requires (1.1b), (1.2) to hold for a suitable measurable selection of $\boldsymbol{b}$. As it occurs in the classical finite-dimensional case, the differential-inclusion approach is suitable to describe the dynamics of control systems, when the velocity vector field involved in the continuity equation depends on a control parameter.

The Explicit Euler method

It seems natural to approximate solutions of (1.1c,d) by a measure-theoretic version of the Explicit Euler scheme. Choosing a step size $\tau > 0$ and a partition $\{0, \tau, \ldots, n\tau, \ldots, N\tau\}$ of the interval $[0, T]$, with $N := \lceil T/\tau \rceil$, we construct a sequence $M^n_\tau \in \mathcal{P}_b(\mathsf{X})$, $n = 0, \ldots, N$, by the algorithm

$$M^0_\tau := \mu_0, \quad M^{n+1}_\tau := (\boldsymbol{i}_{\mathsf{X}} + \tau \boldsymbol{b}^n_\tau)_\sharp M^n_\tau, \quad \boldsymbol{b}^n_\tau \in \boldsymbol{b}[M^n_\tau], \tag{1.3}$$

where $\boldsymbol{i}_{\mathsf{X}}(x) := x$ is the identity map and $\boldsymbol{r}_\sharp \mu$ denotes the push forward of $\mu \in \mathcal{P}(\mathsf{X})$ induced by a Borel map $\boldsymbol{r} : \mathsf{X} \to \mathsf{X}$ and defined by $\boldsymbol{r}_\sharp \mu(B) := \mu(\boldsymbol{r}^{-1}(B))$ for every Borel set $B \subset \mathsf{X}$. If $\bar{M}_\tau$ is the piecewise constant interpolation of the discrete values $(M^n_\tau)^N_{n=0}$, one can then study the convergence of $\bar{M}_\tau$ as $\tau \downarrow 0$, hoping to obtain a solution to (1.1c,d) in the limit.

It is then natural to investigate a few relevant questions:

⟨E.1⟩ What is the most general framework where the Explicit Euler scheme can be implemented?
⟨E.2⟩ What are the structural conditions ensuring its convergence?
⟨E.3⟩ How to characterize the limit solutions and their properties?

Concerning the first question ⟨E.1⟩, one immediately realizes that each iteration of (1.3) actually depends on the probability distribution on the tangent bundle $\mathsf{TX} = \mathsf{X} \times \mathsf{X}$, where the second component plays the role of velocity, in the sense that

$$\Phi^n_\tau := (\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{b}^n_\tau)_\sharp M^n_\tau \in \mathcal{P}(\mathsf{TX})$$

whose first marginal is M^n_τ. If we denote by $\mathsf{x}, \mathsf{v} : \mathsf{TX} \to \mathsf{X}$ the projections

$$\mathsf{x}(x, v) := x, \qquad \mathsf{v}(x, v) := v,$$

and by $\mathsf{exp}^\tau : \mathsf{TX} \to \mathsf{X}$ the exponential map in the flat space X, defined by

$$\mathsf{exp}^\tau(x, v) := x + \tau v,$$

we recover M^{n+1}_τ by a single step of “free motion” driven by Φ^n_τ and given by

$$M^{n+1}_\tau = \mathsf{exp}^\tau_\sharp \Phi^n_\tau = (\mathsf{x} + \tau \mathsf{v})_\sharp \Phi^n_\tau.$$

This operation does not depend on the fact that Φ^n_τ is concentrated on the graph of a map (in this case $\boldsymbol{b}^n_\tau \in \boldsymbol{b}[M^n_\tau]$): one can more generally assign a multivalued map $\mathbf{F} : \mathcal{P}_b(\mathsf{X}) \rightrightarrows \mathcal{P}_b(\mathsf{TX})$ such that for every $\mu \in \mathcal{P}_b(\mathsf{X})$, every measure $\Phi \in \mathbf{F}[\mu] \in \mathcal{P}_b(\mathsf{TX})$ has first marginal $\mu = \mathsf{x}_\sharp \Phi$. We call $\mathbf{F}$ a *multivalued probability vector field* (MPVF in the following), which is in good analogy with the Riemannian interpretation of $\mathcal{P}_b(\mathsf{TX})$. The disintegration $\Phi_x \in \mathcal{P}_b(\mathsf{X})$ of Φ with respect to μ provides a (unique up to μ-negligible sets) Borel family of probability measures on the space of velocities such that $\Phi = \int_{\mathsf{X}} \Phi_x \, \mathrm{d}\mu(x)$. In particular, Φ is induced by a vector field $\boldsymbol{b}$ only if $\Phi_x = \delta_{\boldsymbol{b}(x)}$ is a Dirac mass for μ-a.e. x. In the general case, (1.3) reads as

$$M^0_\tau := \mu_0, \quad M^{n+1}_\tau := \mathsf{exp}^\tau_\sharp \Phi^n_\tau = (\mathsf{x} + \tau \mathsf{v})_\sharp \Phi^n_\tau, \quad \Phi^n_\tau \in \mathbf{F}[M^n_\tau]. \tag{1.4}$$

In addition to its greater generality, this point of view has other advantages: working with the joint distribution $\mathbf{F}[\mu]$ instead of the disintegrated vector field $\boldsymbol{b}[\mu]$ potentially allows for the weakening of the continuity assumption with respect to μ. This relaxation corresponds to the introduction of Young’s measures to study the limit behaviour of weakly converging maps [13]. Adopting this viewpoint, the classical discontinuous example in $\mathbb{R}$ (see [16]), where $\boldsymbol{b}(x) = -\mathrm{sign}(x)$, admits a natural closed realization as MPVF given by

$$\Phi \in \mathbf{F}[\mu] \quad \Leftrightarrow \quad \Phi_x = \begin{cases} \delta_{\boldsymbol{b}(x)} & \text{if } x \neq 0 \\ (1-\theta)\delta_{-1} + \theta\delta_1 & \text{if } x = 0 \end{cases} \quad \text{for some } \theta \in [0, 1].$$

In particular, $\mathbf{F}[\delta_0] = \{\delta_0 \otimes ((1-\theta)\delta_{-1} + \theta\delta_1) \mid \theta \in [0,1]\}$ (see also [9, Example 6.2]).

The study of measure-driven differential equations/inclusions is not new in the literature [15, 34]. However, these studies, devoted to the description of impulsive control systems [8] and mainly motivated by applications in rational mechanics and engineering, have been used to describe evolutions in $\mathbb{R}^d$ rather than in the space of measures.

A second advantage in considering a MPVF is the consistency with the theory of Wasserstein gradient flows generated by geodesically convex functionals introduced in [3] (Wasserstein subdifferentials are particular examples of MPVFs) and with the multivalued version of the notion of probability vector fields introduced in [26, 27], whose originating idea was indeed to describe the uncertainty affecting not only the state of the system, but possibly also the distribution of the vector field itself.

A third advantage is to allow for a more intrinsic geometric viewpoint, inspired by Otto's non-smooth Riemannian interpretation of the Wasserstein space: probability vector fields provide an appropriate description of infinitesimal deformations of probability measures, which should be measured by, e.g., the L^2-Kantorovich-Rubinstein-Wasserstein distance

$$W_2^2(\mu,\nu) := \min\left\{\int_{\mathsf{X}\times\mathsf{X}} |x-y|^2 \,\mathrm{d}\boldsymbol{\gamma}(x,y) : \boldsymbol{\gamma} \in \Gamma(\mu,\nu)\right\}, \tag{1.5}$$

where $\Gamma(\mu,\nu)$ is the set of couplings with marginals μ and ν respectively. It is well known [3, 32, 35] that if μ,ν belong to the space $\mathcal{P}_2(\mathsf{X})$ of Borel probability measures with finite second moment

$$\mathsf{m}_2^2(\mu) := \int_{\mathsf{X}} |x|^2 \,\mathrm{d}\mu(x) < \infty,$$

then the minimum in (1.5) is attained in a compact convex set $\Gamma_o(\mu,\nu)$ and $(\mathcal{P}_2(\mathsf{X}), W_2)$ is a complete and separable metric space. Adopting this viewpoint and proceeding by analogy with the theory of dissipative operators in Hilbert spaces, a natural class of MPVFs for evolutionary problems should at least satisfy a λ-dissipativity condition, with $\lambda \in \mathbb{R}$, such as

$$\begin{aligned} &W_2(\mathsf{exp}^\tau_\sharp \Phi, \mathsf{exp}^\tau_\sharp \Psi) \le (1+\lambda\tau) W_2(\mu,\nu) + o(\tau) \\ &\quad \text{as } \tau \downarrow 0, \text{ for every } (\Phi,\Psi) \in \mathbf{F}[\mu]\times\mathbf{F}[\nu],\ \mu \ne \nu. \end{aligned} \tag{1.6}$$

Metric dissipativity

Condition (1.6) in the simple case $\lambda = 0$ has a clear interpretation in terms of one step of the Explicit Euler method: it is an asymptotic contraction as the time step goes to 0. By using the properties of the Wasserstein distance, we will first compute the right derivative of its square along the deformation exp^τ as follows

$$
\begin{aligned}
[\Phi, \Psi]_r &:= \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}\tau} W_2^2(\exp_\sharp^\tau \Phi, \exp_\sharp^\tau \Psi)\Big|_{\tau=0+} \\
&= \min\left\{ \int_{\mathsf{TX}\times\mathsf{TX}} \langle w - v, y - x\rangle \,\mathrm{d}\boldsymbol{\Theta}(x, v; y, w) : \boldsymbol{\Theta} \in \Gamma(\Phi, \Psi),\ (\mathsf{x}, \mathsf{y})_\sharp \boldsymbol{\Theta} \in \Gamma_o(\mu, \nu) \right\}
\end{aligned}
\tag{1.7}
$$

and we will show that (1.6) admits the equivalent characterization

$$
[\Phi, \Psi]_r \le \lambda W_2^2(\mu, \nu) \quad \text{for every } (\Phi, \Psi) \in \mathbf{F}[\mu] \times \mathbf{F}[\nu]. \tag{1.8}
$$

If we interpret the left hand side of (1.8) as a sort of Wasserstein pseudo-scalar product of Φ and Ψ along the direction of an optimal coupling between μ and ν, (1.8) is in perfect analogy with the canonical definition of λ-dissipativity (also called one-sided Lipschitz condition) for a multivalued map $\mathrm{F} : \mathsf{X} \rightrightarrows \mathsf{X}$, which reads as

$$
\langle w - v, y - x\rangle \le \lambda |x - y|^2 \quad \text{for every } (v, w) \in \mathrm{F}[x] \times \mathrm{F}[y]. \tag{1.9}
$$

It turns out that the (opposite of the) Wasserstein subdifferential $\boldsymbol{\partial}\mathcal{F}$ [3, Sect. 10.3] of a geodesically $(-\lambda)$-convex functional $\mathcal{F} : \mathcal{P}_2(\mathsf{X}) \to (-\infty, +\infty]$ is a MPVF and satisfies a condition equivalent to (1.6) and (1.8). We also notice that (1.8) reduces to (1.9) in the particular case when $\Phi = \delta_{(x,v)}$, $\Psi = \delta_{(y,w)}$ are Dirac masses in TX.

Conditional convergence of the Explicit Euler method

Contrary to the Implicit Euler method, however, even if a MPVF satisfies (1.8), every step of the Explicit Euler scheme (1.4) affects the distance by a further quadratic correction according to the formula

$$
W_2^2(\exp_\sharp^\tau \Phi, \exp_\sharp^\tau \Psi) \le W_2^2(\mu, \nu) + 2\tau\, [\Phi, \Psi]_r + \tau^2\Big(|\Phi|_2^2 + |\Psi|_2^2\Big),
$$
$$
|\Phi|_2^2 := \int_{\mathsf{TX}} |v|^2 \,\mathrm{d}\Phi(x, v),
$$

which depends on the order of magnitude of Φ and Ψ, and thus of $\mathbf{F}$, at μ and ν.

Our first main result (Theorems 6.5 and 6.7), which provides an answer to question ⟨E.2⟩, states that if $\mathbf{F}$ is a λ-dissipative MPVF according to (1.8) then every family of discrete solutions $(\bar{M}_\tau)_{\tau>0}$ of (1.4) in an interval $[0, T]$ satisfying the abstract stability condition

$$
|\Phi_\tau^n|_2 \le L \quad \text{if } 0 \le n \le N := \lceil T/\tau \rceil\,, \tag{1.10}
$$

is uniformly converging to a Lipschitz continuous limit curve $\mu : [0, T] \to \mathcal{P}_2(\mathsf{X})$ starting from μ_0, with a uniform error estimate

$$
W_2(\mu_t, \bar{M}_\tau(t)) \le CL\sqrt{\tau(t + \tau)}\,\mathrm{e}^{\lambda_+ t} \tag{1.11}
$$

for every $t \in [0, T]$, and a universal constant $C \le 14$. Apart from the precise value of C, the estimate (1.11) is sharp [31] and reproduces in the measure-theoretic framework the celebrated Crandall-Liggett estimate [12] for the generation of dissipative semi-groups in Banach spaces. We derive it by adapting to the metric-Wasserstein setting the relaxation and doubling variable techniques of [23], strongly inspired by the ideas of Kružkov [21] and Crandall-Evans [11].

This crucial result does not require any bound on the support of the measures and no local compactness of the underlying space X, so that we will prove it in a general Hilbert space, possibly with infinite dimension. Moreover, if μ, ν are two limit solutions starting from μ_0, ν_0 we show that

$$W_2(\mu_t, \nu_t) \le W_2(\mu_0, \nu_0)\mathrm{e}^{\lambda t} \quad \text{for every } t \in [0, T],$$

as it happens in the case of gradient flows of $(-\lambda)$-convex functions. Once one has these building blocks, it is not too difficult to construct a local and global existence theory, mimicking the standard arguments for ODEs.

Metric characterization of the limit solution

As we stated in question $\langle$E.3$\rangle$, a further important point is to get an effective characterization of the solution μ obtained as limit of the approximation scheme.

As a first property, considered in [26, 27] in the case of a single-valued PVF, one could hope that μ satisfies the continuity equation (1.1a) coupled with the barycentric condition, thus replacing (1.1b) with

$$\boldsymbol{v}_t(x) = \int_{\mathsf{TX}} v \,\mathrm{d}\Phi_t(x, v), \quad \Phi_t \in \mathbf{F}[\mu_t]. \tag{1.12}$$

This is in fact true, as shown in [26, 27] in the finite dimensional case, if $\mathbf{F}$ is single valued and satisfies a stronger Lipschitz dependence w.r.t. μ (see (H1) in Sect. 7.5).

In the framework of dissipative MPVFs, we will replace (1.12) with its relaxation *à la Filippov* (see e.g. [36, Chapter 2] and [2, Chapter 10]) given by

$$\boldsymbol{v}_t(x) = \int_{\mathsf{TX}} v \,\mathrm{d}\Phi_t(x, v) \quad \text{for some} \quad \Phi_t \in \overline{\mathrm{co}}(\mathrm{cl}(\mathbf{F})[\mu_t]),$$

where $\mathrm{cl}(\mathbf{F})$ is the sequential closure of the graph of $\mathbf{F}$ in the strong-weak topology of $\mathcal{P}_2^{sw}(\mathsf{TX})$ and $\overline{\mathrm{co}}(\mathrm{cl}(\mathbf{F})[\mu])$ denotes the closed convex hull of the given section $\mathrm{cl}(\mathbf{F})[\mu]$. We refer to [25] and Sect. 2.2 for more details on the mentioned strong-weak topology; in fact, a more restrictive "directional" closure could be considered, see Sect. 5.5 and in particular Theorem 5.27.

However, even in the case of a single valued map, (1.12) is not enough to characterize the limit solution, as it has been shown by an interesting example in [9, 27] (see also the gradient flow of Example 7.7).

From a Wasserstein viewpoint, one could consider the differential inclusion

$$(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}_t^W)_\sharp \mu_t \in \mathbf{F}[\mu_t], \quad \text{for a.e. } t \in [0, T], \tag{1.13}$$

where $\boldsymbol{v}^W$ is the Wasserstein metric velocity field associated to μ (see Theorem 2.10). However, while this property was appropriate to characterize limit solution μ in the case of gradient flows, it is not reasonable for a general MPVF $\mathbf{F}$. Indeed, the given MPVF $\mathbf{F}$, even if regular, could have no relation with the tangent space $\mathrm{Tan}_{\mu_t}\mathcal{P}_2(\mathsf{X})$ where $\boldsymbol{v}_t^W$ lies.

In order to address the problem of characterizing the limit solution μ, here we follow the metric viewpoint adopted in [3] for gradient flows and we will characterize the limit solutions by a suitable Evolution Variational Inequality satisfied by the squared distance function from given test measures. As a byproduct (see Theorem 5.4), this interpretation will be reflected in a relaxed formulation of the inclusion (1.13) with respect to a suitable extension $\hat{\mathbf{F}}$ of $\mathbf{F}$ introduced in Sect. 4.3. This approach is also strongly influenced by the Bénilan notion of integral solutions to dissipative evolutions in Banach spaces [4]. The main idea is that any differentiable solution to $\dot{x}(t) \in \mathrm{F}[x(t)]$ driven by a λ-dissipative operator in a Hilbert space as in (1.9) satisfies

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|x(t)-y|^2 &= \langle \dot{x}(t), x(t)-y\rangle \\ &= \langle \dot{x}(t)-w, x(t)-y\rangle + \langle w, x(t)-y\rangle \\ &\le \lambda|x(t)-y|^2 - \langle w, y-x(t)\rangle\end{aligned}$$

for every $w \in \mathrm{F}[y]$. In the framework of $\mathcal{P}_2(\mathsf{X})$, we replace $w \in \mathrm{F}[y]$ with $\Psi \in \mathbf{F}[\nu]$ and the scalar product $\langle w, y-x(t)\rangle$ with

$$[\Psi, \mu_t]_r := \min\left\{\int_{\mathsf{TX}\times\mathsf{X}} \langle w, y-x\rangle \,\mathrm{d}\boldsymbol{\Theta}(y, w; x) : \boldsymbol{\Theta} \in \Gamma(\Psi, \mu_t),\ (\mathsf{y}, \mathsf{x})_\sharp\boldsymbol{\Theta} \in \Gamma_o(\nu, \mu_t)\right\},$$

as in (1.7). According to this formal heuristic, we consider the λ-EVI characterization of a limit curve μ as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}W_2^2(\mu_t, \nu) \le \lambda W_2^2(\mu_t, \nu) - [\Psi, \mu_t]_r \quad \text{for every } \Psi \in \mathbf{F}[\nu]. \tag{λ-EVI}$$

As for Bénilan integral solutions, we can considerably relax the apriori smoothness assumptions on μ, just imposing that μ is continuous and (λ-EVI) holds in the sense of distributions in $(0, T)$. In this way, we obtain a robust characterization, which is stable under uniform convergence (cf. Proposition 5.6) and also allows for solutions taking values in the closure of the domain of $\mathbf{F}$. This is particularly important when $\mathbf{F}$ involves drift terms with superlinear growth (see Example 7.5).

The crucial point of this approach relies on a general error estimate, which extends the validity of (1.11) to a general λ-EVI solution μ and therefore guarantees its uniqueness, whenever the Explicit Euler method is solvable, at least locally in time (see Sect. 5.3).

Combining local in time existence with suitable global confinement conditions (see e.g. Theorem 5.32) we can eventually obtain a robust theory for the generation of a λ-flow, i.e. a semigroup $(\mathrm{S}_t)_{t\ge 0}$ in a suitable subset D of $\mathcal{P}_2(\mathsf{X})$ such that $\mathrm{S}_t[\mu_0]$ is the unique λ-EVI solution starting from μ_0 and for every $\mu_0, \mu_1 \in D$

$$W_2(\mathrm{S}_t[\mu_0], \mathrm{S}_t[\mu_1]) \le W_2(\mu_0, \mu_1)\mathrm{e}^{\lambda t} \quad \text{for every } t \ge 0,$$

as in the case of Wasserstein gradient flows of geodesically $(-\lambda)$-convex functionals.

Explicit vs Implicit Euler method

In the framework of contraction semigroups generated by λ-dissipative operators in Hilbert or Banach spaces, a crucial role is played by the Implicit Euler scheme, which has the advantage to be unconditionally stable, and thus avoids any apriori restriction on the local bound of the operator, as we did in (1.10). In Hilbert spaces, it is well known that the solvability of the Implicit Euler scheme is equivalent to the maximality of the graph of the operator.

In the case of a Wasserstein gradient flow of a geodesically convex $\mathcal{F} : \mathcal{P}_2(\mathsf{X}) \to (-\infty, +\infty]$, every step of the Implicit Euler method (also called JKO/Minimizing Movement scheme [3, 20]) can be solved by a variational approach: M_τ^{n+1} has to be selected among the solutions of

$$\underset{M \in \mathcal{P}_2(\mathsf{X})}{\arg\min} \frac{1}{2\tau} W_2^2(M, M_\tau^n) + \mathcal{F}(M). \tag{1.14}$$

Notice, however, that in this case the MPVF $\boldsymbol{\partial}\mathcal{F}$ is defined implicitely in terms of $\mathcal{F}$ and each step of (1.14) provides a suitable variational selection in $\boldsymbol{\partial}\mathcal{F}$, leading in the limit to the minimal selection principle.

In the case of more general dissipative evolutions, it is not at all clear how to solve the Implicit Euler scheme, in particular when $\mathbf{F}[\mu]$ is not concentrated on a map, and to characterize the maximal extension of $\mathbf{F}$ (in the Hilbertian case the maximal extension of a dissipative operator F is explicitly computable at least when the domain of F has not empty interior, see the Theorems of Robert and Bénilan in [30]). The analogy with the Hilbertian theory does not extend to some properties: in particular, a dissipative MPVF $\mathbf{F}$ in $\mathcal{P}_2(\mathsf{X})$ is not locally bounded in the interior of its domain (see Example 7.3) and maximality may fail also for single-valued continuous PVFs (see Example 7.4). Even more remarkably, in the Hilbertian case a crucial equivalent characterization of dissipativity reads as

$$|x - y| \le |(x - \tau v) - (y - \tau w)| \quad \text{for every } (v, w) \in \mathrm{F}[x] \times \mathrm{F}[y],$$

which implies that the resolvent operators $(\boldsymbol{i}_{\mathsf{X}} - \tau \mathrm{F})^{-1}$ – and thus every single step of the Implicit Euler scheme – are contractions on X. On the contrary, if we assume the forward characterizations (1.6) and (1.8) of dissipativity in $\mathcal{P}_2(\mathsf{X})$ (with $\lambda = 0$) we *cannot* conclude in general that

$$W_2(\mu, \nu) \le W_2(\exp_\sharp^{-\tau} \Phi, \exp_\sharp^{-\tau} \Psi) \quad \text{for every } (\Phi, \Psi) \in \mathbf{F}[\mu] \times \mathbf{F}[\nu], \tag{1.15}$$

since the squared distance map $f(t) := W_2^2(\exp^t_\sharp\Phi, \exp^t_\sharp\Psi)$, $t \in \mathbb{R}$, is not convex in general (see e.g. [3, Example 9.1.5]) and the fact that its right derivative at $t = 0$ (corresponding to $[\Phi, \Psi]_r$) is ≤ 0 according to (1.8) does not imply that $f(0) \leq f(t)$ for $t < 0$ (corresponding to (1.15) for $t = -\tau$).

For these reasons, we decided to approach the investigation of dissipative evolutions in $\mathcal{P}_2(\mathsf{X})$ by the Explicit Euler method, and we defer the study of the implicit one to a forthcoming paper.

Plan of the paper

As already mentioned, our theory works for a general separable Hilbert space X, and we recollect some preliminary material concerning the Wasserstein distance in Hilbert spaces and the properties of strong-weak topology for $\mathcal{P}_2(\mathsf{TX})$ in Sect. 2.

In Sect. 3, we will study the semi-concavity properties of W_2 along general deformations induced by the exponential map $\exp^\tau$ and we introduce and study the pairings $[\cdot,\cdot]_r$, $[\cdot,\cdot]_l$. We will apply such tools to derive the precise expressions of the left and right derivatives of W_2 along absolutely continuous curves in $\mathcal{P}_2(\mathsf{X})$ in Sect. 3.2.

In Sect. 4, we will introduce and study the notion of λ-dissipative MPVF, in particular its behaviour along geodesics (Sect. 4.2) and its extension properties (Sect. 4.3).

Sections 5 and 6 contain the core of our results. Section 5 is devoted to the notion of λ-EVI solutions and to their properties: local uniqueness, stability and regularity in Sect. 5.3, global existence in Sect. 5.4 and barycentric characterizations in Sect. 5.5. Section 6 contains the main estimates for the Explicit Euler scheme: the Cauchy estimates between two discrete solutions corresponding to different step sizes in Sect. 6.2 and the uniform error estimates between a discrete and a λ-EVI solution in Sect. 6.3.

Finally, a few examples are collected in Sect. 7.

2 Preliminaries

In this section, we introduce the main concepts and results of Optimal Transport theory that will be extensively used in the rest of the paper. We start by listing the adopted notation.

$\boldsymbol{b}_\Phi$	the barycenter of $\Phi \in \mathcal{P}(\mathsf{TX})$ as in Definition 3.1
$\mathrm{B}_X(x, r)$	the open ball with radius $r > 0$ centered at $x \in X$
$\mathrm{C}(X; Y)$	the set of continuous functions from X to Y
$\mathrm{C}_b(X)$	the set of bounded continuous real valued functions defined in X
$\mathrm{C}_c(X)$	the set of continuous real valued functions with compact support
$\mathrm{Cyl}(\mathsf{X})$	the space of cylindrical functions on X, see Definition 2.9
$\mathrm{cl}(\mathbf{F})$, $\mathrm{co}(\mathbf{F})[\mu]$	the sequential closure and convexification of $\mathbf{F}$, see Sect. 4.3
$\overline{\mathrm{co}}(\mathbf{F})[\mu]$, $\hat{\mathbf{F}}$	sequential closure of convexification and extension of $\mathbf{F}$, see Sect. 4.3
$\frac{\mathrm{d}^+}{\mathrm{d}t}\zeta$, $\frac{\mathrm{d}}{\mathrm{d}t_+}\zeta$	the right upper/lower Dini derivatives of ζ, see (5.3)
$\mathrm{D}(\mathbf{F})$	the proper domain of a set-valued function as in Definition 4.1

$\mathscr{E}(\mu_0, \tau, T, L), \mathscr{M}(\mu_0, \tau, T, L)$	the sets associated to the Explicit Euler scheme (EE) defined in (5.12)
$f_\sharp \nu$	the push-forward of $\nu \in \mathcal{P}(X)$ through the map $f : X \to Y$
$\Gamma(\mu, \nu)$	the set of admissible couplings between μ, ν, see (2.1)
$\Gamma_o(\mu, \nu)$	the set of optimal couplings between μ, ν, see Definition 2.5
$\Gamma_o^i(\mu_0, \mu_1\|\mathbf{F}),\ i = 0, 1$	the set of optimal couplings conditioned to $\mathbf{F}$, see (4.12)
$\mathcal{I}$	an interval of $\mathbb{R}$
$\boldsymbol{i}_X(\cdot)$	the identity function on a set X
$\mathrm{I}(\boldsymbol{\mu}\|\mathbf{F})$	the set of time instants t s.t. $\mathsf{x}^t_\sharp \boldsymbol{\mu}$ belongs to $\mathrm{D}(\mathbf{F})$, see (4.7)
λ_+	the positive part of $\lambda \in \mathbb{R}$, given by $\lambda_+ = \max\{\lambda, 0\}$
Λ, Λ_o	the sets of couplings as in Definition 3.8 and Theorem 3.9
$\mathcal{L}$	the 1-dimensional Lebesgue measure
$\mathsf{m}_2(\nu)$	the 2-nd moment of $\nu \in \mathcal{P}(X)$ as in Definition 2.5
$\|\Phi\|_2$	the 2-nd moment of $\Phi \in \mathcal{P}(\mathsf{TX})$ as in (3.2)
$\|\mathbf{F}\|_2(\mu)$	the 2-nd moment of $\mathbf{F}$ at μ as in (5.20)
$\|\dot{\mu}_t\|$	the metric derivative at t of a locally absolutely continuous curve μ
$\mathcal{P}(X)$	the set of Borel probability measures on the topological space X
$\mathcal{P}_b(X)$	the set of Borel probability measures with bounded support
$\mathcal{P}_2(X)$	the subset of measures in $\mathcal{P}(X)$ with finite quadratic moments
$\mathcal{P}_2^{sw}(\mathsf{X} \times \mathsf{Y})$	the space $\mathcal{P}_2(\mathsf{X} \times \mathsf{Y})$ endowed with a weaker topology as in Definition 2.14
$\mathcal{P}(\mathsf{TX}\|\mu)$	the subset of $\mathcal{P}_2(\mathsf{TX})$ with fixed first marginal μ as in (3.3)
$[\cdot, \cdot]_r, [\cdot, \cdot]_l$	the pseudo scalar products as in Definition 3.5
$[\Phi_t, \boldsymbol{\vartheta}]_{b,t}, [\Phi_t, \boldsymbol{\vartheta}]_{r,t}, [\Phi_t, \boldsymbol{\vartheta}]_{l,t}$	the duality pairings as in Definition 3.18
$[\mathbf{F}, \boldsymbol{\mu}]_{r,t}, [\mathbf{F}, \boldsymbol{\mu}]_{l,t}$	the duality pairings as in Definition 4.8
$[\mathbf{F}, \boldsymbol{\mu}]_{0+}, [\mathbf{F}, \boldsymbol{\mu}]_{1-}$	the limiting duality pairings as in Definition 4.11
$\mathrm{supp}(\nu)$	the support of $\nu \in \mathcal{P}(X)$
$\mathrm{Tan}_\mu \mathcal{P}_2(X)$	the tangent space defined in Theorem 2.10
$W_2(\mu, \nu)$	the L^2-Wasserstein distance between μ and ν, see Definition 2.5
X	a separable Hilbert space
TX	the tangent bundle to X, usually endowed with the strong-weak topology
$\mathsf{x}, \mathsf{v}, \mathsf{exp}^t, \mathsf{s}$	the projection, exponential and reversion maps defined in (3.1) and (3.26)
x^t	the evaluation map defined in (3.4)
$\lfloor \cdot \rfloor, \lceil \cdot \rceil$	the floor and ceiling functions, see (5.8)

In the present paper we will mostly deal with Borel probability measures defined in (subsets of) some separable Hilbert space endowed with the strong or a weaker topology. The convenient setting is therefore provided by Polish/Lusin and completely regular topological spaces.

Recall that a topological space X is Polish (resp. Lusin) if its topology is induced by a complete and separable metric (resp. is coarser than a Polish topology). We will denote by $\mathcal{P}(X)$ the set of Borel probability measures on X. If X is Lusin, every measure $\mu \in \mathcal{P}(X)$ is also a Radon measure, i.e. it satisfies

$$\forall\, B \subset X \text{ Borel}, \forall\, \varepsilon > 0 \quad \exists\, K \subset B \text{ compact s.t. } \mu(B \setminus K) < \varepsilon.$$

X is *completely regular* if it is Hausdorff and for every closed set C and point $x \in X \setminus C$ there exists a continuous function $f : X \to [0, 1]$ s.t. $f(x) = 0$ and $f(C) = \{1\}$.

Given X and Y Lusin spaces, $\mu \in \mathcal{P}(X)$ and a Borel function $f : X \to Y$, there is a canonical way to transfer the measure μ from X to Y through f. This is called the *push* *forward* of μ through f, denoted by $f_\sharp\mu$ and defined by $(f_\sharp\mu)(B) := \mu(f^{-1}(B))$ for every Borel set B in Y, or equivalently

$$\int_Y \varphi \, \mathrm{d}(f_\sharp\mu) = \int_X \varphi \circ f \, \mathrm{d}\mu$$

for every φ bounded (or nonnegative) real valued Borel function on Y. A particular case occurs if $X = X_1 \times X_2$, $Y = X_i$ and $f = \pi^i$ is the projection on the i-th component, $i = 1, 2$. In this case, f is usually denoted with π^i or π^{X_i}, and $\pi^{X_i}_\sharp \mu$ is called the i-th marginal of μ.

This notation is particularly useful when dealing with transport plans: given X_1 and X_2 two completely regular spaces and $\mu \in \mathcal{P}(X_1)$, $\nu \in \mathcal{P}(X_2)$, we define

$$\Gamma(\mu, \nu) := \left\{ \boldsymbol{\gamma} \in \mathcal{P}(X_1 \times X_2) \mid \pi^1_\sharp \boldsymbol{\gamma} = \mu \,,\, \pi^2_\sharp \boldsymbol{\gamma} = \nu \right\}, \tag{2.1}$$

i.e. the set of probability measures on the product space having μ and ν as marginals. On $\mathcal{P}(X)$ we consider the so-called *narrow topology* which is the coarsest topology on $\mathcal{P}(X)$ s.t. the maps $\mu \mapsto \int_X \varphi \, \mathrm{d}\mu$ are continuous for every $\varphi \in \mathrm{C}_b(X)$, the space of real valued and bounded continuous functions on X. In this way a net $(\mu_\alpha)_{\alpha \in \mathbb{A}} \subset \mathcal{P}(X)$ indexed by a directed set $\mathbb{A}$ is said to converge narrowly to $\mu \in \mathcal{P}(X)$, and we write $\mu_\alpha \to \mu$ in $\mathcal{P}(X)$, if

$$\lim_\alpha \int_X \varphi \, \mathrm{d}\mu_\alpha = \int_X \varphi \, \mathrm{d}\mu \quad \text{for every } \varphi \in \mathrm{C}_b(X).$$

We recall the well known Prokhorov's theorem in the context of completely regular topological spaces (see [33, Appendix]).

Theorem 2.1 (Prokhorov) *Let X be a completely regular topological space and let $\mathcal{F} \subset \mathcal{P}(X)$ be a tight subset i.e.*

$$\textit{for all } \varepsilon > 0 \textit{ there exists } K_\varepsilon \subset X \textit{ compact s.t. } \sup_{\mu \in \mathcal{F}} \mu(X \setminus K_\varepsilon) < \varepsilon.$$

Then $\mathcal{F}$ is relatively compact in $\mathcal{P}(X)$ w.r.t. the narrow topology.

It is then relevant to know when a given $\mathcal{F} \subset \mathcal{P}(X)$ is tight. If X is a Lusin completely regular topological space, then the set $\mathcal{F} = \{\mu\} \subset \mathcal{P}(X)$ is tight. Another trivial criterion for tightness is the following: if $\mathcal{F} \subset \mathcal{P}(X_1 \times X_2)$ is s.t. $\mathcal{F}_i := \{\pi^i_\sharp \boldsymbol{\gamma} \mid \boldsymbol{\gamma} \in \mathcal{F}\} \subset \mathcal{P}(X_i)$ are tight for $i = 1, 2$, then also $\mathcal{F}$ is tight. We also recall the following useful proposition (see [3, Remark 5.1.5]).

Proposition 2.2 *Let X be a Lusin completely regular topological space and let $\mathcal{F} \subset \mathcal{P}(X)$. Then $\mathcal{F}$ is tight if and only if there exists $\varphi : X \to [0, +\infty]$ with compact sublevels s.t.*

$$\sup_{\mu \in \mathcal{F}} \int_X \varphi \, \mathrm{d}\mu < +\infty.$$

We recall the so-called *disintegration theorem* (see e.g. [3, Theorem 5.3.1]).

Theorem 2.3 *Let $\mathbb{X}$, X be Lusin completely regular topological spaces, $\boldsymbol{\mu} \in \mathcal{P}(\mathbb{X})$ and $r : \mathbb{X} \to X$ a Borel map. Denote with $\mu = r_\sharp \boldsymbol{\mu} \in \mathcal{P}(X)$. Then there exists a μ-a.e. uniquely determined Borel family of probability measures $\{\boldsymbol{\mu}_x\}_{x \in X} \subset \mathcal{P}(\mathbb{X})$ such that $\boldsymbol{\mu}_x(\mathbb{X} \setminus r^{-1}(x)) = 0$ for μ-a.e. $x \in X$, and*

$$\int_{\mathbb{X}} \varphi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{\mu}(\boldsymbol{x}) = \int_X \left(\int_{r^{-1}(x)} \varphi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{\mu}_x(\boldsymbol{x}) \right) \mathrm{d}\mu(x)$$

for every bounded Borel map $\varphi : \mathbb{X} \to \mathbb{R}$.

Remark 2.4 When $\mathbb{X} = X_1 \times X_2$ and $r = \pi^1$, we can canonically identify the disintegration $\{\boldsymbol{\mu}_x\}_{x \in X_1} \subset \mathcal{P}(\mathbb{X})$ of $\boldsymbol{\mu} \in \mathcal{P}(X_1 \times X_2)$ w.r.t. $\mu = \pi^1_\sharp \boldsymbol{\mu}$ with a family of probability measures $\{\mu_{x_1}\}_{x_1 \in X_1} \subset \mathcal{P}(X_2)$. We write $\boldsymbol{\mu} = \int_{X_1} \mu_{x_1} \, \mathrm{d}\mu(x_1)$.

2.1 Wasserstein distance in Hilbert spaces

Let X be a separable (possibly infinite dimensional) Hilbert space. We will denote by X^s (respt. X^w) the Hilbert space endowed with its strong (resp. weak) topology. Notice that X^w is a Lusin completely regular space. The spaces X^s and X^w share the same class of Borel sets and therefore of Borel probability measures, which we will simply denote by $\mathcal{P}(X)$, using $\mathcal{P}(X^s)$ and $\mathcal{P}(X^w)$ only when we will refer to the corresponding topology. Finally, if X has finite dimension then the two topologies coincide.

We now list some properties of Wasserstein spaces and we refer to [3, § 7] for a complete account of this matter.

Definition 2.5 Given $\mu \in \mathcal{P}(X)$ we define

$$\mathsf{m}_2^2(\mu) := \int_X |x|^2 \, \mathrm{d}\mu(x) \qquad \text{and} \qquad \mathcal{P}_2(X) := \Big\{ \mu \in \mathcal{P}(X) \mid \mathsf{m}_2(\mu) < +\infty \Big\}.$$

The L^2-Wasserstein distance between $\mu, \mu' \in \mathcal{P}_2(X)$ is defined as

$$W_2^2(\mu, \mu') := \inf \left\{ \int_{X \times X} |x - y|^2 \, \mathrm{d}\boldsymbol{\gamma}(x, y) \mid \boldsymbol{\gamma} \in \Gamma(\mu, \mu') \right\}. \tag{2.2}$$

The set of elements of $\Gamma(\mu, \mu')$ realizing the infimum in (2.2) is denoted with $\Gamma_o(\mu, \mu')$. We say that a measure $\boldsymbol{\gamma} \in \mathcal{P}_2(X \times X)$ is optimal if $\boldsymbol{\gamma} \in \Gamma_o(\pi^1_\sharp \boldsymbol{\gamma}, \pi^2_\sharp \boldsymbol{\gamma})$.

We will denote by $\mathrm{B}(\mu, \varrho)$ the open ball centered at μ with radius ϱ in $\mathcal{P}_2(X)$. The metric space $(\mathcal{P}_2(X), W_2)$ enjoys many interesting properties: here we only recall that it is a complete and separable metric space and that W_2-convergence (sometimes denoted with $\xrightarrow{W_2}$) is stronger than the narrow convergence. In particular, given

$(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{P}_2(X)$ and $\mu \in \mathcal{P}_2(X)$, we have [3, Remark 7.1.11] that

$$\mu_n \overset{W_2}{\to} \mu, \text{ as } n \to +\infty \quad \Longleftrightarrow \quad \begin{cases} \mu_n \to \mu \text{ in } \mathcal{P}(X^s), \\ \mathsf{m}_2(\mu_n) \to \mathsf{m}_2(\mu), \end{cases} \quad \text{as } n \to +\infty. \tag{2.3}$$

Finally, we recall that sequences converging in $(\mathcal{P}_2(X), W_2)$ are tight. More precisely we have the following characterization of compactness in $\mathcal{P}_2(X)$.

Lemma 2.6 (Relative compactness in $\mathcal{P}_2(X)$) *A subset $\mathcal{K} \subset \mathcal{P}_2(X)$ is relatively compact w.r.t. the W_2-topology if and only if*

(1) *$\mathcal{K}$ is tight w.r.t. X^s,*
(2) *$\mathcal{K}$ is uniformly 2-integrable, i.e.*

$$\lim_{k\to\infty} \sup_{\mu\in\mathcal{K}} \int_{|x|\geq k} |x|^2 \, \mathrm{d}\mu = 0. \tag{2.4}$$

Proof Tightness is clearly a necessary condition; concerning (2.4) let us notice that the maps

$$F_k : \mathcal{P}_2(X) \to [0, \infty), \quad F_k(\mu) := \int_{|x|\geq k} |x|^2 \, \mathrm{d}\mu$$

are upper semicontinuous, are decreasing w.r.t. k, and converge pointwise to 0 for every $\mu \in \mathcal{P}_2(X)$. Therefore, if $\mathcal{K}$ is relatively compact, they converge uniformly to 0 thanks to Dini's Theorem.

In order to prove that (1) and (2) are also sufficient for relative compactness, it is sufficient to check that every sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{K}$ has a convergent subsequence. Applying Prokhorov Theorem 2.1, we can find $\mu \in \mathcal{P}(X)$ and a convergent subsequence $k \mapsto \mu_{n_k}$ such that $\mu_{n_k} \to \mu$ in $\mathcal{P}(X^s)$. Since $\mathsf{m}_2(\mu_n)$ is uniformly bounded, then $\mu \in \mathcal{P}_2(X)$. Applying [3, Lemma 5.1.7], we also get

$$\lim_{k\to\infty} \mathsf{m}_2(\mu_{n_k}) = \mathsf{m}_2(\mu)$$

so that, by (2.3), we conclude

$$\lim_{k\to\infty} W_2(\mu_{n_k}, \mu) = 0.$$

□

Definition 2.7 (Geodesics) A curve $\mu : [0, 1] \to \mathcal{P}_2(X)$ is said to be a *(constant speed) geodesic* if for all $0 \leq s \leq t \leq 1$ we have

$$W_2(\mu_s, \mu_t) = (t - s) W_2(\mu_0, \mu_1),$$

where μ_t denotes the evaluation at time $t \in [0, 1]$ of μ. We also say that μ is a *geodesic from μ_0 to μ_1*.

We say that $A \subset \mathcal{P}_2(X)$ is a *geodesically convex* set if for any pair $\mu_0, \mu_1 \in A$ there exists a geodesic μ from μ_0 to μ_1 such that $\mu_t \in A$ for every $t \in [0, 1]$.

We recall also the following useful properties of geodesics (see [3, Theorem 7.2.1, Theorem 7.2.2]).

Theorem 2.8 (Properties of geodesics) *Let $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ and $\boldsymbol{\mu} \in \Gamma_o(\mu_0, \mu_1)$. Then $\mu : [0, 1] \to \mathcal{P}_2(X)$, defined by*

$$\mu_t := (\mathsf{x}^t)_\sharp \boldsymbol{\mu}, \quad t \in [0, 1], \tag{2.5}$$

is a (constant speed) geodesic from μ_0 to μ_1, where $\mathsf{x}^t : X^2 \to X$ is given by

$$\mathsf{x}^t(x_0, x_1) := (1 - t)x_0 + t x_1.$$

Conversely, any (constant speed) geodesic μ from μ_0 to μ_1 admits the representation (2.5) for a suitable plan $\boldsymbol{\mu} \in \Gamma_o(\mu_0, \mu_1)$.
Finally, if μ is a geodesic connecting μ_0 to μ_1, then for every $t \in (0, 1)$ there exists a unique optimal plan between μ_0 and μ_t (resp. between μ_t and μ_1) and it is concentrated on a map.

We define the counterpart of $\mathrm{C}_c^\infty(\mathbb{R}^d)$ when we have X in place of $\mathbb{R}^d$.

Definition 2.9 ($\mathrm{Cyl}(X)$) We denote by $\Pi_d(X)$ the space of linear maps $\pi : X \to \mathbb{R}^d$ of the form $\pi(x) = (\langle x, e_1\rangle, \ldots, \langle x, e_d\rangle)$ for an orthonormal set $\{e_1, \ldots, e_d\}$ of X. A function $\varphi : X \to \mathbb{R}$ belongs to the space of cylindrical functions on X, $\mathrm{Cyl}(X)$, if it is of the form

$$\varphi = \psi \circ \pi$$

where $\pi \in \Pi_d(X)$ and $\psi \in \mathrm{C}_c^\infty(\mathbb{R}^d)$.

We recall the following result (see [3, Theorem 8.3.1, Proposition 8.4.5 and Proposition 8.4.6]) characterizing locally absolutely continuous curves in $\mathcal{P}_2(X)$ defined in a (bounded or unbounded) open interval $\mathcal{I} \subset \mathbb{R}$. We use the notation μ_t for the evaluation at time $t \in \mathcal{I}$ of a map $\mu : \mathcal{I} \to \mathcal{P}_2(X)$.

Theorem 2.10 (Wasserstein velocity field) *Let $\mu : \mathcal{I} \to \mathcal{P}_2(X)$ be a locally absolutely continuous curve defined in an open interval $\mathcal{I} \subset \mathbb{R}$. There exists a Borel vector field $\boldsymbol{v} : \mathcal{I} \times X \to X$ and a set $A(\mu) \subset \mathcal{I}$ with $\mathcal{L}(\mathcal{I} \setminus A(\mu)) = 0$ such that the following hold*

(1) $\boldsymbol{v}_t \in \mathrm{Tan}_{\mu_t} \mathcal{P}_2(X) := \overline{\{\nabla\varphi \mid \varphi \in \mathrm{Cyl}(X)\}}^{L^2_{\mu_t}(X;X)}$*, for every $t \in A(\mu)$;*
(2) $\int_X |\boldsymbol{v}_t|^2 \,\mathrm{d}\mu_t = |\dot{\mu}_t|^2 := \lim_{h\to 0} \frac{W_2^2(\mu_{t+h}, \mu_t)}{h^2}$*, for every $t \in A(\mu)$;*
(3) *the continuity equation*

$$\partial_t \mu_t + \nabla \cdot (\boldsymbol{v}_t \mu_t) = 0$$

holds in the sense of distributions in $\mathcal{I} \times X$.

Moreover, $\boldsymbol{v}_t$ *is uniquely determined in* $L^2_{\mu_t}(X;X)$ *for* $t \in A(\mu)$ *and*

$$\lim_{h\to 0} \frac{W_2((\boldsymbol{i}_X + h\boldsymbol{v}_t)_\sharp\mu_t, \mu_{t+h})}{|h|} = 0 \quad \textit{for every } t \in A(\mu). \tag{2.6}$$

We conclude this section with a useful property concerning the upper derivative of the Wasserstein distance, which in fact holds in every metric space.

Lemma 2.11 *Let* $\mu : \mathcal{I} \to \mathcal{P}_2(X)$, $\nu \in \mathcal{P}_2(X)$, $t \in \mathcal{I}$, $\boldsymbol{\sigma}_t \in \Gamma_o(\mu_t, \nu)$, *and consider the constant speed geodesic* $\nu^t : [0,1] \to \mathcal{P}_2(X)$ *defined by* $\nu^t_s := (\mathsf{x}^s)_\sharp\boldsymbol{\sigma}_t$ *for every* $s \in [0,1]$. *The upper right and left Dini derivatives* $b^\pm : (0,1] \to \mathbb{R}$ *defined by*

$$\begin{aligned} b^+(s) :=& \frac{1}{2s} \limsup_{h\downarrow 0} \frac{W_2^2(\mu_{t+h}, \nu^t_s) - W_2^2(\mu_t, \nu^t_s)}{h}, \\ b^-(s) :=& \frac{1}{2s} \limsup_{h\downarrow 0} \frac{W_2^2(\mu_t, \nu^t_s) - W_2^2(\mu_{t-h}, \nu^t_s)}{h} \end{aligned}$$

are respectively decreasing and increasing in $(0,1]$.

Proof Take $0 \le s' < s \le 1$. Since $\nu^t : [0,1] \to \mathcal{P}_2(X)$ is a constant speed geodesic from μ_t to ν, we have

$$W_2(\mu_t, \nu^t_s) = W_2(\mu_t, \nu^t_{s'}) + W_2(\nu^t_{s'}, \nu^t_s),$$

then, by triangular inequality

$$\begin{aligned} W_2(\mu_{t+h}, \nu^t_s) - W_2(\mu_t, \nu^t_s) &\le W_2(\mu_{t+h}, \nu^t_{s'}) + W_2(\nu^t_{s'}, \nu^t_s) - W_2(\mu_t, \nu^t_s) \\ &= W_2(\mu_{t+h}, \nu^t_{s'}) - W_2(\mu_t, \nu^t_{s'}). \end{aligned}$$

Dividing by $h > 0$ and passing to the limit as $h \downarrow 0$ we obtain that the function $a : [0,1] \to \mathbb{R}$ defined by

$$a^+(s) := \limsup_{h\downarrow 0} \frac{W_2(\mu_{t+h}, \nu^t_s) - W_2(\mu_t, \nu^t_s)}{h}$$

is decreasing. It is then sufficient to observe that for $s > 0$

$$b^+(s) = a^+(s)\frac{W_2(\mu_t, \nu^t_s)}{s} = a^+(s)\, W_2(\mu_t, \nu).$$

The monotonicity property of b^- follows by the same argument. □

2.2 A strong-weak topology on measures in product spaces

Let us consider the case where $X = \mathsf{X} \times \mathsf{Y}$ where X, Y are separable Hilbert spaces. The space X is naturally endowed with the product Hilbert norm and $\mathcal{P}_2(X)$ with the corresponding topology induced by the L^2-Wasserstein distance. However, it will be extremely useful to endow $\mathcal{P}_2(X)$ with a weaker topology which is related to the strong-weak topology on X, i.e. the product topology of $\mathsf{X}^s \times \mathsf{Y}^w$. We follow the approach of [25], to which we refer for the proofs of the results presented in this section.

In order to define the topology, we consider the space $\mathrm{C}_2^{sw}(\mathsf{X} \times \mathsf{Y})$ of test functions $\zeta : \mathsf{X} \times \mathsf{Y} \to \mathbb{R}$ such that

$$\zeta \text{ is sequentially continuous in } \mathsf{X}^s \times \mathsf{Y}^w, \tag{2.7}$$

$$\forall\, \varepsilon > 0\; \exists\, A_\varepsilon \geq 0 : |\zeta(x, y)| \leq A_\varepsilon(1 + |x|_{\mathsf{X}}^2) + \varepsilon |y|_{\mathsf{Y}}^2 \quad \forall\, (x, y) \in \mathsf{X} \times \mathsf{Y}. \tag{2.8}$$

Notice in particular that functions in $\mathrm{C}_2^{sw}(\mathsf{X} \times \mathsf{Y})$ have quadratic growth. We endow $\mathrm{C}_2^{sw}(X)$ with the norm

$$\|\zeta\|_{\mathrm{C}_2^{sw}(X)} := \sup_{(x,y)\in X} \frac{|\zeta(x, y)|}{1 + |x|_{\mathsf{X}}^2 + |y|_{\mathsf{Y}}^2}.$$

Remark 2.12 When Y is finite dimensional, (2.7) is equivalent to the continuity of ζ.

Lemma 2.13 $(\mathrm{C}_2^{sw}(\mathsf{X} \times \mathsf{Y}), \|\cdot\|_{\mathrm{C}_2^{sw}(\mathsf{X}\times\mathsf{Y})})$ *is a Banach space.*

Definition 2.14 (Topology of $\mathcal{P}_2^{sw}(\mathsf{X} \times \mathsf{Y})$, [25]) We denote by $\mathcal{P}_2^{sw}(\mathsf{X} \times \mathsf{Y})$ the space $\mathcal{P}_2(\mathsf{X} \times \mathsf{Y})$ endowed with the coarsest topology which makes the following functions continuous

$$\boldsymbol{\mu} \mapsto \int \zeta(x, y)\, \mathrm{d}\boldsymbol{\mu}(x, y), \quad \zeta \in \mathrm{C}_2^{sw}(\mathsf{X} \times \mathsf{Y}).$$

It is obvious that the topology of $\mathcal{P}_2(\mathsf{X} \times \mathsf{Y})$ is finer than the topology of $\mathcal{P}_2^{sw}(\mathsf{X} \times \mathsf{Y})$ and the latter is finer than the topology of $\mathcal{P}(\mathsf{X}^s \times \mathsf{Y}^w)$. It is worth noticing that any bounded bilinear form $B : \mathsf{X} \times \mathsf{Y} \to \mathbb{R}$ belongs to $\mathrm{C}_2^{sw}(\mathsf{X} \times \mathsf{Y})$, so that for every net $(\boldsymbol{\mu}_\alpha)_{\alpha\in\mathbb{A}} \subset \mathcal{P}(\mathsf{X} \times \mathsf{Y})$ indexed by a directed set $\mathbb{A}$, we have

$$\lim_{\alpha\in\mathbb{A}} \boldsymbol{\mu}_\alpha = \boldsymbol{\mu} \quad \text{in } \mathcal{P}_2^{sw}(\mathsf{X} \times \mathsf{Y}) \quad \Rightarrow \quad \lim_{\alpha\in\mathbb{A}} \int B\, \mathrm{d}\boldsymbol{\mu}_\alpha = \int B\, \mathrm{d}\boldsymbol{\mu}. \tag{2.9}$$

The following proposition justifies the interest in the $\mathcal{P}_2^{sw}(\mathsf{X} \times \mathsf{Y})$-topology.

Proposition 2.15 (1) *Assume that* $(\boldsymbol{\mu}_\alpha)_{\alpha\in\mathbb{A}} \subset \mathcal{P}_2(\mathsf{X} \times \mathsf{Y})$ *is a net indexed by the directed set* $\mathbb{A}$, $\boldsymbol{\mu} \in \mathcal{P}_2(\mathsf{X} \times \mathsf{Y})$ *and they satisfy*

(a) $\boldsymbol{\mu}_\alpha \to \boldsymbol{\mu}$ *in* $\mathcal{P}(\mathsf{X}^s \times \mathsf{Y}^w)$,

(b) $\lim_{\alpha\in\mathbb{A}} \int |x|^2_{\mathsf{X}} \,\mathrm{d}\boldsymbol{\mu}_\alpha(x,y) = \int |x|^2_{\mathsf{X}} \,\mathrm{d}\boldsymbol{\mu}(x,y)$,

(c) $\sup_{\alpha\in\mathbb{A}} \int |y|^2_{\mathsf{Y}} \,\mathrm{d}\boldsymbol{\mu}_\alpha(x,y) < \infty$,

then $\boldsymbol{\mu}_\alpha \to \boldsymbol{\mu}$ *in* $\mathcal{P}_2^{sw}(\mathsf{X}\times\mathsf{Y})$. *The converse property holds for sequences: if* $\mathbb{A} = \mathbb{N}$ *and* $\boldsymbol{\mu}_n \to \boldsymbol{\mu}$ *in* $\mathcal{P}_2^{sw}(\mathsf{X}\times\mathsf{Y})$ *as* $n \to \infty$ *then properties (a), (b), (c) hold.*

(2) *For every compact set* $\mathcal{K} \subset \mathcal{P}_2(\mathsf{X}^s)$ *and every constant* $c < \infty$ *the sets*

$$\mathcal{K}_c := \left\{\boldsymbol{\mu} \in \mathcal{P}_2(\mathsf{X}\times\mathsf{Y}) : \pi^{\mathsf{X}}_\sharp \boldsymbol{\mu} \in \mathcal{K}, \quad \int |y|^2_{\mathsf{Y}} \,\mathrm{d}\boldsymbol{\mu}(x,y) \le c\right\}$$

are compact and metrizable in $\mathcal{P}_2^{sw}(\mathsf{X}\times\mathsf{Y})$ *(in particular they are sequentially compact).*

It is worth noticing that the topology $\mathcal{P}_2^{ws}(\mathsf{X}\times\mathsf{Y})$ is strictly weaker than $\mathcal{P}_2(\mathsf{X}\times\mathsf{Y})$ even when Y is finite dimensional. In fact, $\mathrm{C}_2^{sw}(\mathsf{X}\times\mathsf{Y})$ does not contain the quadratic function $(x,y) \mapsto |y|^2_{\mathsf{Y}}$, so that convergence of the quadratic moment w.r.t. y is not guaranteed.

3 Directional derivatives and probability measures on the tangent bundle

From now on, we will denote by X a separable Hilbert space with norm $|\cdot|$ and scalar product $\langle\cdot,\cdot\rangle$. We denote by TX the tangent bundle to X, which is identified with the set $\mathsf{X}\times\mathsf{X}$ with the induced norm $|(x,v)| := \left(|x|^2+|v|^2\right)^{1/2}$ and the strong-weak topology of $\mathsf{X}^s\times\mathsf{X}^w$ (i.e. the product of the strong topology on the first component and the weak topology on the second one). We will denote by $\mathsf{x},\mathsf{v} : \mathsf{TX} \to \mathsf{X}$ the projection maps and by $\mathsf{exp}^t : \mathsf{TX} \to \mathsf{X}$ the exponential map defined by

$$\mathsf{x}(x,v) := x, \quad \mathsf{v}(x,v) := v, \quad \mathsf{exp}^t(x,v) := x + tv. \tag{3.1}$$

The set $\mathcal{P}(\mathsf{TX})$ is defined thanks to the identification of TX with $\mathsf{X}\times\mathsf{X}$ and is endowed with the narrow topology induced by the strong-weak topology in TX. For $\Phi \in \mathcal{P}(\mathsf{TX})$ we define

$$|\Phi|_2^2 := \int_{\mathsf{TX}} |v|^2 \,\mathrm{d}\Phi(x,v). \tag{3.2}$$

We denote by $\mathcal{P}_2(\mathsf{TX})$ the subset of $\mathcal{P}(\mathsf{TX})$ of measures for which $\int \left(|x|^2+|v|^2\right) \mathrm{d}\Phi < \infty$ endowed with the topology of $\mathcal{P}_2^{sw}(\mathsf{TX})$ as in Sect. 2.2. If $\mu \in \mathcal{P}(\mathsf{X})$ we will also consider

$$\mathcal{P}(\mathsf{TX}|\mu) := \left\{\Phi \in \mathcal{P}(\mathsf{TX}) \mid \mathsf{x}_\sharp\Phi = \mu\right\}, \quad \mathcal{P}_2(\mathsf{TX}|\mu) := \{\Phi \in \mathcal{P}(\mathsf{TX}|\mu) : |\Phi|_2 < \infty\}. \tag{3.3}$$

When we deal with the product space X^2, we will use the notation

$$\mathsf{x}^t : \mathsf{X}^2 \to \mathsf{X}, \quad \mathsf{x}^t(x_0, x_1) := (1-t)x_0 + tx_1, \quad t \in [0,1]. \tag{3.4}$$

If $\boldsymbol{v} \in L^2_\mu(\mathsf{X};\mathsf{X})$ we can consider the probability measure

$$\Phi = (\boldsymbol{i}_\mathsf{X}, \boldsymbol{v})_\sharp \mu \in \mathcal{P}_2(\mathsf{TX}|\mu). \tag{3.5}$$

In this case we will say that Φ is concentrated on the graph of the map $\boldsymbol{v}$. More generally, given a Borel family of probability measures $(\Phi_x)_{x\in\mathsf{X}} \subset \mathcal{P}_2(\mathsf{X})$ satisfying

$$\int \Big(\int |v|^2 \, \mathrm{d}\Phi_x(v) \Big) \, \mathrm{d}\mu(x) < \infty \tag{3.6}$$

we can consider the probability measure

$$\Phi = \int_\mathsf{X} \Phi_x \, \mathrm{d}\mu(x) \in \mathcal{P}_2(\mathsf{TX}|\mu). \tag{3.7}$$

Conversely, every $\Phi \in \mathcal{P}_2(\mathsf{TX}|\mu)$ can be disintegrated into a Borel family $(\Phi_x)_{x\in\mathsf{X}} \subset \mathcal{P}_2(\mathsf{X})$ satisfying (3.6) and (3.7). The measure Φ can be associated with a vector field $\boldsymbol{v} \in L^2_\mu(\mathsf{X};\mathsf{X})$ if and only if for μ-a.e. $x \in \mathsf{X}$ we have $\Phi_x = \delta_{\boldsymbol{v}(x)}$. Recalling the disintegration Theorem 2.3 and Remark 2.4, we give the following definition.

Definition 3.1 Given $\Phi \in \mathcal{P}_2(\mathsf{TX}|\mu)$, the *barycenter of* Φ is the function $\boldsymbol{b}_\Phi \in L^2_\mu(\mathsf{X};\mathsf{X})$ defined by

$$\boldsymbol{b}_\Phi(x) := \int_\mathsf{X} v \, \mathrm{d}\Phi_x(v) \quad \text{for } \mu\text{-a.e. } x \in \mathsf{X},$$

where $\{\Phi_x\}_{x\in\mathsf{X}} \subset \mathcal{P}_2(\mathsf{X})$ is the disintegration of Φ w.r.t. μ.

Remark 3.2 Notice that, by the linearity of the scalar product, we get the following identity which will be useful later

$$\int_\mathsf{X} \langle \boldsymbol{\zeta}(x), \boldsymbol{b}_\Phi(x) \rangle \, \mathrm{d}\mu(x) = \int_\mathsf{TX} \langle \boldsymbol{\zeta}(x), v \rangle \, \mathrm{d}\Phi(x, v) \tag{3.8}$$

for all $\boldsymbol{\zeta} \in L^2_\mu(\mathsf{X};\mathsf{X})$.

3.1 Directional derivatives of the Wasserstein distance and duality pairings

Our starting point is a relevant semi-concavity property of the function

$$f(s,t) := \frac{1}{2} W_2^2(\mathsf{exp}^s_\sharp \Phi_0, \mathsf{exp}^t_\sharp \Phi_1), \quad s, t \in \mathbb{R}, \tag{3.9}$$

with $\Phi_0, \Phi_1 \in \mathcal{P}_2(\mathsf{TX})$. We first state an auxiliary result, whose proof is based on [3, Proposition 7.3.1].

Lemma 3.3 *Let $\Phi_0, \Phi_1 \in \mathcal{P}_2(\mathsf{TX})$, $s, t \in \mathbb{R}$, and let $\boldsymbol{\vartheta}^{s,t} \in \Gamma(\mathsf{exp}^s_\sharp \Phi_0, \mathsf{exp}^t_\sharp \Phi_1)$. Then there exists $\boldsymbol{\Theta}^{s,t} \in \Gamma(\Phi_0, \Phi_1)$ such that $(\mathsf{exp}^s, \mathsf{exp}^t)_\sharp \boldsymbol{\Theta}^{s,t} = \boldsymbol{\vartheta}^{s,t}$.*

Proof Define, for every $r, s, t \in \mathbb{R}$,

$$\begin{aligned}
&\Sigma^r : \mathsf{TX} \to \mathsf{TX}, \quad \Sigma^r(x, v) := (\mathsf{exp}^r(x, v), v);\\
&\Lambda^{s,t} : \mathsf{TX} \times \mathsf{TX} \to \mathsf{TX} \times \mathsf{TX}, \quad \Lambda^{s,t} := (\Sigma^s, \Sigma^t).
\end{aligned}$$

Consider the probabilities $(\Sigma^s)_\sharp \Phi_0$, $(\Sigma^t)_\sharp \Phi_1$ and $\boldsymbol{\vartheta}^{s,t}$. They are constructed in such a way that there exists $\boldsymbol{\Psi}^{s,t} \in \mathcal{P}(\mathsf{TX} \times \mathsf{TX})$ s.t.

$$(\mathsf{x}^0, \mathsf{v}^0)_\sharp \boldsymbol{\Psi}^{s,t} = (\Sigma^s)_\sharp \Phi_0, \quad (\mathsf{x}^1, \mathsf{v}^1)_\sharp \boldsymbol{\Psi}^{s,t} = (\Sigma^t)_\sharp \Phi_1, \quad (\mathsf{x}^0, \mathsf{x}^1)_\sharp \boldsymbol{\Psi}^{s,t} = \boldsymbol{\vartheta}^{s,t},$$

where we adopted the notation $\mathsf{x}^i(x_0, v_0, x_1, v_1) := x_i$ and $\mathsf{v}^i(x_0, v_0, x_1, v_1) := v_i$, $i = 0, 1$. We conclude by taking $\boldsymbol{\Theta}^{s,t} := (\Lambda^{-s,-t})_\sharp \boldsymbol{\Psi}^{s,t}$. □

Proposition 3.4 *Let $\Phi_0, \Phi_1 \in \mathcal{P}_2(\mathsf{TX})$ with $\mu_1 = \mathsf{x}_\sharp \Phi_1$ and $\varphi^2 := |\Phi_0|_2^2 + |\Phi_1|_2^2$, let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by (3.9) and let $h, g : \mathbb{R} \to \mathbb{R}$ be defined by*

$$\begin{aligned}
h(s) &:= f(s, s) = \frac{1}{2} W_2^2(\mathsf{exp}^s_\sharp \Phi_0, \mathsf{exp}^s_\sharp \Phi_1),\\
g(s) &:= f(s, 0) = \frac{1}{2} W_2^2(\mathsf{exp}^s_\sharp \Phi_0, \mu_1), \quad s \in \mathbb{R}.
\end{aligned} \tag{3.10}$$

(1) *The function $(s, t) \mapsto f(s, t) - \frac{1}{2}\varphi^2(s^2 + t^2)$ is concave, i.e. it holds*

$$\begin{aligned}
f((1-\alpha)s_0 + \alpha s_1, (1-\alpha)t_0 + \alpha t_1) &\geq (1-\alpha) f(s_0, t_0) + \alpha f(s_1, t_1)\\
&- \frac{1}{2}\alpha(1-\alpha)\Big[(s_1 - s_0)^2 + (t_1 - t_0)^2\Big]\varphi^2
\end{aligned} \tag{3.11}$$

for every $s_0, s_1, t_0, t_1 \in \mathbb{R}$ and every $\alpha \in [0, 1]$.

(2) *The function $s \mapsto h(s) - \varphi^2 s^2$ is concave.*

(3) *the function $s \mapsto g(s) - \frac{1}{2} s^2 |\Phi_0|_2^2$ is concave.*

Proof Let us first prove (3.11). We set $s := (1-\alpha)s_0 + \alpha s_1$, $t := (1-\alpha)t_0 + \alpha t_1$ and we apply Lemma 3.3 to find $\boldsymbol{\Theta} \in \Gamma(\Phi_0, \Phi_1)$ such that $(\mathsf{exp}^s, \mathsf{exp}^t)_\sharp \boldsymbol{\Theta} \in \Gamma_o(\mathsf{exp}^s_\sharp \Phi_0, \mathsf{exp}^t_\sharp \Phi_1)$. Then, recalling the Hilbertian identity

$$|(1-\alpha)a + \alpha b|^2 = (1-\alpha)|a|^2 + \alpha|b|^2 - \alpha(1-\alpha)|a - b|^2, \quad a, b \in \mathsf{X},$$

we have

$$
\begin{aligned}
&W_2^2(\exp_\sharp^s \Phi_0, \exp_\sharp^t \Phi_1) \\
&\quad = \int |x_0 + s v_0 - (x_1 + t v_1)|^2 \, \mathrm{d}\boldsymbol{\Theta} \\
&\quad = \int |(1-\alpha)(x_0 + s_0 v_0) + \alpha(x_0 + s_1 v_0) - (1-\alpha)(x_1 + t_0 v_1) - \alpha(x_1 + t_1 v_1)|^2 \, \mathrm{d}\boldsymbol{\Theta} \\
&\quad = (1-\alpha) \int |x_0 + s_0 v_0 - (x_1 + t_0 v_1)|^2 \, \mathrm{d}\boldsymbol{\Theta} + \alpha \int |x_0 + s_1 v_0 - (x_1 + t_1 v_1)|^2 \, \mathrm{d}\boldsymbol{\Theta} \\
&\qquad - \alpha(1-\alpha) \int |(s_1 - s_0) v_0 + (t_1 - t_0) v_1|^2 \, \mathrm{d}\boldsymbol{\Theta} \\
&\quad \geq (1-\alpha) W_2^2(\exp_\sharp^{s_0} \Phi_0, \exp_\sharp^{t_0} \Phi_1) + \alpha W_2^2(\exp_\sharp^{s_1} \Phi_0, \exp_\sharp^{t_1} \Phi_1) \\
&\qquad - \alpha(1-\alpha)\Big((s_1 - s_0)^2 + (t_1 - t_0)^2\Big)\Big(\int |v_0|^2 \, \mathrm{d}\Phi_0 + \int |v_1|^2 \, \mathrm{d}\Phi_1\Big).
\end{aligned}
$$

which is the thesis. Claims (2) and (3) follow as particular cases when $t = s$ or $t = 0$. □

Semi-concavity is a useful tool to guarantee the existence of one-sided partial derivatives at (0, 0): for every $\alpha, \beta \in \mathbb{R}$ we have (see e.g. [19, Ch. VI, Prop. 1.1.2]) that

$$
\begin{aligned}
f_r'(\alpha, \beta) &= \lim_{\varrho \downarrow 0} \frac{f(\alpha\varrho, \beta\varrho) - f(0,0)}{\varrho} = \sup_{\varrho > 0} \frac{f(\alpha\varrho, \beta\varrho) - f(0,0)}{\varrho} - \frac{\varrho\varphi^2}{2}(\alpha^2 + \beta^2), \\
f_l'(\alpha, \beta) &= \lim_{\varrho \downarrow 0} \frac{f(0,0) - f(-\alpha\varrho, -\beta\varrho)}{\varrho} \\
&= \inf_{\varrho > 0} \frac{f(0,0) - f(-\alpha\varrho, -\beta\varrho)}{\varrho} + \frac{\varrho\varphi^2}{2}(\alpha^2 + \beta^2).
\end{aligned}
$$

f_r' (resp. f_l') is a concave (resp. convex) and positively 1-homogeneous function, i.e. a superlinear (resp. sublinear) function. They satisfy

$$
f_r'(-\alpha, -\beta) = -f_l'(\alpha, \beta) \quad \text{for every } \alpha, \beta \in \mathbb{R}, \tag{3.12}
$$

$$
f_l'(\alpha, \beta) \geq f_r'(\alpha, \beta) \quad \text{for every } \alpha, \beta \in \mathbb{R}, \tag{3.13}
$$

$$
f_r'(\alpha, \beta) \geq \alpha f_r'(1,0) + \beta f_r'(0,1) \quad \text{for every } \alpha, \beta \geq 0,
$$

$$
f(s,t) \leq f(0,0) + f_r'(s,t) - \frac{\varphi^2}{2}(s^2 + t^2) \quad \text{for every } s, t \in \mathbb{R}. \tag{3.14}
$$

Notice moreover that

$$
f_r'(1,0) = g_r'(0) = \lim_{\varrho \downarrow 0} \frac{g(\varrho) - g(0)}{\varrho}
$$

where g is the function defined in (3.10); a similar representation holds for $f_l'(1,0)$. We introduce the following notation for f_r', f_l', g_r' and g_l'.

Definition 3.5 Let $\mu_0, \mu_1 \in \mathcal{P}_2(\mathsf{X})$, $\Phi_0 \in \mathcal{P}_2(\mathsf{TX}|\mu_0)$ and $\Phi_1 \in \mathcal{P}_2(\mathsf{TX}|\mu_1)$. We define

$$[\Phi_0, \mu_1]_r := \lim_{s \downarrow 0} \frac{W_2^2(\mathsf{exp}_\sharp^s \Phi_0, \mu_1) - W_2^2(\mu_0, \mu_1)}{2s},$$
$$[\Phi_0, \mu_1]_l := \lim_{s \downarrow 0} \frac{W_2^2(\mu_0, \mu_1) - W_2^2(\mathsf{exp}_\sharp^{-s} \Phi_0, \mu_1)}{2s},$$

and analogously

$$[\Phi_0, \Phi_1]_r := \lim_{t \downarrow 0} \frac{W_2^2(\mathsf{exp}_\sharp^t \Phi_0, \mathsf{exp}_\sharp^t \Phi_1) - W_2^2(\mu_0, \mu_1)}{2t},$$
$$[\Phi_0, \Phi_1]_l := \lim_{t \downarrow 0} \frac{W_2^2(\mu_0, \mu_1) - W_2^2(\mathsf{exp}_\sharp^{-t} \Phi_0, \mathsf{exp}_\sharp^{-t} \Phi_1)}{2t}.$$

Recalling the definitions of f and g given by (3.9) and (3.10), with Φ_0 and Φ_1 as above, we notice that

$$\begin{aligned}
[\Phi_0, \mu_1]_r &= g_r'(0) = f_r'(1, 0), \\
[\Phi_0, \mu_1]_l &= g_l'(0) = f_l'(1, 0), \\
[\Phi_0, \Phi_1]_r &= f_r'(1, 1), \\
[\Phi_0, \Phi_1]_l &= f_l'(1, 1).
\end{aligned}$$

Remark 3.6 Notice that $[\Phi_0, \mu_1]_r = \left[\Phi_0, \Phi_{\mu_1}\right]_r$ and $[\Phi_0, \mu_1]_l = \left[\Phi_0, \Phi_{\mu_1}\right]_l$, where

$$\Phi_{\mu_1} = (\boldsymbol{i}_{\mathsf{X}}, 0)_\sharp \mu_1 \in \mathcal{P}_2(\mathsf{TX}).$$

Moreover, given $\Phi \in \mathcal{P}(\mathsf{TX})$ and using the notation

$$-\Phi := J_\sharp \Phi, \quad \text{with } J(x, v) := (x, -v), \tag{3.15}$$

we have

$$[-\Phi_0, -\Phi_1]_r = -[\Phi_0, \Phi_1]_l, \quad \text{and} \quad [-\Phi_0, \mu_1]_r = -[\Phi_0, \mu_1]_l.$$

In particular, the properties of $[\cdot, \cdot]_l$ (in $\mathcal{P}_2(\mathsf{TX}) \times \mathcal{P}_2(\mathsf{TX})$ or $\mathcal{P}_2(\mathsf{TX}) \times \mathcal{P}_2(\mathsf{X})$) and the ones of $[\cdot, \cdot]_r$ in $\mathcal{P}_2(\mathsf{TX}) \times \mathcal{P}_2(\mathsf{X})$ can be easily derived by the corresponding ones of $[\cdot, \cdot]_r$ in $\mathcal{P}_2(\mathsf{TX}) \times \mathcal{P}_2(\mathsf{TX})$.

Recalling (3.14) and (3.12) we obtain the following result.

Corollary 3.7 *For every $\mu_0, \mu_1 \in \mathcal{P}_2(\mathsf{X})$ and for every $\Phi_0 \in \mathcal{P}_2(\mathsf{TX}|\mu_0)$, $\Phi_1 \in \mathcal{P}_2(\mathsf{TX}|\mu_1)$, it holds*

$$[\Phi_0, \mu_1]_r + [\Phi_1, \mu_0]_r \le [\Phi_0, \Phi_1]_r \quad \textit{and} \quad [\Phi_0, \mu_1]_l + [\Phi_1, \mu_0]_l \ge [\Phi_0, \Phi_1]_l.$$

Let us now show an important equivalent characterization of the quantities we have just introduced. As usual we will denote by $\mathsf{x}^0, \mathsf{v}^0, \mathsf{x}^1 : \mathsf{TX} \times \mathsf{X} \to \mathsf{X}$ the projection maps of a point (x_0, v_0, x_1) in $\mathsf{TX} \times \mathsf{X}$ (and similarly for $\mathsf{TX} \times \mathsf{TX}$ with $\mathsf{x}^0, \mathsf{v}^0, \mathsf{x}^1, \mathsf{v}^1$).

First of all we introduce the following sets.

Definition 3.8 For every $\Phi_0 \in \mathcal{P}(\mathsf{TX})$ with $\mu_0 = \mathsf{x}_\sharp \Phi_0$ and $\mu_1 \in \mathcal{P}_2(\mathsf{X})$ we set

$$\Lambda(\Phi_0, \mu_1) := \left\{ \boldsymbol{\sigma} \in \Gamma(\Phi_0, \mu_1) \mid (\mathsf{x}^0, \mathsf{x}^1)_\sharp \boldsymbol{\sigma} \in \Gamma_o(\mu_0, \mu_1) \right\}.$$

Analogously, for every $\Phi_0, \Phi_1 \in \mathcal{P}(\mathsf{TX})$ with $\mu_0 = \mathsf{x}_\sharp \Phi_0$ and $\mu_1 = \mathsf{x}_\sharp \Phi_1$ in $\mathcal{P}_2(\mathsf{X})$ we set

$$\Lambda(\Phi_0, \Phi_1) := \left\{ \boldsymbol{\Theta} \in \Gamma(\Phi_0, \Phi_1) \mid (\mathsf{x}^0, \mathsf{x}^1)_\sharp \boldsymbol{\Theta} \in \Gamma_o(\mu_0, \mu_1) \right\}.$$

In the following proposition and subsequent corollary, we provide a useful characterization of the pairings $[\cdot, \cdot]_r$ and $[\cdot, \cdot]_l$. Similar results with analogous proofs can be found also in [18, Theorem 4.2] and [14, Corollary 3.18] where X is a smooth compact Riemannian manifold.

Theorem 3.9 *For every* $\Phi_0, \Phi_1 \in \mathcal{P}_2(\mathsf{TX})$ *and* $\mu_1 \in \mathcal{P}_2(\mathsf{X})$ *we have*

$$[\Phi_0, \mu_1]_r = \min \left\{ \int_{\mathsf{TX} \times \mathsf{X}} \langle x_0 - x_1, v_0 \rangle \, \mathrm{d}\boldsymbol{\sigma} \mid \boldsymbol{\sigma} \in \Lambda(\Phi_0, \mu_1) \right\}, \tag{3.16}$$

$$[\Phi_0, \Phi_1]_r = \min \left\{ \int_{\mathsf{TX} \times \mathsf{TX}} \langle x_0 - x_1, v_0 - v_1 \rangle \, \mathrm{d}\boldsymbol{\Theta} \mid \boldsymbol{\Theta} \in \Lambda(\Phi_0, \Phi_1) \right\}. \tag{3.17}$$

We denote by $\Lambda_o(\Phi_0, \mu_1)$ *(resp.* $\Lambda_o(\Phi_0, \Phi_1)$*) the subset of* $\Lambda(\Phi_0, \mu_1)$ *(resp.* $\Lambda(\Phi_0, \Phi_1)$*) where the minimum in* (3.16) *(resp.* (3.17)*) is attained.*

Proof First, we recall that the minima in the right hand side are attained since $\Lambda(\Phi_0, \mu_1)$ and $\Lambda(\Phi_0, \Phi_1)$ are compact subsets of $\mathcal{P}_2(\mathsf{TX} \times \mathsf{X})$ and $\mathcal{P}_2(\mathsf{TX} \times \mathsf{TX})$ respectively by Lemma 2.6 and the integrands are continuous functions with quadratic growth. Thanks to Remark 3.6, we only need to prove the equality (3.17). For every $\boldsymbol{\Theta} \in \Lambda(\Phi_0, \Phi_1)$ and setting $\mu_0 = \mathsf{x}_\sharp \Phi_0$, $\mu_1 = \mathsf{x}_\sharp \Phi_1$, we have

$$\begin{aligned}
&W_2^2(\mathsf{exp}^s_\sharp(\Phi_0), \mathsf{exp}^s_\sharp(\Phi_1)) \\
&\quad \le \int_{\mathsf{TX} \times \mathsf{TX}} |(x_0 - x_1) + s(v_0 - v_1)|^2 \, \mathrm{d}\boldsymbol{\Theta} \\
&\quad = \int_{\mathsf{X}^2} |x_0 - x_1|^2 \, \mathrm{d}(\mathsf{x}^0, \mathsf{x}^1)_\sharp \boldsymbol{\Theta} \\
&\qquad + 2s \int_{\mathsf{TX} \times \mathsf{TX}} \langle x_0 - x_1, v_0 - v_1 \rangle \, \mathrm{d}\boldsymbol{\Theta} + s^2 \int_{\mathsf{X}^2} |v_0 - v_1|^2 \, \mathrm{d}\boldsymbol{\Theta} \\
&\quad = W_2^2(\mu_0, \mu_1) + 2s \int_{\mathsf{TX} \times \mathsf{TX}} \langle x_0 - x_1, v_0 - v_1 \rangle \, \mathrm{d}\boldsymbol{\Theta} + s^2 \int_{\mathsf{X}^2} |v_0 - v_1|^2 \, \mathrm{d}\boldsymbol{\Theta}.
\end{aligned}$$

and this immediately implies

$$[\Phi_0, \Phi_1]_r \leq \min \left\{ \int_{\mathsf{TX}\times\mathsf{TX}} \langle x_0 - x_1, v_0 - v_1 \rangle \, \mathrm{d}\boldsymbol{\Theta} \mid \boldsymbol{\Theta} \in \Lambda(\Phi_0, \Phi_1) \right\}.$$

In order to prove the converse inequality, thanks to Lemma 3.3, for every $s > 0$ we can find $\boldsymbol{\Theta}_s \in \Gamma(\Phi_0, \Phi_1)$ s.t.

$$(\mathsf{exp}^s, \mathsf{exp}^s)_\sharp \boldsymbol{\Theta}_s \in \Gamma_o(\mathsf{exp}^s_\sharp \Phi_0, \mathsf{exp}^s_\sharp \Phi_1).$$

Then

$$\begin{aligned}
\frac{W_2^2(\mathsf{exp}^s_\sharp \Phi_0, \mathsf{exp}^s_\sharp \Phi_1) - W_2^2(\mu_0, \mu_1)}{2s} &\geq \frac{1}{2s} \int_{\mathsf{TX}\times\mathsf{TX}} |(x_0 - x_1) + s(v_0 - v_1)|^2 \, \mathrm{d}\boldsymbol{\Theta}_s \\
&\quad - \frac{1}{2s} \int_{\mathsf{TX}\times\mathsf{TX}} |x_0 - x_1|^2 \, \mathrm{d}\boldsymbol{\Theta}_s \\
&\geq \int_{\mathsf{TX}\times\mathsf{TX}} \langle x_0 - x_1, v_0 - v_1 \rangle \, \mathrm{d}\boldsymbol{\Theta}_s.
\end{aligned} \tag{3.18}$$

Since $\Gamma(\Phi_0, \Phi_1)$ is compact in $\mathcal{P}_2(\mathsf{TX} \times \mathsf{TX})$, there exists a vanishing sequence $k \mapsto s_k$ and $\boldsymbol{\Theta} \in \Gamma(\Phi_0, \Phi_1)$ s.t. $\boldsymbol{\Theta}_{s_k} \to \boldsymbol{\Theta}$ in $\mathcal{P}_2(\mathsf{TX} \times \mathsf{TX})$. Moreover it holds $(\mathsf{exp}^{s_k}, \mathsf{exp}^{s_k})_\sharp \boldsymbol{\Theta}_{s_k} \to (\mathsf{x}^0, \mathsf{x}^1)_\sharp \boldsymbol{\Theta}$ in $\mathcal{P}(\mathsf{TX} \times \mathsf{TX})$ so that $(\mathsf{x}^0, \mathsf{x}^1)_\sharp \boldsymbol{\Theta} \in \Gamma_o(\mu_0, \mu_1)$, and therefore $\boldsymbol{\Theta} \in \Lambda(\Phi_0, \Phi_1)$. The convergence in $\mathcal{P}_2(\mathsf{TX} \times \mathsf{TX})$ yields

$$\lim_k \int_{\mathsf{TX}\times\mathsf{TX}} \langle x_0 - x_1, v_0 - v_1 \rangle \, \mathrm{d}\boldsymbol{\Theta}_{s_k} = \int_{\mathsf{TX}\times\mathsf{TX}} \langle x_0 - x_1, v_0 - v_1 \rangle \, \mathrm{d}\boldsymbol{\Theta},$$

so that, passing to the limit in (3.18) along the sequence s_k, we obtain

$$[\Phi_0, \Phi_1]_r \geq \int_{\mathsf{TX}\times\mathsf{TX}} \langle x_0 - x_1, v_0 - v_1 \rangle \, \mathrm{d}\boldsymbol{\Theta}$$

for some $\boldsymbol{\Theta} \in \Lambda(\Phi_0, \Phi_1)$. □

Corollary 3.10 *Let $\Phi_0, \Phi_1 \in \mathcal{P}_2(\mathsf{TX})$ and $\mu_1 \in \mathcal{P}_2(\mathsf{X})$, then*

$$\begin{aligned}
[\Phi, \mu_1]_l &= \max \left\{ \int_{\mathsf{TX}\times\mathsf{X}} \langle x_0 - x_1, v_0 \rangle \, \mathrm{d}\boldsymbol{\sigma} \mid \boldsymbol{\sigma} \in \Lambda(\Phi_0, \mu_1) \right\}, \\
[\Phi_0, \Phi_1]_l &= \max \left\{ \int_{\mathsf{TX}\times\mathsf{TX}} \langle x_0 - x_1, v_0 - v_1 \rangle \, \mathrm{d}\boldsymbol{\Theta} \mid \boldsymbol{\Theta} \in \Lambda(\Phi_0, \Phi_1) \right\}.
\end{aligned} \tag{3.19}$$

3.2 Right and left derivatives of the Wasserstein distance along a.c. curves

Let us now discuss the differentiability of the map $\mathcal{I} \ni t \mapsto \frac{1}{2} W_2^2(\mu_t, \nu)$ along a locally absolutely continuous curve $\mu : \mathcal{I} \to \mathcal{P}_2(\mathsf{X})$, with $\mathcal{I}$ an open interval of $\mathbb{R}$ and $\nu \in \mathcal{P}_2(\mathsf{X})$.

Theorem 3.11 *Let $\mu : \mathcal{I} \to \mathcal{P}_2(\mathsf{X})$ be a locally absolutely continuous curve and let $\boldsymbol{v} : \mathcal{I} \times \mathsf{X} \to \mathsf{X}$ and $A(\mu)$ be as in Theorem 2.10. Then, for every $\nu \in \mathcal{P}_2(\mathsf{X})$ and every $t \in A(\mu)$, it holds*

$$\lim_{h\downarrow 0} \frac{W_2^2(\mu_{t+h},\nu) - W_2^2(\mu_t,\nu)}{2h} = \left[(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}_t)_\sharp \mu_t, \nu\right]_r,$$
$$\lim_{h\uparrow 0} \frac{W_2^2(\mu_{t+h},\nu) - W_2^2(\mu_t,\nu)}{2h} = \left[(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}_t)_\sharp \mu_t, \nu\right]_l, \tag{3.20}$$

so that the map $s \mapsto W_2^2(\mu_s,\nu)$ is left and right differentiable at every $t \in A(\mu)$. In particular,

1. *if $t \in A(\mu)$ and $\nu \in \mathcal{P}_2(\mathsf{X})$ are s.t. there exists a unique optimal transport plan between μ_t and ν, then the map $s \mapsto W_2^2(\mu_s,\nu)$ is differentiable at t;*
2. *there exists a subset $A(\mu,\nu) \subset A(\mu)$ of full Lebesgue measure such that $s \mapsto W_2^2(\mu_s,\nu)$ is differentiable in $A(\mu,\nu)$ and*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} W_2^2(\mu_t,\nu) = \left[(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}_t)_\sharp \mu_t, \nu\right]_r = \left[(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}_t)_\sharp \mu_t, \nu\right]_l$$
$$= \int \langle \boldsymbol{v}_t(x_1), x_1 - x_2\rangle \,\mathrm{d}\boldsymbol{\mu}(x_1,x_2)$$

for every $\boldsymbol{\mu} \in \Gamma_o(\mu_t,\nu)$, $t \in A(\mu,\nu)$.

Proof Let $\nu \in \mathcal{P}_2(\mathsf{X})$ and for every $t \in \mathcal{I}$ we set $\Phi_t := (\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}_t)_\sharp \mu_t \in \mathcal{P}_2(\mathsf{TX})$. By Theorem 3.9, we have

$$\lim_{h\downarrow 0} \frac{W_2^2(\exp_\sharp^h \Phi_t,\nu) - W_2^2(\mu_t,\nu)}{2h} = \left[(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}_t)_\sharp \mu_t, \nu\right]_r,$$
$$\lim_{h\uparrow 0} \frac{W_2^2(\exp_\sharp^h \Phi_t,\nu) - W_2^2(\mu_t,\nu)}{2h} = \left[(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}_t)_\sharp \mu_t, \nu\right]_l.$$

Since $\exp_\sharp^h \Phi_t = (\boldsymbol{i}_{\mathsf{X}} + h\boldsymbol{v}_t)_\sharp \mu_t$, then thanks to Theorem 2.10 we have that the above limits coincide respectively with the limits in the statement, for all $t \in A(\mu)$.

Claim (1) comes by the characterizations given in Theorem 3.9 and Corollary 3.10. Indeed, if there exists a unique optimal transport plan between μ_t and ν, then $\left[(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}_t)_\sharp \mu_t, \nu\right]_r = \left[(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}_t)_\sharp \mu_t, \nu\right]_l$.

Claim (2) is a simple consequence of the fact that $s \mapsto W_2^2(\mu_s,\nu)$ is differentiable a.e. in $\mathcal{I}$. □

Remark 3.12 In Theorem 3.11 we can actually replace $\boldsymbol{v}$ with any Borel velocity field $\boldsymbol{w}$ solving the continuity equation for μ and s.t. $\|\boldsymbol{w}_t\|_{L^2_{\mu_t}} \in L^1_{loc}(\mathcal{I})$. Indeed, we notice that by [3, Lemma 5.3.2],

$$\Lambda((\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}_t)_\sharp\mu_t, \nu) = \{(\mathsf{x}^0, \boldsymbol{v}_t \circ \mathsf{x}^0, \mathsf{x}^1)_\sharp\boldsymbol{\gamma} \mid \boldsymbol{\gamma} \in \Gamma_o(\mu_t, \nu)\},$$
$$\Lambda((\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{w}_t)_\sharp\mu_t, \nu) = \{(\mathsf{x}^0, \boldsymbol{w}_t \circ \mathsf{x}^0, \mathsf{x}^1)_\sharp\boldsymbol{\gamma} \mid \boldsymbol{\gamma} \in \Gamma_o(\mu_t, \nu)\},$$

so that, by [3, Proposition 8.5.4], we get

$$\left[(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}_t)_\sharp\mu_t, \nu\right]_r = \left[(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{w}_t)_\sharp\mu_t, \nu\right]_r,$$
$$\left[(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}_t)_\sharp\mu_t, \nu\right]_l = \left[(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{w}_t)_\sharp\mu_t, \nu\right]_l.$$

Remark 3.13 In general, if $\mu : \mathcal{I} \to \mathcal{P}_2(\mathsf{X})$ is a locally absolutely continuous curve and $\nu \in \mathcal{P}_2(\mathsf{X})$, then the map $\mathcal{I} \ni s \mapsto W_2^2(\mu_s, \nu)$ is locally absolutely continuous and thus differentiable in a set of full measure $A(\mu, \nu) \subset \mathcal{I}$ which, in principle, depends both on μ and ν. What Theorem 3.11 shows is that, independently of ν, there is a full measure set $A(\mu)$, depending only on μ, where this map is left and right differentiable. If moreover ν and $t \in A(\mu)$ are such that there is a unique optimal transport plan between them, we can actually conclude that such a map is differentiable at t. We refer in particular to Appendix A for a concrete example showing the optimality of the result stated in Theorem 3.11.

Theorem 3.14 *Let $\mu^1, \mu^2 : \mathcal{I} \to \mathcal{P}_2(\mathsf{X})$ be locally absolutely continuous curves and let $\boldsymbol{v}^1, \boldsymbol{v}^2 : \mathcal{I} \times \mathsf{X} \to \mathsf{X}$ be the corresponding Wasserstein velocity fields satisfying (2.6) in $A(\mu^1)$ and $A(\mu^2)$ respectively. Then, for every $t \in A(\mu^1) \cap A(\mu^2)$, it holds*

$$\lim_{h\downarrow 0} \frac{W_2^2(\mu^1_{t+h}, \mu^2_{t+h}) - W_2^2(\mu^1_t, \mu^2_t)}{2h} = \left[(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}^1_t)_\sharp\mu^1_t, (\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}^2_t)_\sharp\mu^2_t\right]_r,$$
$$\lim_{h\uparrow 0} \frac{W_2^2(\mu^1_{t+h}, \mu^2_{t+h}) - W_2^2(\mu^1_t, \mu^2_t)}{2h} = \left[(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}^1_t)_\sharp\mu^1_t, (\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}^2_t)_\sharp\mu^2_t\right]_l.$$

In particular, there exists a subset $A \subset A(\mu^1) \cap A(\mu^2)$ of full Lebesgue measure such that $s \mapsto W_2^2(\mu^1_s, \mu^2_s)$ is differentiable in A and

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} W_2^2(\mu^1_t, \mu^2_t) &= \left[(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}^1_t)_\sharp\mu^1_t, (\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}^2_t)_\sharp\mu^2_t\right]_r \\ &= \left[(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}^1_t)_\sharp\mu^1_t, (\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}^2_t)_\sharp\mu^2_t\right]_l \\ &= \int \langle \boldsymbol{v}^1_t - \boldsymbol{v}^2_t, x_1 - x_2\rangle \,\mathrm{d}\boldsymbol{\mu}(x_1, x_2)\end{aligned} \tag{3.21}$$

for every $\boldsymbol{\mu} \in \Gamma_o(\mu^1_t, \mu^2_t)$, $t \in A$.

The proof of Theorem 3.14 follows by the same argument of the proof of Theorem 3.11.

3.3 Convexity and semicontinuity of duality parings

We want now to investigate the semicontinuity and convexity properties of the functionals $[\cdot,\cdot]_r$ and $[\cdot,\cdot]_l$.

Lemma 3.15 *Let $(\Phi_n)_{n\in\mathbb{N}} \subset \mathcal{P}_2(\mathsf{TX})$ be converging to Φ in $\mathcal{P}_2^{sw}(\mathsf{TX})$, and let $(\nu_n)_{n\in\mathbb{N}} \subset \mathcal{P}_2(\mathsf{X})$ be converging to ν in $\mathcal{P}_2(\mathsf{X})$. Then*

$$\liminf_n [\Phi_n, \nu_n]_r \geq [\Phi, \nu]_r \quad \text{and} \quad \limsup_n [\Phi_n, \nu_n]_l \leq [\Phi, \nu]_l \,. \tag{3.22}$$

Finally, if $(\Phi_n^i)_{n\in\mathbb{N}}$, $i = 0, 1$, are sequences converging to Φ^i in $\mathcal{P}_2^{sw}(\mathsf{TX})$ then

$$\liminf_{n\to\infty} \left[\Phi_n^0, \Phi_n^1\right]_r \geq \left[\Phi^0, \Phi^1\right]_r , \qquad \limsup_{n\to\infty} \left[\Phi_n^0, \Phi_n^1\right]_l \geq \left[\Phi^0, \Phi^1\right]_l . \tag{3.23}$$

Proof We just consider the proof of the first inequality (3.22); the other statements follow by similar arguments and by Remark 3.6.

We can extract a subsequence of $(\Phi_n)_{n\in\mathbb{N}}$ (not relabeled) s.t. the lim inf is achieved as a limit. We have to prove that

$$\lim_n [\Phi_n, \nu_n]_r \geq [\Phi, \nu]_r \,. \tag{3.24}$$

For every $n \in \mathbb{N}$ take $\boldsymbol{\sigma}_n \in \Lambda_o(\Phi_n, \nu_n)$ and $\bar{\boldsymbol{\vartheta}}_n = (\mathsf{x}^0, \mathsf{x}^1)_\sharp \boldsymbol{\sigma}_n$. Since the marginals of $\bar{\boldsymbol{\vartheta}}_n$ are converging w.r.t. W_2, the family $(\bar{\boldsymbol{\vartheta}}_n)_{n\in\mathbb{N}}$ is relatively compact in $\mathcal{P}_2(\mathsf{X}^2)$. Hence, $(\boldsymbol{\sigma}_n)_{n\in\mathbb{N}}$ is relatively compact in $\mathcal{P}_2^{sws}(\mathsf{TX} \times \mathsf{X})$ by Proposition 2.15, since the moments $\int |v_0|^2 \,\mathrm{d}\boldsymbol{\sigma}_n(x_0, v_0, x_1) = |\Phi_n|_2^2$ are uniformly bounded by assumption. Thus, possibly passing to a further subsequence, we have that $(\boldsymbol{\sigma}_n)_{n\in\mathbb{N}}$ converges to some $\boldsymbol{\sigma}$ in $\mathcal{P}_2^{sws}(\mathsf{TX}\times\mathsf{X})$. In particular $\boldsymbol{\sigma} \in \Lambda(\Phi, \nu)$ since optimality of the X^2 marginals is preserved by narrow convergence. Indeed, it sufficies to use [3, Proposition 7.1.3] noting that

$$\int |x_0 - x_1|^2 \,\mathrm{d}\boldsymbol{\sigma}_n \leq 2\,\mathsf{m}_2^2(\mathsf{x}_\sharp \Phi_n) + 2\,\mathsf{m}_2^2(\nu_n) \leq K,$$

for some $K \geq 0$.

The relation in (2.9) then yields

$$\lim_{n\to\infty} [\Phi_n, \nu_n]_r = \lim_{n\to\infty} \int \langle v_0, x_0 - x_1\rangle \,\mathrm{d}\boldsymbol{\sigma}_n = \int \langle v_0, x_0 - x_1\rangle \,\mathrm{d}\boldsymbol{\sigma}$$

which yields (3.24) since the r.h.s. is larger than $[\Phi, \nu]_r$ by Theorem 3.9. □

Remark 3.16 Notice that in the special case in which $\Lambda(\Phi, \nu)$ is a singleton, then the limit exists and it holds

$$\lim_{n\to\infty} [\Phi_n, \nu_n]_r = [\Phi, \nu]_r \,, \qquad \lim_{n\to\infty} [\Phi_n, \nu_n]_l = [\Phi, \nu]_l \,.$$

Lemma 3.17 *For every $\mu, \nu \in \mathcal{P}_2(\mathsf{X})$ the maps $\Phi \mapsto [\Phi, \nu]_r$ and $(\Phi, \Psi) \mapsto [\Phi, \Psi]_r$ (resp. $\Phi \mapsto [\Phi, \nu]_l$ and $(\Phi, \Psi) \mapsto [\Phi, \Psi]_l$) are convex (resp. concave) in $\mathcal{P}_2(\mathsf{TX}|\mu)$ and $\mathcal{P}_2(\mathsf{TX}|\mu) \times \mathcal{P}_2(\mathsf{TX}|\nu)$.*

Proof We prove the convexity of $(\Phi,\Psi)\mapsto[\Phi,\Psi]_r$ in $\mathcal{P}_2(\mathsf{TX}|\mu)\times\mathcal{P}_2(\mathsf{TX}|\nu)$; the argument of the proofs of the other statements are completely analogous.

Let $\Phi_k\in\mathcal{P}_2(\mathsf{TX}|\mu)$, $\Psi_k\in\mathcal{P}_2(\mathsf{TX}|\nu)$, and let $\beta_k\geq 0$, with $\sum_k\beta_k=1$, $k=1,\ldots,K$. We set $\Phi=\sum_{k=1}^K\beta_k\Phi_k$, $\Psi=\sum_{k=1}^K\beta_k\Psi_k$, For every k let us select $\boldsymbol{\Theta}_k\in\Lambda(\Phi_k,\Psi_k)$ such that

$$[\Phi_k,\Psi_k]_r=\int\langle v_1-v_0,x_1-x_0\rangle\,\mathrm{d}\boldsymbol{\Theta}_k.$$

It is not difficult to check that $\boldsymbol{\Theta}:=\sum_k\beta_k\boldsymbol{\Theta}_k\in\Lambda(\Phi,\Psi)$ so that

$$\begin{aligned}[\Phi,\Psi]_r&\leq\int\langle v_1-v_0,x_1-x_0\rangle\,\mathrm{d}\boldsymbol{\Theta}\\&=\sum_k\beta_k\int\langle v_1-v_0,x_1-x_0\rangle\,\mathrm{d}\boldsymbol{\Theta}_k=\sum_k\beta_k\,[\Phi_k,\Psi_k]_r\,.\end{aligned}$$

□

3.4 Behaviour of duality pairings along geodesics

We have seen that the duality pairings $[\cdot,\cdot]_r$ and $[\cdot,\cdot]_l$ may differ when the collection of optimal plans $\Gamma_o(\mu_0,\mu_1)$ contains more than one element. It is natural to expect a simpler behaviour along geodesics. We will introduce the following definition, where we use the notation

$$\mathsf{x}^t(x_0,x_1):=(1-t)x_0+tx_1,\quad \mathsf{v}^0(x_0,v_0,x_1):=v_0$$

for every $(x_0,v_0,x_1)\in\mathsf{TX}\times\mathsf{X}$, $t\in[0,1]$.

Definition 3.18 For $\boldsymbol{\vartheta}\in\mathcal{P}_2(\mathsf{X}\times\mathsf{X})$, $t\in[0,1]$, $\vartheta_t=\mathsf{x}^t_\sharp\boldsymbol{\vartheta}$ and $\Phi_t\in\mathcal{P}_2(\mathsf{TX}|\vartheta_t)$, we set

$$\Gamma_t(\Phi_t,\boldsymbol{\vartheta}):=\left\{\boldsymbol{\sigma}\in\mathcal{P}_2(\mathsf{TX}\times\mathsf{X})\mid(\mathsf{x}^0,\mathsf{x}^1)_\sharp\boldsymbol{\sigma}=\boldsymbol{\vartheta}\ \text{ and }\ (\mathsf{x}^t\circ(\mathsf{x}^0,\mathsf{x}^1),\mathsf{v}^0)_\sharp\boldsymbol{\sigma}=\Phi_t\right\},\tag{3.25}$$

which is not empty since $\vartheta_t=\mathsf{x}^t_\sharp\boldsymbol{\vartheta}=\mathsf{x}_\sharp\Phi_t$. We set

$$\begin{aligned}[\Phi_t,\boldsymbol{\vartheta}]_{b,t}&:=\int\Big\langle x_0-x_1,\boldsymbol{b}_{\Phi_t}(\mathsf{x}^t(x_0,x_1))\Big\rangle\,\mathrm{d}\boldsymbol{\vartheta}(x_0,x_1),\\ [\Phi_t,\boldsymbol{\vartheta}]_{r,t}&:=\min\left\{\int\langle x_0-x_1,v_0\rangle\,\mathrm{d}\boldsymbol{\sigma}(x_0,v_0,x_1)\mid\boldsymbol{\sigma}\in\Gamma_t(\Phi_t,\boldsymbol{\vartheta})\right\},\\ [\Phi_t,\boldsymbol{\vartheta}]_{l,t}&:=\max\left\{\int\langle x_0-x_1,v_0\rangle\,\mathrm{d}\boldsymbol{\sigma}(x_0,v_0,x_1)\mid\boldsymbol{\sigma}\in\Gamma_t(\Phi_t,\boldsymbol{\vartheta})\right\}.\end{aligned}$$

If moreover $\Phi_0 \in \mathcal{P}_2(\mathsf{TX}|\vartheta_0)$, $\Phi_1 \in \mathcal{P}_2(\mathsf{TX}|\vartheta_1)$, $\boldsymbol{\vartheta} \in \Gamma(\vartheta_0, \vartheta_1)$, we define

$$[\Phi_0, \Phi_1]_{r,\boldsymbol{\vartheta}} := [\Phi_0, \boldsymbol{\vartheta}]_{r,0} - [\Phi_1, \boldsymbol{\vartheta}]_{l,1},$$
$$[\Phi_0, \Phi_1]_{l,\boldsymbol{\vartheta}} := [\Phi_0, \boldsymbol{\vartheta}]_{l,0} - [\Phi_1, \boldsymbol{\vartheta}]_{r,1}.$$

Notice that, if $(\Phi_t)_x$ is the disintegration of Φ_t with respect to $\vartheta_t = \mathsf{x}_\sharp \Phi_t$, we can consider the barycentric coupling $\boldsymbol{\sigma}_t := \int_{\mathsf{X}\times\mathsf{X}} (\Phi_t)_{\mathsf{x}^t} \, \mathrm{d}\boldsymbol{\vartheta} \in \Gamma_t(\Phi_t, \boldsymbol{\vartheta})$, i.e.

$$\int \psi(x_0, v_0, x_1) \, \mathrm{d}\boldsymbol{\sigma}_t = \int \Big[\int \psi(x_0, v_0, x_1) \, \mathrm{d}(\Phi_t)_{(1-t)x_0+tx_1}(v_0) \Big] \, \mathrm{d}\boldsymbol{\vartheta}(x_0, x_1)$$

so that $[\Phi_t, \boldsymbol{\vartheta}]_{b,t} = \int \langle v_0, x_0 - x_1 \rangle \, \mathrm{d}\boldsymbol{\sigma}_t$ and

$$[\Phi_t, \boldsymbol{\vartheta}]_{r,t} \le [\Phi_t, \boldsymbol{\vartheta}]_{b,t} \le [\Phi_t, \boldsymbol{\vartheta}]_{l,t}.$$

If we define the reversion map

$$\mathsf{s} : \mathsf{X}^2 \to \mathsf{X}^2, \quad \mathsf{s}(x_0, x_1) := (x_1, x_0), \tag{3.26}$$

with a similar definition for $\mathsf{TX} \times \mathsf{X}$, given by $\mathsf{s}(x_0, v_0, x_1) := (x_1, v_0, x_0)$, it is easy to check that

$$\boldsymbol{\sigma} \in \Gamma_t(\Phi_t, \boldsymbol{\vartheta}) \quad \Leftrightarrow \quad \mathsf{s}_\sharp \boldsymbol{\sigma} \in \Gamma_{1-t}(\Phi_t, \mathsf{s}_\sharp \boldsymbol{\vartheta})$$

so that

$$[\Phi_t, \boldsymbol{\vartheta}]_{r,t} = -[\Phi_t, \mathsf{s}_\sharp \boldsymbol{\vartheta}]_{l,1-t}, \quad [\Phi_t, \boldsymbol{\vartheta}]_{l,t} = -[\Phi_t, \mathsf{s}_\sharp \boldsymbol{\vartheta}]_{r,1-t}. \tag{3.27}$$

We point out that (3.16) and (3.19) have simpler versions in two particular cases, which will be explained in the next remark.

Remark 3.19 (Particular cases) Suppose that $\boldsymbol{\vartheta} \in \mathcal{P}_2(\mathsf{X}^2)$, $t \in [0, 1]$, $\vartheta_t = \mathsf{x}^t_\sharp \boldsymbol{\vartheta}$, $\Phi_t \in \mathcal{P}_2(\mathsf{TX}|\vartheta_t)$ and $\mathsf{x}^t : \mathsf{X}^2 \to \mathsf{X}$ is $\boldsymbol{\vartheta}$-essentially injective so that $\boldsymbol{\vartheta}$ is concentrated on a Borel map

$$(X_t^0, X_t^1) : \mathsf{X} \to \mathsf{X} \times \mathsf{X}, \text{ i.e. } \boldsymbol{\vartheta} = (X_t^0, X_t^1)_\sharp \vartheta_t.$$

In this case $\Gamma_t(\Phi_t, \boldsymbol{\vartheta})$ contains a unique element given by $(X_t^0 \circ \mathsf{x}, \mathsf{v}, X_t^1 \circ \mathsf{x})_\sharp \Phi_t$ and

$$[\Phi_t, \boldsymbol{\vartheta}]_{r,t} = [\Phi_t, \boldsymbol{\vartheta}]_{l,t} = [\Phi_t, \boldsymbol{\vartheta}]_{b,t} = \int \langle v, X_t^0(x) - X_t^1(x) \rangle \, \mathrm{d}\Phi_t(x, v)$$
$$= \int \langle \boldsymbol{b}_{\Phi_t}, X_t^0 - X_t^1 \rangle \, \mathrm{d}\vartheta_t, \tag{3.28}$$

where in the last formula we have applied the barycentric reduction (3.8). When $t = 0$ and $\boldsymbol{\vartheta}$ is the unique element of $\Gamma_o(\vartheta_0, \vartheta_1)$ then $X_t^0(x) = x$ and we obtain

$$\begin{aligned}[\Phi_t, \vartheta_1]_r &= [\Phi_t, \vartheta_1]_l = [\Phi_t, \boldsymbol{\vartheta}]_{r,0} = [\Phi_t, \boldsymbol{\vartheta}]_{l,0} \\ &= \int \langle v, x - X_t^1(x)\rangle \,\mathrm{d}\Phi_t(x, v) = \int \langle \boldsymbol{b}_{\Phi_t}, x - X_t^1(x)\rangle \,\mathrm{d}\vartheta_0(x).\end{aligned}$$

Another simple case is when

$$\Phi_t = (\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{w})_\sharp \vartheta_t$$

for some vector field $\boldsymbol{w} \in L^2_{\vartheta_t}(\mathsf{X}; \mathsf{X})$ as in (3.5) (i.e. its disintegration Φ_x w.r.t. ϑ_t takes the form $\delta_{\boldsymbol{w}(x)}$ and $\boldsymbol{w} = \boldsymbol{b}_{\Phi_t}$). We have

$$[\Phi_t, \boldsymbol{\vartheta}]_{r,t} = [\Phi_t, \boldsymbol{\vartheta}]_{l,t} = \int \Big\langle \boldsymbol{w}((1-t)x_0 + tx_1), x_0 - x_1 \Big\rangle \,\mathrm{d}\boldsymbol{\vartheta}(x_0, x_1).$$

In particular we get

$$[\Phi_t, \vartheta_1]_r = \min \left\{ \int \langle \boldsymbol{w}(x), x_0 - x_1\rangle \,\mathrm{d}\boldsymbol{\vartheta}(x_0, x_1) \mid \boldsymbol{\vartheta} \in \Gamma_o(\vartheta_0, \vartheta_1) \right\}.$$

An important case in which the previous Remark 3.19 applies is that of geodesics in $\mathcal{P}_2(\mathsf{X})$.

Lemma 3.20 *Let $\mu_0, \mu_1 \in \mathcal{P}_2(\mathsf{X})$, $\mu : [0, 1] \to \mathcal{P}_2(\mathsf{X})$ be a constant speed geodesic induced by an optimal plan $\boldsymbol{\mu} \in \Gamma_o(\mu_0, \mu_1)$ by the relation*

$$\mu_t = \mathsf{x}^t_\sharp \boldsymbol{\mu}, \quad t \in [0, 1], \quad \textit{where} \quad \mathsf{x}^t(x_0, x_1) = (1-t)x_0 + tx_1.$$

If $t \in (0, 1)$, $\Phi_t \in \mathcal{P}_2(\mathsf{TX}|\mu_t)$, $\hat{\boldsymbol{\mu}} = \mathsf{s}_\sharp \boldsymbol{\mu} \in \Gamma_o(\mu_1, \mu_0)$, with s the reversion map in (3.26), then

$$\begin{aligned}\frac{1}{1-t}[\Phi_t, \mu_1]_r &= \frac{1}{1-t}[\Phi_t, \mu_1]_l \\ &= [\Phi_t, \boldsymbol{\mu}]_{r,t} \\ &= [\Phi_t, \boldsymbol{\mu}]_{l,t} \\ &= -\frac{1}{t}[\Phi_t, \mu_0]_r \\ &= -\frac{1}{t}[\Phi_t, \mu_0]_l \\ &= -[\Phi_t, \hat{\boldsymbol{\mu}}]_{r,1-t} \\ &= -[\Phi_t, \hat{\boldsymbol{\mu}}]_{l,1-t}.\end{aligned} \tag{3.29}$$

Proof The crucial fact is that $\mathsf{x}^t : \mathsf{X}^2 \to \mathsf{X}$ is injective on $\mathrm{supp}(\boldsymbol{\mu})$ and thus a bijection on its image $\mathrm{supp}(\mu_t)$. Indeed, take $(x_0, x_1), (x_0', x_1') \in \mathrm{supp}(\boldsymbol{\mu})$, then

$$\begin{aligned}
\left|\mathsf{x}^t(x_0, x_1) - \mathsf{x}^t(x_0', x_1')\right|^2 &= (1-t)^2|x_0 - x_0'|^2 + t^2|x_1 - x_1'|^2 \\
&\quad + 2t(1-t)\langle x_0 - x_0', x_1 - x_1'\rangle \\
&\geq (1-t)^2|x_0 - x_0'|^2 + t^2|x_1 - x_1'|^2
\end{aligned}$$

thanks to the cyclical monotonicity of $\mathrm{supp}(\boldsymbol{\mu})$ (see [3, Remark 7.1.2]).

Then, for every $x \in \mathrm{supp}(\mu_t)$, there exists a unique couple $(x_0, x_1) = (X_t^0(x), X_t^1(x)) \in \mathrm{supp}(\boldsymbol{\mu})$ s.t. $x = (1-t)x_0 + tx_1$, where we refer to Remark 3.19 for the definitions of X_t^0, X_t^1 (cf. also [32, Theorem 5.29]). Hence, in the following diagram all maps are bijections:

$$\begin{tikzcd}
\mathrm{supp}(\boldsymbol{\mu}_{t0}) & \mathrm{supp}(\boldsymbol{\mu}) \arrow[l, "(\mathsf{x}^t,\mathsf{x}^0)"'] \arrow[r, "(\mathsf{x}^t,\mathsf{x}^1)"] \arrow[d, "\mathsf{x}^t"] & \mathrm{supp}(\boldsymbol{\mu}_{t1}) \\
& \mathrm{supp}(\mu_t) \arrow[ul, "(\boldsymbol{i}_{\mathsf{X}}, X_t^0)"] \arrow[ur, "(\boldsymbol{i}_{\mathsf{X}}, X_t^1)"'] &
\end{tikzcd}$$

where $\boldsymbol{\mu}_{t1} = (\mathsf{x}^t, \mathsf{x}^1)_\sharp\boldsymbol{\mu} = (\boldsymbol{i}_{\mathsf{X}}, X_t^1)_\sharp\mu_t$ is the unique element of $\Gamma_o(\mu_t, \mu_1)$ and $\boldsymbol{\mu}_{t0} = (\mathsf{x}^t, \mathsf{x}^0)_\sharp\boldsymbol{\mu} = (\boldsymbol{i}_{\mathsf{X}}, X_t^0)_\sharp\mu_t = (\mathsf{x}^{1-t}, \mathsf{x}^1)_\sharp\hat{\boldsymbol{\mu}}$ is the unique element of $\Gamma_o(\mu_t, \mu_0)$ (see Theorem 2.8). Since

$$\frac{x - X_t^1(x)}{1-t} = \frac{x - x_1}{1-t} = x_0 - x_1 = -\frac{x - x_0}{t} = -\frac{x - X_t^0(x)}{t},$$

and $\Lambda(\Phi_t, \mu_1) = \{(\boldsymbol{i}_{\mathsf{TX}}, X_t^1 \circ \mathsf{x})_\sharp\Phi_t\}$ thanks to Theorem 2.8, by Theorem 3.9 and Corollary 3.10 we have

$$[\Phi_t, \mu_1]_r = [\Phi_t, \mu_1]_l = \int_{\mathsf{TX}} \langle v, x - X_t^1(x)\rangle \,\mathrm{d}\Phi_t(x, v).$$

Analogously, $\Lambda(\Phi_t, \mu_0) = \{(\boldsymbol{i}_{\mathsf{TX}}, X_t^0 \circ \mathsf{x})_\sharp\Phi_t\}$. Hence

$$[\Phi_t, \mu_0]_r = [\Phi_t, \mu_0]_l = \int_{\mathsf{TX}} \langle v, x - X_t^0(x)\rangle \,\mathrm{d}\Phi_t(x, v).$$

Also recalling (3.27) and (3.28) we conclude. □

4 Dissipative probability vector fields: the metric viewpoint

4.1 Multivalued probability vector fields and $\boldsymbol{\lambda}$-dissipativity

Definition 4.1 (Multivalued Probability Vector Field - MPVF) A *multivalued probability vector field* $\mathbf{F}$ is a nonempty subset of $\mathcal{P}_2(\mathsf{TX})$ with domain $\mathrm{D}(\mathbf{F}) := \mathsf{x}_\sharp(\mathbf{F}) =$

$\{\mathsf{x}_\sharp\Phi : \Phi \in \mathbf{F}\}$. Given $\mu \in \mathcal{P}_2(\mathsf{X})$, we define the *section* $\mathbf{F}[\mu]$ of $\mathbf{F}$ as

$$\mathbf{F}[\mu] := (\mathsf{x}_\sharp)^{-1}(\mu) \cap \mathbf{F} = \left\{\Phi \in \mathbf{F} \mid \mathsf{x}_\sharp\Phi = \mu\right\}.$$

A *selection* $\mathbf{F}'$ of $\mathbf{F}$ is a subset of $\mathbf{F}$ such that $\mathrm{D}(\mathbf{F}') = \mathrm{D}(\mathbf{F})$. We call $\mathbf{F}$ a *probability vector field* (PVF) if $\mathsf{x}_\sharp$ is injective in $\mathbf{F}$, i.e. $\mathbf{F}[\mu]$ contains a unique element for every $\mu \in \mathrm{D}(\mathbf{F})$. A MPVF $\mathbf{F}$ is a vector field if for every $\mu \in \mathrm{D}(\mathbf{F})$, the section $\mathbf{F}[\mu]$ contains a unique element Φ concentrated on a map, i.e. $\Phi = (\boldsymbol{i}_\mathsf{X}, \boldsymbol{b}_\Phi)_\sharp\mu$.

Remark 4.2 We can equivalently formulate Definition 4.1 by considering $\mathbf{F}$ as a multifunction, as in the case, e.g., of the Wasserstein subdifferential $\boldsymbol{\partial}\mathcal{F}$ of a function $\mathcal{F} : \mathcal{P}_2(\mathsf{X}) \to (-\infty, +\infty]$, see [3, Ch. 10] and the next Sect. 7.1. According to this viewpoint, a MPVF is a set-valued map $\mathbf{F} : \mathcal{P}_2(\mathsf{X}) \supset \mathrm{D}(\mathbf{F}) \rightrightarrows \mathcal{P}_2(\mathsf{TX})$ such that $\mathsf{x}_\sharp\Phi = \mu$ for all $\Phi \in \mathbf{F}[\mu]$. In this way, each section $\mathbf{F}[\mu]$ is nothing but the image of $\mu \in \mathrm{D}(\mathbf{F})$ through $\mathbf{F}$. In this case, *probability vector fields* correspond to single valued maps: this notion has been used in [27] with the aim of describing a sort of velocity field on $\mathcal{P}(\mathsf{X})$, and later in [26] dealing with Multivalued Probability Vector Fields (called Probability Multifunctions).

Definition 4.3 (Metrically λ-dissipative MPVF) A MPVF $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$ is (metrically) λ-*dissipative*, with $\lambda \in \mathbb{R}$, if

$$[\Phi_0, \Phi_1]_r \le \lambda W_2^2(\mu_0, \mu_1) \quad \text{for every } \Phi_0, \Phi_1 \in \mathbf{F},\ \mu_i = \mathsf{x}_\sharp\Phi_i. \tag{4.1}$$

We say that $\mathbf{F}$ is (metrically) λ-*accretive* if $-\mathbf{F} = \{-\Phi : \Phi \in \mathbf{F}\}$ (recall (3.15)) is $-\lambda$-dissipative, i.e.

$$[\Phi_0, \Phi_1]_l \ge \lambda W_2^2(\mu_0, \mu_1) \quad \text{for every } \Phi_0, \Phi_1 \in \mathbf{F},\ \mu_i = \mathsf{x}_\sharp\Phi_i.$$

In Sect. 7 we collect explicit examples of λ-dissipative MPVFs.

Remark 4.4 Notice that (4.1) is equivalent to asking for the existence of a coupling $\boldsymbol{\Theta} \in \Lambda(\Phi_0, \Phi_1)$ (thus $(\mathsf{x}^0, \mathsf{x}^1)_\sharp\boldsymbol{\Theta}$ is optimal between $\mu_0 = \mathsf{x}_\sharp\Phi_0$ and $\mu_1 = \mathsf{x}_\sharp\Phi_1$) such that

$$\int \langle v_1 - v_0, x_1 - x_0\rangle \,\mathrm{d}\boldsymbol{\Theta} \le \lambda W_2^2(\mu_0, \mu_1) = \lambda \int |x_1 - x_0|^2 \,\mathrm{d}\boldsymbol{\Theta}.$$

As anticipated in the Introduction, dealing with (1.6) and (1.8), the λ-dissipativity condition (4.1) has a natural metric interpretation: if $\Phi_0, \Phi_1 \in \mathbf{F}$ with $\mu_0 = \mathsf{x}_\sharp\Phi_0$, $\mu_1 = \mathsf{x}_\sharp\Phi_1$, performing a first order Taylor expansion of the map

$$t \mapsto \frac{1}{2} W_2^2(\mathsf{exp}^t \Phi_0, \mathsf{exp}^t \Phi_1)$$

at $t = 0$, recalling Definition 3.5, we have

$$W_2^2(\mathsf{exp}^t \Phi_0, \mathsf{exp}^t \Phi_1) \le (1 + 2\lambda t) W_2^2(\mu_0, \mu_1) + o(t) \quad \text{as } t \downarrow 0.$$

Remark 4.5 Thanks to Corollary 3.7, (4.1) implies the weaker condition

$$[\Phi_0, \mu_1]_r + [\Phi_1, \mu_0]_r \le \lambda W_2^2(\mu_0, \mu_1) \quad \text{for every } \Phi_0, \Phi_1 \in \mathbf{F},\ \mu_i = \mathsf{x}_\sharp \Phi_i. \tag{4.2}$$

It is clear that the inequality of (4.2) implies the inequality of (4.1) whenever $\Gamma_o(\mu_0, \mu_1)$ contains only one element. More generally, we will see in Corollary 4.13 that (4.2) is in fact equivalent to (4.1) when $\mathrm{D}(\mathbf{F})$ is geodesically convex (according to Definition 2.7).

As in the standard Hilbert case, λ-dissipativity can be reduced to dissipativity (meaning 0-dissipativity) by a simple transformation as shown in Lemma 4.6. Let us introduce the map

$$L^\lambda : \mathsf{TX} \to \mathsf{TX}, \quad L^\lambda(x, v) := (x, v - \lambda x).$$

Lemma 4.6 *$\mathbf{F}$ is a λ-dissipative* MPVF *(resp. satisfies (4.2)) if and only if $\mathbf{F}^\lambda := L^\lambda_\sharp(\mathbf{F}) = \{L^\lambda_\sharp \Phi \mid \Phi \in \mathbf{F}\}$ is dissipative (resp. satisfies (4.2) with $\lambda = 0$).*

Proof Let us first check the case of (4.2). If $\boldsymbol{\sigma} \in \mathcal{P}_2(\mathsf{TX} \times \mathsf{X})$ with $(\mathsf{x}^i)_\sharp \boldsymbol{\sigma} = \mu_i, i = 0, 1$, the transformed plan $\boldsymbol{\sigma}^\lambda := (L^\lambda, \boldsymbol{i}_\mathsf{X})_\sharp \boldsymbol{\sigma}$ satisfies

$$\begin{aligned}\int \langle v_0, x_0 - x_1\rangle \,\mathrm{d}\boldsymbol{\sigma}^\lambda &= \int \langle v_0 - \lambda x_0, x_0 - x_1\rangle \,\mathrm{d}\boldsymbol{\sigma} \\ &= \int \langle v_0, x_0 - x_1\rangle \,\mathrm{d}\boldsymbol{\sigma} - \frac{\lambda}{2}\int |x_0 - x_1|^2 \,\mathrm{d}\boldsymbol{\sigma} \\ &\quad + \frac{\lambda}{2}\Big(\mathsf{m}_2^2(\mu_1) - \mathsf{m}_2^2(\mu_0)\Big).\end{aligned} \tag{4.3}$$

Since $\boldsymbol{\sigma} \in \Lambda_o(\Phi_0, \mu_1)$ if and only if $\boldsymbol{\sigma}^\lambda \in \Lambda_o(L^\lambda_\sharp \Phi_0, \mu_1)$, (4.3) yields

$$\int \langle v_0, x_0 - x_1\rangle \,\mathrm{d}\boldsymbol{\sigma}^\lambda = \int \langle v_0, x_0 - x_1\rangle \,\mathrm{d}\boldsymbol{\sigma} - \frac{\lambda}{2}\Big(\mathsf{m}_2^2(\mu_0) - \mathsf{m}_2^2(\mu_1) + W_2^2(\mu_0, \mu_1)\Big)$$

and therefore

$$\Big[L^\lambda_\sharp \Phi_0, \mu_1\Big]_r = [\Phi_0, \mu_1]_r - \frac{\lambda}{2}\Big(\mathsf{m}_2^2(\mu_0) - \mathsf{m}_2^2(\mu_1) + W_2^2(\mu_0, \mu_1)\Big). \tag{4.4}$$

Using the corresponding identity for $\Big[L^\lambda_\sharp \Phi_1, \mu_0\Big]_r$ we obtain that $\mathbf{F}^\lambda$ is dissipative.

Similarly, if $\boldsymbol{\Theta} \in \mathcal{P}_2(\mathsf{TX} \times \mathsf{TX})$ with $\mathsf{x}^i_\sharp \boldsymbol{\Theta} = \mu_i$, the plan $\boldsymbol{\Theta}^\lambda := (L^\lambda, L^\lambda)_\sharp \boldsymbol{\Theta}$ satisfies

$$\begin{aligned}\int \langle v_0 - v_1, x_0 - x_1\rangle \,\mathrm{d}\boldsymbol{\Theta}^\lambda &= \int \langle v_0 - v_1 - \lambda(x_0 - x_1), x_0 - x_1\rangle \,\mathrm{d}\boldsymbol{\Theta} \\ &= \int \langle v_0 - v_1, x_0 - x_1\rangle \,\mathrm{d}\boldsymbol{\Theta} - \lambda \int |x_0 - x_1|^2 \,\mathrm{d}\boldsymbol{\Theta}.\end{aligned} \tag{4.5}$$

Reasoning with a similar argument as for the case of assumption (4.2), using the identity (4.5), we get the equivalence between the λ-dissipativity of $\mathbf{F}$ and the dissipativity of $\mathbf{F}^{\lambda}$. □

Let us conclude this section by showing that λ-dissipativity can be deduced from a Lipschitz like condition similar to the one considered in [27] (see Sect. 7.5).

Lemma 4.7 *Suppose that the* MPVF $\mathbf{F}$ *satisfies*

$$\mathcal{W}_2(\mathbf{F}[\nu], \mathbf{F}[\nu']) \le L W_2(\nu, \nu') \quad \textit{for every } \nu, \nu' \in \mathrm{D}(\mathbf{F}),$$

where $\mathcal{W}_2 : \mathcal{P}_2(\mathsf{TX}) \times \mathcal{P}_2(\mathsf{TX}) \to [0, +\infty)$ *is defined by*

$$\mathcal{W}_2^2(\Phi_0, \Phi_1) = \inf \left\{ \int_{\mathsf{TX}\times\mathsf{TX}} |v_0 - v_1|^2 \, \mathrm{d}\boldsymbol{\Theta}(x_0, v_0, x_1, v_1) : \boldsymbol{\Theta} \in \Lambda(\Phi_0, \Phi_1) \right\},$$

with $\Lambda(\cdot, \cdot)$ *as in Definition* 3.8*. Then* $\mathbf{F}$ *is λ-dissipative according to* (4.1)*, for* $\lambda := \frac{1}{2}(1 + L^2)$

Proof Let $\nu', \nu'' \in \mathrm{D}(\mathbf{F})$, then by Theorem 3.9 and Young's inequality, we have

$$\begin{aligned}
\left[\mathbf{F}[\nu'], \mathbf{F}[\nu'']\right]_r &= \min \left\{ \int_{\mathsf{TX}\times\mathsf{TX}} \langle x' - x'', v' - v'' \rangle \, \mathrm{d}\boldsymbol{\Theta} \ : \ \boldsymbol{\Theta} \in \Lambda(\mathbf{F}[\nu'], \mathbf{F}[\nu'']) \right\} \\
&\le \frac{1}{2} \left(W_2^2(\nu', \nu'') + \mathcal{W}_2^2(\mathbf{F}[\nu'], \mathbf{F}[\nu'']) \right) \\
&\le \frac{L^2 + 1}{2} \, W_2^2(\nu', \nu'').
\end{aligned}$$

□

4.2 Behaviour of λ-dissipative MPVF along geodesics

Let us now study the behaviour of a MPVF $\mathbf{F}$ along geodesics. Recall that in the case of a dissipative map $\mathrm{F} : \mathsf{H} \to \mathsf{H}$ in a Hilbert space H, it is quite immediate to prove that the real function

$$f(t) := \langle F(x_t), x_0 - x_1 \rangle, \quad x_t = (1 - t)x_0 + t x_1, \quad t \in [0, 1] \tag{4.6}$$

is monotone increasing. This property has a natural counterpart in the case of measures.

Let $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$, $\mu_0, \mu_1 \in \overline{\mathrm{D}(\mathbf{F})}$, $\boldsymbol{\mu} \in \Gamma_o(\mu_0, \mu_1)$. In order to compute the measure-theoretic analogue of the scalar product in (4.6), we need to define the set

$$\mathrm{I}(\boldsymbol{\mu}|\mathbf{F}) := \left\{ t \in [0, 1] : \mathsf{x}^t_\sharp \boldsymbol{\mu} \in \mathrm{D}(\mathbf{F}) \right\}, \tag{4.7}$$

since we can evaluate the MPVF $\mathbf{F}$ along geodesics only for time instants $t \in [0, 1]$ at which they lie inside the domain.

Definition 4.8 Let $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$ be a MPVF. Let $\mu_0, \mu_1 \in \overline{\mathrm{D}(\mathbf{F})}$, $\boldsymbol{\mu} \in \Gamma_o(\mu_0, \mu_1)$ and let $\mu_t := \mathsf{x}^t_\sharp \boldsymbol{\mu}$, $t \in [0, 1]$. For every $t \in \mathrm{I}(\boldsymbol{\mu}|\mathbf{F})$ we define

$$[\mathbf{F}, \boldsymbol{\mu}]_{r,t} := \sup \Big\{ [\Phi_t, \boldsymbol{\mu}]_{r,t} \mid \Phi_t \in \mathbf{F}[\mu_t] \Big\}, \quad [\mathbf{F}, \boldsymbol{\mu}]_{l,t} := \inf \Big\{ [\Phi_t, \boldsymbol{\mu}]_{l,t} \mid \Phi_t \in \mathbf{F}[\mu_t] \Big\}.$$

Theorem 4.9 *Let us suppose that the* MPVF $\mathbf{F}$ *satisfies* (4.2), *let* $\mu_0, \mu_1 \in \overline{\mathrm{D}(\mathbf{F})}$, *and let* $\boldsymbol{\mu} \in \Gamma_o(\mu_0, \mu_1)$. *Then the following properties hold*

(1) $[\mathbf{F}, \boldsymbol{\mu}]_{l,t} \le [\mathbf{F}, \boldsymbol{\mu}]_{r,t}$ *for every* $t \in (0, 1) \cap \mathrm{I}(\boldsymbol{\mu}|\mathbf{F})$*;*
(2) $[\mathbf{F}, \boldsymbol{\mu}]_{r,s} \le [\mathbf{F}, \boldsymbol{\mu}]_{l,t} + \lambda(t - s)\, W_2^2(\mu_0, \mu_1)$ *for every* $s, t \in \mathrm{I}(\boldsymbol{\mu}|\mathbf{F})$, $s < t$*;*
(3) *the maps*

$$t \mapsto [\mathbf{F}, \boldsymbol{\mu}]_{r,t} + \lambda t\, W_2^2(\mu_0, \mu_1) \quad \textit{and} \quad t \mapsto [\mathbf{F}, \boldsymbol{\mu}]_{l,t} + \lambda t\, W_2^2(\mu_0, \mu_1)$$

are increasing respectively in $\mathrm{I}(\boldsymbol{\mu}|\mathbf{F}) \setminus \{1\}$ *and in* $\mathrm{I}(\boldsymbol{\mu}|\mathbf{F}) \setminus \{0\}$*;*
(4) *if* t_0 *is a right accumulation point of* $\mathrm{I}(\boldsymbol{\mu}|\mathbf{F})$, *then*

$$\lim_{t \downarrow t_0} [\mathbf{F}, \boldsymbol{\mu}]_{r,t} = \lim_{t \downarrow t_0} [\mathbf{F}, \boldsymbol{\mu}]_{l,t} \tag{4.8}$$

and these right limits exist. If, instead, t_0 *is a left accumulation point of* $\mathrm{I}(\boldsymbol{\mu}|\mathbf{F})$, *the same holds with the right limits in (4.8) replaced by the left limits at* t_0*;*
(5) $[\mathbf{F}, \boldsymbol{\mu}]_{l,t} = [\mathbf{F}, \boldsymbol{\mu}]_{r,t}$ *at every interior point* t *of* $\mathrm{I}(\boldsymbol{\mu}|\mathbf{F})$ *where one of them is continuous.*

Proof Throughout all the proof we set

$$f_r(t) := [\mathbf{F}, \boldsymbol{\mu}]_{r,t} \quad \text{and} \quad f_l(t) := [\mathbf{F}, \boldsymbol{\mu}]_{l,t}. \tag{4.9}$$

Thanks to Lemma 4.6 and in particular to (4.4), it is easy to check that it is sufficient to consider the dissipative case $\lambda = 0$.

(1) It is a direct consequence of Lemma 3.20 and the definitions of f_r and f_l.
(2) We prove that for every $\Phi_s \in \mathbf{F}[\mu_s]$ and $\Phi'_t \in \mathbf{F}[\mu_t]$ it holds

$$[\Phi_s, \boldsymbol{\mu}]_{r,s} \le [\Phi'_t, \boldsymbol{\mu}]_{l,t}. \tag{4.10}$$

The thesis will follow immediately passing to the sup over $\Phi_s \in \mathbf{F}[\mu_s]$ in the l.h.s. and to the inf over $\Phi'_t \in \mathbf{F}[\mu_t]$ in the r.h.s.. It is enough to prove (4.10) in case at least one between s, t belongs to $(0, 1)$. Let us define the map $L : \mathcal{P}_2(\mathsf{TX} \times \mathsf{X}) \to \mathbb{R}$ as

$$L(\gamma) := \int_{\mathsf{TX} \times \mathsf{X}} \langle v_0, x_0 - x_1 \rangle \, \mathrm{d}\gamma(x_0, v_0, x_1) \quad \gamma \in \mathcal{P}_2(\mathsf{TX} \times \mathsf{X}).$$

Observe that, since it never happens that $s = 0$ and $t = 1$ at the same time, the map $T_{s,t} : \Gamma_s(\Phi_s, \boldsymbol{\mu}) \to \Lambda(\Phi_s, \mu_t)$, with $\Gamma_s(\cdot, \cdot)$ as in (3.25) and $\Lambda(\cdot, \cdot)$ as in

Definition 3.8, defined as

$$T_{s,t}(\boldsymbol{\sigma}) := \left(\mathsf{x}^s \circ (\mathsf{x}^0, \mathsf{x}^1), \mathsf{v}^0, \mathsf{x}^t \circ (\mathsf{x}^0, \mathsf{x}^1)\right)_\sharp \boldsymbol{\sigma}$$

is a bijection s.t. $(t-s)L(\boldsymbol{\sigma}) = L(T_{s,t}(\boldsymbol{\sigma}))$ for every $\boldsymbol{\sigma} \in \Gamma_s(\Phi_s, \boldsymbol{\mu})$. This immediately gives that

$$(t-s)[\Phi_s, \boldsymbol{\mu}]_{r,s} = [\Phi_s, \mu_t]_r \, .$$

In the same way we can deduce that

$$(s-t)[\Phi'_t, \boldsymbol{\mu}]_{l,t} = \left[\Phi'_t, \mu_s\right]_r \, .$$

Thanks to the dissipativity assumption (4.2) of $\mathbf{F}$, we get

$$(t-s)[\Phi_s, \boldsymbol{\mu}]_{r,s} - (t-s)[\Phi'_t, \boldsymbol{\mu}]_{l,t} = [\Phi_s, \mu_t]_r + \left[\Phi'_t, \mu_s\right]_r \le 0.$$

(3) Combining (1) and (2) we have that for every $s, t \in \mathrm{I}(\boldsymbol{\mu}|\mathbf{F})$ with $0 < s < t < 1$ it holds

$$f_l(s) \le f_r(s) \le f_l(t) \le f_r(t), \tag{4.11}$$

with f_r, f_l as in (4.9). This implies that both f_l and f_r are increasing in $\mathrm{I}(\boldsymbol{\mu}|\mathbf{F}) \cap (0, 1)$. Observe that, again combining (1) and (2), it also holds

$$\begin{aligned} f_r(0) &\le f_l(t) \le f_r(t), \\ f_l(t) &\le f_r(t) \le f_l(1) \end{aligned}$$

for every $t \in \mathrm{I}(\boldsymbol{\mu}|\mathbf{F}) \setminus \{0, 1\}$, and then f_r is increasing in $\mathrm{I}(\boldsymbol{\mu}|\mathbf{F}) \setminus \{1\}$ and f_l is increasing in $\mathrm{I}(\boldsymbol{\mu}|\mathbf{F}) \setminus \{0\}$.

(4) It is an immediate consequence of (4.11).

(5) It is a straightforward consequence of (4).

□

Thanks to Theorem 4.9(4), we have

$$\begin{aligned} \lim_{t \downarrow 0} [\mathbf{F}, \boldsymbol{\mu}]_{r,t} &= \lim_{t \downarrow 0} [\mathbf{F}, \boldsymbol{\mu}]_{l,t}, \\ \lim_{t \uparrow 1} [\mathbf{F}, \boldsymbol{\mu}]_{r,t} &= \lim_{t \uparrow 1} [\mathbf{F}, \boldsymbol{\mu}]_{l,t}, \end{aligned}$$

and those limits exist whenever the starting time $t_0 = 0$ and the final time $t_1 = 1$ are accumulation points of $\mathrm{I}(\boldsymbol{\mu}|\mathbf{F})$, respectively. Due to the importance played by these objects in Sect. 5, we give the following definitions. These are intended to weaken the requirement for the operator's domain $\mathrm{D}(\mathbf{F})$ to be open or geodesically convex.

Definition 4.10 Let $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$, $\mu_0, \mu_1 \in \overline{\mathrm{D}(\mathbf{F})}$, $\boldsymbol{\mu} \in \Gamma_o(\mu_0, \mu_1)$. We define the sets

$$\Gamma_o^i(\mu_0, \mu_1|\mathbf{F}) := \{\boldsymbol{\mu} \in \Gamma_o(\mu_0, \mu_1) : i \text{ is an accumulation point of } \mathrm{I}(\boldsymbol{\mu}|\mathbf{F})\}, \; i = 0, 1 \tag{4.12}$$

$$\Gamma_o^{01}(\mu_0, \mu_1|\mathbf{F}) := \Gamma_o^0(\mu_0, \mu_1|\mathbf{F}) \cap \Gamma_o^1(\mu_0, \mu_1|\mathbf{F}). \tag{4.13}$$

Notice that these sets depend on $\mathbf{F}$ just through $\mathrm{D}(\mathbf{F})$. In particular, if $\mu_0, \mu_1 \in \mathrm{D}(\mathbf{F})$ and $\mathrm{D}(\mathbf{F})$ is open or geodesically convex according to Definition 2.7 then $\Gamma_o^{01}(\mu_0, \mu_1|\mathbf{F}) \neq \emptyset$.

By the previous discussion, the next definition is well posed.

Definition 4.11 Let us suppose that the MPVF $\mathbf{F}$ satisfies (4.2), let $\mu_0, \mu_1 \in \overline{\mathrm{D}(\mathbf{F})}$.

$$\text{If } \boldsymbol{\mu} \in \Gamma_o^0(\mu_0, \mu_1|\mathbf{F}) \text{ we set } \quad [\mathbf{F}, \boldsymbol{\mu}]_{0+} := \lim_{t \downarrow 0} [\mathbf{F}, \boldsymbol{\mu}]_{r,t} = \lim_{t \downarrow 0} [\mathbf{F}, \boldsymbol{\mu}]_{l,t}$$

$$\text{If } \boldsymbol{\mu} \in \Gamma_o^1(\mu_0, \mu_1|\mathbf{F}) \text{ we set } \quad [\mathbf{F}, \boldsymbol{\mu}]_{1-} := \lim_{t \uparrow 1} [\mathbf{F}, \boldsymbol{\mu}]_{r,t} = \lim_{t \uparrow 1} [\mathbf{F}, \boldsymbol{\mu}]_{l,t}.$$

In the following statements, we make use of the objects introduced in Definition 4.10 in order to get refined dissipativity conditions involving the limiting pseudo-scalar products of Definition 4.11. These results will be useful in the sequel: in Proposition 4.17 they allow to get a dissipativity property of a suitable notion of extension $\hat{\mathbf{F}}$ of $\mathbf{F}$; in Sect. 5 (see in particular Lemma 5.3) they are relevant to study the properties of so-called λ-EVI solutions for a λ-dissipative MPVF $\mathbf{F}$.

Corollary 4.12 *Let us keep the same notation of Theorem 4.9 and let $s \in \mathrm{I}(\boldsymbol{\mu}|\mathbf{F}) \cap (0, 1)$ with $\Phi \in \mathbf{F}[\mu_s]$.*

(1) *If $\boldsymbol{\mu} \in \Gamma_o^0(\mu_0, \mu_1|\mathbf{F})$, we have that*

$$[\mathbf{F}, \boldsymbol{\mu}]_{0+} \leq [\Phi, \boldsymbol{\mu}]_{l,s} + \lambda s W^2 = [\Phi, \boldsymbol{\mu}]_{r,s} + \lambda s W^2; \tag{4.14}$$

if moreover $\Phi_0 \in \mathbf{F}[\mu_0]$ then

$$[\Phi_0, \mu_1]_r \leq [\Phi_0, \boldsymbol{\mu}]_{r,0} \leq [\mathbf{F}, \boldsymbol{\mu}]_{0+}. \tag{4.15}$$

(2) *If $\boldsymbol{\mu} \in \Gamma_o^1(\mu_0, \mu_1|\mathbf{F})$, we have that*

$$[\Phi, \boldsymbol{\mu}]_{l,s} - \lambda(1-s) W^2 = [\Phi, \boldsymbol{\mu}]_{r,s} - \lambda(1-s) W^2 \leq [\mathbf{F}, \boldsymbol{\mu}]_{1-};$$

if moreover $\Phi_1 \in \mathbf{F}[\mu_1]$ then

$$[\mathbf{F}, \boldsymbol{\mu}]_{1-} \leq [\Phi_1, \boldsymbol{\mu}]_{l,1} \leq -[\Phi_1, \mu_0]_r \tag{4.16}$$

(3) *In particular, for every $\Phi_0 \in \mathbf{F}[\mu_0]$, $\Phi_1 \in \mathbf{F}[\mu_1]$ and $\boldsymbol{\mu} \in \Gamma_o^{01}(\mu_0, \mu_1|\mathbf{F})$ we obtain*

$$[\Phi_0, \Phi_1]_{r,\boldsymbol{\mu}} \leq [\mathbf{F}, \boldsymbol{\mu}]_{0+} - [\mathbf{F}, \boldsymbol{\mu}]_{1-} \leq \lambda W_2^2(\mu_0, \mu_1). \tag{4.17}$$

(4.17) immediately yields the following property.

Corollary 4.13 *Suppose that a* MPVF $\mathbf{F}$ *satisfies*

$$\text{for every } \mu_0, \mu_1 \in \mathrm{D}(\mathbf{F}) \text{ the set } \Gamma_o^{01}(\mu_0, \mu_1|\mathbf{F}) \text{ of (4.13) is not empty} \tag{4.18}$$

(e.g. if $\mathrm{D}(\mathbf{F})$ *is open or geodesically convex), then* $\mathbf{F}$ *is* λ*-dissipative according to* (4.1) *if and only if it satisfies* (4.2)*.*

Proposition 4.14 *Let* $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$ *be a* MPVF *satisfying* (4.2)*, let* $\mu_0 \in \overline{\mathrm{D}(\mathbf{F})}$ *and let* $\Phi \in \mathcal{P}_2(\mathsf{TX}|\mu_0)$*. Consider the following statements*

- (P1) $[\Phi, \mu]_r + [\Psi, \mu_0]_r \le \lambda W_2^2(\mu_0, \mu)$ *for every* $\Psi \in \mathbf{F}$ *with* $\mu = \mathsf{x}_\sharp \Psi$*;*
- (P2) *for every* $\mu \in \mathrm{D}(\mathbf{F})$ *there exists* $\Psi \in \mathbf{F}[\mu]$ *s.t.* $[\Phi, \mu]_r + [\Psi, \mu_0]_r \le \lambda W_2^2(\mu_0, \mu)$*;*
- (P3) $[\Phi, \boldsymbol{\mu}]_{r,0} \le [\mathbf{F}, \boldsymbol{\mu}]_{0+}$ *for every* $\mu_1 \in \overline{\mathrm{D}(\mathbf{F})}$*,* $\boldsymbol{\mu} \in \Gamma_o^0(\mu_0, \mu_1|\mathbf{F})$*;*
- (P4) $[\Phi, \boldsymbol{\mu}]_{r,0} \le [\mathbf{F}, \boldsymbol{\mu}]_{0+}$ *for every* $\mu_1 \in \mathrm{D}(\mathbf{F})$*,* $\boldsymbol{\mu} \in \Gamma_o^0(\mu_0, \mu_1|\mathbf{F})$*;*
- (P5) $[\Phi, \boldsymbol{\mu}]_{r,0} \le \lambda W_2^2(\mu_0, \mu_1) + [\mathbf{F}, \boldsymbol{\mu}]_{1-}$ *for every* $\mu_1 \in \overline{\mathrm{D}(\mathbf{F})}$*,* $\boldsymbol{\mu} \in \Gamma_o^1(\mu_0, \mu_1|\mathbf{F})$*;*
- (P6) $[\Phi, \boldsymbol{\mu}]_{r,0} \le \lambda W_2^2(\mu_0, \mu_1) + [\mathbf{F}, \boldsymbol{\mu}]_{1-}$ *for every* $\mu_1 \in \mathrm{D}(\mathbf{F})$*,* $\boldsymbol{\mu} \in \Gamma_o^1(\mu_0, \mu_1|\mathbf{F})$*.*

Then the following hold

1. (P1) $\Rightarrow$ (P2) $\Rightarrow$ (P3) $\Rightarrow$ (P4)*;*
2. (P1) $\Rightarrow$ (P2) $\Rightarrow$ (P5) $\Rightarrow$ (P6)*;*
3. *if for every* $\mu_1 \in \mathrm{D}(\mathbf{F})$ $\Gamma_o^0(\mu_0, \mu_1|\mathbf{F}) \neq \emptyset$*, then* (P4) $\Rightarrow$ (P1) *(in particular,* (P1)*,* (P2)*,* (P3)*,* (P4) *are equivalent);*
4. *if for every* $\mu_1 \in \mathrm{D}(\mathbf{F})$ $\Gamma_o^1(\mu_0, \mu_1|\mathbf{F}) \neq \emptyset$*, then* (P6) $\Rightarrow$ (P1) *(in particular,* (P1)*,* (P2)*,* (P5)*,* (P6) *are equivalent).*

Proof We first prove that (P2) $\Rightarrow$ (P3),(P5). Let us choose an arbitrary $\mu_1 \in \overline{\mathrm{D}(\mathbf{F})}$; by the definition of $[\mathbf{F}, \boldsymbol{\mu}]_{r,t}$ and arguing as in the proof of Theorem 4.9(2), for all $\boldsymbol{\mu} \in \Gamma_o(\mu_0, \mu_1)$ and $t \in \mathrm{I}(\boldsymbol{\mu}|\mathbf{F})$ there exists $\Psi_t \in \mathbf{F}[\mu_t]$ such that

$$\begin{aligned}
[\Phi, \boldsymbol{\mu}]_{r,0} &= \frac{1}{t}[\Phi, \mu_t]_r \\
&\le -\frac{1}{t}[\Psi_t, \mu_0]_r + t\lambda W_2^2(\mu_0, \mu_1) \\
&= [\Psi_t, \boldsymbol{\mu}]_{r,t} + t\lambda W_2^2(\mu_0, \mu_1) \\
&\le [\mathbf{F}, \boldsymbol{\mu}]_{r,t} + t\lambda W_2^2(\mu_0, \mu_1),
\end{aligned}$$

where we also used (3.29). If $\boldsymbol{\mu} \in \Gamma_o^0(\mu_0, \mu_1|\mathbf{F})$, by passing to the limit as $t \downarrow 0$ we get (P3).

In the second case, assuming that $\boldsymbol{\mu} \in \Gamma_o^1(\mu_0, \mu_1|\mathbf{F})$, we can pass to the limit as $t \uparrow 1$ and we get (P5).

We now prove item (3). Let $\mu_1 \in \mathrm{D}(\mathbf{F})$, $\Psi \in \mathbf{F}[\mu_1]$, $\boldsymbol{\mu} \in \Gamma_o^0(\mu_0, \mu_1|\mathbf{F})$, $s \in \mathrm{I}(\boldsymbol{\mu}|\mathbf{F}) \cap (0, 1)$, $\Phi_s \in \mathbf{F}[\mu_s]$, with $\mu_s = \mathsf{x}^s_\sharp \boldsymbol{\mu}$. Assuming (P4) and using (4.15), (4.14), (3.29) and (4.2), we have

$$\begin{aligned}[\Phi, \mu_1]_r \le [\Phi, \boldsymbol{\mu}]_{r,0} \le [\mathbf{F}, \boldsymbol{\mu}]_{0+} \le [\Phi_s, \boldsymbol{\mu}]_{r,s} + \lambda s W_2^2(\mu_0, \mu_1) \\ = \frac{1}{1-s}[\Phi_s, \mu_1]_r + \lambda s W_2^2(\mu_0, \mu_1) \\ \le -\frac{1}{1-s}[\Psi, \mu_s]_r + \lambda(1+s) W_2^2(\mu_0, \mu_1).\end{aligned}$$

By Lemma 3.15, letting $s \downarrow 0$ we get (P1). Item (4) follows by (4.15), (4.16). □

4.3 Extensions of dissipative MPVF

Let us briefly study a few simple properties about extensions of λ-dissipative MPVFs. The first one concerns the sequential closure in $\mathcal{P}_2^{sw}(\mathsf{TX})$ (the sequential closure may be smaller than the topological closure, but see Proposition 2.15): given $A \subset \mathcal{P}_2(\mathsf{TX})$, we will denote by $\mathrm{cl}(A)$ its sequential closure defined by

$$\mathrm{cl}(A) := \left\{\Phi \in \mathcal{P}_2(\mathsf{TX}) : \exists\, (\Phi_n)_{n\in\mathbb{N}} \subset A : \Phi_n \to \Phi \text{ in } \mathcal{P}_2^{sw}(\mathsf{TX})\right\}.$$

Proposition 4.15 *If* $\mathbf{F}$ *is a* λ*-dissipative* MPVF *according to* (4.1)*, then its sequential closure* $\mathrm{cl}(\mathbf{F})$ *is* λ*-dissipative as well according to* (4.1).

Proof If Φ^i, $i = 0, 1$, belong to $\mathrm{cl}(\mathbf{F})$, we can find sequences $(\Phi^i_n)_{n\in\mathbb{N}} \subset \mathbf{F}$ such that $\Phi^i_n \to \Phi^i$ in $\mathcal{P}_2^{sw}(\mathsf{TX})$ as $n \to \infty$, $i = 0, 1$. It is then sufficient to pass to the limit in the inequality

$$\left[\Phi^0_n, \Phi^1_n\right]_r \le \lambda W_2^2(\mu^0_n, \mu^1_n), \quad \mu^i_n = \mathsf{x}_\sharp \Phi^i_n$$

using the lower semicontinuity property (3.23) and the fact that convergence in $\mathcal{P}_2^{sw}(\mathsf{TX})$ yields $\mu^i_n \to \mathsf{x}_\sharp \Phi^i$ in $\mathcal{P}_2(\mathsf{X})$ as $n \to \infty$. □

A second result concerns the convexification of the sections of $\mathbf{F}$. For every $\mu \in \mathrm{D}(\mathbf{F})$ we set

$$\mathrm{co}(\mathbf{F})[\mu] := \text{the convex hull of } \mathbf{F}[\mu] = \left\{\sum_k \alpha_k \Phi_k : \Phi_k \in \mathbf{F}[\mu], \alpha_k \ge 0, \sum_k \alpha_k = 1\right\}, \tag{4.19}$$

$$\overline{\mathrm{co}}(\mathbf{F})[\mu] := \mathrm{cl}(\mathrm{co}(\mathbf{F})[\mu]). \tag{4.20}$$

Notice that if $\mathbf{F}[\mu]$ is bounded in $\mathcal{P}_2(\mathsf{TX})$ then $\overline{\mathrm{co}}(\mathbf{F})[\mu]$ coincides with the closed convex hull of $\mathbf{F}[\mu]$.

Proposition 4.16 *If* $\mathbf{F}$ *is* λ*-dissipative according to* (4.1)*, then* $\mathrm{co}(\mathbf{F})$ *and* $\overline{\mathrm{co}}(\mathbf{F})$ *are* λ*-dissipative as well according to* (4.1).

Proof By Proposition 4.15 and noting that $\overline{\mathrm{co}}(\mathbf{F}) \subset \mathrm{cl}(\mathrm{co}(\mathbf{F}))$, it is sufficient to prove that $\mathrm{co}(\mathbf{F})$ is λ-dissipative. By Lemma 4.6 it is not restrictive to assume $\lambda = 0$. Let

$\Phi^i \in \mathrm{co}(\mathbf{F})[\mu_i]$, $i = 0, 1$; there exist positive coefficients α^i_k, $k = 1, \ldots, K$, with $\sum_k \alpha^i_k = 1$, and elements $\Phi^i_k \in \mathbf{F}[\mu^i]$, $i = 0, 1$, such that $\Phi^i = \sum_{k=1}^K \alpha^i_k \Phi^i_k$. Setting $\beta_{h,k} := \alpha^0_h \alpha^1_k$, we can apply Lemma 3.17 and we obtain

$$\left[\Phi^0, \Phi^1\right]_r = \left[\sum_{h,k} \beta_{h,k} \Phi^0_h, \sum_{h,k} \beta_{h,k} \Phi^1_k\right]_r \le \sum_{h,k} \beta_{h,k} \left[\Phi^0_h, \Phi^1_k\right]_r \le 0. \qquad \square$$

We recall that in the Hilbertian case (cf. e.g. [7]), a fundamental role is played by the notion of maximality for a dissipative operator $\mathrm{F} \subset \mathsf{H} \times \mathsf{H}$. Indeed, this notion enables to extablish the existence and uniqueness of solutions of the corresponding evolution equation and to get crucial properties of the resolvent operator. Moreover, if F is maximal, in order to prove that an element $(x, v) \in \mathsf{H} \times \mathsf{H}$ belongs to F it is enough to verify that it satisfies the dissipativity inequality

$$\langle v - w, x - y\rangle \le 0 \quad \text{for every } (y, w) \in \mathrm{F}. \tag{4.21}$$

For these reasons, if F is not maximal it is important to study its maximal extension, whose elements (x, v) must satisfy (4.21).

By analogy with the Hilbertian framework, it is interesting to study the properties of the *extended* MPVF defined by

$$\hat{\mathbf{F}} := \left\{ \Phi \in \mathcal{P}_2(\mathsf{TX}) : \begin{array}{l} \mu = \mathsf{x}_\sharp \Phi \in \overline{\mathrm{D}(\mathbf{F})}, \\ [\Phi, \nu]_r + [\Psi, \mu]_r \le \lambda W_2^2(\mu, \nu) \quad \forall\, \Psi \in \mathbf{F},\ \nu = \mathsf{x}_\sharp \Psi \end{array} \right\}. \tag{4.22}$$

This notion of extension $\hat{\mathbf{F}}$ of a MPVF $\mathbf{F}$ will be involved later in Sect. 5 dealing with differential inclusions in Wasserstein spaces, in particular in Theorem 5.4 and in Sect. 5.5.

It is obvious that $\mathbf{F} \subset \hat{\mathbf{F}}$; if the domain of $\mathbf{F}$ satisfies the geometric condition (4.24), the following result shows that $\hat{\mathbf{F}}$ provides the maximal λ-dissipative extension of $\mathbf{F}$.

Proposition 4.17 *Let* $\mathbf{F}$ *be a* λ*-dissipative* MPVF *according to* (4.1).

(a) *If* $\mathbf{F}' \supset \mathbf{F}$ *is* λ*-dissipative according to* (4.1)*, with* $\mathrm{D}(\mathbf{F}') \subset \overline{\mathrm{D}(\mathbf{F})}$*, then* $\mathbf{F}' \subset \hat{\mathbf{F}}$*. In particular* $\overline{\mathrm{co}}(\mathrm{cl}(\mathbf{F})) \subset \hat{\mathbf{F}}$.
(b) $\widehat{\mathrm{cl}(\mathbf{F})} = \hat{\mathbf{F}}$ *and* $\widehat{\mathrm{co}(\mathbf{F})} = \hat{\mathbf{F}}$.
(c) $\hat{\mathbf{F}}$ *is sequentially closed and* $\mathbf{F}[\hat{\mu}]$ *is convex for every* $\mu \in \mathrm{D}(\hat{\mathbf{F}})$.
(d) *If* $\mathrm{D}(\mathbf{F})$ *satisfies* (4.18)*, then the restriction of* $\hat{\mathbf{F}}$ *to* $\mathrm{D}(\mathbf{F})$ *is* λ*-dissipative according to* (4.1) *and for every* $\mu_0, \mu_1 \in \mathrm{D}(\mathbf{F})$ *it holds*

$$[\mathbf{F}, \boldsymbol{\mu}]_{0+} = [\hat{\mathbf{F}}, \boldsymbol{\mu}]_{0+}, \quad [\mathbf{F}, \boldsymbol{\mu}]_{1-} = [\hat{\mathbf{F}}, \boldsymbol{\mu}]_{1-} \quad \textit{for every } \boldsymbol{\mu} \in \Gamma^{01}_o(\mu_0, \mu_1|\mathbf{F}). \tag{4.23}$$

(e) *If* $\mu_0 \in \overline{\mathrm{D}(\mathbf{F})}$, $\mu_1 \in \mathrm{D}(\mathbf{F})$ *and* $\Gamma^1_o(\mu_0, \mu_1|\mathbf{F}) \neq \emptyset$ *then*

$$\Phi_i \in \mathbf{F}[\mu_i]^{\hat{}} \;\Rightarrow\; [\Phi_0, \Phi_1]_r \le \lambda W_2^2(\mu_0, \mu_1).$$

(f) *If*

$$\text{for every } \mu_0, \mu_1 \in \overline{\mathrm{D}(\mathbf{F})} \text{ the set } \Gamma_o^{01}(\mu_0, \mu_1|\mathbf{F}) \text{ is not empty,} \tag{4.24}$$

then $\hat{\mathbf{F}}$ *is* λ*-dissipative as well according to* (4.1) *and for every* $\mu_0, \mu_1 \in \overline{\mathrm{D}(\mathbf{F})}$ (4.23) *holds.*

Proof Claim (a) is obvious since every λ-dissipative extension $\mathbf{F}'$ of $\mathbf{F}$ in $\overline{\mathrm{D}(\mathbf{F})}$ satisfies $\mathbf{F}' \subset \hat{\mathbf{F}}$.

(b) Let us prove that if $\Phi \in \hat{\mathbf{F}}$ then $\Phi \in \widehat{\mathrm{cl}(\mathbf{F})}$. If $\Psi \in \mathrm{cl}(\mathbf{F})$ we can find a sequence $(\Psi_n)_{n\in\mathbb{N}} \subset \mathbf{F}$ converging to Ψ in $\mathcal{P}_2^{sw}(\mathsf{TX})$ as $n \to \infty$. We can then pass to the limit in the inequalities

$$[\Phi, \nu_n]_r + [\Phi_n, \mu]_r \le \lambda W_2^2(\mu, \nu_n), \quad \mu = \mathsf{x}_\sharp\Phi,\ \nu_n = \mathsf{x}_\sharp\Psi_n,$$

using the lower semicontinuity results of Lemma 3.15. We conclude since $\overline{\mathrm{D}(\mathbf{F})} = \overline{\mathrm{D}(\mathrm{cl}(\mathbf{F}))}$.

In order to prove that $\Phi \in \hat{\mathbf{F}} \Rightarrow \Phi \in \widehat{\mathrm{co}(\mathbf{F})}$ we take $\Psi = \sum \alpha_k \Psi_k \in \mathrm{co}(\mathbf{F})$; for some $\Psi_k \in \mathbf{F}[\nu]$, $\nu = \mathsf{x}_\sharp\Psi \in \mathrm{D}(\mathbf{F})$, and positive coefficients α_k, $k = 1, \ldots, K$, with $\sum_k \alpha_k = 1$. Taking a convex combination of the inequalities

$$[\Phi, \nu]_r + [\Psi_k, \mu]_r \le \lambda W_2^2(\mu, \nu), \quad \text{for every } k = 1, \ldots, K,$$

and using Lemma 3.17 we obtain

$$[\Phi, \nu]_r + [\Psi, \mu]_r \le \sum_k \alpha_k \Big([\Phi, \nu]_r + [\Psi_k, \mu]_r \Big) \le \lambda W_2^2(\mu, \nu).$$

The proof of claim (c) follows by a similar argument.

(d) Let $\mu_i \in \mathrm{D}(\mathbf{F})$, $\Phi_i \in \mathbf{F}[\hat{\mu}_i]$, $i = 0, 1$, and $\boldsymbol{\mu} \in \Gamma_o^{01}(\mu_0, \mu_1|\mathbf{F})$. The implication (P1)⇒(P4) of Proposition 4.14 applied to $\boldsymbol{\mu}$ and to $\mathsf{s}_\sharp\boldsymbol{\mu}$, with s the reversion map in (3.26), yields

$$[\Phi_0, \boldsymbol{\mu}]_{r,0} \le [\mathbf{F}, \boldsymbol{\mu}]_{0+}, \quad [\Phi_1, \mathsf{s}_\sharp\boldsymbol{\mu}]_{r,0} \le [\mathbf{F}, \mathsf{s}_\sharp\boldsymbol{\mu}]_{0+} = -[\mathbf{F}, \boldsymbol{\mu}]_{1-}$$

so that (4.17) yields

$$[\Phi_0, \Phi_1]_r \le [\Phi_0, \boldsymbol{\mu}]_{r,0} + [\Phi_1, \mathsf{s}_\sharp\boldsymbol{\mu}]_{r,0} \le [\mathbf{F}, \boldsymbol{\mu}]_{0+} - [\mathbf{F}, \boldsymbol{\mu}]_{1-} \le \lambda W_2^2(\mu_0, \mu_1).$$

In order to prove (4.23) we observe that $\mathbf{F} \subset \hat{\mathbf{F}}$ so that, for every $\boldsymbol{\mu} \in \Gamma_o^{01}(\mu_0, \mu_1|\mathbf{F})$ and every $t \in \mathrm{I}(\boldsymbol{\mu}|\mathbf{F})$, we have $[\mathbf{F}, \boldsymbol{\mu}]_{r,t} \le [\hat{\mathbf{F}}, \boldsymbol{\mu}]_{r,t}$ and $[\mathbf{F}, \boldsymbol{\mu}]_{l,t} \ge [\hat{\mathbf{F}}, \boldsymbol{\mu}]_{l,t}$, hence (4.23) is a consequence of Definition 4.11 and Theorem 4.9.

The proof of claim (f) follows by the same argument.

In the case of claim (e), we use the implication (P1)⇒(P6) of Proposition 4.14 applied to $\boldsymbol{\mu}$ and the implication (P1)⇒(P3) applied to $\mathsf{s}_\sharp\boldsymbol{\mu}$, obtaining

$$[\Phi_0, \boldsymbol{\mu}]_{r,0} \le \lambda W_2^2(\mu_0, \mu_1) + [\mathbf{F}, \boldsymbol{\mu}]_{1-}, \quad [\Phi_1, \mathsf{s}_\sharp\boldsymbol{\mu}]_{r,0} \le [\mathbf{F}, \mathsf{s}_\sharp\boldsymbol{\mu}]_{0+} = -[\mathbf{F}, \boldsymbol{\mu}]_{1-}$$

and then

$$[\Phi_0, \Phi_1]_r \le [\Phi_0, \boldsymbol{\mu}]_{r,0} + [\Phi_1, \mathsf{s}_\sharp\boldsymbol{\mu}]_{r,0} \le \lambda W_2^2(\mu_0, \mu_1).$$

□

5 Solutions to measure differential inclusions

5.1 Metric characterization and EVI

Let $\mathcal{I}$ denote an arbitrary (bounded or unbounded) interval in $\mathbb{R}$.

The aim of this section is to study a suitable notion of solution to the following differential inclusion in the L^2-Wasserstein space of probability measures

$$\dot{\mu}_t \in \mathbf{F}[\mu_t], \qquad t \in \mathcal{I}, \tag{5.1}$$

driven by a MPVF $\mathbf{F}$ as in Definition 4.1. In particular, we will address the usual Cauchy problem when (5.1) is supplemented by a given initial condition.

Measure Differential Inclusions have been introduced in [26] extending to the multi-valued framework the theory of Measure Differential Equations developed in [27]. In these papers, the author aims to describe the evolution of curves in the space of probability measures under the action of a so called *probability vector field* $\mathbf{F}$ (see Definition 4.1 and Remark 4.2). However, as exploited also in [9], the definition of solution to (5.1) given in [9, 26, 27] is too weak and it does not enjoy uniqueness property which is recovered only at the level of the semigroup through an approximation procedure.

From the Wasserstein viewpoint, the simplest way to interpret (5.1) is to ask for a locally absolutely continuous curve $\mu : \mathcal{I} \to \mathcal{P}_2(\mathsf{X})$ to satisfy

$$(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}_t)_\sharp\mu_t \in \mathbf{F}[\mu_t] \quad \text{for a.e. } t \in \mathcal{I}, \tag{5.2}$$

where $\boldsymbol{v}$ is the Wasserstein metric velocity vector associated to μ (see Theorem 2.10). Even in the case of a regular PVF, however, (5.2) is too strong, since there is no reason why a given $\mathbf{F}[\mu_t]$ should be associated to a vector field of the tangent space $\mathrm{Tan}_{\mu_t}\mathcal{P}_2(\mathsf{X})$. Starting from (5.2), we thus introduce a weaker definition of solution to (5.1), modeled on the so-called EVI formulation for gradient flows, which will eventually suggest, as a natural formulation of (5.1), the relaxed version of (5.2) as a differential inclusion with respect to the extension $\hat{\mathbf{F}}$ of $\mathbf{F}$ introduced in (4.22).

We start from this simple remark: whenever $\mathbf{F}$ is λ-dissipative according to (4.1), recalling Theorem 3.11 and Remark 4.5, one easily sees that every locally absolutely continuous solution according to the above definition (5.2) also satisfies the Evolution Variational Inequality (λ-EVI)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}W_2^2(\mu_t,\nu) \le \lambda W_2^2(\mu_t,\nu) - [\Phi,\mu_t]_r \quad \text{in } \mathscr{D}'\big(\operatorname{int}(\mathfrak{I})\big), \tag{λ-EVI}$$

for every $\nu \in \mathrm{D}(\mathbf{F})$ and every $\Phi \in \mathbf{F}[\nu]$, where $[\cdot,\cdot]_r$ is the functional pairing in Definition 3.5 and the writing $\mathscr{D}'\big(\operatorname{int}(\mathfrak{I})\big)$ means that the expression has to be understood in the distributional sense over $\operatorname{int}(\mathfrak{I})$ (in fact, (λ-EVI) holds a.e. in $\mathfrak{I}$). This provides a heuristic motivation for the following definition.

Definition 5.1 (λ-EVI solution) Let $\mathbf{F}$ be a MPVF and let $\lambda \in \mathbb{R}$. We say that a continuous curve $\mu : \mathfrak{I} \to \overline{\mathrm{D}(\mathbf{F})}$ is a λ-EVI *solution* to (5.1) for the MPVF $\mathbf{F}$ if (λ-EVI) holds for every $\nu \in \mathrm{D}(\mathbf{F})$ and every $\Phi \in \mathbf{F}[\nu]$.
A λ-EVI solution μ is said to be a *strict solution* if $\mu_t \in \mathrm{D}(\mathbf{F})$ for every $t \in \mathfrak{I}, t > \inf \mathfrak{I}$.
A λ-EVI solution μ is said to be a *global solution* if $\sup \mathfrak{I} = +\infty$.

In Example 7.5 we will clarify the interest of imposing no more than continuity in the above definition.

Recall that the right upper and lower Dini derivatives of a function $\zeta : \mathfrak{I} \to \mathbb{R}$ are defined for every $t \in \mathfrak{I}$, $t < \sup \mathfrak{I}$ by

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\zeta(t) := \limsup_{h\downarrow 0}\frac{\zeta(t+h)-\zeta(t)}{h}, \qquad \frac{\mathrm{d}}{\mathrm{d}t_+}\zeta(t) := \liminf_{h\downarrow 0}\frac{\zeta(t+h)-\zeta(t)}{h}. \tag{5.3}$$

Remark 5.2 Arguing as in [22, Lemma A.1] and using the lower semicontinuity of the map $t \mapsto [\Phi,\mu_t]_r$, the distributional inequality of (λ-EVI) can be equivalently reformulated in terms of the right upper or lower Dini derivatives of the squared distance function and requiring the condition to hold for every $t \in \operatorname{int}(\mathfrak{I})$:

$$\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t}W_2^2(\mu_t,\nu) \le \lambda W_2^2(\mu_t,\nu) - [\Phi,\mu_t]_r \quad \text{for every } t \in \operatorname{int}(\mathfrak{I}),\ \Phi \in \mathbf{F},\ \nu = \mathsf{x}_\sharp\Phi, \tag{λ-EVI$_1$}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t_+}W_2^2(\mu_t,\nu) \le \lambda W_2^2(\mu_t,\nu) - [\Phi,\mu_t]_r \quad \text{for every } t \in \operatorname{int}(\mathfrak{I}),\ \Phi \in \mathbf{F},\ \nu = \mathsf{x}_\sharp\Phi. \tag{λ-EVI$_2$}$$

A further equivalent formulation [22, Theorem 3.3] involves the difference quotients: for every $s, t \in \mathfrak{I}$, $s < t$

$$\begin{aligned} &\mathrm{e}^{-2\lambda(t-s)}\,W_2^2(\mu_t,\nu) - W_2^2(\mu_s,\nu) \\ &\quad \le -2\int_s^t \mathrm{e}^{-2\lambda(r-s)}\,[\Phi,\mu_r]_r\,\mathrm{d}r \quad \text{for every } \Phi \in \mathbf{F},\ \nu = \mathsf{x}_\sharp\Phi. \end{aligned} \tag{λ-EVI$_3$}$$

Finally, if μ is also locally absolutely continuous, then (λ-EVI$_1$) and (λ-EVI$_2$) are also equivalent to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}W_2^2(\mu_t,\nu) \le \lambda W_2^2(\mu_t,\nu) - [\Phi,\mu_t]_r \qquad \text{for a.e. } t \in \mathcal{I} \text{ and every } \Phi \in \mathbf{F},\ \nu = \mathsf{x}_\sharp\Phi.$$

The following lemma discusses further properties of λ-EVI solutions. We refer respectively to (4.7), (4.12) and Definition 4.11 for the definitions of $\mathrm{I}(\boldsymbol{\mu}|\mathbf{F})$, $\Gamma_o^i(\cdot,\cdot|\mathbf{F})$, with $i = 0, 1$, and for the definitions of $[\mathbf{F},\boldsymbol{\mu}]_{0+}$ and $[\mathbf{F},\boldsymbol{\mu}]_{1-}$.

Lemma 5.3 *Let* $\mathbf{F}$ *be a* λ*-dissipative* MPVF *according to* (4.1) *and let* $\mu : \mathcal{I} \to \overline{\mathrm{D}(\mathbf{F})}$ *be a continuous* λ-EVI *solution to* (5.1)*. We have*

$$\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t}W_2^2(\mu_t,\nu) \le [\mathbf{F},\boldsymbol{\mu}_t]_{0+} \quad \textit{for every } \nu \in \overline{\mathrm{D}(\mathbf{F})},\ t \in \operatorname{int}(\mathcal{I}),\ \boldsymbol{\mu}_t \in \Gamma_o^0(\mu_t,\nu|\mathbf{F}), \tag{5.4a}$$

$$\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t}W_2^2(\mu_t,\nu) \le \lambda W_2^2(\mu_t,\nu) + [\mathbf{F},\boldsymbol{\mu}_t]_{1-} \quad \textit{for every } \nu \in \overline{\mathrm{D}(\mathbf{F})},\ t \in \operatorname{int}(\mathcal{I}),\ \boldsymbol{\mu}_t \in \Gamma_o^1(\mu_t,\nu|\mathbf{F}). \tag{5.4b}$$

If moreover μ *is locally absolutely continuous with Wasserstein velocity field* $\boldsymbol{v}$ *satisfying* (2.6) *for every* t *in the subset* $A(\mu) \subset \mathcal{I}$ *of full Lebesgue measure, then*

$$\left[(\boldsymbol{i}_\mathsf{X},\boldsymbol{v}_t)_\sharp\mu_t,\nu\right]_r \le \lambda W_2^2(\mu_t,\nu) - [\Phi,\mu_t]_r \qquad \textit{if } t \in A(\mu),\ \Phi \in \mathbf{F},\ \nu = \mathsf{x}_\sharp\Phi, \tag{5.5a}$$

$$[(\boldsymbol{i}_\mathsf{X},\boldsymbol{v}_t)_\sharp\mu_t,\boldsymbol{\mu}_t]_{r,0} \le [\mathbf{F},\boldsymbol{\mu}_t]_{0+} \qquad \textit{if } t \in A(\mu),\ \nu \in \overline{\mathrm{D}(\mathbf{F})},\ \boldsymbol{\mu}_t \in \Gamma_o^0(\mu_t,\nu|\mathbf{F}), \tag{5.5b}$$

$$[(\boldsymbol{i}_\mathsf{X},\boldsymbol{v}_t)_\sharp\mu_t,\boldsymbol{\mu}_t]_{r,0} \le \lambda W_2^2(\mu_t,\nu) + [\mathbf{F},\boldsymbol{\mu}_t]_{1-} \qquad \textit{if } t \in A(\mu),\ \nu \in \overline{\mathrm{D}(\mathbf{F})},\ \boldsymbol{\mu}_t \in \Gamma_o^1(\mu_t,\nu|\mathbf{F}). \tag{5.5c}$$

Proof In order to check (5.5a) it is sufficient to combine (3.20) of Theorem 3.11 with (λ-EVI$_1$). (5.5b) and (5.5c) then follow applying Proposition 4.14. Let us now prove (5.4a): fix $\nu \in \overline{\mathrm{D}(\mathbf{F})}$ and $t \in \operatorname{int}(\mathcal{I})$. Take $\boldsymbol{\mu}_t \in \Gamma_o(\mu_t,\nu)$ and define the constant speed geodesic $\nu^t : [0,1] \to \mathcal{P}_2(\mathsf{X})$ by $\nu_s^t := (\mathsf{x}^s)_\sharp\boldsymbol{\mu}_t$, thus in particular $\nu_0^t = \mu_t$ and $\nu_1^t = \nu$. Then by Lemma 2.11, for every $s \in \mathrm{I}(\boldsymbol{\mu}|\mathbf{F}) \cap (0,1)$ and $\Phi_s \in \mathbf{F}(\nu_s^t)$ we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t}W_2^2(\mu_t,\nu) &\le \frac{1}{2s}\frac{\mathrm{d}^+}{\mathrm{d}t}W_2^2(\mu_t,\nu_s^t)\\ &\le -\frac{1}{s}[\Phi_s,\mu_t]_r + \frac{\lambda}{s}W_2^2(\mu_t,\nu_s^t)\\ &\le [\mathbf{F},\boldsymbol{\mu}_t]_{r,s} + \lambda s W_2^2(\mu_t,\nu),\end{aligned}$$

where the second inequality comes from (λ-EVI$_1$). Taking $\boldsymbol{\mu}_t \in \Gamma_o^0(\mu_t,\nu|\mathbf{F})$ and passing to the limit as $s \downarrow 0$ we get (5.4a). Analogously for (5.4b). □

We can now give an interpretation of absolutely continuous λ-EVI solutions in terms of differential inclusions.

Theorem 5.4 *Let* $\mathbf{F}$ *be a* λ*-dissipative* MPVF *according to* (4.1) *and let* $\mu : \mathcal{I} \to \overline{\mathrm{D}(\mathbf{F})}$ *be a locally absolutely continuous curve.*

(1) *If* μ *satisfies the differential inclusion* (5.2) *driven by any* λ*-dissipative extension of* $\mathbf{F}$ *in* $\mathrm{D}(\mathbf{F})$*, then* μ *is also a* λ-EVI *solution to* (5.1) *for* $\mathbf{F}$.
(2) μ *is a* λ-EVI *solution of* (5.1) *for* $\mathbf{F}$ *if and only if*

$$(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}_t)_\sharp \mu_t \in \hat{\mathbf{F}}[\mu_t] \quad \textit{for a.e. } t \in \mathcal{I}. \tag{5.6}$$

(3) *If* $\mathrm{D}(\mathbf{F})$ *satisfies* (4.18) *and* $\mu_t \in \mathrm{D}(\mathbf{F})$ *for a.e.* $t \in \mathcal{I}$*, then the following properties are equivalent:*
   - μ *is a* λ-EVI *solution to* (5.1) *for* $\mathbf{F}$.
   - μ *satisfies* (5.5b).
   - μ *is a* λ-EVI *solution to* (5.1) *for the restriction of* $\hat{\mathbf{F}}$ *to* $\mathrm{D}(\mathbf{F})$.

(4) *If* $\mathbf{F}$ *satisfies* (4.24) *then* μ *is a* λ-EVI *solution to* (5.1) *for* $\mathbf{F}$ *if and only if it is a* λ-EVI *solution to* (5.1) *for* $\hat{\mathbf{F}}$.

Proof (1) It is sufficient to apply Theorem 3.11 and the definition of λ-dissipativity.

The left-to-right implication $\Rightarrow$ of (2) follows by (5.5a) of Lemma 5.3 and the definition of $\hat{\mathbf{F}}$.

Conversely, if μ satisfies (5.6), $\nu \in \mathrm{D}(\mathbf{F})$, $\Phi \in \mathbf{F}[\nu]$, then Theorem 3.11 and the definition of $\hat{\mathbf{F}}$ yield

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} W_2^2(\mu_t, \nu) = \left[(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}_t)_\sharp \mu_t, \nu\right]_r \le \lambda W_2^2(\mu_t, \nu) - [\Phi, \mu_t]_r \quad \text{a.e. in } \mathcal{I}.$$

Claim (3) is an immediate consequence of Lemma 5.3, Proposition 4.17(d) and Proposition 4.14.

Claim (4) is a consequence of Proposition 4.17(f) and the λ-dissipativity of $\hat{\mathbf{F}}$. $\square$

The result stated in Theorem 5.4 suggests a compatibility between the notion of EVI solution for a dissipative MPVF and the notion of gradient flow for a convex functional in $\mathcal{P}_2(\mathsf{X})$. This correspondence is analysed in Sect. 7.1, where we consider the particular case where the MPVF is the opposite of the Fréchet subdifferential of a proper, lower semicontinuous and convex functional $\mathcal{F} : \mathcal{P}_2(\mathsf{X}) \to (-\infty, +\infty]$ (see Proposition 7.2).

We derive a further useful a priori bound for λ-EVI solutions.

Proposition 5.5 *Let* $\mathbf{F}$ *be a* λ*-dissipative* MPVF *according to* (4.1) *and let* $T \in (0, +\infty]$. *Every* λ-EVI *solution* $\mu : [0, T) \to \overline{\mathrm{D}(\mathbf{F})}$ *with initial datum* $\mu_0 \in \mathrm{D}(\mathbf{F})$ *satisfies the a priori bound*

$$W_2(\mu_t, \mu_0) \le 2|\mathbf{F}|_2(\mu_0) \int_0^t \mathrm{e}^{\lambda s}\,\mathrm{d}s \tag{5.7}$$

for all $t \in [0, T)$*, where*

$$|\mathbf{F}|_2(\mu) := \inf\left\{|\Phi|_2 : \Phi \in \mathbf{F}[\mu]\right\}$$

for every $\mu \in \mathrm{D}(\mathbf{F})$.

Proof Let $\Phi \in \mathbf{F}(\mu_0)$. Then ($\lambda$-EVI) with $\nu := \mu_0$ yields

$$\frac{\mathrm{d}^+}{\mathrm{d}t} W_2^2(\mu_t, \mu_0) - 2\lambda W_2^2(\mu_t, \mu_0) \le -2\,[\Phi, \mu_t]_r \le 2|\Phi|_2\, W_2(\mu_t, \mu_0)$$

for every $t \in [0, T)$. We can then apply the estimate of Lemma B.1 to obtain

$$\mathrm{e}^{-\lambda t} W_2(\mu_t, \mu_0) \le 2|\Phi|_2 \int_0^t \mathrm{e}^{-\lambda s}\,\mathrm{d}s$$

for all $t \in [0, T)$, which in turn yields (5.7). □

We conclude this section with a result showing the robustness of the notion of λ-EVI solution.

Proposition 5.6 *If* $\mu_n : \mathfrak{I} \to \overline{\mathrm{D}(\mathbf{F})}$ *is a sequence of* λ-EVI *solutions locally uniformly converging to* μ *as* $n \to \infty$*, then* μ *is a* λ-EVI *solution.*

Proof μ is a continuous curve defined in $\mathfrak{I}$ with values in $\overline{\mathrm{D}(\mathbf{F})}$. Using pointwise convergence, the lower semicontinuity of $\mu \mapsto [\Phi, \mu]_r$ of Lemma 3.15, and Fatou's Lemma, it is easy to pass to the limit in the equivalent characterization (λ-EVI$_3$) of λ-EVI solutions, written for μ_n. □

5.2 Local existence of λ-EVI solutions by the Explicit Euler Scheme

In order to prove the existence of a λ-EVI solution to (5.1), our strategy is to employ an approximation argument through an Explicit Euler scheme as it occurs for ODEs. In the following $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the floor and the ceiling functions respectively, i.e.

$$\lfloor t \rfloor := \max\{m \in \mathbb{Z} \mid m \le t\} \quad \text{and} \quad \lceil t \rceil := \min\{m \in \mathbb{Z} \mid m \ge t\}, \tag{5.8}$$

for any $t \in \mathbb{R}$.

Definition 5.7 (Explicit Euler Scheme) Let $\mathbf{F}$ be a MPVF and suppose we are given a step size $\tau > 0$, an initial datum $\mu_0 \in \mathrm{D}(\mathbf{F})$, a bounded interval $[0, T]$, corresponding to the final step $\mathrm{N}(T, \tau) := \lceil T/\tau \rceil$, and a stability bound $L > 0$. A sequence $(M_\tau^n, \Phi_\tau^n)_{0 \le n \le \mathrm{N}(T,\tau)} \subset \mathrm{D}(\mathbf{F}) \times \mathbf{F}$ is a *L-stable solution to the Explicit Euler Scheme in* $[0, T]$ *starting from* $\mu_0 \in \mathrm{D}(\mathbf{F})$ if

$$\begin{cases} M_\tau^0 = \mu_0, \\ \Phi_\tau^n \in \mathbf{F}[M_\tau^n],\ |\Phi_\tau^n|_2 \le L & 0 \le n < \mathrm{N}(T, \tau), \\ M_\tau^n = (\mathsf{exp}^\tau)_\sharp \Phi_\tau^{n-1} & 1 \le n \le \mathrm{N}(T, \tau). \end{cases} \tag{EE}$$

We define the following two different interpolations of the sequence (M_τ^n, Φ_τ^n):

- the affine interpolation:

$$M_\tau(t) := (\exp^{t-n\tau})_\sharp \Phi^n_\tau \quad \text{if } t \in [n\tau, (n+1)\tau] \text{ for some } n \in \mathbb{N},\ 0 \le n < \mathrm{N}(T,\tau), \tag{5.9}$$

- the piecewise constant interpolation:

$$\bar{M}_\tau(t) := M_\tau^{\lfloor t/\tau \rfloor}, \quad t \in [0,T], \tag{5.10}$$

$$\boldsymbol{F}_\tau(t) := \Phi_\tau^{\lfloor t/\tau \rfloor}, \quad t \in [0,T]. \tag{5.11}$$

We define the following (possibly empty) sets

$$\begin{aligned} \mathscr{E}(\mu_0,\tau,T,L) &:= \{(M_\tau,\boldsymbol{F}_\tau) \mid M_\tau, \boldsymbol{F}_\tau \text{ are the curves given by (5.9), (5.11) respectively}\} \\ \mathscr{M}(\mu_0,\tau,T,L) &:= \{M_\tau \mid M_\tau \text{ is the curve given by (5.9)}\}. \end{aligned} \tag{5.12}$$

Remark 5.8 We immediately notice that, if $(M_\tau, \boldsymbol{F}_\tau) \in \mathscr{E}(\mu_0,\tau,T,L)$ and $\bar{M}_\tau(\cdot)$ is as in (5.10), then the following holds for any $0 \le s \le t \le T$:

(1) the affine interpolation can be trivially written as

$$M_\tau(t) = \left(\exp^{t-\lfloor t/\tau\rfloor\tau}\right)_\sharp (\boldsymbol{F}_\tau(t))\,;$$

(2) M_τ satisfies the uniform Lipschitz bound

$$W_2(M_\tau(t), M_\tau(s)) \le L|t-s|; \tag{5.13}$$

(3) we have the following estimate

$$W_2(\bar{M}_\tau(t), M_\tau(t)) = W_2\left(M_\tau\left(\left\lfloor \frac{t}{\tau}\right\rfloor \tau\right), M_\tau(t)\right) \le L\tau. \tag{5.14}$$

The estimate (5.14) shows that the stability and convergence results stated for the affine interpolation (see Theorem 5.9) can be easily adapted to the piecewise constant one.

Notice that, since in general $\mathbf{F}[\mu]$ is not reduced to a singleton, the sets $\mathscr{E}(\mu_0,\tau,T,L)$ and $\mathscr{M}(\mu_0,\tau,T,L)$ may contain more than one element (or may be empty). Stable solutions to the Explicit Euler scheme generated by a λ-dissipative MPVF exhibit a nice behaviour, which is clarified by the following important result, which will be proved in Sect. 6 (see Proposition 6.3 and Theorems 6.4, 6.5 and 6.7), with explicit estimates of the error constants $A(\delta)$. We stress that in the next statement $A(\delta)$ solely depend on δ (in particular, it is independent of $\lambda, L, T, \tau, \eta, M_\tau, M_\eta$).

Theorem 5.9 *Let* $\mathbf{F}$ *be a λ-dissipative* MPVF *according to* (4.1).

(1) *For every* $\mu_0, \mu_0' \in \mathrm{D}(\mathbf{F})$, *every* $M_\tau \in \mathscr{M}(\mu_0, \tau, T, L)$, $M_\tau' \in \mathscr{M}(\mu_0', \tau, T, L)$ *with* $\tau\lambda_+ \le 2$ *we have*

$$W_2(M_\tau(t), M_\tau'(t)) \le \mathrm{e}^{\lambda t} W_2(\mu_0, \mu_0') + 8L\sqrt{t\tau}\left(1 + |\lambda|\sqrt{t\tau}\right)\mathrm{e}^{\lambda_+ t} \quad (5.15)$$

for every $t \in [0, T]$.

(2) *For every* $\delta > 1$ *there exists a constant* $A(\delta)$ *such that if* $M_\tau \in \mathscr{M}(M_\tau^0, \tau, T, L)$ *and* $M_\eta \in \mathscr{M}(M_\eta^0, \eta, T, L)$ *with* $\lambda_+(\tau + \eta) \le 1$ *then*

$$W_2(M_\tau(t), M_\eta(t)) \le \left(\delta\, W_2(M_\tau^0, M_\eta^0) + A(\delta)L\sqrt{(\tau+\eta)(t+\tau+\eta)}\right)\mathrm{e}^{\lambda_+ t}$$

for every $t \in [0, T]$.

(3) *For every* $\delta > 1$ *there exists a constant* $A(\delta)$ *such that if* $\mu : [0, T] \to \overline{\mathrm{D}(\mathbf{F})}$ *is a* λ-EVI *solution and* $M_\tau \in \mathscr{M}(M_\tau^0, \tau, T, L)$ *then*

$$W_2(\mu_t, M_\tau(t)) \le \left(\delta\, W_2(\mu_0, M_\tau^0) + A(\delta)L\sqrt{\tau(t+\tau)}\right)\mathrm{e}^{\lambda_+ t} \quad (5.16)$$

for every $t \in [0, T]$.

(4) *If* $n \mapsto \tau(n)$ *is a vanishing sequence of time steps,* $(\mu_{0,n})_{n\in\mathbb{N}}$ *is a sequence in* $\mathrm{D}(\mathbf{F})$ *converging to* $\mu_0 \in \overline{\mathrm{D}(\mathbf{F})}$ *in* $\mathcal{P}_2(\mathsf{X})$ *and* $M_n \in \mathscr{M}(\mu_{0,n}, \tau(n), T, L)$, *then* M_n *is uniformly converging to a Lipschitz continuous limit curve* $\mu : [0, T] \to \overline{\mathrm{D}(\mathbf{F})}$ *which is a* λ-EVI *solution starting from* μ_0.

Definition 5.10 (Local and global solvability of (EE)) We say that the Explicit Euler Scheme (EE) associated to a MPVF $\mathbf{F}$ is *locally solvable* at $\mu_0 \in \mathrm{D}(\mathbf{F})$ if there exist strictly positive constants $\boldsymbol{\tau}, T, L$ such that $\mathscr{E}(\mu_0, \tau, T, L)$ is not empty for every $\tau \in (0, \boldsymbol{\tau})$. We say that (EE) is *globally solvable* at $\mu_0 \in \mathrm{D}(\mathbf{F})$ if for every $T > 0$ there exist strictly positive constants $\boldsymbol{\tau}, L$ such that $\mathscr{E}(\mu_0, \tau, T, L)$ is not empty for every $\tau \in (0, \boldsymbol{\tau})$.

If we assume that the Explicit Euler scheme is locally solvable, Theorem 5.9 provides a crucial tool to obtain local existence and uniqueness of λ-EVI solutions.

Let us now state the main existence result for λ-EVI solutions. Given $T \in (0, +\infty]$ and $\mu : [0, T) \to \mathcal{P}_2(\mathsf{X})$ we denote by $|\dot\mu_t|_+$ the right upper metric derivative

$$|\dot\mu_t|_+ := \limsup_{h\downarrow 0} \frac{W_2(\mu_{t+h}, \mu_t)}{h}.$$

Theorem 5.11 (Local existence and uniqueness) *Let* $\mathbf{F}$ *be a* λ*-dissipative* MPVF *according to* (4.1).

(a) *If the Explicit Euler Scheme is locally solvable at* $\mu_0 \in \mathrm{D}(\mathbf{F})$, *then there exists* $T > 0$ *and a unique Lipschitz continuous* λ-EVI *solution* $\mu : [0, T] \to \overline{\mathrm{D}(\mathbf{F})}$ *starting from* μ_0, *satisfying*

$$t \mapsto \mathrm{e}^{-\lambda t}|\dot\mu_t|_+ \quad \textit{is decreasing in } [0, T). \quad (5.17)$$

If $\mu' : [0, T'] \to \overline{\mathrm{D}(\mathbf{F})}$ *is any other* λ-EVI *solution starting from* μ_0 *then* $\mu_t = \mu'_t$ *if* $0 \le t \le \min\{T, T'\}$.

(b) *If the Explicit Euler Scheme is locally solvable in* $\mathrm{D}(\mathbf{F})$ *and*

$$\begin{aligned} &\textit{for any local } \lambda - \text{EVI } \textit{solution } \mu \textit{ starting from } \mu_0 \in \mathrm{D}(\mathbf{F}) \\ &\textit{there exists } \delta > 0 : \quad t \in [0, \delta] \quad \Rightarrow \quad \mu_t \in \mathrm{D}(\mathbf{F}), \end{aligned} \tag{5.18}$$

then for every $\mu_0 \in \mathrm{D}(\mathbf{F})$ *there exist a unique maximal time* $T \in (0, \infty]$ *and a unique strict locally Lipschitz continuous* λ-EVI *solution* $\mu : [0, T) \to \mathrm{D}(\mathbf{F})$ *starting from* μ_0*, which satisfies* (5.17) *and*

$$T < \infty \quad \Rightarrow \quad \lim_{t \uparrow T} \mu_t \notin \mathrm{D}(\mathbf{F}). \tag{5.19}$$

Any other λ-EVI *solution* $\mu' : [0, T') \to \overline{\mathrm{D}(\mathbf{F})}$ *starting from* μ_0 *coincides with* μ *in* $[0, \min\{T, T'\})$.

Proof (a) Let $\boldsymbol{\tau}, T, L$ positive constants such that $\mathscr{E}(\mu_0, \tau, T, L)$ is not empty for every $\tau \in (0, \boldsymbol{\tau})$. Thanks to Theorem 5.9(2), the family $M_\tau \in \mathscr{E}(\mu_0, \tau, T, L)$ satisfies the Cauchy condition in $\mathrm{C}([0, T]; \mathcal{P}_2(\mathsf{X}))$ so that there exists a unique limit curve

$$\mu = \lim_{\tau \downarrow 0} M_\tau$$

which is also Lipschitz in time, thanks to the a-priori bound (5.13). Theorem 5.9(4) shows that μ is a λ-EVI solution starting from μ_0 and the estimate (5.16) of Theorem 5.9(3) shows that any other λ-EVI solution in an interval $[0, T']$ starting from μ_0 should coincide with μ in the interval $[0, \min\{T', T\}]$.

Let us now check (5.17): we fix s, t such that $0 \le s < t < T$ and $h \in (0, T - t)$, and we set

$$s_\tau := \tau \lfloor s/\tau \rfloor \quad \text{and} \quad h_\tau := \tau \lfloor h/\tau \rfloor .$$

The curves

$$r \mapsto M_\tau(s_\tau + r) \quad \text{and} \quad r \mapsto M_\tau(s_\tau + h_\tau + r)$$

belong to $\mathscr{M}(M_\tau(s_\tau), \tau, t - s, L)$ and $\mathscr{M}(M_\tau(s_\tau + h_\tau), \tau, t - s, L)$, so that (5.15) yields

$$W_2(M_\tau(s_\tau + t - s), M_\tau(s_\tau + h_\tau + (t - s))) \le \mathrm{e}^{\lambda(t-s)} W_2(M_\tau(s_\tau), M_\tau(s_\tau + h_\tau)) + B\sqrt{\tau},$$

for $B = B(\lambda, L, \boldsymbol{\tau}, T)$. Passing to the limit as $\tau \downarrow 0$ we get

$$W_2(\mu_t, \mu_{t+h}) \le \mathrm{e}^{\lambda(t-s)} W_2(\mu_s, \mu_{s+h}).$$

Dividing by h and passing to the limit as $h \downarrow 0$ we get (5.17).

(b) Let us call $\mathcal{S}$ the collection of λ-EVI solutions $\mu : [0, S) \to \mathrm{D}(\mathbf{F})$ starting from μ_0 with values in $\mathrm{D}(\mathbf{F})$ and defined in some interval $[0, S)$, $S = S(\mu)$. Thanks to (5.18) and the previous claim the set $\mathcal{S}$ is not empty.

It is also easy to check that two curves $\mu', \mu'' \in \mathcal{S}$ coincide in the common domain $[0, S)$ with

$$S := \min\left\{S(\mu'), S(\mu'')\right\}.$$

Indeed, the set

$$\left\{t \in [0, S) : \mu'_r = \mu''_r \text{ if } 0 \le r \le t\right\}$$

contains $t = 0$, is closed since μ', μ'' are continuous, and it is also open since, if $\mu' = \mu''$ in $[0, t]$, then the previous claim and the fact that $\mu'_t = \mu''_t \in \mathrm{D}(\mathbf{F})$ show that $\mu' = \mu''$ also in a right neighborhood of t. Since $[0, S)$ is connected, we conclude that $\mu' = \mu''$ in $[0, S)$.

We can thus define

$$T := \sup\left\{S(\mu) : \mu \in \mathcal{S}\right\},$$

obtaining that there exists a unique λ-EVI solution μ starting from μ_0 and defined in $[0, T)$ with values in $\mathrm{D}(\mathbf{F})$.

If $T < \infty$, since μ is Lipschitz in $[0, T)$ thanks to (5.17), we know that there exists the limit

$$\bar{\mu} := \lim_{t \uparrow T} \mu_t$$

in $\mathcal{P}_2(\mathsf{X})$. If $\bar{\mu} \in \mathrm{D}(\mathbf{F})$ we can extend μ to a λ-EVI solution with values in $\mathrm{D}(\mathbf{F})$ and defined in an interval $[0, T')$ with $T' > T$, which contradicts the maximality of T. □

Recall that a set A in a metric space X is locally closed if every point of A has a neighborhood U such that $A \cap U = \bar{A} \cap U$. Equivalently, A is the intersection of an open and a closed subset of X. In particular, open or closed sets are locally closed.

We refer to Definition 5.1 for the notion of strict EVI solutions, used in the following.

Corollary 5.12 *Let* $\mathbf{F}$ *be a* λ*-dissipative* MPVF *according to* (4.1) *for which the Explicit Euler Scheme is locally solvable in* $\mathrm{D}(\mathbf{F})$*. If* $\mathrm{D}(\mathbf{F})$ *is locally closed then for every* $\mu_0 \in \mathrm{D}(\mathbf{F})$ *there exists a unique maximal strict and locally Lipschitz continuous* λ-EVI *solution* $\mu : [0, T) \to \mathrm{D}(\mathbf{F})$, $T \in (0, +\infty]$, *satisfying* (5.19).

Let us briefly discuss the question of local solvability of the Explicit Euler scheme. The main constraints of the Explicit Euler construction relies on the a priori stability bound and in the condition $M^n_\tau \in \mathrm{D}(\mathbf{F})$ for every step $0 \le n \le \mathrm{N}(T, \tau)$. This constraint is feasible if at each measure M^n_τ, $0 \le n < \mathrm{N}(T, \tau)$, the set $\mathrm{Adm}_{\tau,L}(M^n_\tau)$

defined by

$$\mathrm{Adm}_{\tau,L}(\mu) := \left\{\Phi \in \mathbf{F}[\mu] : |\Phi|_2 \le L \quad \text{and} \quad \exp_\sharp^\tau \Phi \in \mathrm{D}(\mathbf{F})\right\}$$

is not empty. If $\mathrm{D}(\mathbf{F})$ is open and $\mathbf{F}$ is locally bounded, then it is easy to check that the Explicit Euler scheme is locally solvable (see Lemma 5.13). We will adopt the following notation:

$$|\mathbf{F}|_2(\mu) := \inf\{|\Phi|_2 : \Phi \in \mathbf{F}[\mu]\} \quad \text{for every } \mu \in \mathrm{D}(\mathbf{F}), \tag{5.20}$$

and we will also introduce the upper semicontinuous envelope $|\mathbf{F}|_{2\star}$ of the function $|\mathbf{F}|_2$: i.e.

$$\begin{aligned}|\mathbf{F}|_{2\star}(\mu) &:= \inf_{\delta>0} \sup\{|\mathbf{F}|_2(\nu) : \nu \in \mathrm{D}(\mathbf{F}),\ W_2(\nu,\mu) \le \delta\}\\ &= \sup\left\{\limsup_{k\to\infty} |\mathbf{F}|_2(\mu_k) : \mu_k \in \mathrm{D}(\mathbf{F}),\ \mu_k \to \mu \text{ in } \mathcal{P}_2(\mathsf{X})\right\}.\end{aligned}$$

Lemma 5.13 *If* $\mathbf{F}$ *is a* λ*-dissipative* MPVF *according to* (4.1), $\mu_0 \in \mathrm{Int}(\mathrm{D}(\mathbf{F}))$ *and* $\mathbf{F}$ *is bounded in a neighborhood of* μ_0*, i.e. there exists* $\varrho > 0$ *such that* $|\mathbf{F}|_2$ *is bounded in* $\mathrm{B}(\mu_0, \varrho)$*, then the Explicit Euler scheme is locally solvable at* μ_0 *and the locally Lipschitz continuous solution* μ *given by Theorem* 5.11*(a) satisfies*

$$|\dot\mu_t|_+ \le e^{\lambda t} |\mathbf{F}|_{2\star}(\mu_0) \quad \textit{for all } t \in [0, T). \tag{5.21}$$

In particular, if $\mathrm{D}(\mathbf{F})$ *is open and* $\mathbf{F}$ *is locally bounded, for every* $\mu_0 \in \mathrm{D}(\mathbf{F})$ *there exists a unique maximal locally Lipschitz continuous* λ-EVI *solution* $\mu : [0, T) \to \mathcal{P}_2(\mathsf{X})$ *satisfying* (5.19) *and* (5.21).

Proof Let $\mu_0 \in \mathrm{Int}(\mathrm{D}(\mathbf{F}))$ and let $\varrho, L > 0$ so that $|\mathbf{F}|_2(\mu) < L$ for every $\mu \in \mathrm{B}(\mu_0, \varrho)$. We set

$$T := \varrho/(2L) \quad \text{and} \quad \boldsymbol{\tau} := \min\{T, 1\}$$

and we perform a simple induction argument to prove that

$$W_2(M_\tau^n, \mu_0) \le Ln\tau < \varrho$$

if $n \le \mathrm{N}(T, \tau)$, so that we can always find an element $\Phi_\tau^n \in \mathrm{Adm}_{\tau,L}(M_\tau^n)$. In fact, if $W_2(M_\tau^n, \mu_0) < Ln\tau$ and $n < \mathrm{N}(T, \tau)$ then

$$W_2(M_\tau^{n+1}, \mu_0) \le W_2(M_\tau^{n+1}, M_\tau^n) + W_2(M_\tau^n, \mu_0) \le L(n+1)\tau.$$

The property in (5.17) shows that $|\dot\mu_t|_+ \le Le^{\lambda t}$ for every $L > |\mathbf{F}|_{2\star}(\mu_0)$, so that we obtain (5.21). $\square$

More refined estimates will be discussed in the next sections. Here we will show another example, tailored to the case of measures with bounded support.

Proposition 5.14 *Let* $\mathbf{F}$ *be a* λ*-dissipative* MPVF *according to* (4.1). *Assume that* $\mathrm{D}(\mathbf{F}) \subset \mathcal{P}_b(\mathsf{X})$ *and for every* $\mu_0 \in \mathrm{D}(\mathbf{F})$ *there exist* $\varrho > 0$, $L > 0$ *such that, for every* $\mu \in \mathcal{P}_b(\mathsf{X})$ *with* $\mathrm{supp}(\mu) \subset \mathrm{supp}(\mu_0) + \mathrm{B}_{\mathsf{X}}(\varrho)$*, there exists* $\Phi \in \mathbf{F}[\mu]$ *such that*

$$\mathrm{supp}(\mathsf{v}_\sharp \Phi) \subset \mathrm{B}_{\mathsf{X}}(L).$$

Then for every $\mu_0 \in \mathrm{D}(\mathbf{F})$ *there exists* $T \in (0, +\infty]$ *and a unique maximal strict and locally Lipschitz continuous* λ-EVI *solution* $\mu : [0, T) \to \mathrm{D}(\mathbf{F})$ *satisfying* (5.19).

Proof Arguing as in the proof of Lemma 5.13, it is easy to check that setting $T := \varrho/4L$, $\boldsymbol{\tau} = \min\{T, 1\}$, we can find a discrete solution $(M_\tau, \boldsymbol{F}_\tau) \in \mathscr{E}(\mu_0, \tau, T, L)$ satisfying the more restrictive condition

$$\mathrm{supp}(M_\tau^n) \subset \mathrm{supp}(\mu_0) + \mathrm{B}_{\mathsf{X}}(Ln\tau) \subset \mathrm{supp}(\mu_0) + \mathrm{B}_{\mathsf{X}}(\varrho/2), \quad \text{and} \quad \mathrm{supp}(\mathsf{v}_\sharp \Phi_\tau^n) \subset \mathrm{B}_{\mathsf{X}}(L).$$

So that the Explicit Euler scheme is locally solvable and M_τ satisfies the uniform bound

$$\mathrm{supp}(M_\tau(t)) \subset \mathrm{supp}(\mu_0) + \mathrm{B}_{\mathsf{X}}(\varrho/2) \tag{5.22}$$

for every $t \in [0, T]$. Theorem 5.11 then yields the existence of a local solution, and Theorem 5.9(3) shows that the local solution satisfies the same bound (5.22) on the support, so that (5.18) holds. □

5.3 Stability and uniqueness

In the following theorem we prove a stability result for λ-EVI solutions of (5.1), as it occurs in the classical Hilbert case. We distinguish three cases: the first one assumes that the Explicit Euler scheme is locally solvable in $\mathrm{D}(\mathbf{F})$.

Theorem 5.15 (Uniqueness and Stability) *Let* $\mathbf{F}$ *be a* λ*-dissipative* MPVF *according to* (4.1) *such that the Explicit Euler scheme is locally solvable in* $\mathrm{D}(\mathbf{F})$*, and let* $\mu^1, \mu^2 : [0, T) \to \overline{\mathrm{D}(\mathbf{F})}$*,* $T \in (0, +\infty]$*, be* λ-EVI *solutions to (5.1). If* μ^1 *is strict, then*

$$W_2(\mu_t^1, \mu_t^2) \le W_2(\mu_0^1, \mu_0^2)\, \mathrm{e}^{\lambda_+ t} \quad \textit{for every } t \in [0, T). \tag{5.23}$$

In particular, if $\mu_0^1 = \mu_0^2$ *then* $\mu^1 \equiv \mu^2$ *in* $[0, T)$.
If μ^1, μ^2 *are both strict, then*

$$W_2(\mu_t^1, \mu_t^2) \le W_2(\mu_0^1, \mu_0^2)\, \mathrm{e}^{\lambda t} \quad \textit{for every } t \in [0, T). \tag{5.24}$$

Proof In order to prove (5.23), let us fix $t \in (0, T)$. Since the Explicit Euler scheme is locally solvable and $\mu_t^1 \in \mathrm{D}(\mathbf{F})$, there exist $\boldsymbol{\tau}, \delta, L$ such that $\mathscr{M}(\mu_t^1, \tau, \delta, L)$ is not empty for every $\tau \in (0, \boldsymbol{\tau})$. If $M_\tau^1 \in \mathscr{M}(\mu_t^1, \tau, \delta, L)$, then (5.16) yields

$$\begin{aligned} W_2(\mu_{t+h}^1, \mu_{t+h}^2) &\le W_2(M_\tau^1(h), \mu_{t+h}^2) + W_2(M_\tau^1(h), \mu_{t+h}^1) \\ &\le \delta\, W_2(\mu_t^1, \mu_t^2)\mathrm{e}^{\lambda_+ h} + B\sqrt{\tau} \quad \text{if } 0 \le h \le \delta, \end{aligned}$$

for $B = B(\lambda, L, \boldsymbol{\tau}, \delta)$ Passing to the limit as $\tau \downarrow 0$ we obtain

$$W_2(\mu_{t+h}^1, \mu_{t+h}^2) \le \delta\, W_2(\mu_t^1, \mu_t^2)\mathrm{e}^{\lambda_+ h}$$

and a further limit as $\delta \downarrow 1$ yields

$$W_2(\mu_{t+h}^1, \mu_{t+h}^2) \le W_2(\mu_t^1, \mu_t^2)\mathrm{e}^{\lambda_+ h}$$

for every $h \in [0, \delta]$, which implies that the map $t \mapsto \mathrm{e}^{-\lambda_+ t} W_2(\mu_t^1, \mu_t^2)$ is decreasing in $[t, t+\delta]$. Since t is arbitrary, we obtain (5.23).

In order to prove the estimate (5.24) (which is better than (5.23) when $\lambda < 0$), we argue in a similar way, using (5.15).

As before, for a given $t \in (0, T)$, since the Explicit Euler scheme is locally solvable and $\mu_t^1, \mu_t^2 \in \mathrm{D}(\mathbf{F})$, there exist $\boldsymbol{\tau}, \delta, L$ such that $\mathscr{M}(\mu_t^1, \tau, \delta, L)$ and $\mathscr{M}(\mu_t^2, \tau, \delta, L)$ are not empty for every $\tau \in (0, \boldsymbol{\tau})$. If $M_\tau^i \in \mathscr{M}(\mu_t^i, \tau, \delta, L)$, for $i = 1, 2$, (5.15) and (5.16) then yield

$$\begin{aligned} W_2(\mu_{t+h}^1, \mu_{t+h}^2) &\le W_2(\mu_{t+h}^1, M_\tau^1(h)) + W_2(M_\tau^1(h), M_\tau^2(h)) + W_2(\mu_{t+h}^2, M_\tau^2(h)) \\ &\le \mathrm{e}^{\lambda h} W_2(\mu_t^1, \mu_t^2) + B\sqrt{\tau} \end{aligned}$$

if $0 \le h \le \delta$, with $B = B(\lambda, L, \boldsymbol{\tau}, \delta)$. Passing to the limit as $\tau \downarrow 0$ we obtain

$$W_2(\mu_{t+h}^1, \mu_{t+h}^2) \le \mathrm{e}^{\lambda h} W_2(\mu_t^1, \mu_t^2)$$

which implies that the map $t \mapsto \mathrm{e}^{-\lambda t} W_2(\mu_t^1, \mu_t^2)$ is decreasing in $(0, T)$. □

It is possible to prove (5.24) by a direct argument depending on the definition of λ-EVI solution and a geometric condition on $\mathrm{D}(\mathbf{F})$. The simplest situation deals with absolutely continuous curves.

Theorem 5.16 (Stability for absolutely continuous solutions) *Let* $\mathbf{F}$ *be a* λ*-dissipative* MPVF *according to* (4.1) *and let* $\mu^1, \mu^2 : [0, T) \to \overline{\mathrm{D}(\mathbf{F})}$, $T \in (0, +\infty]$, *be* locally absolutely continuous λ-EVI *solutions to* (5.1). *If* $\Gamma_o^0(\mu_t^1, \mu_t^2|\mathbf{F}) \neq \emptyset$ *for a.e.* $t \in (0, T)$, *then* (5.24) *holds. In particular, if* $\mu_0^1 = \mu_0^2$ *then* $\mu^1 \equiv \mu^2$ *in* $[0, T)$.

Proof Since μ^1, μ^2 are locally absolutely continuous curves, we can apply Theorem 3.14 and find a subset $A \subset A(\mu^1) \cap A(\mu^2)$ of full Lebesgue measure such that (3.21) holds and $\Gamma_o^0(\mu_t^1, \mu_t^2|\mathbf{F}) \neq \emptyset$ for every $t \in A$. Selecting $\boldsymbol{\mu}_t \in \Gamma_o^0(\mu_t^1, \mu_t^2|\mathbf{F})$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}W_2^2(\mu_t^1,\mu_t^2)=\int\langle \boldsymbol{v}_t^1(x_1),x_1-x_2\rangle\,\mathrm{d}\boldsymbol{\mu}_t(x_1,x_2)+\int\langle \boldsymbol{v}_t^2(x_2),x_2-x_1\rangle\,\mathrm{d}\boldsymbol{\mu}_t(x_1,x_2).$$

Note that

$$\Gamma_0\left((\boldsymbol{i}_{\mathsf{X}},\boldsymbol{v}_t^1)_\sharp\mu_t^1,\boldsymbol{\mu}_t\right)=\Lambda\left((\boldsymbol{i}_{\mathsf{X}},\boldsymbol{v}_t^1)_\sharp\mu_t^1,\mu_t^2\right)=\left\{(\mathsf{x}^0,\boldsymbol{v}_t^1\circ\mathsf{x}^0,\mathsf{x}^1)_\sharp\boldsymbol{\mu}_t\right\},$$
$$\Gamma_0\left((\boldsymbol{i}_{\mathsf{X}},\boldsymbol{v}_t^2)_\sharp\mu_t^2,\mathsf{s}_\sharp\boldsymbol{\mu}_t\right)=\Lambda\left((\boldsymbol{i}_{\mathsf{X}},\boldsymbol{v}_t^2)_\sharp\mu_t^2,\mu_t^1\right)=\left\{(\mathsf{x}^1,\boldsymbol{v}_t^2\circ\mathsf{x}^1,\mathsf{x}^0)_\sharp\boldsymbol{\mu}_t\right\}$$

by [3, Lemma 5.3.2], where $\Gamma_0(\cdot,\cdot)$ is the set defined in (3.25) with $t=0$ and $\Lambda(\cdot,\cdot)$ is defined in Definition 3.8. Hence, using (5.5b), (5.5c) and recalling the definition of reversion map s in (3.26), for every $t\in A$ we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}W_2^2(\mu_t^1,\mu_t^2)&=[(\boldsymbol{i}_{\mathsf{X}},\boldsymbol{v}_t^1)_\sharp\mu_t^1,\boldsymbol{\mu}_t]_{r,0}+[(\boldsymbol{i}_{\mathsf{X}},\boldsymbol{v}_t^2)_\sharp\mu_t^2,\mathsf{s}_\sharp\boldsymbol{\mu}_t]_{r,0}\\&\le[\mathbf{F},\boldsymbol{\mu}_t]_{0+}+\lambda W_2^2(\mu_t^1,\mu_t^2)+[\mathbf{F},\mathsf{s}_\sharp\boldsymbol{\mu}_t]_{1-}\\&=\lambda W_2^2(\mu_t^1,\mu_t^2),\end{aligned}$$

where we also used the property

$$[\mathbf{F},\mathsf{s}_\sharp\boldsymbol{\mu}_t]_{1-}=-[\mathbf{F},\boldsymbol{\mu}_t]_{0+}.$$

□

The last situation deals with the comparison between an absolutely continuous and a merely continuous λ-EVI solution. The argument is technically more involved and takes inspiration from the proof of [23, Theorem 1.1]: we refer to the Introduction of [23] for an explanation of the heuristic idea.

Theorem 5.17 (Refined stability) *Let $T>0$ and $\mathbf{F}$ be a λ-dissipative* MPVF *according to* (4.1). *Let*

(i) $\mu^1:[0,T]\to\overline{\mathrm{D}(\mathbf{F})}$ *be an absolutely continuous* λ-EVI *solution for* $\mathbf{F}$, *with* $\mu_0^1\in\mathrm{D}(\mathbf{F})$*;*
(ii) $\mu^2:[0,T]\to\overline{\mathrm{D}(\mathbf{F})}$ *be* λ-EVI *solution for* $\mathbf{F}$.

If at least one of the following properties hold:

(1) $\Gamma_o^0(\mu_r^1,\mu_s^2|\mathbf{F})\neq\emptyset$ *for every* $s\in(0,T)$ *and* $r\in[0,T)\setminus N$ *with* $N\subset(0,T)$, $\mathcal{L}(N)=0$*;*
(2) μ^1 *satisfies* (5.2),

then

$$W_2(\mu_t^1,\mu_t^2)\le e^{\lambda t}W_2(\mu_0^1,\mu_0^2)\quad\textit{for every } t\in[0,T].$$

Proof We extend μ^1 in $(-\infty,0)$ with the constant value μ_0^1, denote by $\boldsymbol{v}$ the Wasserstein velocity field associated to μ^1 (and extended to 0 outside $A(\mu^1)$) and define the functions $w,f,h:(-\infty,T]\times[0,T]\to\mathbb{R}$ by

$$w(r,s) := W_2(\mu_r^1, \mu_s^2)$$
$$f(r,s) := \begin{cases} 2|\mathbf{F}|_2(\mu_0^1) w(0,s) & \text{if } r < 0, \\ 0 & \text{if } r \ge 0, \end{cases} \qquad h(r,s) := \begin{cases} 0 & \text{if } r < 0, \\ 2\left[(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}_r)_\sharp \mu_r^1, \mu_s^2\right]_r & \text{if } r \ge 0. \end{cases}$$

Theorem 3.11 yields

$$\frac{\partial}{\partial r} w^2(r,s) = h(r,s) \quad \text{in } \mathscr{D}'(-\infty, T), \text{ for every } s \in [0,T]. \tag{5.25}$$

In case (1) holds, writing (5.4b) for μ^2 with $\nu = \mu_r^1$ and $r \in (-\infty, T]\setminus N$, then for every $\boldsymbol{\mu}_{rs} \in \Gamma_o^0(\mu_r^1, \mu_s^2|\mathbf{F})$ we obtain

$$\frac{\mathrm{d}^+}{\mathrm{d}s} w^2(r,s) \le 2\lambda w^2(r,s) - 2[\mathbf{F}, \boldsymbol{\mu}_{rs}]_{0+} \quad \text{for } s \in (0,T) \text{ and } r \in (-\infty, T)\setminus N. \tag{5.26}$$

On the other hand (5.5b) yields

$$\begin{aligned} -2[\mathbf{F}, \boldsymbol{\mu}_{rs}]_{0+} &\le -2[(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}_r)_\sharp \mu_r^1, \boldsymbol{\mu}_{rs}]_{r,0} \\ &\le -2\left[(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}_r)_\sharp \mu_r^1, \mu_s^2\right]_r \quad \text{for every } r \in A(\mu^1)\setminus N, \\ -2[\mathbf{F}, \boldsymbol{\mu}_{rs}]_{0+} &\le 2|\mathbf{F}|_2(\mu_0^1) w(0,s) = f(r,s) \quad \text{for every } r < 0. \end{aligned} \tag{5.27}$$

Combining (5.26) and (5.27) we obtain

$$\frac{\mathrm{d}^+}{\mathrm{d}s} w^2(r,s) \le 2\lambda w^2(r,s) + f(r,s) - h(r,s) \quad \text{for } s \in (0,T),\ r \in (-\infty, 0] \cup A(\mu^1)\setminus N.$$

Since, recalling Theorem 2.10, we have $|h(r,s)| \le 2|\dot{\mu}_r^1|\, w(r,s)$, then applying Lemma B.4 we get

$$\frac{\partial}{\partial s} w^2(r,s) \le 2\lambda w^2(r,s) + f(r,s) - h(r,s) \quad \text{in } \mathscr{D}'(0,T), \text{ for a.e. } r \in (-\infty, T]. \tag{5.28}$$

The expression in (5.28) can also be deduced in case (2) using (5.2).

By multiplying both inequalities (5.25) and (5.28) by $e^{-2\lambda s}$ we get

$$\begin{aligned} \frac{\partial}{\partial r}\left(e^{-2\lambda s} w^2(r,s)\right) &= e^{-2\lambda s} h(r,s) \quad \text{in } \mathscr{D}'(-\infty, T) \text{ and every } s \in [0,T], \\ \frac{\partial}{\partial s}\left(e^{-2\lambda s} w^2(r,s)\right) &\le e^{-2\lambda s}\big(f(r,s) - h(r,s)\big) \quad \text{in } \mathscr{D}'(0,T) \text{ and a.e. } r \in (-\infty, T]. \end{aligned}$$

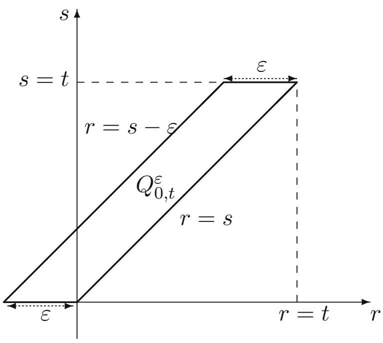

Fig. 1 Strip $Q^{\varepsilon}_{0,t}$ corresponding to penalization about the diagonal $\{r = s\}$

We fix $t \in [0, T]$ and $\varepsilon > 0$ and we apply the Divergence theorem in [23, Lemma 6.15] on the two-dimensional strip $Q^{\varepsilon}_{0,t}$ as in Fig. 1,

$$Q^{\varepsilon}_{0,t} := \{(r, s) \in \mathbb{R}^2 \mid 0 \le s \le t\, ,\ s - \varepsilon \le r \le s\}, \tag{5.29}$$

and we get

$$\int_{t-\varepsilon}^{t} e^{-2\lambda t} w^2(r, t)\, \mathrm{d}r \le \int_{-\varepsilon}^{0} w^2(r, 0)\, \mathrm{d}r + \iint_{Q^{\varepsilon}_{0,t}} e^{-2\lambda s} f(r, s)\, \mathrm{d}r\, \mathrm{d}s.$$

Using

$$w(t, t) \le \int_{r}^{t} |\dot{\mu}^1_u|\, \mathrm{d}u + w(r, t) \le \int_{t-\varepsilon}^{t} |\dot{\mu}^1_u|\, \mathrm{d}u + w(r, t) \quad \text{if } t - \varepsilon \le r \le t,$$

then, for every $\delta, \delta_\star > 1$ conjugate coefficients ($\delta_\star = \delta/(\delta - 1)$), we get

$$w^2(t, t) \le \delta w^2(r, t) + \delta_\star \left(\int_{t-\varepsilon}^{t} |\dot{\mu}^1_u|\, \mathrm{d}u \right)^2. \tag{5.30}$$

Integrating (5.30) w.r.t. r in the interval $(t - \varepsilon, t)$, we obtain

$$e^{-2\lambda t} w^2(t, t) \le \frac{\delta}{\varepsilon} \int_{t-\varepsilon}^{t} e^{-2\lambda t} w^2(r, t)\, \mathrm{d}r + \delta_\star \left(\int_{t-\varepsilon}^{t} |\dot{\mu}^1_u|\, \mathrm{d}u \right)^2 \max\{1, e^{2|\lambda|T}\}. \tag{5.31}$$

Finally, we have the following inequality

$$\varepsilon^{-1} \iint_{Q^{\varepsilon}_{0,t}} e^{-2\lambda s} f(r, s)\, \mathrm{d}r\, \mathrm{d}s \le 2|\mathbf{F}|_2(\mu^1_0) \int_{0}^{\varepsilon} e^{-2\lambda s} w(0, s)\, \mathrm{d}s. \tag{5.32}$$

Summing up (5.31) and (5.32) we obtain

$$
\begin{aligned}
e^{-2\lambda t} w^2(t) \le \delta \left(w^2(0) + 2|\mathbf{F}|_2(\mu_0^1) \int_0^{\varepsilon} e^{-2\lambda s} w(0,s)\,\mathrm{d}s \right) \\
+\delta_\star \left(\int_{t-\varepsilon}^{t} |\dot{\mu}_u^1|\,\mathrm{d}u \right)^2 \max\{1, e^{2|\lambda|T}\}.
\end{aligned}
$$

where we have used the notation $w(s) = w(s,s)$. Taking the limit as $\varepsilon \downarrow 0$ and $\delta \downarrow 1$, we obtain the thesis. □

Corollary 5.18 (Local Lipschitz estimate) *Let* $\mathbf{F}$ *be a* λ*-dissipative* MPVF *according to* (4.1) *and let* $\mu : (0,T) \to \overline{\mathrm{D}(\mathbf{F})}$*,* $T \in (0, +\infty]$*, be a* λ-EVI *solution to* (5.1)*. If at least one of the following two conditions holds*

(a) μ *is strict and* (EE) *is locally solvable in* $\mathrm{D}(\mathbf{F})$*,*
(b) μ *is locally absolutely continuous and* (4.24) *holds,*

then μ *is locally Lipschitz and*

$$
t \mapsto \mathrm{e}^{-\lambda t} |\dot{\mu}_t|_+ \quad \textit{is decreasing in } (0,T). \tag{5.33}
$$

Proof Since for every $h > 0$ the curve $t \mapsto \mu_{t+h}$ is a λ-EVI solution, (5.24) yields

$$
\mathrm{e}^{-\lambda(t-s)} W_2(\mu_{t+h}, \mu_t) \le W_2(\mu_{s+h}, \mu_s)
$$

for every $0 < s < t$. Dividing by h and taking the limsup as $h \downarrow 0$, we get (5.33), which in turn shows the local Lipschitz character of μ. □

5.4 Global existence and generation of λ-flows

We collect here a few simple results on the existence of global solutions and the generation of a λ-flow. A first result can be deduced from the global solvability of the Explicit Euler scheme.

Theorem 5.19 (Global existence) *Let* $\mathbf{F}$ *be a* λ*-dissipative* MPVF *according to* (4.1)*. If the Explicit Euler Scheme is globally solvable at* $\mu_0 \in \mathrm{D}(\mathbf{F})$*, then there exists a unique global and locally Lipschitz continuous* λ-EVI *solution* $\mu : [0,\infty) \to \overline{\mathrm{D}(\mathbf{F})}$ *starting from* μ_0*.*

Proof We can argue as in the proof of Theorem 5.11(a), observing that the global solvability of (EE) allows for the construction of a limit solution on every interval $[0,T]$, $T > 0$. □

Let us provide a simple condition ensuring global solvability, whose proof is deferred to Sect. 6.

Proposition 5.20 *Let* $\mathbf{F}$ *be a* λ*-dissipative* MPVF *according to* (4.1). *Assume that for every* $R > 0$ *there exist* $M = \mathrm{M}(R) > 0$ *and* $\bar{\tau} = \bar{\tau}(R) > 0$ *such that, for every* $\mu \in \mathrm{D}(\mathbf{F})$ *with* $\mathsf{m}_2(\mu) \le R$ *and every* $0 < \tau \le \bar{\tau}$,

$$\textit{there exists } \Phi \in \mathbf{F}[\mu] \textit{ s.t. } |\Phi|_2 \le \mathrm{M}(R) \textit{ and } \exp^\tau_\sharp \Phi \in \mathrm{D}(\mathbf{F}). \tag{5.34}$$

Then the Explicit Euler scheme is globally solvable in $\mathrm{D}(\mathbf{F})$.

Global existence of λ-EVI solution is also related to the existence of a λ-flow.

Definition 5.21 We say that the λ-dissipative MPVF $\mathbf{F}$, according to (4.1), *generates a* λ*-flow* if for every $\mu_0 \in \overline{\mathrm{D}(\mathbf{F})}$ there exists a unique λ-EVI solution $\mu = \mathrm{S}[\mu_0]$ starting from μ_0 and the maps $\mu_0 \mapsto \mathrm{S}_t[\mu_0] = (\mathrm{S}[\mu_0])_t$ induce a semigroup of Lipschitz transformations $(\mathrm{S}_t)_{t \ge 0}$ of $\overline{\mathrm{D}(\mathbf{F})}$ satisfying

$$W_2(\mathrm{S}_t[\mu_0], \mathrm{S}_t[\mu_1]) \le \mathrm{e}^{\lambda t} W_2(\mu_0, \mu_1) \quad \text{for every } t \ge 0. \tag{5.35}$$

Theorem 5.22 (Generation of a λ-flow) *Let* $\mathbf{F}$ *be a* λ*-dissipative* MPVF *according to* (4.1). *If at least one of the following properties is satisfied:*

(a) *the Explicit Euler Scheme is globally solvable for every* μ_0 *in a dense subset of* $\mathrm{D}(\mathbf{F})$*;*
(b) *the Explicit Euler Scheme is locally solvable in* $\mathrm{D}(\mathbf{F})$ *and, for every* μ_0 *in a dense subset of* $\mathrm{D}(\mathbf{F})$*, there exists a strict global* λ-EVI *solution starting from* μ_0*;*
(c) *the Explicit Euler Scheme is locally solvable in* $\mathrm{D}(\mathbf{F})$ *and* $\mathrm{D}(\mathbf{F})$ *is closed;*
(d) *for every* $\mu_0 \in \mathrm{D}(\mathbf{F})$*,* $\mu_1 \in \overline{\mathrm{D}(\mathbf{F})}$ *we have* $\Gamma^0_o(\mu_0, \mu_1|\mathbf{F}) \neq \emptyset$ *and, for every* μ_0 *in a dense subset of* $\mathrm{D}(\mathbf{F})$*, there exists a locally absolutely continuous strict global* λ-EVI *solution starting from* μ_0*;*
(e) *for every* μ_0 *in a dense subset of* $\mathrm{D}(\mathbf{F})$*, there exists a locally absolutely continuous solution of* (5.2) *starting from* μ_0*,*

then $\mathbf{F}$ *generates a* λ*-flow.*

Proof (a) Let D be the dense subset of $\mathrm{D}(\mathbf{F})$ for which (EE) is globally solvable. For every $\mu_0 \in D$ we define $\mathrm{S}_t[\mu_0]$, $t \ge 0$, as the value at time t of the unique λ-EVI solution starting from μ_0, whose existence is guaranteed by Theorem 5.19.
If $\mu_0, \mu_1 \in D$, $T > 0$, we can find $\boldsymbol{\tau}, L$ such that $\mathscr{M}(\mu_0, \tau, T, L)$ and $\mathscr{M}(\mu_1, \tau, T, L)$ are not empty for every $\tau \in (0, \boldsymbol{\tau})$. We can then pass to the limit in the uniform estimate (5.15) for every choice of $M^i_\tau \in \mathscr{M}(\mu_i, \tau, T, L)$, $i = 0, 1$, obtaining (5.35) for every $\mu_0, \mu_1 \in D$.
We can then extend the map S_t to $\overline{D} = \overline{\mathrm{D}(\mathbf{F})}$ still preserving the same property. Proposition 5.6 shows that for every $\mu_0 \in \overline{\mathrm{D}(\mathbf{F})}$ the continuous curve $t \mapsto \mathrm{S}_t[\mu_0]$ is a λ-EVI solution starting from μ_0.
Finally, if $\mu : [0, T') \to \overline{\mathrm{D}(\mathbf{F})}$ is any λ-EVI solution starting from μ_0, we can apply (5.16) to get

$$W_2(\mu_t, M^1_\tau(t)) \le \Big(2W_2(\mu_0, \mu_1) + C(\boldsymbol{\tau}, L, T)\sqrt{\tau}\Big)\mathrm{e}^{\lambda_+ t} \tag{5.36}$$

for every $t \in [0, T]$, $T < T'$ and $\tau < \boldsymbol{\tau}$, where $C(\boldsymbol{\tau}, L, T) > 0$ is a suitable constant. Passing to the limit as $\tau \downarrow 0$ in (5.36) we obtain

$$W_2(\mu_t, \mathrm{S}_t[\mu_1]) \le 2W_2(\mu_0, \mu_1)\mathrm{e}^{\lambda_+ t} \quad \text{for every } t \in [0, T]. \tag{5.37}$$

Choosing now a sequence $\mu_{1,n}$ in D converging to μ_0 and observing that we can choose arbitrary $T < T'$, we eventually get $\mu_t = \mathrm{S}_t[\mu_0]$ for every $t \in [0, T')$.

(b) Let D be the dense subset of $\mathrm{D}(\mathbf{F})$ such that there exists a global strict λ-EVI solution starting from D. By Theorem 5.15 such a solution is unique and the corresponding family of solution maps $\mathrm{S}_t : D \to \overline{\mathrm{D}(\mathbf{F})}$ satisfy (5.35). Arguing as in the previous claim, we can extend S_t to $\overline{\mathrm{D}(\mathbf{F})}$ still preserving (5.35) and the fact that $t \mapsto \mathrm{S}_t[\mu_0]$ is a λ-EVI solution.
If μ is λ-EVI solution starting from μ_0, Theorem 5.15 shows that (5.37) holds for every $\mu_1 \in D$. By approximation we conclude that $\mu_t = \mathrm{S}_t[\mu_0]$.

(c) Corollary 5.12 shows that for every initial datum $\mu_0 \in \mathrm{D}(\mathbf{F})$ there exists a global λ-EVI solution. We can then apply Claim (b).

(d) Let D be the dense subset of $\mathrm{D}(\mathbf{F})$ such that there exists a locally absolutely continuous strict global λ-EVI solution starting from D. By Theorem 5.16 such a solution is the unique locally absolutely continuous solution starting from μ_0 and the corresponding family of solution maps $\mathrm{S}_t : D \to \overline{\mathrm{D}(\mathbf{F})}$ satisfy (5.35). Arguing as in the previous claim (b), we can extend S_t to $\overline{\mathrm{D}(\mathbf{F})}$ still preserving (5.35) (again thanks to Theorem 5.16) and the fact that $t \mapsto \mathrm{S}_t[\mu_0]$ is a λ-EVI solution.
If μ is a λ-EVI solution starting from $\mu_0 \in \overline{\mathrm{D}(\mathbf{F})}$ and $(\mu_0^n)_{n\in\mathbb{N}} \subset D$ is a sequence converging to μ_0, we can apply Theorem 5.17(1) and conclude that $\mu_t = \mathrm{S}_t[\mu_0]$.

(e) The proof follows by the same argument of the previous claim, eventually applying Theorem 5.17(2). □

By Lemma 5.13 we immediately get the following result.

Corollary 5.23 *If* $\mathbf{F}$ *is locally bounded* λ*-dissipative* MPVF *according to* (4.1), *with* $\mathrm{D}(\mathbf{F}) = \mathcal{P}_2(\mathsf{X})$, *then for every* $\mu_0 \in \mathcal{P}_2(\mathsf{X})$ *there exists a unique global* λ-EVI *solution starting from* μ_0.

We conclude this section by showing a consistency result with the Hilbertian theory, related to the example of Sect. 7.2.

Corollary 5.24 (Consistency with the theory of contraction semigroups in Hilbert spaces) *Let* $F \subset \mathsf{X} \times \mathsf{X}$ *be a dissipative maximal subset generating the semigroup* $(R_t)_{t\ge 0}$ *of nonlinear contractions* [7, Theorem 3.1]. *Let* $\mathbf{F}$ *be the dissipative* MPVF *according to* (4.1), *defined by*

$$\mathbf{F} := \{\Phi \in \mathcal{P}_2(\mathsf{TX}) \mid \Phi \textit{ is concentrated on } F\}.$$

The semigroup $\mu_0 \mapsto \mathrm{S}_t[\mu_0] := (R_t)_\sharp\mu_0$, $t \ge 0$, *is the* 0*-flow generated by* $\mathbf{F}$ *in* $\overline{\mathrm{D}(\mathbf{F})}$.

Proof Let D be the set of discrete measures $\frac{1}{n}\sum_{j=1}^n \delta_{x_j}$ with $x_j \in \mathrm{D}(F)$. Since every $\mu_0 \in \overline{\mathrm{D}(\mathbf{F})}$ is supported in $\overline{\mathrm{D}(F)}$, D is dense in $\overline{\mathrm{D}(\mathbf{F})}$. Our thesis follows by applying

Theorem 5.22(e) if we show that for every $\mu_0^n = \frac{1}{n}\sum_{j=1}^n \delta_{x_{j,0}} \in D$ there exists a locally absolutely continuous solution $\mu^n : [0,\infty) \to D$ of (5.2) starting from μ_0^n.

It can be directly checked that

$$\mu_t^n := (R_t)_\sharp \mu_0^n = \frac{1}{n}\sum_{j=1}^n \delta_{x_{j,t}}, \quad x_{j,t} := R_t(x_{j,0})$$

satisfies the continuity equation with Wasserstein velocity vector $\boldsymbol{v}_t$ (defined on the finite support of μ_t^n) satisfying

$$\boldsymbol{v}_t(x_{j,t}) = \dot{x}_{j,t} = F^\circ(x_{j,t}) \quad \text{and} \quad |\boldsymbol{v}_t(x_{j,t})| \le |F^\circ(x_{j,0})|$$

for every $j = 1, \dots, n$, and a.e. $t > 0$, where F° is the minimal selection of F. It follows that

$$(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}_t)_\sharp \mu_t^n \in \mathbf{F}[\mu_t^n] \quad \text{for a.e. } t > 0,$$

so that μ^n is a Lipschitz EVI solution for $\mathbf{F}$ starting from μ_0^n. We can thus conclude observing that the map $\mu_0 \mapsto (R_t)_\sharp\mu_0$ is a contraction in $\mathcal{P}_2(\mathsf{X})$ and the curve $\mu_t^n = (R_t)_\sharp \mu_0^n$ is continuous with values in $\overline{\mathrm{D}(\mathbf{F})}$. □

5.5 Barycentric property

If we assume that the MPVF $\mathbf{F}$ is a sequentially closed subset of $\mathcal{P}_2^{sw}(\mathsf{TX})$ with convex sections, we are able to provide a stronger result showing a particular property satisfied by the solutions of (5.1) (see Theorem 5.27). This is called *barycentric property* and it is strictly connected with the weaker definition of solution discussed in [9, 26, 27].

We first introduce a directional closure of $\mathbf{F}$ along smooth cylindrical deformations. We set

$$\exp^\varphi(x) := x + \nabla\varphi(x)$$

for every $\varphi \in \mathrm{Cyl}(\mathsf{X})$, and

$$\overline{\mathbf{F}}[\mu] := \left\{ \Phi \in \mathcal{P}_2(\mathsf{X}) \;\middle|\; \begin{array}{l} \exists\, \varphi \in \mathrm{Cyl}(\mathsf{X}),\ (r_n)_{n\in\mathbb{N}} \subset [0,+\infty),\ r_n \downarrow 0, \\ \Phi_n \in \mathbf{F}[\exp_\sharp^{r_n\varphi}\mu] : \ \Phi_n \to \Phi \text{ in } \mathcal{P}_2^{sw}(\mathsf{TX}) \end{array} \right\}. \tag{5.38}$$

Definition 5.25 (Barycentric property) Let $\mathbf{F}$ be a MPVF. We say that a locally absolutely continuous curve $\mu : \mathcal{I} \to \mathrm{D}(\mathbf{F})$ satisfies the *barycentric property* (resp. the *relaxed barycentric property*) if for a.e. $t \in \mathcal{I}$ there exists $\Phi_t \in \mathbf{F}[\mu_t]$ (resp. $\Phi_t \in \overline{\mathrm{co}}(\overline{\mathbf{F}}[\mu_t])$) such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathsf{X}} \varphi(x)\,\mathrm{d}\mu_t(x) = \int_{\mathsf{TX}} \langle \nabla\varphi(x), v\rangle\,\mathrm{d}\Phi_t(x,v) \quad \text{for every } \varphi \in \mathrm{Cyl}(\mathsf{X}). \tag{5.39}$$

Notice that $\mathbf{F} \subset \overline{\mathbf{F}} \subset \mathrm{cl}(\mathbf{F})$ and $\overline{\mathbf{F}} = \mathbf{F}$ if $\mathbf{F}$ is sequentially closed in $\mathcal{P}_2^{sw}(\mathsf{TX})$. Recalling Proposition 4.17(a) we also get

$$\overline{\mathrm{co}}(\overline{\mathbf{F}}) \subset \hat{\mathbf{F}},$$

so that the relaxed barycentric property implies the corresponding property for the extended MPVF $\hat{\mathbf{F}}$ defined in (4.22). In particular, considering the directional closure $\overline{\mathbf{F}}$ in place of the sequential closure $\mathrm{cl}(\mathbf{F})$ not only allows us to obtain a finer result, but it could be easier to compute when one considers specific examples, being $\bar{\mathbf{F}}$ the closure of $\mathbf{F}$ along regular directions.

Remark 5.26 If $\mathsf{X} = \mathbb{R}^d$, the property stated in Definition 5.25 coincides with the weak definition of solution to (5.1) given in [26].

The aim is to prove that the λ-EVI solution of (5.1) enjoys the barycentric property of Definition 5.25, under suitable mild conditions on $\mathbf{F}$. This is strictly related to the behaviour of $\mathbf{F}$ along the family of smooth deformations induced by cylindrical functions. Let us denote by $\mathbf{pr}_\mu$ the orthogonal projection in $L^2_\mu(\mathsf{X};\mathsf{X})$ onto the tangent space $\mathrm{Tan}_\mu\,\mathcal{P}_2(\mathsf{X})$ and by $\boldsymbol{b}_\Phi$ the barycenter of Φ as in Definition 3.1.

Theorem 5.27 *Let* $\mathbf{F}$ *be a* λ*-dissipative* MPVF *according to* (4.1). *Assume that for every* $\mu \in \mathrm{D}(\mathbf{F})$ *there exist constants* $M, \varepsilon > 0$ *such that*

$$\exp_\sharp^\varphi \mu \in \mathrm{D}(\mathbf{F}) \quad \textit{and} \quad |\mathbf{F}|_2(\exp_\sharp^\varphi \mu) \le M \tag{5.40}$$

for every $\varphi \in \mathrm{Cyl}(\mathsf{X})$ *such that* $\sup_{\mathsf{X}} |\nabla\varphi| \le \varepsilon$. *If* $\mu : \mathcal{I} \to \mathrm{D}(\mathbf{F})$ *is a locally absolutely continuous* λ-EVI *solution of* (5.1) *with Wasserstein velocity field* $\boldsymbol{v}$ *satisfying* (2.6) *for every* t *in the subset* $A(\mu) \subset \mathcal{I}$ *of full Lebesgue measure, then*

$$\textit{for every } t \in A(\mu) \textit{ there exists } \Phi_t \in \overline{\mathrm{co}}(\overline{\mathbf{F}})[\mu_t] \textit{ such that } \quad \boldsymbol{v}_t = \mathbf{pr}_{\mu_t} \circ \boldsymbol{b}_{\Phi_t}. \tag{5.41}$$

In particular, μ *satisfies the relaxed barycentric property.*

If moreover $\overline{\mathbf{F}} = \mathbf{F}$ *and, for every* $\nu \in \mathrm{D}(\mathbf{F})$, *the section* $\mathbf{F}[\nu]$ *is a convex subset of* $\mathcal{P}_2(\mathsf{TX})$, *i.e.*

$$\mathbf{F}[\nu] = \mathrm{co}(\mathbf{F})[\nu],$$

then μ *satisfies the barycentric property* (5.39).

Proof We divide the proof of (5.41) into two steps.

Claim 1 *Let* $t \in A(\mu)$ *and* $M = M_t$ *be the constant associated to the measure* μ_t *in* (5.40). *Then* $\boldsymbol{v}_t \in \overline{\mathrm{co}}(K_t)$, *where*

$$K_t := \left\{ \mathbf{pr}_{\mu_t}(\boldsymbol{b}_\Phi) \ : \ \Phi \in \overline{\mathbf{F}}[\mu_t],\ |\Phi|_2 \le M_t \right\} \subset \mathrm{Tan}_{\mu_t}\,\mathcal{P}_2(\mathsf{X}). \tag{5.42}$$

Proof of Claim 1 For every $\zeta \in \mathrm{Cyl}(\mathsf{X})$ there exists $\delta = \delta(\zeta) > 0$ such that $\nu^\zeta := \exp_\sharp^{-\delta\zeta} \mu_t \in \mathrm{D}(\mathbf{F})$ and $\boldsymbol{\sigma}^\zeta := (\boldsymbol{i}_\mathsf{X}, \exp^{-\delta\zeta})_\sharp \mu_t \in \Gamma_o^{01}(\mu_t, \nu^\zeta|\mathbf{F})$ is the unique optimal transport plan between μ_t and ν^ζ.

Thanks to Theorem 3.11, the map $s \mapsto W_2^2(\mu_s, \nu^\zeta)$ is differentiable at $s = t$, moreover by employing also (5.5b), it holds

$$\delta \int_\mathsf{X} \langle \boldsymbol{v}_t(x), \nabla\zeta(x)\rangle \,\mathrm{d}\mu_t(x) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2} W_2^2(\mu_t, \nu^\zeta) \le [\mathbf{F}, \boldsymbol{\sigma}^\zeta]_{0+} = \lim_{s\downarrow 0} [\mathbf{F}, \boldsymbol{\sigma}^\zeta]_{l,s}. \quad (5.43)$$

We can choose a decreasing vanishing sequence $(s_k)_{k\in\mathbb{N}} \subset (0, 1)$, measures $\nu_k^\zeta := \mathsf{x}_\sharp^{s_k} \boldsymbol{\sigma}^\zeta$ and $\Phi_k^\zeta \in \mathbf{F}[\nu_k^\zeta]$ such that $\sup_k |\Phi_k^\zeta|_2 \le M_t$ and $\Phi_k^\zeta \to \Phi^\zeta$ in $\mathcal{P}_2^{sw}(\mathsf{TX})$. Then, by (5.16), we get $\Phi^\zeta \in \overline{\mathbf{F}}[\mu_t]$ with $|\Phi^\zeta|_2 \le M_t$ and by (5.43) and the upper semicontinuity of $[\cdot, \cdot]_l$ (see Lemma 3.15) we get

$$\delta \int_\mathsf{X} \langle \boldsymbol{v}_t(x), \nabla\zeta(x)\rangle \,\mathrm{d}\mu_t(x) \le \left[\Phi^\zeta, \nu^\zeta\right]_l = \delta \int_{\mathsf{TX}} \langle v, \nabla\zeta(x)\rangle \,\mathrm{d}\Phi^\zeta(x, v). \quad (5.44)$$

Indeed, notice that, by [3, Lemma 5.3.2], we have $\Lambda(\Phi^\zeta, \nu^\zeta) = \{\Phi^\zeta \otimes \nu^\zeta\}$ with $(\mathsf{x}^0, \mathsf{x}^1)_\sharp(\Phi^\zeta \otimes \nu^\zeta) = \boldsymbol{\sigma}^\zeta$.

By means of the identity highlighted in Remark 3.2, the expression in (5.44) can be written as follows

$$\langle \boldsymbol{v}_t, \nabla\zeta\rangle_{L^2_{\mu_t}(\mathsf{X};\mathsf{X})} \le \langle \boldsymbol{b}_{\Phi^\zeta}, \nabla\zeta\rangle_{L^2_{\mu_t}(\mathsf{X};\mathsf{X})} = \langle \mathbf{pr}_{\mu_t}(\boldsymbol{b}_{\Phi^\zeta}), \nabla\zeta\rangle_{L^2_{\mu_t}(\mathsf{X};\mathsf{X})}$$

so that

$$\langle \boldsymbol{v}_t, \nabla\zeta\rangle_{L^2_{\mu_t}(\mathsf{X};\mathsf{X})} \le \sup_{\boldsymbol{b}\in K_t} \langle \boldsymbol{b}, \nabla\zeta\rangle_{L^2_{\mu_t}(\mathsf{X};\mathsf{X})}$$

for all $\zeta \in \mathrm{Cyl}(\mathsf{X})$, with K_t as in (5.42). Applying Lemma B.3 in $\mathrm{Tan}_{\mu_t} \mathcal{P}_2(\mathsf{X}) \subset L^2_{\mu_t}(\mathsf{X};\mathsf{X})$ we obtain that $\boldsymbol{v}_t \in \overline{\mathrm{co}}(K_t)$.

Claim 2 *For every $\boldsymbol{w} \in \overline{\mathrm{co}}(K_t)$ there exists $\Psi \in \overline{\mathrm{co}}(\overline{\mathbf{F}})[\mu_t]$ such that $\boldsymbol{w} = \mathbf{pr}_{\mu_t} \circ \boldsymbol{b}_\Psi$.*

Proof of Claim 2 Notice that an element $\boldsymbol{w} \in \mathrm{Tan}_\mu \mathcal{P}_2(\mathsf{X})$ coincides with $\mathbf{pr}_\mu(\boldsymbol{b}_\Psi)$ for $\Psi \in \mathcal{P}_2(\mathsf{TX}|\mu)$ if and only if

$$\int \langle \boldsymbol{w}, \nabla\zeta\rangle \,\mathrm{d}\mu = \int \langle v, \nabla\zeta\rangle \,\mathrm{d}\Psi(x, v) \quad (5.45)$$

for every $\zeta \in \mathrm{Cyl}(\mathsf{X})$. It is easy to check that any element $\boldsymbol{w} \in \mathrm{co}(K)_t$ can be represented as $\mathbf{pr}_{\mu_t}(\boldsymbol{b}_\Psi)$ (and thus as in (5.45)) for some $\Psi \in \mathrm{co}(\overline{\mathbf{F}}[\mu_t])$. If $\boldsymbol{w} \in \overline{\mathrm{co}}(K_t)$ we can find a sequence $(\Psi_n)_{n\in\mathbb{N}} \subset \mathrm{co}(\overline{\mathbf{F}}[\mu_t])$ such that $|\Psi_n|_2 \le M_t$ and $\boldsymbol{w}_n = \mathbf{pr}_{\mu_t}(\boldsymbol{b}_{\Psi_n}) \to \boldsymbol{w}$ in $L^2_{\mu_t}(\mathsf{X};\mathsf{X})$. Since the sequence $(\Psi_n)_{n\in\mathbb{N}}$ is relatively compact in $\mathcal{P}_2^{sw}(\mathsf{TX})$ by Proposition 2.15(2), we can extract a (not relabeled) subsequence converging to a limit Ψ in $\mathcal{P}_2^{sw}(\mathsf{TX})$, as $n \to +\infty$. By definition $\Psi \in \overline{\mathrm{co}}(\overline{\mathbf{F}}[\mu_t])$ with $|\Psi|_2 \le M_t$. We can eventually pass to the limit in (5.45) written for $\boldsymbol{w}_n$ and Ψ_n thanks to $\mathcal{P}_2^{sw}(\mathsf{TX})$ convergence, obtaining the corresponding identity for $\boldsymbol{w}$ and Ψ in the limit.

The thesis (5.41) follows by Claim 1 and Claim 2.

Finally, being μ locally absolutely continuous, it satisfies the continuity equation driven by $\boldsymbol{v}$ in the sense of distributions (see Theorem 2.10), so that by (5.41) we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathsf{X}}\zeta(x)\,\mathrm{d}\mu_t(x) &= \int_{\mathsf{X}}\langle\nabla\zeta(x), \boldsymbol{v}_t(x)\rangle\,\mathrm{d}\mu_t(x)\\ &= \int_{\mathsf{TX}}\langle\nabla\zeta(x), v\rangle\,\mathrm{d}\Phi_t(x,v) \quad \text{for all } \zeta\in \mathrm{Cyl}(\mathsf{X}),\end{aligned}$$

for all $t \in A(\mu)$. ☐

Remark 5.28 We notice that it is always possible to estimate the value of M_t in (5.42) by $|\mathbf{F}|_{2\star}(\mu_t)$.

Remark 5.29 Using a standard approximation argument (see for example the proof of Lemma 5.1.12(f) in [3]) it is possible to show that actually the barycentric property (5.39) holds for every $\varphi \in \mathrm{C}^{1,1}(\mathsf{X};\mathbb{R})$ (indeed, in this case, $\nabla\varphi \in \mathrm{Tan}_\mu\,\mathcal{P}_2(\mathsf{X})$ for every $\mu \in \mathcal{P}_2(\mathsf{X})$).

Remark 5.30 We point out that the result stated in Theorem 5.27 is still valid if we replace the convex hull of $\mathbf{F}$ defined in (4.19) using the “flat” structure of $\mathcal{P}_2(\mathsf{TX})$, with the following one which makes use of plan interpolations

$$\widetilde{\mathrm{co}}(\mathbf{F})(\nu) := \left\{ \left(\mathsf{x}, \sum_{k_1}^{N}\alpha_k \mathsf{v}_k\right)_\sharp \boldsymbol{\Phi} \;\middle|\; \begin{array}{l}\boldsymbol{\Phi}\in\mathcal{P}(\mathsf{X}^{N+1}),\ (\mathsf{x},\mathsf{v}_k)_\sharp\boldsymbol{\Phi} = \Phi_k,\ \Phi_k \in \mathbf{F}[\nu],\\ \alpha_k \ge 0,\ k=1,\dots,N,\ \sum_{k=1}^N \alpha_k = 1,\ N\in\mathbb{N}\end{array}\right\},$$

for any $\nu \in \mathrm{D}(\mathbf{F})$, where

$$\mathsf{x}(x, v_1, \dots, v_N) = x \quad\text{and}\quad \mathsf{v}_k(x, v_1,\dots,v_N) = v_k,\ \ k = 1,\dots,N.$$

Indeed, $\mathrm{co}(\mathbf{F})(\nu)$ and $\widetilde{\mathrm{co}}(\mathbf{F})(\nu)$ share the same barycentric projection. However, while $\mathrm{co}(\mathbf{F})$ preserves dissipativity as proved in Proposition 4.16, $\widetilde{\mathrm{co}}(\mathbf{F})(\nu)$ does not satisfy this property in general, as highlighted in the following example: let $\mathsf{X} = \mathbb{R}$ and consider the PVF $\mathbf{F}$, with domain $\mathrm{D}(\mathbf{F}) = \left\{\delta_0, \frac{1}{2}\delta_1 + \frac{1}{2}\delta_0\right\}$, defined by

$$\mathbf{F}[\delta_0] := \frac{1}{2}\delta_{(0,3)} + \frac{1}{2}\delta_{(0,-3)}, \qquad \mathbf{F}\left[\frac{1}{2}\delta_1 + \frac{1}{2}\delta_0\right] := \frac{1}{2}\delta_{(1,2)} + \frac{1}{2}\delta_{(0,1)}.$$

Then $\mathbf{F}$ is dissipative, indeed

$$\left[\mathbf{F}[\delta_0], \mathbf{F}\left[\frac{1}{2}\delta_1 + \frac{1}{2}\delta_0\right]\right]_r \le -1 \le 0.$$

However, $\widetilde{\mathrm{co}}(\mathbf{F})$ is not dissipative, indeed, if we take $\delta_{(0,0)} \in \widetilde{\mathrm{co}}(\mathbf{F})[\delta_0]$, we have

$$\left[\delta_{(0,0)}, \mathbf{F}\left[\frac{1}{2}\delta_1 + \frac{1}{2}\delta_0\right]\right]_r = 2 > 0.$$

As a complement to the studies investigated in this section, we prove the converse characterization of Theorem 5.27 in the particular case of *regular measures* or *regular vector fields*. We refer to [3, Definitions 6.2.1, 6.2.2] for the definition of $\mathcal{P}_2^r(\mathsf{X})$, that is the space of regular measures on X. When $\mathsf{X} = \mathbb{R}^d$ has finite dimension, $\mathcal{P}_2^r(\mathsf{X})$ is just the subset of measures in $\mathcal{P}_2(\mathsf{X})$ which are absolutely continuous w.r.t. the d-dimensional Lebesgue measure $\mathcal{L}^d$.

Theorem 5.31 *Let* $\mathbf{F}$ *be a* λ*-dissipative* MPVF *according to* (4.1). *Let* $\mu : \mathcal{I} \to \mathrm{D}(\mathbf{F})$ *be a locally absolutely continuous curve satisfying the relaxed barycentric property of Definition 5.25. If for a.e.* $t \in \mathcal{I}$ *at least one of the following properties holds:*

(1) $\mu_t \in \mathcal{P}_2^r(\mathsf{X})$,
(2) $\overline{\mathbf{F}}[\mu_t]$ *contains a unique element* Φ_t *concentrated on a map, i.e.* $\Phi_t = (\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{b}_{\Phi_t})_\sharp \mu_t$

then μ *is* λ-EVI *solution of* (5.1).

Proof Take $\varphi \in \mathrm{Cyl}(\mathsf{X})$ and observe that, since μ has the relaxed barycentric property, then for a.e. $t \in \mathcal{I}$ (recall Theorem 3.11) there exists $\Phi_t \in \overline{\mathrm{co}}(\overline{\mathbf{F}}[\mu_t])$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathsf{X}} \varphi(x)\,\mathrm{d}\mu_t(x) = \int_{\mathsf{TX}} \langle \nabla\varphi(x), v\rangle\,\mathrm{d}\Phi_t = \int_{\mathsf{X}} \langle \nabla\varphi, \mathbf{pr}_{\mu_t} \circ \boldsymbol{b}_{\Phi_t}\rangle\,\mathrm{d}\mu_t = \int_{\mathsf{X}} \langle \boldsymbol{v}_t, \nabla\varphi\rangle\,\mathrm{d}\mu_t,$$

hence μ solves the continuity equation $\partial_t \mu_t + \mathrm{div}(\boldsymbol{v}_t \mu_t) = 0$, with $\boldsymbol{v}_t = \mathbf{pr}_{\mu_t} \circ \boldsymbol{b}_{\Phi_t} \in \mathrm{Tan}_{\mu_t} \mathcal{P}_2(\mathsf{X})$. By Theorem 3.11, we also know that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t, \nu) = \int_{\mathsf{X}^2} \langle \boldsymbol{v}_t(x_0), x_0 - x_1\rangle\,\mathrm{d}\boldsymbol{\gamma}_t(x_0, x_1) \tag{5.46}$$

for any $t \in A(\mu, \nu)$, $\boldsymbol{\gamma}_t \in \Gamma_o(\mu_t, \nu)$, $\nu \in \mathcal{P}_2(\mathsf{X})$. Possibly disregarding a Lebesgue negligible set, we can decompose the set $A(\mu, \nu)$ in the union $A_1 \cup A_2$, where A_1, A_2 correspond to the times t for which the properties (1) and (2) hold.

If $t \in A_1$ and $\nu \in \mathrm{D}(\mathbf{F})$, then by [3, Theorem 6.2.10], since $\mu_t \in \mathcal{P}_2^r(\mathsf{X})$, there exists a unique $\boldsymbol{\gamma}_t \in \Gamma_o(\mu_t, \nu)$ and $\boldsymbol{\gamma}_t = (\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{r}_t)_\sharp \mu_t$ for some map $\boldsymbol{r}_t$ s.t. $\boldsymbol{i}_{\mathsf{X}} - \boldsymbol{r}_t \in \mathrm{Tan}_{\mu_t} \mathcal{P}_2(\mathsf{X}) \subset L^2_{\mu_t}(\mathsf{X}; \mathsf{X})$ (recall [3, Proposition 8.5.2]), so that

$$\begin{aligned}
\int_{\mathsf{X}^2} \langle \boldsymbol{v}_t(x_0), x_0 - x_1\rangle\,\mathrm{d}\boldsymbol{\gamma}_t(x_0, x_1) &= \int_{\mathsf{X}} \langle \boldsymbol{v}_t(x_0), x_0 - \boldsymbol{r}_t(x_0)\rangle\,\mathrm{d}\mu_t(x_0) \\
&= \int_{\mathsf{X}} \langle \boldsymbol{b}_{\Phi_t}, x_0 - \boldsymbol{r}_t(x_0)\rangle\,\mathrm{d}\mu_t(x_0) \\
&= \int_{\mathsf{TX}} \langle v, x - \boldsymbol{r}_t(x)\rangle\,\mathrm{d}\Phi_t(x, v) \\
&= [\Phi_t, \nu]_r\,,
\end{aligned} \tag{5.47}$$

where we also applied Theorem 3.9 and Remark 3.19, recalling that in this case $\Lambda(\Phi_t, \nu)$ is a singleton.

If $t \in A_2$ we can select the optimal plan $\boldsymbol{\gamma}_t \in \Gamma_o(\mu_t, \nu)$ along which

$$[\Phi_t, \nu]_r = [\Phi_t, \boldsymbol{\gamma}_t]_{r,0} = \int_{\mathsf{X}} \langle \boldsymbol{b}_{\Phi_t}(x_0), x_0 - x_1 \rangle \, \mathrm{d}\boldsymbol{\gamma}_t(x_0, x_1).$$

If $\boldsymbol{r}_t$ is the barycenter of $\boldsymbol{\gamma}_t$ with respect to its first marginal μ_t, recalling that $\boldsymbol{i}_{\mathsf{X}} - \boldsymbol{r}_t \in \mathrm{Tan}_{\mu_t} \mathcal{P}_2(\mathsf{X})$ (see also the proof of [3, Thm. 12.4.4]) we also get

$$\begin{aligned}
\int_{\mathsf{X}^2} \langle \boldsymbol{v}_t(x_0), x_0 - x_1 \rangle \, \mathrm{d}\boldsymbol{\gamma}_t(x_0, x_1) &= \int_{\mathsf{X}} \langle \boldsymbol{v}_t(x_0), x_0 - \boldsymbol{r}_t(x_0) \rangle \, \mathrm{d}\mu_t(x_0) \\
&= \int_{\mathsf{X}} \langle \boldsymbol{b}_{\Phi_t}(x_0), x_0 - \boldsymbol{r}_t(x_0) \rangle \, \mathrm{d}\mu_t(x_0) \\
&= \int_{\mathsf{X}} \langle \boldsymbol{b}_{\Phi_t}(x_0), x_0 - x_1 \rangle \, \mathrm{d}\boldsymbol{\gamma}_t(x_0, x_1) \\
&= [\Phi_t, \nu]_r \,, \qquad (5.48)
\end{aligned}$$

where we still applied Theorem 3.9 and Remark 3.19.

Combining (5.46) with (5.47) and (5.48) we eventually get

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} W_2^2(\mu_t, \nu) = [\Phi_t, \nu]_r \le -[\Psi, \mu_t]_r + \lambda W_2^2(\mu_t, \nu) \quad \text{for every } \Psi \in \mathbf{F}[\nu],$$

by definition of $\hat{\mathbf{F}}$ and the fact that $\overline{\mathrm{co}}(\overline{\mathbf{F}})[\mu_t] \subset \hat{\mathbf{F}}[\mu_t]$. $\square$

Thanks to Theorem 5.31, we can apply to barycentric solutions the uniqueness and approximation results of the previous Sections. We conclude this section with a general result on the existence of a λ-flow for λ-dissipative MPVFs, which is the natural refinement of Proposition 5.14

Theorem 5.32 (Generation of λ-flow) *Let* $\mathbf{F}$ *be a* λ*-dissipative* MPVF *according to* (4.1). *Assume that* $\mathcal{P}_b(\mathsf{X}) \subset \mathrm{D}(\mathbf{F})$ *and for every* $\mu_0 \in \mathcal{P}_b(\mathsf{X})$ *there exist* $\varrho > 0$ *and* $L > 0$ *such that, for every* μ *with* $\mathrm{supp}(\mu) \subset \mathrm{supp}(\mu_0) + \mathrm{B}_{\mathsf{X}}(\varrho)$*,*

$$\textit{there exists } \Phi \in \mathbf{F}[\mu] \textit{ s.t. } \mathrm{supp}(\mathsf{v}_\sharp \Phi) \subset \mathrm{B}_{\mathsf{X}}(L). \qquad (5.49)$$

Let $\mathbf{F}_b := \mathbf{F} \cap \mathcal{P}_b(\mathsf{TX})$*. If there exists* $a \ge 0$ *such that for every* $\Phi \in \mathbf{F}_b$

$$\mathrm{supp}(\Phi) \subset \left\{ (x, v) \in \mathsf{TX} : \langle v, x \rangle \le a(1 + |x|^2) \right\}, \qquad (5.50)$$

then $\mathbf{F}$ *generates a* λ*-flow.*

Proof It is enough to prove that $\mathbf{F}_b$ generates a λ-flow. Applying Proposition 5.14 to the MPVF $\mathbf{F}_b$, we know that for every $\mu_0 \in \mathrm{D}(\mathbf{F}_b)$ there exists a unique maximal strict and locally Lipschitz continuous λ-EVI solution $\mu : [0, T) \to \mathcal{P}_b(\mathsf{X})$ driven by

$\mathbf{F}_b$ and satisfying (5.19). We argue by contradiction, and we assume that $T < +\infty$. Notice that by (5.49) $\mathbf{F}$ satisfies (5.40), so that μ is a relaxed barycentric solution for $\mathbf{F}_b$. Since $\mu_0 \in \mathcal{P}_b(\mathsf{X})$, we know that $\operatorname{supp}(\mu_0) \subset \mathrm{B}_{\mathsf{X}}(r_0)$ for some $r_0 > 1$.

It is easy to check that (5.50) holds also for every $\Phi \in \overline{\mathrm{co}}(\overline{\mathbf{F}}_b)$. Moreover, setting $b := 2a$, condition (5.50) yields

$$\langle v, x \rangle \le b|x|^2 \quad \text{for every} \quad (x, v) \in \operatorname{supp} \Phi \in \mathbf{F}_b, \ |x| \ge 1. \tag{5.51}$$

Let $\phi(r) : \mathbb{R} \to \mathbb{R}$ be any smooth increasing function such that $\phi(r) = 0$ if $r \le r_0$ and $\phi(r) = 1$ if $r \ge r_0 + 1$, and let $\varphi(t, x) := \phi(|x|\mathrm{e}^{-bt})$. Clearly $\varphi \in \mathrm{C}^{1,1}(\mathsf{X} \times [0, +\infty))$, with

$$\begin{aligned}
\nabla\varphi(t, x) &= \frac{x}{|x|}\phi'(|x|\mathrm{e}^{-bt})\mathrm{e}^{-bt} \ \text{ if } x \ne 0,\\
\nabla\varphi(t, 0) &= 0,\\
\partial_t\varphi(t, x) &= -b\phi'(|x|\mathrm{e}^{-bt})|x|\mathrm{e}^{-bt}.
\end{aligned}$$

We thus have for a.e. $t \in [0, T)$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathsf{X}} \varphi(t, x)\,\mathrm{d}\mu_t &= \mathrm{e}^{-bt}\int_{\mathsf{TX}} \Big(- b\phi'(|x|\mathrm{e}^{-bt})|x| + \langle v, x\rangle |x|^{-1}\phi'(|x|\mathrm{e}^{-bt})\Big)\mathrm{d}\Phi_t(v, x)\\
&\le \mathrm{e}^{-bt}\int_{\mathsf{TX}} \Big(- b\phi'(|x|\mathrm{e}^{-bt})|x| + b|x|\phi'(|x|\mathrm{e}^{-bt})\Big)\mathrm{d}\Phi_t(v, x) = 0
\end{aligned}$$

where in the last inequality we used (5.51) and the fact that the integrand vanishes if $|x| \le 1$. We get

$$\int_{\mathsf{X}} \varphi(t, x)\,\mathrm{d}\mu_t = 0 \quad \text{in } [0, T);$$

this implies that $\operatorname{supp}(\mu_t) \subset \mathrm{B}_{\mathsf{X}}((r_0 + 1)\mathrm{e}^{bt})$ so that the limit measure μ_T belongs to $\mathcal{P}_b(\mathsf{X})$ as well, leading to a contradiction with (5.19) for $\mathbf{F}_b$.

We deduce that μ is a global strict λ-EVI solution for $\mathbf{F}_b$. We can then apply Theorem 5.22(b) to $\mathbf{F}_b$. □

6 Explicit Euler scheme

In this section, we collect all the main estimates concerning the Explicit Euler scheme (EE) of Definition 5.7. For the sequel, we recall the notations

$$M_\tau(\cdot) \quad \text{and} \quad \bar{M}_\tau(\cdot)$$

for the affine and piecewise constant interpolations, respectively, of the sequence (M_τ^n, Φ_τ^n) in (EE). We also recall the notations

$$\mathscr{E}(\mu_0, \tau, T, L) \quad \text{and} \quad \mathscr{M}(\mu_0, \tau, T, L)$$

for the (possibly empty) set of all the curves $(M_\tau, \boldsymbol{F}_\tau)$ and M_τ, respectively, arising from the solution of (EE).

6.1 The Explicit Euler scheme: preliminary estimates

Our first step is to prove simple a priori estimates and a discrete version of (λ-EVI) as a consequence of Proposition 3.4.

Proposition 6.1 *Every solution $(M_\tau, \boldsymbol{F}_\tau) \in \mathscr{E}(\mu_0, \tau, T, L)$ of (EE) satisfies*

$$W_2(M_\tau(t), \mu_0) \le Lt, \quad |\boldsymbol{F}_\tau(t)|_2 \le L \quad \textit{for every } t \in [0, T], \tag{6.1}$$

$$W_2(M_\tau(t), M_\tau(s)) \le L|t-s| \quad \textit{for every } s, t \in [0, T], \tag{6.2}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(M_\tau(t), \nu) \le [\boldsymbol{F}_\tau(t), \nu]_r + \tau|\boldsymbol{F}_\tau(t)|_2^2 \le [\boldsymbol{F}_\tau(t), \nu]_r + \tau L^2 \tag{IEVI}$$

for every $t \in [0, T]$ and $\nu \in \mathcal{P}_2(\mathsf{X})$, with possibly countable exceptions. In particular

$$\frac{1}{2}W_2^2(M_\tau^{n+1}, \nu) - \frac{1}{2}W_2^2(M_\tau^n, \nu) \le \tau\left[\Phi_\tau^n, \nu\right]_r + \frac{1}{2}\tau^2 L^2 \tag{6.3}$$

for every $0 \le n < \mathrm{N}(T, \tau)$ and $\nu \in \mathcal{P}_2(\mathsf{X})$.

Proof The second inequality of (6.1) is a trivial consequence of the definition of $\mathscr{E}(\mu_0, \tau, T, L)$, the first inequality is a particular case of (6.2). The estimate (6.2) is immediate if $n\tau \le s < t \le (n+1)\tau$ since

$$\begin{aligned} W_2(M_\tau(s), M_\tau(t)) &= W_2((\exp^{s-n\tau})_\sharp \Phi_\tau^n, (\exp^{t-n\tau})_\sharp \Phi_\tau^n) \\ &\le \sqrt{\int_{\mathsf{TX}} |(t-s)v)|^2 \,\mathrm{d}\Phi_\tau^n} \\ &= (t-s)\sqrt{\int_{\mathsf{TX}} |v|^2 \,\mathrm{d}\Phi_\tau^n} \\ &\le (t-s)L. \end{aligned}$$

This implies that the metric velocity of M_τ is bounded by L in $[0, T]$ and therefore M_τ is L-Lipschitz.

Let us recall that for every $\nu \in \mathcal{P}_2(\mathsf{X})$ and $\Phi \in \mathcal{P}_2(\mathsf{TX})$ the function $g(t) := \frac{1}{2}W_2^2(\mathsf{exp}^t_\sharp \Phi, \nu)$ satisfies

$$
\begin{aligned}
&t \mapsto g(t) - \frac{1}{2}t^2|\Phi|_2^2 \text{ is concave}, \quad g_r'(0) = [\Phi, \nu]_r\,, \\
&g'(t) \le [\Phi, \nu]_r + t|\Phi|_2^2 \quad \text{for } t \ge 0,
\end{aligned}
\tag{6.4}
$$

by Definition 3.5 and Proposition 3.4. In particular, the concavity yields the differentiability of g with at most countable exceptions. Thus, taking any $n \in \mathbb{N}$, $0 \le n < \mathrm{N}(T, \tau)$, $t \in [n\tau, (n+1)\tau)$ and $\Phi = \Phi^n_\tau$ so that $\mathsf{exp}^t_\sharp \Phi = M_\tau(t)$, (6.4) yields (IEVI). The inequality in (6.3) follows by integration in each interval $[n\tau, (n+1)\tau]$. □

In the following, we prove a uniform bound on curves $M_\tau \in \mathscr{M}(\mu_0, \tau, T, L)$ which is useful to prove global solvability of the Explicit Euler scheme, as stated in Proposition 5.20. We will use the Gronwall estimates of Lemma B.1 and Lemma B.2.

Proposition 6.2 *Let* $\mathbf{F}$ *be a* λ*-dissipative* MPVF *according to* (4.1)*. Assume that for every* $R > 0$ *there exist* $M = \mathrm{M}(R) > 0$ *and* $\bar\tau = \bar\tau(R) > 0$ *such that, for every* $\mu \in \mathrm{D}(\mathbf{F})$ *with* $\mathsf{m}_2(\mu) \le R$ *and every* $0 < \tau \le \bar\tau$,

$$
\textit{there exists } \Phi \in \mathbf{F}[\mu] \textit{ s.t. } |\Phi|_2 \le \mathrm{M}(R) \textit{ and } \mathsf{exp}^\tau_\sharp \Phi \in \mathrm{D}(\mathbf{F}). \tag{6.5}
$$

Then the Explicit Euler scheme is globally solvable in $\mathrm{D}(\mathbf{F})$*. More precisely, if for a given* $\mu_0 \in \mathrm{D}(\mathbf{F})$ *with* $\Psi_0 \in \mathbf{F}[\mu_0]$, $\mathsf{m}_0 := \mathsf{m}_2(\mu_0)$, *and we set*

$$
R := \mathsf{m}_0 + \Big(|\Psi_0|_2 + 1\Big)\sqrt{2T}\mathrm{e}^{(1+2\lambda_+)T}, \quad L := \mathrm{M}(R), \quad \boldsymbol{\tau} = \min\left\{\frac{1}{L^2}, \bar\tau(R), T\right\}, \tag{6.6}
$$

then for every $\tau \in (0, \boldsymbol{\tau}]$ *the set* $\mathscr{E}(\mu_0, \tau, T, L)$ *is not empty.*

Proof We want to prove by induction that for every integer $N \le \mathrm{N}(T, \tau)$, (EE) has a solution up to the index N satisfying the upper bound

$$
\mathsf{m}_2(M^N_\tau) \le R, \tag{6.7}
$$

corresponding to the constants R, L given by (6.6). For $N = 0$ the statement is trivially satisfied. Assuming that $0 \le N < \mathrm{N}(T, \tau)$ and elements (M^n_τ, Φ^n_τ), $0 \le n < N$, M^N_τ, are given satisfying (EE) and (6.7), we want to show that we can perform a further step of the Euler Scheme so that (EE) is solvable up to the index $N+1$ and $\mathsf{m}_2(M^{N+1}_\tau) \le R$.

Notice that by the induction hypothesis, for $n = 0, \ldots, N-1$, we have $|\Phi^n_\tau|_2 \le L$; since $\mathsf{m}_2(M^N_\tau) \le R$, by (6.5) we can select $\Phi^N_\tau \in \mathbf{F}[M^N_\tau]$ with $|\Phi^N_\tau|_2 \le L$ such that $M^{N+1}_\tau = \mathsf{exp}^\tau_\sharp \Phi^N_\tau \in \mathrm{D}(\mathbf{F})$. Using (6.3) with $\nu = \mu_0$, the λ-dissipativity with $\Psi_0 \in \mathbf{F}[\mu_0]$

$$
\big[\Phi^n_\tau, \mu_0\big]_r \le \lambda W_2^2(M^n_\tau, \mu_0) - \big[\Psi_0, M^n_\tau\big]_r\,,
$$

and the bound

$$-\left[\Psi_0, M_\tau^n\right]_r \le \frac{1}{2} W_2^2(M_\tau^n, \mu_0) + \frac{1}{2}|\Psi_0|_2^2,$$

we end up with

$$\frac{1}{2} W_2^2(M_\tau^{n+1}, \mu_0) - \frac{1}{2} W_2^2(M_\tau^n, \mu_0) \le \frac{\tau^2}{2} L^2 + \tau \left(\frac{1}{2} + \lambda_+\right) W_2^2(M_\tau^n, \mu_0) + \frac{\tau}{2}|\Psi_0|_2^2,$$

for every $n \le N$. Using the Gronwall estimate of Lemma B.2 we get

$$W_2(M_\tau^n, \mu_0) \le \sqrt{T+\tau}\Big(|\Psi_0|_2 + \sqrt{\tau} L\Big) \mathrm{e}^{(\frac{1}{2}+\lambda_+)(T+\tau)} \le \sqrt{2T}\Big(|\Psi_0|_2 + 1\Big) \mathrm{e}^{(1+2\lambda_+)T}$$

for every $n \le N+1$, so that

$$\mathsf{m}_2(M_\tau^{N+1}) \le \mathsf{m}_0 + \sqrt{2T}\Big(|\Psi_0|_2 + 1\Big) \mathrm{e}^{(1+2\lambda_+)T} \le R.$$

□

We conclude this section by proving the stability estimate (5.15) of Theorem 5.9. We introduce the notation

$$I_\kappa(t) := \int_0^t \mathrm{e}^{\kappa r}\, \mathrm{d}r = \frac{1}{\kappa}(\mathrm{e}^{\kappa t} - 1) \quad \text{if } \kappa \ne 0; \quad I_0(t) := t.$$

Notice that for every $t \ge 0$

$$I_\kappa(t) \le t\mathrm{e}^{\kappa t} \quad \text{if } \kappa \ge 0. \tag{6.8}$$

Proposition 6.3 *Let $M_\tau \in \mathscr{M}(\mu_0, \tau, T, L)$ and $M'_\tau \in \mathscr{M}(\mu'_0, \tau, T, L)$. If $\lambda_+ \tau \le 2$ then*

$$W_2(M_\tau(t), M'_\tau(t)) \le W_2(\mu_0, \mu'_0)\mathrm{e}^{\lambda t} + 8L\sqrt{t\tau}\Big(1 + |\lambda|\sqrt{t\tau}\Big)\mathrm{e}^{\lambda_+ t}$$

for every $t \in [0, T]$.

Proof Let us set $w(t) := W_2(M_\tau(t), M'_\tau(t))$. Since by Proposition 3.4(2), in every interval $[n\tau, (n+1)\tau]$ the function $t \mapsto w^2(t) - 4L^2(t - n\tau)^2$ is concave, with

$$\frac{\mathrm{d}}{\mathrm{d}t} w^2(t)\bigg|_{t=n\tau+} = 2\left[\boldsymbol{F}_\tau(t), \boldsymbol{F}'_\tau(t)\right]_r \le 2\lambda W_2^2(\bar{M}_\tau(t), \bar{M}'_\tau(t)),$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} w^2(t) \le 2\lambda W_2^2(\bar{M}_\tau(t), \bar{M}'_\tau(t)) + 8L^2\tau$$

for every $t \in [0, T]$, with possibly countable exceptions. Using the identity

$$a^2 - b^2 = 2b(a-b) + |a-b|^2$$

with $a = W_2(\bar{M}_\tau(t), \bar{M}'_\tau(t))$ and $b = W_2(M_\tau(t), M'_\tau(t))$ and observing that

$$|a-b| \le W_2(\bar{M}_\tau(t), M_\tau(t)) + W_2(\bar{M}'_\tau(t), M'_\tau(t)) \le 2L\tau,$$

we eventually get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} w^2(t) &\le 2\lambda w^2(t) + 8L^2\tau + 8|\lambda| L\tau w(t) + \lambda_+ 8L^2\tau^2 \\ &\le 2\lambda w^2(t) + 8|\lambda| L\tau w(t) + 24L^2\tau,\end{aligned}$$

since $\lambda_+\tau \le 2$ by assumption. The Gronwall estimate in Lemma B.1 and (6.8) yield

$$\begin{aligned}w(t) &\le \Big(w^2(0)\mathrm{e}^{2\lambda t} + 24L^2\tau \mathrm{I}_{2\lambda}(t)\Big)^{1/2} + 8|\lambda| L\tau \mathrm{I}_\lambda(t) \\ &\le w(0)\mathrm{e}^{\lambda t} + 8L\sqrt{t\tau}\Big(1 + |\lambda|\sqrt{t\tau}\Big)\mathrm{e}^{\lambda_+ t}.\end{aligned}$$

□

6.2 Error estimates for the Explicit Euler Scheme

This subsection is devoted to the proof of the core of Theorem 5.9. In particular, we prove a Cauchy estimate for the affine interpolant of the Explicit Euler Scheme under different step sizes and a uniform (optimal, see [31]) error estimate between the affine interpolation and the λ-EVI solution for $\mathbf{F}$. We stress that the results obtained for the affine interpolant of the sequence generated by the Explicit Euler Scheme in Definition 5.7 can be adjusted for the piecewise contant interpolant $\bar{M}_\tau(\cdot)$ thanks to (5.14).

Theorem 6.4 *Let* $\mathbf{F}$ *be a* λ*-dissipative* MPVF *according to* (4.1). *If* $M_\tau \in \mathscr{M}(M^0_\tau, \tau, T, L)$, $M_\eta \in \mathscr{M}(M^0_\eta, \eta, T, L)$ *with* $\lambda\sqrt{T(\tau+\eta)} \le 1$, *then for every* $\delta > 1$ *there exists a constant* $C(\delta)$ *such that*

$$W_2(M_\tau(t), M_\eta(t)) \le \Big(\sqrt{\delta} W_2(M^0_\tau, M^0_\eta) + C(\delta) L\sqrt{(\tau+\eta)(t+\tau+\eta)}\Big)\mathrm{e}^{\lambda_+ t}$$

for every $t \in [0, T]$.

Proof We argue as in the Proof of Theorem 5.17. Since λ-dissipativity implies λ'-dissipativity for $\lambda' \ge \lambda$, it is not restrictive to assume $\lambda > 0$. We set $\sigma := \tau + \eta$. We will extensively use the a priori bounds (6.1) and (6.2); in particular,

$$W_2(M_\tau(t), \bar{M}_\tau(t)) \le L\tau, \quad W_2(M_\eta(t), \bar{M}_\eta(t)) \le L\eta.$$

We will also extend M_τ and $\bar{M}_\tau$ for negative times by setting

$$M_\tau(t) = \bar{M}_\tau(t) = M_\tau^0, \quad \boldsymbol{F}_\tau(t) = M_\tau^0 \otimes \delta_0 \quad \text{if } t < 0. \tag{6.9}$$

The proof is divided into several steps.

1. Doubling variables.

We fix a final time $t \in [0, T]$ and two variables $r, s \in [0, t]$ together with the functions

$$\begin{aligned} w(r,s) &:= W_2(M_\tau(r), M_\eta(s)), & w_\tau(r,s) &:= W_2(\bar{M}_\tau(r), M_\eta(s)), \\ w_\eta(r,s) &:= W_2(M_\tau(r), \bar{M}_\eta(s)), & w_{\tau,\eta}(r,s) &:= W_2(\bar{M}_\tau(r), \bar{M}_\eta(s)), \end{aligned} \tag{6.10}$$

observing that

$$\max\left\{|w - w_\tau|, |w_\eta - w_{\tau,\eta}|\right\} \le L\tau, \quad \max\left\{|w - w_\eta|, |w_\tau - w_{\tau,\eta}|\right\} \le L\eta. \tag{6.11}$$

By Proposition 6.1, we can write (IEVI) for M_τ and get

$$\frac{\partial}{\partial r}\frac{1}{2}W_2^2(M_\tau(r), \nu_1) \le \tau|\boldsymbol{F}_\tau(r)|_2^2 + [\boldsymbol{F}_\tau(r), \nu_1]_r \quad \text{for every } \nu_1 \in \mathcal{P}_2(\mathsf{X}); \tag{IEVI$_\tau$}$$

and for M_η obtaining

$$\begin{aligned} &\frac{\partial}{\partial s}\frac{1}{2}W_2^2(M_\eta(s), \nu_2) \\ &\quad \le \eta|\boldsymbol{F}_\eta(s)|_2^2 + \left[\boldsymbol{F}_\eta(s), \nu_2\right]_r \\ &\quad \le \eta|\boldsymbol{F}_\eta(s)|_2^2 + \lambda W_2^2(\bar{M}_\eta(s), \nu_2) - \left[\Phi, \bar{M}_\eta(s)\right]_r \quad \text{for every } \Phi \in \mathbf{F}[\nu_2],\ \nu_2 \in \mathrm{D}(\mathbf{F}). \end{aligned} \tag{IEVI$_\eta$}$$

Apart from possible countable exceptions, (IEVI$_\tau$) holds for $r \in (-\infty, t]$ and (IEVI$_\eta$) for $s \in [0, t]$. Taking $\nu_1 = \bar{M}_\eta(s)$, $\nu_2 = \bar{M}_\tau(r)$, $\Phi = \boldsymbol{F}_\tau(\max\{r, 0\}) \in \mathbf{F}[\bar{M}_\tau(r)]$, summing the two inequalities (IEVI$_{\tau,\eta}$), setting

$$f(r,s) := \begin{cases} 2LW_2(\bar{M}_\eta(s), M_\tau(0)) = 2Lw_\eta(0,s) & \text{if } r < 0, \\ 0 & \text{if } r \ge 0, \end{cases}$$

using (6.1) and the λ-dissipativity of $\mathbf{F}$, we obtain

$$\frac{\partial}{\partial r}w_\eta^2(r,s) + \frac{\partial}{\partial s}w_\tau^2(r,s) \le 2\lambda w_{\tau,\eta}^2(r,s) + 2L^2\sigma + f(r,s)$$

in $(-\infty, t] \times [0, t]$ (see also [23, Lemma 6.15]). By multiplying both sides by $e^{-2\lambda s}$, we have

$$\frac{\partial}{\partial r}\mathrm{e}^{-2\lambda s}w_\eta^2 + \frac{\partial}{\partial s}\mathrm{e}^{-2\lambda s}w_\tau^2 \le \left(2\lambda\left(w_{\tau,\eta}^2 - w_\tau^2\right) + f + 2L^2\sigma\right)\mathrm{e}^{-2\lambda s}. \tag{6.12}$$

Using (6.11), the inequalities

$$
\begin{aligned}
&w_{\tau,\eta} + w_\tau = w_{\tau,\eta} - w_\tau + 2(w_\tau - w) + 2w \leq 2L\sigma + 2w,\\
&|w(r,s) - w(s,s)| \leq L|r-s|
\end{aligned}
$$

and the elementary inequality $a^2 - b^2 \leq |a-b||a+b|$, we get

$$
2\big(w^2_{\tau,\eta}(r,s) - w^2_\tau(r,s)\big) \leq R_{r,s}, \quad \text{if } r,s \leq t,
$$

where $R_{r,s} := 4L^2\sigma(\sigma + |r-s|) + 4L\sigma w(s,s)$. Thus (6.12) becomes

$$
\frac{\partial}{\partial r} \mathrm{e}^{-2\lambda s} w^2_\eta + \frac{\partial}{\partial s} \mathrm{e}^{-2\lambda s} w^2_\tau \leq Z_{r,s}, \tag{6.13}
$$

where $Z_{r,s} := \Big(R\lambda + f + 2L^2\sigma\Big)\mathrm{e}^{-2\lambda s}$.

2. Penalization.

We fix any $\varepsilon > 0$ and apply the Divergence Theorem to the inequality (6.13) in the two-dimensional strip $Q^\varepsilon_{0,t}$ as in (5.29) and we get

$$
\begin{aligned}
&\int_{t-\varepsilon}^t \mathrm{e}^{-2\lambda t} w^2_\tau(r,t)\,\mathrm{d}r \leq \int_{-\varepsilon}^0 w^2_\tau(r,0)\,\mathrm{d}r +\\
&\quad + \int_0^t \mathrm{e}^{-2\lambda s}\Big(w^2_\tau(s,s) - w^2_\eta(s,s)\Big)\,\mathrm{d}s + \int_0^t \mathrm{e}^{-2\lambda s}\Big(w^2_\eta(s-\varepsilon,s) - w^2_\tau(s-\varepsilon,s)\Big)\,\mathrm{d}s\\
&\quad + \iint_{Q^\varepsilon_{0,t}} Z_{r,s}\,\mathrm{d}r\mathrm{d}s.
\end{aligned}
\tag{6.14}
$$

3. Estimates of the r.h.s..

We want to estimate the integrals (say I_0, I_1, I_2, I_3) of the right hand side of (6.14) in terms of

$$
w(s) := w(s,s) \quad \text{and} \quad W(t) := \sup_{0\leq s\leq t} \mathrm{e}^{-\lambda s} w(s).
$$

We easily get

$$
I_0 = \int_{-\varepsilon}^0 w^2_\tau(r,0)\,\mathrm{d}r = \varepsilon w^2(0).
$$

(6.11) yields

$$
|w_\tau(s,s) - w_\eta(s,s)| \leq L(\tau + \eta) = L\sigma
$$

and

$$
|w^2_\tau(s,s) - w^2_\eta(s,s)| \leq L\sigma\Big(L\sigma + 2w(s)\Big);
$$

after an integration,

$$I_1 \le L^2\sigma^2 t + 2L\sigma \int_0^t e^{-2\lambda s} w(s)\,\mathrm{d}s \le L^2\sigma^2 t + 2L\sigma t W(t).$$

Performing the same computations for the third integral term at the r.h.s. of (6.14) we end up with

$$\begin{aligned} I_2 &= \int_0^t e^{-2\lambda s}\left(w_\eta^2(s-\varepsilon, s) - w_\tau^2(s-\varepsilon, s)\right)\mathrm{d}s \\ &\le L^2 t\sigma^2 + 2L\sigma \int_0^t e^{-2\lambda s} w(s-\varepsilon, s)\,\mathrm{d}s \\ &\le L^2\sigma^2 t + 2L^2\sigma\varepsilon t + 2L\sigma \int_0^t e^{-2\lambda s} w(s)\,\mathrm{d}s \\ &\le L^2\sigma^2 t + 2L^2\sigma\varepsilon t + 2L\sigma t W(t). \end{aligned}$$

Eventually, using the elementary inequalities,

$$\iint_{Q^\varepsilon_{0,t}} \lambda \mathrm{e}^{-2\lambda s}\,\mathrm{d}r\,\mathrm{d}s \le \frac{\varepsilon}{2}, \quad \iint_{Q^\varepsilon_{0,t}} \mathrm{e}^{-2\lambda s} w(s,s)\,\mathrm{d}r\,\mathrm{d}s = \varepsilon \int_0^t \mathrm{e}^{-2\lambda s} w(s)\,\mathrm{d}s,$$

and $f(r,s) \le 2L^2(\eta + s) + 2Lw(s)$ for $r < 0$ and $f(r,s) = 0$ for $r \ge 0$, we get

$$\begin{aligned} I_3 = \iint_{Q^\varepsilon_{0,t}} Z_{r,s}\,\mathrm{d}r\mathrm{d}s &\le 2L^2\sigma\varepsilon(\sigma+\varepsilon) + 4L\lambda\sigma\varepsilon \int_0^t \mathrm{e}^{-2\lambda s} w(s)\,\mathrm{d}s + 2L^2\sigma\varepsilon t \\ &\quad + 2\iint_{Q^\varepsilon_{0,\min\{\varepsilon,t\}}} (L^2(\eta+s) + Lw(s))\mathrm{e}^{-2\lambda s}\,\mathrm{d}r\mathrm{d}s \\ &\le 2L^2\sigma\varepsilon(\sigma+\varepsilon) + 2L^2\varepsilon^2(\sigma+\varepsilon) + 2L^2\sigma\varepsilon t + 4L\lambda\sigma\varepsilon t W(t) + 2L\varepsilon^2 W(\min\{t,\varepsilon\}). \end{aligned}$$

We eventually get

$$\begin{aligned} \sum_{k=0}^3 I_k \le \varepsilon w^2(0) &+ 2L^2\sigma^2 t + 4L^2\sigma\varepsilon t + 2L^2\varepsilon(\sigma+\varepsilon)^2 \\ &+ 4L\sigma(1+\lambda\varepsilon)tW(t) + 2L\varepsilon^2 W(\min\{t,\varepsilon\}). \end{aligned} \tag{6.15}$$

4. L.h.s. and penalization

We want to use the first integral term in (6.14) to derive a pointwise estimate for $w(t)$; (6.2) and (6.10) yield

$$w(t) = w(t,t) \le L(t-r) + w(r,t) \le L(\tau + |t-r|) + w_\tau(r,t). \tag{6.16}$$

We then square (6.16), use the Young inequality (i.e. $2ab \le \frac{a^2}{\vartheta} + \vartheta b^2$ for any $a, b \ge 0$, $\vartheta > 0$), multiply the resulting inequality by $\frac{e^{-2\lambda t}}{\varepsilon}$ and integrate over the interval $(t-\varepsilon, t)$. So that, for every $\delta, \delta_\star > 1$ conjugate coefficients, we get

$$\begin{aligned} e^{-2\lambda t} w^2(t) &\le \frac{\delta}{\varepsilon} \int_{t-\varepsilon}^{t} e^{-2\lambda t} w_\tau^2(r,t)\,\mathrm{d}r + \delta_\star L^2(\tau+\varepsilon)^2 \\ &\le \frac{\delta}{\varepsilon}(I_0 + I_1 + I_2 + I_3) + \delta_\star L^2(\tau+\varepsilon)^2, \end{aligned}$$

with I_0, I_1, I_2, I_3 as in step 3. Using (6.15) yields

$$\begin{aligned} e^{-2\lambda t} w^2(t) \le (2\delta + \delta_\star) L^2(\sigma+\varepsilon)^2 + \delta\Big(w^2(0) + 2L^2\sigma^2 t/\varepsilon + 4L^2\sigma t\Big) \\ + \frac{4L(1+\lambda\varepsilon)\sigma\delta}{\varepsilon} t W(t) + 2L\varepsilon\delta W(\min\{t,\varepsilon\}). \end{aligned}$$

5. Conclusion.

Choosing $\varepsilon := \sqrt{\sigma \ \max\{\sigma, t\}}$ and assuming $\lambda\sqrt{T\sigma} \le 1$, we obtain

$$e^{-2\lambda t} w^2(t) \le \delta w^2(0) + (14\delta + 4\delta_\star)L^2 \sigma \ \max\{\sigma,t\} + 10\delta L \sqrt{\sigma \ \max\{\sigma,t\}}\, W(t). \tag{6.17}$$

Since the right hand side of (6.17) is an increasing function of t, (6.17) holds even if we substitute the left hand side with $e^{-2\lambda s} w^2(s)$ for every $s \in [0,t]$; we thus obtain the inequality

$$W^2(t) \le \delta w^2(0) + (14\delta + 4\delta_\star)L^2 \sigma \ \max\{\sigma,t\} + 10\delta L\sqrt{\sigma \ \max\{\sigma,t\}}\, W(t).$$

Using the elementary property for positive a, b

$$W^2 \le a + 2bW \quad \Rightarrow \quad W \le b + \sqrt{b^2 + a} \le 2b + \sqrt{a}, \tag{6.18}$$

we eventually obtain

$$\begin{aligned} e^{-\lambda t} w(t) &\le \Big(\delta w^2(0) + (14\delta + 4\delta_\star)L^2 \sigma \ \max\{\sigma,t\}\Big)^{1/2} + 10\delta L\sqrt{\sigma \ \max\{\sigma,t\}} \\ &\le \sqrt{\delta}\, w(0) + C(\delta) L \sqrt{\sigma \ \max\{\sigma,t\}}, \end{aligned}$$

with $C(\delta) := (14\,\delta + 4\,\delta_\star)^{1/2} + 10\,\delta$. □

6.3 Error estimates between discrete and EVI solutions

Theorem 6.5 *Let* $\mathbf{F}$ *be a* λ*-dissipative* MPVF *according to* (4.1)*. If* $\mu : [0,T] \to \overline{\mathrm{D}(\mathbf{F})}$ *is a* λ-EVI *solution and* $M_\tau \in \mathscr{M}(M_\tau^0, \tau, T, L)$*, then for every* $\delta > 1$ *there exists a* *constant* $C(\delta)$ *such that*

$$W_2(\mu_t, M_\tau(t)) \le \Big(\sqrt{\delta}\, W_2(\mu_0, M_\tau^0) + C(\delta)L\sqrt{\tau(t+\tau)}\Big)\mathrm{e}^{\lambda_+ t}$$

for every $t \in [0, T]$.

Remark 6.6 When $\mu_0 = M_\tau^0$ and $\lambda \le 0$ we obtain the optimal error estimate

$$W_2(\mu_t, M_\tau(t)) \le 13L\sqrt{\tau(t+\tau)}.$$

Proof We repeat the same argument of the previous proof, still assuming $\lambda > 0$, extending $M_\tau, \bar{M}_\tau, \boldsymbol{F}_\tau$ as in (6.9) and setting

$$w(r,s) := W_2(M_\tau(r), \mu_s), \quad w_\tau(r,s) := W_2(\bar{M}_\tau(r), \mu_s).$$

We use (λ-EVI) for μ_s with $\nu = \bar{M}_\tau(r)$ and $\Phi = \boldsymbol{F}_\tau(\max\{r, 0\})$ and (IEVI) for $M_\tau(r)$ with $\nu = \mu_s$ obtaining

$$\begin{aligned}
&\frac{\partial}{\partial r}\frac{\mathrm{e}^{-2\lambda s}}{2} W_2^2(M_\tau(r), \mu_s) \le \mathrm{e}^{-2\lambda s}\Big(\tau|\boldsymbol{F}_\tau(r)|_2^2 + [\boldsymbol{F}_\tau(r), \mu_s]_r\Big) && s \in [0,T], r \in (-\infty, T)\\
&\frac{\partial}{\partial s}\frac{\mathrm{e}^{-2\lambda s}}{2} W_2^2(\mu_s, \bar{M}_\tau(r)) \le -\mathrm{e}^{-2\lambda s}\,[\boldsymbol{F}_\tau(\max\{r,0\}), \mu_s]_r && \text{in } \mathscr{D}'(0,T), r \in (-\infty, T).
\end{aligned}$$

Using [23, Lemma 6.15] we can sum the two contributions obtaining

$$\frac{\partial}{\partial r}\mathrm{e}^{-2\lambda s} w^2(r,s) + \frac{\partial}{\partial s}\mathrm{e}^{-2\lambda s} w_\tau^2(r,s) \le Z_{r,s},$$

where $Z_{r,s} := (2L^2\tau + 2f(r,s))\mathrm{e}^{-2\lambda s}$, and

$$f(r,s) := \begin{cases} LW_2(M_\tau(0), \mu_s) = Lw(0,s) & \text{if } r < 0,\\ 0 & \text{if } r \ge 0.\end{cases}$$

Let $t \in [0, T]$ and $\varepsilon > 0$. Applying the Divergence Theorem in $Q_{0,t}^\varepsilon$ (see (5.29) and Figure 1), we get

$$\begin{aligned}
\int_{t-\varepsilon}^{t} \mathrm{e}^{-2\lambda t} w_\tau^2(r,t)\,\mathrm{d}r &\le \int_{-\varepsilon}^{0} w_\tau^2(r,0)\,\mathrm{d}r\\
&+ \int_0^t \mathrm{e}^{-2\lambda s}\left(w_\tau^2(s,s) - w^2(s,s)\right)\mathrm{d}s + \int_0^t \mathrm{e}^{-2\lambda s}\left(w^2(s-\varepsilon,s) - w_\tau^2(s-\varepsilon,s)\right)\mathrm{d}s\\
&+ \iint_{Q_{0,t}^\varepsilon} Z_{r,s}\,\mathrm{d}r\mathrm{d}s. \qquad (6.19)
\end{aligned}$$

Using

$$w(t,t) \le w(r,t) + L(t-r) \le w_\tau(r,t) + L(\tau+\varepsilon) \quad \text{if } t - \varepsilon \le r \le t,$$

we get for every $\delta, \delta_\star > 1$ conjugate coefficients ($\delta_\star = \delta/(\delta-1)$)

$$\mathrm{e}^{-2\lambda t} w^2(t) \le \frac{\delta}{\varepsilon} \int_{t-\varepsilon}^{t} e^{-2\lambda t} w_\tau^2(r,t)\,\mathrm{d}r + \delta_\star L^2(\tau+\varepsilon)^2. \tag{6.20}$$

Similarly to (6.11) we have

$$|w_\tau(s,s) - w(s,s)| \le L\tau, \quad |w_\tau^2(s,s) - w^2(s,s)| \le L\tau\Big(L\tau + 2w(s)\Big)$$

and, after an integration,

$$\int_0^t e^{-2\lambda s}\left(w_\tau^2(s,s) - w^2(s,s)\right)\mathrm{d}s \le L^2 t\tau^2 + 2L\tau \int_0^t e^{-2\lambda s} w(s)\,\mathrm{d}s. \tag{6.21}$$

Performing the same computations for the third integral term at the r.h.s. of (6.19) we end up with

$$\begin{aligned}\int_0^t e^{-2\lambda s}\left(w^2(s-\varepsilon,s) - w_\tau^2(s-\varepsilon,s)\right)\mathrm{d}s &\le L^2 t\tau^2 + 2L\tau\int_0^t e^{-2\lambda s} w(s-\varepsilon,s)\,\mathrm{d}s \\ &\le L^2 t\tau(\tau+2\varepsilon) + 2L\tau \int_0^t e^{-2\lambda s} w(s)\,\mathrm{d}s.\end{aligned} \tag{6.22}$$

Finally, since if $r < 0$ we have $f(r,s) = Lw(0,s) \le L^2 s + Lw(s,s)$, then

$$\begin{aligned}\varepsilon^{-1}\iint_{Q_{0,t}^{\varepsilon}} Z_{r,s}\,\mathrm{d}r\mathrm{d}s &\le 2L^2 t\tau + \varepsilon^{-1}\iint_{Q^{\varepsilon}_{0,\min\{\varepsilon,t\}}} 2f(r,s)\mathrm{e}^{-2\lambda s}\,\mathrm{d}r\mathrm{d}s \\ &\le 2L^2 t\tau + L^2\varepsilon^2 + 2L\varepsilon \sup_{0\le s\le \min\{\varepsilon,t\}} \mathrm{e}^{-\lambda s} w(s).\end{aligned} \tag{6.23}$$

Using (6.21), (6.22), (6.23) in (6.19), we can rewrite the bound in (6.20) as

$$\begin{aligned}\mathrm{e}^{-2\lambda t} w^2(t) \le \delta_\star L^2(\tau+\varepsilon)^2 &+ \delta\Big(w^2(0) + 2L^2 t\tau^2/\varepsilon + 2L^2 t\tau + L^2\varepsilon^2 \\ &+ 2L\varepsilon \sup_{0\le s\le\min\{\varepsilon,t\}} \mathrm{e}^{-\lambda s} w(s)\Big) \\ &+ \frac{4\delta L\tau}{\varepsilon}\int_0^t e^{-2\lambda s} w(s)\,\mathrm{d}s.\end{aligned}$$

Choosing $\varepsilon := \sqrt{\tau\ \max\{\tau,t\}}$ we get

$$\begin{aligned}\mathrm{e}^{-2\lambda t} w^2(t) \le 4\,\delta_\star L^2\,\tau\ \max\{\tau,t\} &+ \delta\Big(w^2(0) + 5L^2\,\tau\ \max\{\tau,t\}\Big) \\ &+ 6\,\delta L\sqrt{\tau\ \max\{\tau,t\}}\ \sup_{0\le s\le t}\ \mathrm{e}^{-\lambda s} w(s).\end{aligned}$$

A further application of (6.18) yields

$$
\begin{aligned}
\mathrm{e}^{-\lambda t} w(t) &\le \Big(\delta w^2(0) + (5\delta + 4\delta_\star) L^2 \tau \, \max\{\tau, t\}\Big)^{1/2} + 6\delta L \sqrt{\tau \, \max\{\tau, t\}} \\
&\le \sqrt{\delta} w(0) + C(\delta) L \sqrt{t+\tau}\sqrt{\tau},
\end{aligned}
$$

with $C(\delta) := (5\delta + 4\delta_\star)^{1/2} + 6\delta$. □

As proved in the following, the limit curve of the interpolants $(M_\tau)_{\tau>0}$ of the Euler Scheme defined in (5.9) is actually a λ-EVI solution of (5.1).

Theorem 6.7 *Let* $\mathbf{F}$ *be a* λ*-dissipative* MPVF *according to* (4.1) *and let* $n \mapsto \tau(n)$ *be a vanishing sequence of time steps, let* $(\mu_{0,n})_{n\in\mathbb{N}}$ *be a sequence in* $\mathrm{D}(\mathbf{F})$ *converging to* $\mu_0 \in \overline{\mathrm{D}(\mathbf{F})}$ *in* $\mathcal{P}_2(\mathsf{X})$ *and let* $M_n \in \mathscr{M}(\mu_{0,n}, \tau(n), T, L)$. *Then* M_n *is uniformly converging to a Lipschitz continuous limit curve* $\mu : [0, T] \to \overline{\mathrm{D}(\mathbf{F})}$ *which is a* λ-EVI *solution starting from* μ_0.

Proof Theorem 6.4 shows that M_n is a Cauchy sequence in $\mathrm{C}([0, T]; \overline{\mathrm{D}(\mathbf{F})})$, so that there exists a unique limit curve μ as $n \to \infty$. Moreover, μ is also L-Lipschitz and, recalling (5.14), we have that μ is also the uniform limit of $\bar{M}_{\tau(n)}$.

Let us fix a reference measure $\nu \in \mathrm{D}(\mathbf{F})$ and $\Phi \in \mathbf{F}[\nu]$. The (IEVI) and the λ-dissipativity of $\mathbf{F}$ yield

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2} W_2^2(M_n(t), \nu) &\le \tau(n) |\boldsymbol{F}_{\tau(n)}(t)|_2^2 + \big[\boldsymbol{F}_{\tau(n)}, \nu\big]_r \\
&\le \tau(n)\, L^2 + \lambda W_2^2(\bar{M}_{\tau(n)}(t), \nu) - \big[\Phi, \bar{M}_{\tau(n)}(t)\big]_r
\end{aligned}
$$

for a.e. $t \in [0, T]$. Integrating the above inequality in an interval $(t, t+h) \subset [0, T]$ we get

$$
\begin{aligned}
\frac{W_2^2(M_n(t+h), \nu) - W_2^2(M_n(t), \nu)}{2h} &\le \tau(n) L^2 \\
&+ \frac{1}{h}\int_t^{t+h} \Big(\lambda W_2^2(\bar{M}_{\tau(n)}(s), \nu) - \big[\Phi, \bar{M}_{\tau(n)}(s)\big]_r\Big)\, \mathrm{d}s.
\end{aligned}
\tag{6.24}
$$

Notice that as $n \to +\infty$, by (5.14), we have

$$
\liminf_{n\to+\infty} \big[\Phi, \bar{M}_{\tau(n)}(s)\big]_r \ge [\Phi, \mu_s]_r
$$

for every $s \in [0, T]$, together with the uniform bound given by

$$
\Big|\big[\Phi, \bar{M}_{\tau(n)}(s)\big]_r\Big| \le \frac{1}{2} W_2^2(\bar{M}_{\tau(n)}(s), \nu) + \frac{1}{2}|\Phi|_2^2
$$

for every $s \in [0, T]$. Thanks to Fatou's Lemma and the uniform convergence given by Theorem 6.4, we can pass to the limit as $n \to +\infty$ in (6.24) obtaining

$$\frac{W_2^2(\mu_{t+h}, \nu) - W_2^2(\mu_t, \nu)}{2h} \le \frac{1}{h} \int_t^{t+h} \Big(\lambda W_2^2(\mu_s, \nu) - [\Phi, \mu_s]_r \Big) \, \mathrm{d}s.$$

A further limit as $h \downarrow 0$ yields

$$\frac{1}{2} \frac{\mathrm{d}^+}{\mathrm{d}t} W_2^2(\mu_t, \nu) \le \lambda W_2^2(\mu_t, \nu) - [\Phi, \mu_t]_r$$

which provides (λ-EVI). □

7 Examples of λ-dissipative MPVFs and λ-flows

In the first part of this section, we present significant examples of λ-dissipative MPVFs which are interesting for applications. In Sect. 7.4, we give some examples of MPVFs generating λ-flows with particular properties. We then conclude with Sect. 7.5, where we compare our framework with that developed in [27], revisiting in particular the splitting particle example in Example 7.11.

7.1 Subdifferentials of λ-convex functionals

Recall that a functional $\mathcal{F} : \mathcal{P}_2(\mathsf{X}) \to (-\infty, +\infty]$ is λ-(geodesically) convex on $\mathcal{P}_2(\mathsf{X})$ (see [3, Definition 9.1.1]) if for any μ_0, μ_1 in the proper domain $D(\mathcal{F}) := \{\mu \in \mathcal{P}_2(\mathsf{X}) \mid \mathcal{F}(\mu) < +\infty\}$ there exists $\boldsymbol{\mu} \in \Gamma_o(\mu_0, \mu_1)$ such that

$$\mathcal{F}(\mu_t) \le (1-t)\mathcal{F}(\mu_0) + t\mathcal{F}(\mu_1) - \frac{\lambda}{2} t(1-t) W_2^2(\mu_0, \mu_1)$$

for every $t \in [0, 1]$, where $\mu : [0, 1] \to \mathcal{P}_2(\mathsf{X})$ is the constant speed geodesic induced by $\boldsymbol{\mu}$, i.e. $\mu_t = \mathsf{x}^t_\sharp \boldsymbol{\mu}$.

The *Fréchet subdifferential* $\boldsymbol{\partial}\mathcal{F}$ of $\mathcal{F}$ [3, Definition 10.3.1] is a MPVF which can be characterized [3, Theorem 10.3.6] by

$$\begin{aligned} \Phi \in \boldsymbol{\partial}\mathcal{F}[\mu] \quad &\Leftrightarrow \quad \mu \in D(\mathcal{F}), \mathcal{F}(\nu) - \mathcal{F}(\mu) \ge -[\Phi, \nu]_l + \frac{\lambda}{2} W_2^2(\mu, \nu) \\ &\text{for every } \nu \in D(\mathcal{F}). \end{aligned}$$

According to the notation introduced in (3.15), we set

$$-\boldsymbol{\partial}\mathcal{F}[\mu] = J_\sharp \boldsymbol{\partial}\mathcal{F}[\mu], \quad \text{with } J(x, v) := (x, -v), \tag{7.1}$$

and we have the following result.

Theorem 7.1 *If* $\mathcal{F} : \mathcal{P}_2(\mathsf{X}) \to (-\infty, +\infty]$ *is a proper, lower semicontinuous and* λ*-convex functional, then* $-\boldsymbol{\partial}\mathcal{F}$ *is a* $(-\lambda)$-dissipative MPVF *according to* (4.1).

In the following proposition, we prove a correspondence between gradient flows for $\mathcal{F}$ and $(-\lambda)$-EVI solutions for the MPVF $-\boldsymbol{\partial}\mathcal{F}$. We refer respectively to (4.7), (4.12) and Definition 4.11 for the definitions of $\mathrm{I}(\boldsymbol{\mu}|\mathbf{F})$, $\Gamma_o^0(\cdot, \cdot|\mathbf{F})$ and $[\mathbf{F}, \boldsymbol{\mu}]_{0+}$.

Proposition 7.2 *Let* $\mathcal{F} : \mathcal{P}_2(\mathsf{X}) \to (-\infty, +\infty]$ *be a proper, lower semicontinuous and* λ*-convex functional and let* $\mu : \mathcal{I} \to \mathrm{D}(\boldsymbol{\partial}\mathcal{F})$ *be a locally absolutely continuous curve, with* $\mathcal{I}$ *a (bounded or unbounded) interval in* $\mathbb{R}$*. Then*

(1) *if* μ *is a Gradient Flow for* $\mathcal{F}$ *i.e.*

$$(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}_t)_\sharp\mu_t \in -\boldsymbol{\partial}\mathcal{F}(\mu_t) \quad a.e.\ t \in \mathcal{I},$$

then μ *is a* $(-\lambda)$-EVI *solution of* (5.1) *for the* MPVF $-\boldsymbol{\partial}\mathcal{F}$ *as in* (7.1)*;*

(2) *if* μ *is a* $(-\lambda)$-EVI *solution of* (5.1) *for the* MPVF $-\boldsymbol{\partial}\mathcal{F}$ *and the domain of* $\boldsymbol{\partial}\mathcal{F}$ *satisfies*

$$\textit{for a.e. } t \in \mathcal{I},\ \Gamma_o^0(\mu_t, \nu|\boldsymbol{\partial}\mathcal{F}) \neq \emptyset \quad \textit{for every } \nu \in \mathrm{D}(\boldsymbol{\partial}\mathcal{F}),$$

then μ *is a Gradient Flow for* $\mathcal{F}$.

Proof The first assertion is a consequence Theorem 5.4(1). We prove the second claim; by (5.5b) we have that for a.e. $t \in \mathcal{I}$ it holds

$$\left[(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}_t)_\sharp\mu_t, \nu\right]_r \le [(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}_t)_\sharp\mu_t, \boldsymbol{\mu}_t]_{r,0} \le [-\boldsymbol{\partial}\mathcal{F}, \boldsymbol{\mu}_t]_{0+}$$

for every $\nu \in \mathrm{D}(\mathcal{F})$ and $\boldsymbol{\mu}_t \in \Gamma_o^0(\mu_t, \nu|\boldsymbol{\partial}\mathcal{F})$. We show that for every $\nu_0, \nu_1 \in \mathrm{D}(\boldsymbol{\partial}\mathcal{F})$ and every $\boldsymbol{\nu} \in \Gamma_o^0(\nu_0, \nu_1|\mathbf{F})$

$$[-\boldsymbol{\partial}\mathcal{F}, \boldsymbol{\nu}]_{0+} \le \mathcal{F}(\nu_1) - \mathcal{F}(\nu_0) - \frac{\lambda}{2}W_2^2(\nu_0, \nu_1). \tag{7.2}$$

To prove that, we take $s \in \mathrm{I}(\boldsymbol{\nu}|\boldsymbol{\partial}\mathcal{F}) \cap (0, 1)$ and $\Phi_s \in -\boldsymbol{\partial}\mathcal{F}(\nu_s)$, where $\nu_s := \mathsf{x}^s_\sharp\boldsymbol{\nu}$. By definition of subdifferential we have

$$[\Phi_s, \nu_1]_r \le \mathcal{F}(\nu_1) - \mathcal{F}(\nu_s) - \frac{\lambda}{2}W_2^2(\nu_s, \nu_1).$$

Dividing by $(1 - s)$, using (3.29) and passing to the infimum w.r.t. $\Phi_s \in -\boldsymbol{\partial}\mathcal{F}(\nu_s)$ we obtain

$$[-\boldsymbol{\partial}\mathcal{F}, \boldsymbol{\nu}]_{r,s} \le \frac{1}{1-s}\left(\mathcal{F}(\nu_1) - \mathcal{F}(\nu_s)\right) - \frac{\lambda(1-s)}{2}W_2^2(\nu_0, \nu_1).$$

Passing to the limit as $s \downarrow 0$ and using the lower semicontinuity of $\mathcal{F}$ lead to the result. Once that (7.2) is established we have that for a.e. $t \in \mathcal{I}$ it holds

$$\left[(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}_t)_\sharp\mu_t, \nu\right]_r \le \mathcal{F}(\nu) - \mathcal{F}(\mu_t) - \frac{\lambda}{2}W_2^2(\mu_t, \nu) \quad \text{for every } \nu \in \mathrm{D}(\boldsymbol{\partial}\mathcal{F}). \tag{7.3}$$

To conclude it is enough to use the lower semicontinuity of the l.h.s. (see Lemma 3.15) and the fact that $\mathrm{D}(\boldsymbol{\partial}\mathcal{F})$ is dense in $\mathrm{D}(\mathcal{F})$ in energy: indeed we can apply [25, Corollary 4.5] and [3, Lemma 3.1.2] to the proper, lower semicontinuous and convex functional $\mathcal{F}^\lambda : \mathcal{P}_2(\mathsf{X}) \to (-\infty, +\infty]$ defined as

$$\mathcal{F}^\lambda(\nu) = \mathcal{F}(\nu) - \frac{\lambda}{2}\mathsf{m}_2^2(\nu)$$

to get the existence, for every $\nu \in \mathrm{D}(\mathcal{F})$, of a family $(\nu^\tau)_{\tau>0} \subset \mathrm{D}(\mathcal{F}^\lambda) = \mathrm{D}(\mathcal{F})$ s.t.

$$\nu^\tau \to \nu, \quad \mathcal{F}^\lambda(\nu^\tau) \to \mathcal{F}^\lambda(\nu) \quad \text{as } \tau \downarrow 0.$$

Of course $\mathcal{F}(\nu^\tau) \to \mathcal{F}(\nu)$ as $\tau \downarrow 0$ and, applying [3, Lemma 10.3.4], we see that $\nu^\tau \in \mathrm{D}(\boldsymbol{\partial}\mathcal{F}^\lambda)$. However $\boldsymbol{\partial}\mathcal{F}^\lambda = L^\lambda_\sharp \boldsymbol{\partial}\mathcal{F}$ (see (4.4)) so that $\nu^\tau \in \mathrm{D}(\boldsymbol{\partial}\mathcal{F})$. We can thus write (7.3) for ν^τ in place of ν and pass to the limit as $\tau \downarrow 0$, obtaining that, by definition of subdifferential, $(\boldsymbol{i}_\mathsf{X}, \boldsymbol{v}_t)_\sharp \mu_t \in -\boldsymbol{\partial}\mathcal{F}(\mu_t)$ for a.e. $t \in \mathcal{I}$. □

Referring to [3], here we list interesting and explicit examples of $(-\lambda)$-dissipative MPVFs, according to (4.1), induced by proper, lower semicontinuous and λ-convex functionals, focusing on the cases when $\mathrm{D}(\boldsymbol{\partial}\mathcal{F}) = \mathcal{P}_2(\mathsf{X})$.

(1) *Potential energy.* Let $P : \mathsf{X} \to \mathbb{R}$ be a l.s.c. and λ-convex functional satisfying

$$|\partial^o P(x)| \le C(1 + |x|) \quad \text{for every } x \in \mathsf{X},$$

for some constant $C > 0$, where $\partial^o P(x)$ is the element of minimal norm in $\partial P(x)$. By [3, Proposition 10.4.2] the PVF

$$\mathbf{F}[\mu] := (\boldsymbol{i}_\mathsf{X}, -\partial^o P)_\sharp \mu, \quad \mu \in \mathcal{P}_2(\mathsf{X}),$$

is a $(-\lambda)$-dissipative selection of $-\boldsymbol{\partial}\mathcal{F}_P$ for the potential energy functional

$$\mathcal{F}_P(\mu) := \int_\mathsf{X} P \,\mathrm{d}\mu, \quad \mu \in \mathcal{P}_2(\mathsf{X}).$$

(2) *Interaction energy.* If $W : \mathsf{X} \to [0, +\infty)$ is an even, differentiable, and λ-convex function for some $\lambda \in \mathbb{R}$, whose differential has a linear growth, then, by [3, Theorem 10.4.11], the PVF

$$\mathbf{F}[\mu] := (\boldsymbol{i}_\mathsf{X}, (-\nabla W * \mu))_\sharp\, \mu, \quad \mu \in \mathcal{P}_2(\mathsf{X}),$$

is a $(-\lambda)$-dissipative selection of $-\boldsymbol{\partial}\mathcal{F}_W$, the opposite of the Wasserstein subdifferential of the interaction energy functional

$$\mathcal{F}_W(\mu) := \frac{1}{2}\int_{\mathsf{X}^2} W(x - y)\,\mathrm{d}(\mu \otimes \mu)(x, y), \quad \mu \in \mathcal{P}_2(\mathsf{X}).$$

(3) *Opposite Wasserstein distance.* Let $\bar\mu \in \mathcal{P}_2(\mathsf{X})$ be fixed and consider the functional $\mathcal{F}_{\mathrm{Wass}} : \mathcal{P}_2(\mathsf{X}) \to \mathbb{R}$ defined as

$$\mathcal{F}_{\mathrm{Wass}}(\mu) := -\frac{1}{2} W_2^2(\mu, \bar\mu), \quad \mu \in \mathcal{P}_2(\mathsf{X}),$$

which is geodesically (-1)-convex [3, Proposition 9.3.12]. Setting

$$\boldsymbol{b}(\mu) := \arg\min \left\{ \int_{\mathsf{X}} |\boldsymbol{b}(x) - x|^2 \,\mathrm{d}\mu \;:\; \boldsymbol{b} = \boldsymbol{b}_{\boldsymbol{\gamma}} \in L^2_\mu(\mathsf{X}; \mathsf{X}),\ \boldsymbol{\gamma} \in \Gamma_o(\mu, \bar\mu) \right\},$$

the PVF

$$\mathbf{F}[\mu] := (\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{i}_{\mathsf{X}} - \boldsymbol{b}(\mu))_{\#}\mu, \quad \mu \in \mathcal{P}_2(\mathsf{X})$$

is a selection of $-\boldsymbol{\partial}\mathcal{F}_{\mathrm{Wass}}(\mu)$ and it is therefore 1-dissipative according to (4.1).

7.2 MPVF concentrated on the graph of a multifunction

The previous example of Sect. 7.1 has a natural generalization in terms of dissipative graphs in $\mathsf{X} \times \mathsf{X}$ [1, 2, 7]. We consider a (non-empty) λ-dissipative set $\mathrm{F} \subset \mathsf{X} \times \mathsf{X}$, i.e. satisfying

$$\langle v_0 - v_1, x_0 - x_1 \rangle \le \lambda |x_0 - x_1|^2 \quad \text{for every } (x_0, v_0), (x_1, v_1) \in \mathrm{F}.$$

The corresponding MPVF defined as

$$\mathbf{F} := \{\Phi \in \mathcal{P}_2(\mathsf{TX}) \mid \Phi \text{ is concentrated on } \mathrm{F}\}$$

is λ-dissipative as well, according to (4.1). In fact, if $\Phi_0, \Phi_1 \in \mathbf{F}$ with $\nu_i = \mathsf{x}_\sharp \Phi_i$, $i = 0, 1$, and $\boldsymbol{\Theta} \in \Lambda(\Phi_0, \Phi_1)$ then $(x_0, v_0, x_1, v_1) \in \mathrm{F} \times \mathrm{F}$ $\boldsymbol{\Theta}$-a.e., so that

$$\int_{\mathsf{TX}\times\mathsf{TX}} \langle v_0 - v_1, x_0 - x_1 \rangle \,\mathrm{d}\boldsymbol{\Theta}(x_0, v_0, x_1, v_1) \le \lambda \int_{\mathsf{TX}\times\mathsf{TX}} |x_0 - x_1|^2 \,\mathrm{d}\boldsymbol{\Theta} = \lambda W_2^2(\nu_0, \nu_1).$$

since $(\mathsf{x}^0, \mathsf{x}^1)_\sharp \boldsymbol{\Theta} \in \Gamma_o(\nu_0, \nu_1)$. Taking the supremum w.r.t. $\boldsymbol{\Theta} \in \Lambda(\Phi_0, \Phi_1)$ we obtain $[\Phi_0, \Phi_1]_l \le \lambda W_2^2(\nu_0, \nu_1)$ which is even stronger than λ-dissipativity. If $\mathrm{D}(\mathrm{F}) = \mathsf{X}$ then $\mathrm{D}(\mathbf{F})$ contains $\mathcal{P}_c(\mathsf{X})$, the set of Borel probability measures with compact support. If F has also a linear growth, then it is easy to check that $\mathrm{D}(\mathbf{F}) = \mathcal{P}_2(\mathsf{X})$ as well.

Despite the analogy just shown with dissipative operators in Hilbert spaces, there are important differences with the Wasserstein framework, as highlighted in the following examples. In particular, in Sect. 4.2 we showed how dissipativity allows to deduce relevant properties when the MPVF $\mathbf{F}$ is tested against optimal directions. On the contrary, whenever $\mathsf{v}_\sharp \mathbf{F}[\mu]$ is orthogonal to $\mathrm{Tan}_\mu \mathcal{P}_2(\mathsf{X})$, we are not able to deduce informations through the dissipativity assumption, as shown in Example 7.3 and Example 7.4.

Example 7.3 Let $\mathsf{X} = \mathbb{R}^2$, let $B := \{x \in \mathbb{R}^2 \mid |x| \le 1\}$ be the closed unit ball, let $\mathcal{L}_B$ be the (normalized) Lebesgue measure on B, and let $\boldsymbol{r} : \mathbb{R}^2 \to \mathbb{R}^2$, $\boldsymbol{r}(x_1, x_2) = (x_2, -x_1)$ be the anti-clockwise rotation of $\pi/2$ degrees. We define the MPVF

$$\mathbf{F}[\nu] = \begin{cases} (\boldsymbol{i}_{\mathbb{R}^2}, 0)_\sharp \nu, & \text{if } \nu \in \mathcal{P}_2(\mathbb{R}^2) \setminus \{\mathcal{L}_B\}, \\ \left\{ (\boldsymbol{i}_{\mathbb{R}^2}, a\boldsymbol{r})_\sharp \mathcal{L}_B \mid a \in \mathbb{R} \right\}, & \text{if } \nu = \mathcal{L}_B. \end{cases}$$

Observe that $\mathrm{D}(\mathbf{F}) = \mathcal{P}_2(\mathbb{R}^2)$ and $\mathbf{F}$ is obviously unbounded at $\nu = \mathcal{L}_B$, i.e.

$$\sup \{ |\Phi|_2 \ : \ \Phi \in \mathbf{F}[\mathcal{L}_B] \} = +\infty.$$

The MPVF $\mathbf{F}$ is also dissipative with $\lambda = 0$ according to (4.1): indeed, thanks to Remark 3.6 it is enough to check that

$$\left[(\boldsymbol{i}_{\mathbb{R}^2}, a\boldsymbol{r})_\sharp \mathcal{L}_B, \nu \right]_r = 0 \quad \text{for every } \nu \in \mathcal{P}_2(\mathbb{R}^2),\ a \in \mathbb{R}. \tag{7.4}$$

To prove (7.4), we notice that the optimal transport plan from $\mathcal{L}_B$ to ν is concentrated on a map which belongs to the tangent space $\mathrm{Tan}_{\mathcal{L}_B} \mathcal{P}_2(\mathbb{R}^2)$ [3, Prop. 8.5.2]; by Remark 3.19 we have just to check that

$$\int_{\mathbb{R}^2} \langle \boldsymbol{r}(x), \nabla \varphi(x) \rangle \, \mathrm{d}\mathcal{L}_B(x) = 0 \quad \text{for every } \varphi \in \mathrm{C}_c^\infty(\mathbb{R}^2),$$

that is a consequence of the Divergence Theorem on B. This example is in contrast with the Hilbertian theory of dissipative operators according to which an everywhere defined dissipative operator is locally bounded (see [7, Proposition 2.9]).

Example 7.4 In the same setting of the previous example, let us define the MPVF

$$\mathbf{F}[\nu] = (\boldsymbol{i}_{\mathbb{R}^2}, \boldsymbol{r})_\sharp \nu, \quad \boldsymbol{r}(x_1, x_2) = (x_2, -x_1), \quad \nu \in \mathcal{P}_2(\mathbb{R}^2).$$

It is easy to check that $\mathbf{F}$ is dissipative according to (4.1) and Lipschitz continuous (as a map from $\mathcal{P}_2(\mathbb{R}^2)$ to $\mathcal{P}_2(\mathsf{T}\mathbb{R}^2)$). Moreover, arguing as in Example 7.3, we can show that $(\boldsymbol{i}_{\mathbb{R}^d}, 0)_\sharp \mathcal{L}_B \in \hat{\mathbf{F}}[\mathcal{L}_B]$, where $\hat{\mathbf{F}}$ is defined in (4.22). This is again in contrast with the Hilbertian theory of dissipative operators, stating that a single valued, everywhere defined, and continuous dissipative operator coincides with its maximal extension (see [7, Proposition 2.4]).

7.3 Interaction field induced by a dissipative map

Let us consider the Hilbert space $\mathsf{Y} = \mathsf{X}^n$, $n \in \mathbb{N}$, endowed with the scalar product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \frac{1}{n} \sum_{i=1}^n \langle x_i, y_i \rangle$, for every $\boldsymbol{x} = (x_i)_{i=1}^n$, $\boldsymbol{y} = (y_i)_{i=1}^n \in \mathsf{X}^n$. We identify TY with $(\mathsf{TX})^n$ and we denote by x^i, v^i the i-th coordinate maps. Every permutation $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ in $\mathrm{Sym}(n)$ operates on Y by the obvious formula $\sigma(\boldsymbol{x})_i = x_{\sigma(i)}$, $i = 1, \dots, n$, $\boldsymbol{x} \in \mathsf{Y}$.

Let $G : \mathsf{Y} \to \mathsf{Y}$ be a Borel λ-dissipative map bounded on bounded sets (this property is always true if Y has finite dimension) and satisfying

$$\boldsymbol{x} \in \mathrm{D}(G) \quad \Rightarrow \quad \sigma(\boldsymbol{x}) \in \mathrm{D}(G),\ G(\sigma(\boldsymbol{x})) = \sigma(G(\boldsymbol{x})) \quad \text{for every permutation } \sigma. \tag{7.5}$$

Denoting by $(G^1, \dots, G^n)$ the components of G, by x^i the projections from Y to X and by $\mu^{\otimes n} = \bigotimes_{i=1}^n \mu$, we have that the MPVF

$$\mathbf{F}[\mu] := (\mathsf{x}^1, G^1)_\sharp \mu^{\otimes n} \quad \text{with domain } \mathrm{D}(\mathbf{F}) := \mathcal{P}_b(\mathsf{X})$$

is λ-dissipative as well according to (4.1). Indeed, let $\mu, \nu \in \mathrm{D}(\mathbf{F})$, $\boldsymbol{\gamma} \in \Gamma_o(\mu, \nu)$ and let

$$\Phi = (\mathsf{x}^1, G^1)_\sharp \mu^{\otimes n} \quad \text{and} \quad \Psi = (\mathsf{x}^1, G^1)_\sharp \nu^{\otimes n}.$$

We can consider the plan $\boldsymbol{\beta} := P_\sharp \boldsymbol{\gamma}^{\otimes n} \in \Gamma(\mu^{\otimes n}, \nu^{\otimes n})$, where

$$P((x_1, y_1), \dots, (x_n, y_n)) := ((x_1, \dots, x_n), (y_1, \dots, y_n)).$$

Considering the map $H^1(\boldsymbol{x}, \boldsymbol{y}) := (x_1, G^1(\boldsymbol{x}), y_1, G^1(\boldsymbol{y}))$ we have $\boldsymbol{\Theta} := H^1_\sharp \boldsymbol{\beta} \in \Lambda(\Phi, \Psi)$, so that

$$\begin{aligned}
[\Phi, \Psi]_r &\le \int \langle v_1 - w_1, x_1 - y_1 \rangle \,\mathrm{d}\boldsymbol{\Theta}(x_1, v_1, y_1, w_1) \\
&= \int \langle G^1(\boldsymbol{x}) - G^1(\boldsymbol{y}), x_1 - y_1 \rangle \,\mathrm{d}\boldsymbol{\beta}(\boldsymbol{x}, \boldsymbol{y}) \\
&= \frac{1}{n} \sum_{k=1}^n \int \langle G^k(\boldsymbol{x}) - G^k(\boldsymbol{y}), x_k - y_k \rangle \,\mathrm{d}\boldsymbol{\beta}(\boldsymbol{x}, \boldsymbol{y}) \\
&= \int \langle G(\boldsymbol{x}) - G(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y} \rangle \,\mathrm{d}\boldsymbol{\beta}(\boldsymbol{x}, \boldsymbol{y}),
\end{aligned}$$

where we used (7.5) and the invariance of $\boldsymbol{\beta}$ with respect to permutations. The λ-dissipativity of G then yields

$$\begin{aligned}
\int \langle G(\boldsymbol{x}) - G(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y} \rangle \,\mathrm{d}\boldsymbol{\beta}(\boldsymbol{x}, \boldsymbol{y}) &\le \lambda \int |\boldsymbol{x} - \boldsymbol{y}|_{\mathsf{Y}}^2 \,\mathrm{d}\boldsymbol{\beta}(\boldsymbol{x}, \boldsymbol{y}) \\
&= \lambda \frac{1}{n} \sum_{k=1}^n \int |x_k - y_k|_{\mathsf{Y}}^2 \,\mathrm{d}\boldsymbol{\beta}(\boldsymbol{x}, \boldsymbol{y}) \\
&= \lambda \frac{1}{n} \sum_{k=1}^n \int |x_k - y_k|_{\mathsf{Y}}^2 \,\mathrm{d}\boldsymbol{\gamma}(x_k, y_k) = \lambda W_2^2(\mu, \nu).
\end{aligned}$$

A typical example when $n = 2$ is provided by

$$G(x_1, x_2) := (A(x_1 - x_2), A(x_2 - x_1))$$

where $A : \mathsf{X} \to \mathsf{X}$ is a Borel, locally bounded, dissipative and antisymmetric map satisfying $A(-z) = -A(z)$. We easily get

$$\begin{aligned}
&\langle G(\boldsymbol{x}) - G(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y}\rangle \\
&= \frac{1}{2}\Big(\langle A(x_1 - x_2) - A(y_1 - y_2), x_1 - y_1\rangle - \langle A(x_1 - x_2) - A(y_1 - y_2), x_2 - y_2\rangle\Big) \\
&= \frac{1}{2}\langle A(x_1 - x_2) - A(y_1 - y_2), x_1 - x_2 - (y_1 - y_2)\rangle \le 0.
\end{aligned}$$

In this case

$$\mathbf{F}[\mu] = (\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{a}[\mu])_\sharp \mu, \quad \boldsymbol{a}[\mu](x) = \int_{\mathsf{X}} A(x - y)\, \mathrm{d}\mu(y) \quad \text{for every } x \in \mathsf{X}.$$

7.4 A few borderline examples

In this subsection, we collect a few examples which reveal the importance of some of the technical tools we developed in Sect. 5. First of all we exhibit an example of dissipative MPVF generating a 0-flow, for which solutions starting from given initial data are merely continuous. In particular, the nice regularizing effect of gradient flows (see [6] for the Hilbert case and [3, Theorem 4.0.4, Theorem 11.2.1] for the general metric and Wasserstein settings), according to which a solution belongs to the domain of the functional for any $t > 0$ even if the initial datum merely belongs to its closure, does not hold for general dissipative evolutions. This also clarifies the interest in a definition of continuous, not necessarily absolutely continuous, solution given in Definition 5.1.

Example 7.5 (Lifting of dissipative evolutions and lack of regularizing effect) Let us consider the situation of Corollary 5.24, choosing the Hilbert space $\mathsf{X} = \ell^2(\mathbb{N})$. Following [31, Example 3] we can easily find a maximal linear dissipative operator $A : \mathrm{D}(A) \subset \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ whose semigroup does not provide a regularizing effect. We define A as

$$A(x_1, x_2, \ldots, x_{2k-1}, x_{2k}, \ldots) = (-x_2, x_1, \ldots, -kx_{2k}, kx_{2k-1}, \ldots), \quad x \in \mathrm{D}(A),$$

with domain

$$\mathrm{D}(A) := \left\{x \in \ell^2(\mathbb{N}) : \sum_{k=1}^{\infty} k^2 |x_k|^2 < \infty\right\},$$

so that there is no regularizing effect for the semigroup $(R_t)_{t\ge 0}$ generated by (the graph of) A: evolutions starting outside the domain $\mathrm{D}(A)$ stay outside the domain and do not give raise to locally Lipschitz or a.e. differentiable curves. Corollary 5.24 shows that the 0-flow $(S_t)_{t\geq 0}$ generated by $\mathbf{F}$ on $\mathcal{P}_2(X)$ is given by

$$S_t[\mu_0] = (R_t)_\sharp \mu_0 \quad \text{for every } \mu_0 \in \overline{\mathrm{D}(\mathbf{F})} = \mathcal{P}_2(\mathsf{X})$$

so that there is the same lack of regularizing effect on probability measures.

In the next example we show that a constant MPVF generates a barycentric solution.

Example 7.6 (Constant PVF and barycentric evolutions) Given $\theta \in \mathcal{P}_2(\mathsf{X})$, we consider the constant PVF

$$\mathbf{F}[\mu] := \mu \otimes \theta.$$

$\mathbf{F}$ is dissipative according to (4.1): in fact, if $\Phi_i = \mu_i \otimes \theta$, $i = 0, 1$, $\boldsymbol{\mu} \in \Gamma_o(\mu_0, \mu_1)$, and $\boldsymbol{r} : \mathsf{X} \times \mathsf{X} \times X \to \mathsf{TX} \times \mathsf{TX}$ is defined by $\boldsymbol{r}(x_0, x_1, v) := (x_0, v; x_1, v)$, then

$$\Theta = \boldsymbol{r}_\sharp(\boldsymbol{\mu} \otimes \theta) \in \Lambda(\Phi_0, \Phi_1)$$

so that (3.17) yields

$$[\Phi_0, \Phi_1]_r \leq \int \langle x_0 - x_1, v - v\rangle \,\mathrm{d}(\boldsymbol{\mu} \otimes \theta)(x_0, x_1, v) = 0.$$

Applying Proposition 5.20 and Theorem 5.19 we immediately see that $\mathbf{F}$ generates a 0-flow $(\mathrm{S}_t)_{t\geq 0}$ in $\mathcal{P}_2(\mathsf{X})$, obtained as a limit of the Explicit Euler scheme. It is also straightforward to notice that we can apply Theorem 5.27 to $\mathbf{F}$ so that for every $\mu_0 \in \mathcal{P}_2(\mathsf{X})$ the unique EVI solution $\mu_t = \mathrm{S}_t \mu_0$ satisfies the continuity equation

$$\partial_t \mu_t + \nabla \cdot (\boldsymbol{b}\mu_t) = 0, \quad \boldsymbol{b} = \int_{\mathsf{X}} v \,\mathrm{d}\theta(v).$$

Since $\boldsymbol{b}$ is constant, we deduce that S_t acts as a translation with constant velocity $\boldsymbol{b}$, i.e.

$$\mu_t = (\boldsymbol{i}_{\mathsf{X}} + t\boldsymbol{b})_\sharp \mu_0,$$

so that S_t coincides with the semigroup generated by the PVF $\mathbf{F}'[\mu] := (\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{b})_\sharp \mu$.

We conclude this subsection with a 1-dimensional example of a curve which satisfies the barycentric property but it is not an EVI solution.

Example 7.7 Let $\mathsf{X} = \mathbb{R}$. It is well known (see e.g. [24]) that $\mathcal{P}_2(\mathbb{R})$ is isometric to the closed convex subset $\mathcal{K} \subset L^2(0, 1)$ of the (essentially) increasing maps under the action of the isometry $\mathcal{J} : \mathcal{P}_2(\mathbb{R}) \to \mathcal{K}$ which maps each measure $\mu \in \mathcal{P}_2(\mathbb{R})$ into the pseudo inverse of its cumulative distribution function.

It follows that for every $\bar\nu \in \mathcal{P}_2(\mathbb{R})$ the functional $\mathcal{F} : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ defined as

$$\mathcal{F}(\mu) := \frac{1}{2} W_2^2(\mu, \bar\nu)$$

is 1-convex, since it satisfies $\mathcal{F}(\mu) = \mathcal{G}(\mathcal{J}(\mu))$ where $\mathcal{G} : L^2(0, 1) \to \mathbb{R}$ is defined as

$$\mathcal{G}(u) := \begin{cases} \frac{1}{2}\|u - \mathcal{J}(\bar\nu)\|^2 & \text{if } u \in \mathcal{K}, \\ +\infty & \text{otherwise.} \end{cases}$$

Thus $\mathcal{F}$ generates a gradient flow $(\mathrm{S}_t)_{t\ge 0}$ which is a semigroup of contractions in $\mathcal{P}_2(\mathbb{R})$; for every $\mu_0 \in \mathcal{P}_2(\mathbb{R})$, the map $\mathrm{S}_t[\mu_0]$ is the unique (-1)-EVI solution for the MPVF $-\boldsymbol{\partial}\mathcal{F}$ starting from $\mu_0 \in \mathcal{P}_2(\mathbb{R})$ (see Proposition 7.2). Since the notion of gradient flow is purely metric, the gradient flow of $\mathcal{G}$ starting from $\mathcal{J}(\mu_0)$ is just the image through $\mathcal{J}$ of the gradient flow of $\mathcal{F}$ starting from $\mu_0 \in \mathcal{P}_2(\mathbb{R})$. Indeed: let μ be the gradient flow for $\mathcal{F}$ starting from $\mu_0 \in \mathcal{P}_2(\mathbb{R})$, then by e.g. [3, Theorem 11.1.4] we have that μ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t, \nu) \le \mathcal{F}(\nu) - \mathcal{F}(\mu_t) - \frac{1}{2}W_2^2(\mu_t, \nu) \quad \text{for a.e. } t > 0, \text{ for every } \nu \in \mathcal{P}_2(\mathbb{R}),$$

so that we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\mathcal{J}(\mu_t) - \mathcal{J}(\nu)\|^2 \le \mathcal{G}(\mathcal{J}(\nu)) - \mathcal{G}(\mathcal{J}(\mu_t)) - \frac{1}{2}\|\mathcal{J}(\mu_t) - \mathcal{J}(\nu)\|^2,$$

which, recalling the characterization of gradient flows in Hilbert spaces, gives that $u(t) := \mathcal{J}(\mu_t)$ is the gradient flow of $\mathcal{G}$ starting from $\mathcal{J}(\mu_0)$. It is easy to check that

$$u(t) := \mathrm{e}^{-t}\mathcal{J}(\mu_0) + (1 - \mathrm{e}^{-t})\mathcal{J}(\bar\nu)$$

is the gradient flow of $\mathcal{G}$ starting from $u_0 = \mathcal{J}(\mu_0)$. Note that $u(t)$ is the $L^2(0, 1)$ geodesic from $\mathcal{J}(\bar\nu)$ to $\mathcal{J}(\mu_0)$ evaluated at the rescaled time e^{-t}, so that $\mathrm{S}_t[\mu_0]$ must coincide with the evaluation at time e^{-t} of the (unique) geodesic connecting $\bar\nu$ to μ_0 i.e.

$$\mathrm{S}_t[\mu_0] = \mathsf{x}^s_\sharp\boldsymbol{\gamma}, \quad s = \mathrm{e}^{-t} \in (0, 1],$$

where $\boldsymbol{\gamma} \in \Gamma_o(\bar\nu, \mu_0)$.

Let us now consider the particular case $\bar\nu = \frac{1}{2}\delta_{-a} + \frac{1}{2}\delta_a$, where $a > 0$ is a fixed parameter and $\mu_0 = \delta_0$. It is straightforward to see that

$$\mu_t = \mathrm{S}_t[\delta_0] = \frac{1}{2}\delta_{a(1-\mathrm{e}^{-t})} + \frac{1}{2}\delta_{a(\mathrm{e}^{-t}-1)}, \quad t \ge 0$$

so that

$$(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}_t)_\sharp \mu_t = \frac{1}{2}\delta_{((1-e^{-t})a, e^{-t}a)} + \frac{1}{2}\delta_{((e^{-t}-1)a, -e^{-t}a)} \in -\boldsymbol{\partial}\mathcal{F}(\mu_t), \quad \text{a.e. } t > 0,$$

where $\boldsymbol{v}$ is the Wasserstein velocity field of μ_t. On the other hand, [3, Lemma 10.3.8] shows that

$$\delta_0 \otimes \left(\frac{1}{2}\delta_{-a} + \frac{1}{2}\delta_a\right) \in -\boldsymbol{\partial}\mathcal{F}(\delta_0)$$

so that the constant curve $\bar{\mu}_t := \delta_0$ for $t \geq 0$ has the barycentric property for the MPVF $-\boldsymbol{\partial}\mathcal{F}$ but it is not a EVI solution for $-\boldsymbol{\partial}\mathcal{F}$, being different from $\mu_t = \mathrm{S}_t[\delta_0]$.

7.5 Comparison with [27]

In this section, we provide a brief comparison between the assumptions we required in order to develop a strong concept of solution to (5.1) and the hypotheses assumed in [27]. We remind that the relation between our solution and the weaker notion studied in [27] was exploited in Sect. 5.5. Here, we conclude with a further remark coming from the connections between our approximating scheme proposed in (EE) and the schemes proposed in [9] and [27].

We consider a finite time horizon $[0, T]$ with $T > 0$, the space $\mathsf{X} = \mathbb{R}^d$ and we deal with measures in $\mathcal{P}_b(\mathbb{R}^d)$ and in $\mathcal{P}_b(\mathsf{T}\mathbb{R}^d)$, i.e. compactly supported. We also deal with *single-valued* probability vector fields (PVF) for simplicity, which can be considered as everywhere defined maps $\mathbf{F} : \mathcal{P}_b(\mathbb{R}^d) \to \mathcal{P}_b(\mathsf{T}\mathbb{R}^d)$ such that $\mathsf{x}_\sharp \mathbf{F}[\nu] = \nu$. This is indeed the framework examined in [27].

We start by recalling the assumptions required in [27] for a PVF $\mathbf{F} : \mathcal{P}_b(\mathbb{R}^d) \to \mathcal{P}_b(\mathsf{T}\mathbb{R}^d)$.

(H1) there exists a constant $M > 0$ such that for all $\nu \in \mathcal{P}_b(\mathbb{R}^d)$,

$$\sup_{(x,v)\in \mathrm{supp}(\mathbf{F}[\nu])} |v| \leq M \left(1 + \sup_{x\in \mathrm{supp}(\nu)} |x|\right);$$

(H2) $\mathbf{F}$ satisfies the following Lipschitz condition: there exists a constant $L \geq 0$ such that for every $\Phi = \mathbf{F}[\nu]$, $\Phi' = \mathbf{F}[\nu']$ there exists $\boldsymbol{\Theta} \in \Lambda(\Phi, \Phi')$ satisfying

$$\int_{\mathsf{T}\mathbb{R}^d \times \mathsf{T}\mathbb{R}^d} |v_0 - v_1|^2 \, \mathrm{d}\boldsymbol{\Theta}(x_0, v_0, x_1, v_1) \leq L^2 W_2^2(\nu, \nu'),$$

with $\Lambda(\cdot, \cdot)$ as in Definition 3.8.

Remark 7.8 Condition (H1) is (H:bound) in [27], while (H2) corresponds to (H:lip) in [27] in case $p = 2$ (see also Remark 5 in [27]).
We stress that actually in [27] condition (H2) is local, meaning that L is allowed to depend on the radius R of a ball centered at 0 and containing the supports of ν and

ν'. Thanks to assumption (H1), it is easy to show that for every final time T all the discrete solutions of the Explicit Euler scheme and of the scheme of [27] starting from an initial measure with support in $\mathrm{B}(0, R)$ are supported in a ball $\mathrm{B}(0, R')$ where R' solely depends on R and T. We can thus restrict the PVF $\mathbf{F}$ to the (geodesically convex) set of measures with support in $\mathrm{B}(0, R')$ and act as L does not depend on the support of the measures.

Proposition 7.9 *If* $\mathbf{F} : \mathcal{P}_b(\mathbb{R}^d) \to \mathcal{P}_b(\mathsf{T}\mathbb{R}^d)$ *is a* PVF *satisfying* (H2), *then* $\mathbf{F}$ *is* λ-*dissipative according to* (4.1) *for* $\lambda = \frac{L^2+1}{2}$, *the Explicit Euler scheme is globally solvable in* $\mathrm{D}(\mathbf{F})$, *and* $\mathbf{F}$ *generates a* λ-*flow, whose trajectories are the limit of the Explicit Euler scheme in each finite interval* $[0, T]$.

Proof The λ-dissipativity comes from Lemma 4.7. We prove that (5.34) holds. Let $\nu \in \mathrm{D}(\mathbf{F})$ and take $\mathbf{\Theta} \in \Lambda(\mathbf{F}[\nu], \mathbf{F}[\delta_0])$ such that

$$\int_{\mathsf{T}\mathbb{R}^d \times \mathsf{T}\mathbb{R}^d} |v' - v''|^2 \,\mathrm{d}\mathbf{\Theta} \le L^2 W_2^2(\nu, \delta_0) = L^2 \mathsf{m}_2^2(\nu).$$

Since $\mathbf{F}[\delta_0] \in \mathcal{P}_c(\mathsf{T}\mathbb{R}^d)$ by assumption, there exists $D > 0$ such that $\mathrm{supp}(\mathsf{v}_\sharp \mathbf{F}[\delta_0]) \subset B_D(0)$. Hence, we have

$$\begin{aligned} L^2 \mathsf{m}_2^2(\nu) &\ge \int_{\mathsf{T}\mathbb{R}^d \times \mathsf{T}\mathbb{R}^d} |v' - v''|^2 \,\mathrm{d}\mathbf{\Theta} \\ &\ge \int_{\mathsf{T}\mathbb{R}^d \times \mathsf{T}\mathbb{R}^d} [|v'| - D]_+^2 \,\mathrm{d}\mathbf{\Theta} \\ &\ge \int_{\mathsf{T}\mathbb{R}^d} |v'|^2 \,\mathrm{d}\mathbf{F}[\nu] - 2D \int_{\mathsf{T}\mathbb{R}^d} |v'| \,\mathrm{d}\mathbf{F}[\nu], \end{aligned}$$

where $[\,.\,]_+$ denotes the positive part. By the trivial estimate $|v'| \le D + \frac{|v'|^2}{4D}$, we conclude

$$|\mathbf{F}[\nu]|_2^2 \le 2\left(2D^2 + L^2 \mathsf{m}_2^2(\nu)\right).$$

Hence (5.34) and thus the global solvability of the Explicit Euler scheme in $\mathrm{D}(\mathbf{F})$ by Proposition 5.20. To conclude it is enough to apply Theorem 5.22(a) and Theorem 6.7. $\square$

It is immediate to notice that the semi-discrete Lagrangian scheme proposed in [9] coincides with the Explicit Euler Scheme given in Definition 5.7. In particular, we can state the following comparison between the limit obtained by the Explicit Euler scheme (EE) (leading to the λ-EVI solution of (5.1)) and that of the approximating LASs scheme proposed in [27] (leading to a barycentric solution to (5.1) in the sense of Definition 5.25).

Corollary 7.10 *Let* $\mathbf{F}$ *be a* PVF *satisfying* (H1)-(H2), $\mu_0 \in \mathcal{P}_b(\mathbb{R}^d)$ *and let* $T \in (0, +\infty)$. *Let* $(n_k)_{k\in\mathbb{N}}$ *be a sequence such that the LASs scheme* $(\mu^{n_k})_{k\in\mathbb{N}}$ *of* [27, Definition 3.1] *converges uniformly-in-time and let* $(M_{\tau_k})_{k\in\mathbb{N}}$ *be the affine interpolants of* *the Explicit Euler Scheme defined in* (5.9)*, with* $\tau_k = \frac{T}{n_k}$*. Then* $(\mu^{n_k})_{k\in\mathbb{N}}$ *and* $(M_{\tau_k})_{k\in\mathbb{N}}$ *converge to the same limit curve* $\mu : [0, T] \to \mathcal{P}_b(\mathbb{R}^d)$*, which is the unique* λ-EVI *solution of* (5.1) *in* $[0, T]$.

Proof By Proposition 7.9, $\mathbf{F}$ is a $\left(\frac{L^2+1}{2}\right)$-dissipative MPVF according to (4.1) s.t. $M(\mu_0, \tau, T, \tilde{L}) \neq \emptyset$ for every $\tau > 0$, where $\tilde{L} > 0$ is a suitable constant depending on μ_0 and $\mathbf{F}$. Thus by Theorem 6.7, $(M_{\tau_k})_{k\in\mathbb{N}}$ uniformly converges to a λ-EVI solution $\mu : [0, T] \to \mathcal{P}_2(\mathbb{R}^d)$ which is unique since $\mathbf{F}$ generates a $\left(\frac{L^2+1}{2}\right)$-flow. Since we start from a compactly supported μ_0, the semi-discrete Lagrangian scheme of [9] and our Euler Scheme actually coincide. To conclude we apply [9, Theorem 4.1] obtaining that μ is also the limit of the LASs scheme. $\square$

We conclude that among the possibly not-unique (see [9]) barycentric solutions to (5.1) - i.e. the solutions in the sense of [27]/Definition 5.25 - we are *selecting* only one (the λ-EVI solution), which turns out to be the one associated with the LASs approximating scheme.

In light of this observation, we revisit an interesting example studied in [27, Sect 7.1] and [9, Sect. 6].

Example 7.11 (Splitting particle) For every $\nu \in \mathcal{P}_b(\mathbb{R})$ define:

$$B(\nu) := \sup\left\{x : \nu(]-\infty, x]) \leq \frac{1}{2}\right\}, \qquad \eta(\nu) := \nu(]-\infty, B(\nu)]) - \frac{1}{2},$$

so that $\nu(\{B(\nu)\}) = \eta(\nu) + \frac{1}{2} - \nu(]-\infty, B(\nu)[)$. We define the PVF $\mathbf{F}[\nu] := \int \mathbf{F}_x[\nu]\, \mathrm{d}\nu(x)$, by

$$\mathbf{F}_x[\nu] := \begin{cases} \delta_{-1} & \text{if } x < B(\nu) \\ \delta_1 & \text{if } x > B(\nu) \\ \frac{1}{\nu(\{B(\nu)\})}\left(\eta\delta_1 + \left(\frac{1}{2} - \nu(]-\infty, B(\nu)[)\right)\delta_{-1}\right) & \text{if } x = B(\nu),\, \nu(\{B(\nu)\}) > 0. \end{cases}$$

By [27, Proposition 7.2], $\mathbf{F}$ satisfies assumptions (H1)-(H2) with $L = 0$ and the LASs scheme admits a unique limit. Moreover, the solution $\mu : [0, T] \to \mathcal{P}_b(\mathbb{R})$ obtained as limit of LASs, is given by

$$\begin{aligned} \mu_t(A) &= \mu_0((A \cap]-\infty, B(\mu_0) - t[) + t) + \mu_0((A \cap]B(\mu_0) + t, +\infty[) - t) \\ &\quad + \frac{1}{\mu_0(\{B(\mu_0)\})}\left(\eta\delta_{B(\mu_0)+t}(A) + (\frac{1}{2} - \mu_0(]-\infty, B(\mu_0)[))\delta_{B(\mu_0)-t}(A)\right). \end{aligned} \tag{7.6}$$

By Corollary 7.10, (7.6) is the (unique) λ-EVI solution of (5.1). In particular:

(i) if $\mu_0 = \frac{1}{b-a}\mathcal{L}\llcorner_{[a,b]}$, i.e. the normalized Lebesgue measure restricted to $[a, b]$, we get $\mu_t = \frac{1}{b-a}\mathcal{L}\llcorner_{[a-t, \frac{a+b}{2}-t]} + \frac{1}{b-a}\mathcal{L}\llcorner_{[\frac{a+b}{2}+t, b+t]}$;

(ii) if $\mu_0 = \delta_{x_0}$, we get $\mu_t = \frac{1}{2}\delta_{x_0+t} + \frac{1}{2}\delta_{x_0-t}$.

Notice that, in case (i), since $\mu_t \ll \mathcal{L}$ for all $t \in (0, T)$, i.e. $\mu_t \in \mathcal{P}_2^r(\mathbb{R})$, we can also apply Theorem 5.31 to conclude that μ is the λ-EVI solution of (5.1) with $\mu_0 = \frac{1}{b-a}\mathcal{L}\llcorner_{[a,b]}$. Moreover, take $\varepsilon > 0$, and consider case (i) where we denote by μ_0^ε the initial datum and by μ^ε the corresponding λ-EVI solution to (5.1) with $a = x_0 - \varepsilon$, $b = x_0 + \varepsilon$. We can apply (5.35) with $\mu_0 = \mu_0^\varepsilon$ and $\mu_1 = \delta_{x_0}$ in order to give another proof that, for all $t \in [0, T]$, the W_2-limit of $S_t[\mu_0^\varepsilon]$ as $\varepsilon \downarrow 0$, that is $S_t[\delta_{x_0}] = \frac{1}{2}\delta_{x_0+t} + \frac{1}{2}\delta_{x_0-t}$, is a λ-EVI solution starting from δ_{x_0}. Thus we end up with (ii).

Dealing with case (ii), we recall that, if $\mu_0 = \delta_{x_0}$ then also the stationary curve $\bar{\mu}_t = \delta_{x_0}$, for all $t \in [0, T]$, satisfies the barycentric property of Definition 5.25 (see [9, Example 6.1]), thus it is a solution in the sense of [27]. However, $\bar{\mu}$ is not a λ-EVI solution since it does not coincide with the curve given by (ii). This fact can also be checked by a direct calculation as follows: we find $\nu \in \mathcal{P}_b(\mathbb{R})$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\bar{\mu}_t, \nu) > \lambda W_2^2(\bar{\mu}_t, \nu) - [\mathbf{F}[\nu], \bar{\mu}_t]_r \qquad t \in (0, T), \tag{7.7}$$

where $\lambda = \frac{1}{2}$ is the dissipativity constant of the PVF $\mathbf{F}$ coming from the proof of Proposition 7.9. Notice that the l.h.s. of (7.7) is always zero since $t \mapsto \bar{\mu}_t = \delta_0$ is constant. Take $\nu = \mathcal{L}\llcorner_{[0,1]}$ so that we get $\mathbf{F}[\nu] = \int \mathbf{F}_x[\nu]\,\mathrm{d}\nu(x)$, with $\mathbf{F}_x[\nu] = \delta_1$ if $x > \frac{1}{2}$, $\mathbf{F}_x[\nu] = \delta_{-1}$ if $x < \frac{1}{2}$. Noting that $\Lambda(\mathbf{F}[\nu], \delta_0) = \{\mathbf{F}[\nu] \otimes \delta_0\}$, by using the characterization in Theorem 3.9 we compute

$$\begin{aligned}
[\mathbf{F}[\nu], \delta_0]_r &= \int_{\mathsf{TX}} \langle x, v\rangle\,\mathrm{d}\mathbf{F}[\nu] \\
&= \int_0^{1/2} \langle x, v\rangle\,\mathrm{d}\mathbf{F}_x[\nu](v)\,\mathrm{d}x \\
&\quad + \int_{1/2}^1 \langle x, v\rangle\,\mathrm{d}\mathbf{F}_x[\nu](v)\,\mathrm{d}x = \frac{1}{4}.
\end{aligned}$$

Since $W_2^2(\delta_0, \nu) = \mathsf{m}_2^2(\nu) = \frac{1}{3}$, we have

$$\lambda W_2^2(\bar{\mu}_t, \nu) - [\mathbf{F}[\nu], \bar{\mu}_t]_r = \frac{1}{6} - \frac{1}{4} < 0,$$

and thus we obtain the desired inequality (7.7) with $\nu = \mathcal{L}\llcorner_{[0,1]}$.

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Appendix A. Wasserstein differentiability along curves

We want to highlight how the result in Theorem 3.11, emphasized in Remark 3.13, is optimal giving an example of a locally absolutely continuous curve $\mu : [0, +\infty) \to \mathcal{P}_2(\mathbb{R}^2)$ s.t. the full measure set of differentiability points of the map $[0, +\infty) \ni s \mapsto W_2^2(\mu_s, \nu)$ depends also on $\nu \in \mathcal{P}_2(\mathbb{R}^2)$. To do that it is enough to show that

$$\text{for every } t_0 \in A(\mu) \text{ there exist } \nu_0 \in \mathcal{P}_2(\mathbb{R}^2) \text{ and } \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \in \Gamma_o(\mu_{t_0}, \nu_0) \text{ s.t. } L(\boldsymbol{\gamma}_1) \neq L(\boldsymbol{\gamma}_2),$$

where $A(\mu)$ is as in Theorem 2.10 and, for $\boldsymbol{\gamma} \in \mathcal{P}_2(\mathbb{R}^2 \times \mathbb{R}^2)$ s.t. $\mathsf{x}^0_\sharp \boldsymbol{\gamma} = \mu_t$, we define

$$L(\boldsymbol{\gamma}) := \int_{\mathsf{X}^2} \langle \boldsymbol{v}_t(x), x - y \rangle \, \mathrm{d}\boldsymbol{\gamma}(x, y).$$

Indeed this will imply that $\left[(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}_{t_0})_\sharp \mu_{t_0}, \nu_0\right]_r \neq \left[(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}_{t_0})_\sharp \mu_{t_0}, \nu_0\right]_l$, hence the non differentiability at t_0.

Let us consider two regular functions $u : [0, +\infty) \to \mathbb{R}^2$ and $r : [0, +\infty) \to \mathbb{R}$ s.t. $|u_t| = 1$ for every $t \geq 0$. Let $\omega : [0, +\infty) \to \mathbb{R}^2$ be defined as the orthogonal direction to u_t:

$$\omega_t := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} u_t, \qquad t \geq 0.$$

Being the norm of u constant in time, there exists some regular $\lambda : (0, +\infty) \to \mathbb{R}$ s.t. $\dot{u}_t = \lambda_t \omega_t$ for every $t > 0$. Finally we define

$$\begin{aligned} x_1 : [0, +\infty) \to \mathbb{R}^2, &\qquad x_1(t) := r_t u_t, \\ x_2 : [0, +\infty) \to \mathbb{R}^2, &\qquad x_2(t) := -r_t u_t, \\ \mu : [0, +\infty) \to \mathcal{P}_2(\mathbb{R}^2), &\qquad \mu_t := \frac{1}{2}\left(\delta_{x_1(t)} + \delta_{x_2(t)}\right). \end{aligned}$$

Observe that $\dot{x}_1(t) = \dot{r}_t u_t + r_t \dot{u}_t = -\dot{x}_2(t)$ for every $t > 0$. Moreover, for every $\varphi \in \mathrm{C}_c^\infty(\mathbb{R}^2)$ and $t > 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^2} \varphi \, \mathrm{d}\mu_t = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2}\varphi(x_1(t)) + \frac{1}{2}\varphi(x_2(t)) \right)$$

$$
\begin{aligned}
&= \frac{1}{2}\nabla\varphi(x_1(t))\,\dot{x}_1(t) + \frac{1}{2}\nabla\varphi(x_2(t))\,\dot{x}_2(t)\\
&= \int_{\mathbb{R}^2} \langle v_t(x), \nabla\varphi(x)\rangle\,\mathrm{d}\mu_t,
\end{aligned}
$$

where

$$
\boldsymbol{v}_t(x) := \begin{cases} \dot{x}_1(t) & \text{if } x = x_1(t),\\ \dot{x}_2(t) & \text{if } x = x_2(t), \end{cases} \qquad t > 0.
$$

Hence, the above defined vector field $\boldsymbol{v}_t$ solves the continuity equation with μ_t. Let $t_0 \in A(\mu)$ and let us define $\omega_0 := \omega(t_0)$, $\nu_0 := \frac{1}{2}\delta_{\omega_0} + \frac{1}{2}\delta_{-\omega_0}$ and the plans $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \in \Gamma_o(\mu_{t_0}, \nu_0)$ by

$$
\begin{aligned}
\boldsymbol{\gamma}_1 &:= \frac{1}{2}\delta_{x_1(t_0)} \otimes \delta_{\omega_0} + \frac{1}{2}\delta_{x_2(t_0)} \otimes \delta_{-\omega_0},\\
\boldsymbol{\gamma}_2 &:= \frac{1}{2}\delta_{x_2(t_0)} \otimes \delta_{\omega_0} + \frac{1}{2}\delta_{x_1(t_0)} \otimes \delta_{-\omega_0}.
\end{aligned}
$$

Notice that they are optimal since any plan in $\Gamma(\mu_{t_0}, \nu_0)$ has the same cost, being the points $\omega_0, x_1(t_0), x_2(t_0), -\omega_0$ the vertexes of a rhombus. Finally, we compute $L(\boldsymbol{\gamma}_1)$ and $L(\boldsymbol{\gamma}_2)$:

$$
\begin{aligned}
L(\boldsymbol{\gamma}_1) &= \int_{\mathbb{R}^2\times\mathbb{R}^2} \langle x - y, \boldsymbol{v}_t(x)\rangle\,\mathrm{d}\boldsymbol{\gamma}_1(x, y)\\
&= \frac{1}{2}\langle \dot{x}_1(t_0), x_1(t_0) - \omega_0\rangle + \frac{1}{2}\langle \dot{x}_2(t_0), x_2(t_0) + \omega_0\rangle\\
&= \langle \dot{x}_1(t_0), x_1(t_0) - \omega_0\rangle = \langle \dot{r}_{t_0}u_{t_0} + r_{t_0}\dot{u}_{t_0}, r_{t_0}u_{t_0} - \omega_0\rangle = r_{t_0}\dot{r}_{t_0} - r_{t_0}\lambda_{t_0},\\
L(\boldsymbol{\gamma}_2) &= \int_{\mathbb{R}^2\times\mathbb{R}^2} \langle x - y, \boldsymbol{v}_t(x)\rangle\,\mathrm{d}\boldsymbol{\gamma}_2(x, y)\\
&= \frac{1}{2}\langle \dot{x}_2(t_0), x_2(t_0) - \omega_0\rangle + \frac{1}{2}\langle \dot{x}_1(t_0), x_1(t_0) + \omega_0\rangle\\
&= \langle \dot{x}_1(t_0), x_1(t_0) + \omega_0\rangle = \langle \dot{r}_{t_0}u_{t_0} + r_{t_0}\dot{u}_{t_0}, r_{t_0}u_{t_0} + \omega_0\rangle = r_{t_0}\dot{r}_{t_0} + r_{t_0}\lambda_{t_0}.
\end{aligned}
$$

In this way, if $r_{t_0} \neq 0$ and $\lambda_{t_0} \neq 0$ we have $L(\boldsymbol{\gamma}_1) \neq L(\boldsymbol{\gamma}_2)$. A possible choice for u and r satisfying the assumptions is

$$
u_t := (\cos(t), \sin(t)), \qquad r_t = 1, \qquad t \geq 0,
$$

so that $\lambda_t = 1$ for every $t > 0$.

Appendix B. Technical results

We report here two useful versions of the Gronwall Lemma, where the first one is [3, Lemma 4.1.8].

Lemma B.1 *Let $x : [0, +\infty) \to \mathbb{R}$ be a locally absolutely continuous function, let $a, b \in L^1_{loc}([0, +\infty))$ and let $\delta \in \mathbb{R}$ be such that*

$$\frac{\mathrm{d}}{\mathrm{d}t} x^2(t) + 2\delta x^2(t) \leq a(t) + 2b(t)x(t) \quad \textit{for a.e. } t > 0.$$

Then for every $T > 0$ we have

$$\mathrm{e}^{\delta T} |x(T)| \leq \left(x^2(0) + \sup_{t \in [0,T]} \int_0^t \mathrm{e}^{2\delta s} a(s)\, \mathrm{d}s \right)^{1/2} + 2 \int_0^T \mathrm{e}^{\delta t} |b(t)|\, \mathrm{d}t.$$

Lemma B.2 (Discrete Gronwall inequality) *Let $\alpha \geq 0$, $y \geq 0$, $\tau > 0$ and $N \in \mathbb{N}$ with $N > 0$. If a sequence $(x_n)_{n \in \mathbb{N}}$ of positive real numbers satisfies*

$$x_{n+1} - x_n \leq \tau y + \tau \alpha x_n, \tag{B.1}$$

for any $0 \leq n \leq N$, then

$$x_n \leq (x_0 + \tau n y)\mathrm{e}^{\alpha n \tau},$$

for any $0 \leq n \leq N + 1$.

Proof We treat only the non trivial case $n \geq 1$ and $\alpha > 0$; we will repeatedly use the elementary inequality

$$1 + x \leq e^x \tag{B.2}$$

for every $x \in \mathbb{R}$. Multiplying (B.1) written for $n = k \in \{0, \dots, N\}$ by $\mathrm{e}^{-\alpha\tau(k+1)}$, we obtain

$$\mathrm{e}^{-\alpha\tau(k+1)} x_{k+1} \leq \tau y \mathrm{e}^{-\alpha\tau(k+1)} + x_k(1 + \tau\alpha)\mathrm{e}^{-\alpha\tau(k+1)} \leq \tau y \mathrm{e}^{-\alpha\tau(k+1)} + x_k \mathrm{e}^{-\alpha\tau k},$$

where the last inequality comes from (B.2) with $x = \alpha\tau$. Let $n \in \{0, \dots, N + 1\}$; we sum the previous inequality written for $k \in \{0, \dots, n - 1\}$ obtaining

$$\mathrm{e}^{-\alpha\tau n} x_n - x_0 \leq \tau y \mathrm{e}^{-\alpha\tau} \sum_{k=0}^{n-1} \left(\mathrm{e}^{-\alpha\tau}\right)^k = \tau y \mathrm{e}^{-\alpha\tau} \frac{1 - \mathrm{e}^{-\alpha\tau n}}{1 - \mathrm{e}^{-\alpha\tau}}.$$

Then we get

$$\begin{aligned} x_n &\leq x_0 \mathrm{e}^{\alpha\tau n} + \tau y \frac{\mathrm{e}^{\alpha\tau n} - 1}{\mathrm{e}^{\alpha\tau} - 1} \\ &= x_0 \mathrm{e}^{\alpha\tau n} + \tau y n \frac{\mathrm{e}^{\alpha\tau n} - 1}{\alpha\tau n} \frac{\alpha\tau}{\mathrm{e}^{\alpha\tau} - 1} \\ &\leq x_0 \mathrm{e}^{\alpha\tau n} + \tau y n \mathrm{e}^{\alpha\tau n}, \end{aligned}$$

where we used again (B.2) in the last step. □

We recall the following characterization of the closed convex hull $\overline{\mathrm{co}}(C)$ of a set C (i.e. the intersection of all the closed convex sets containing C) in a Banach space.

Lemma B.3 *Let Z be a Banach space and let $C \subset Z$ be nonempty. Then $v \in \overline{\mathrm{co}}(C)$ if and only if*

$$\langle z^*, v\rangle \le \sup_{c\in C} \langle z^*, c\rangle \tag{B.3}$$

for all $z^* \in Z^*$. Moreover if C is bounded, it is enough to have (B.3) holding for every $z^* \in W$, with W a dense subset of Z^*.

Proof The result is a direct consequence of Hahn-Banach theorem.
Concerning the last assertion, observe that the function

$$Z^* \ni z^* \mapsto \sup_{c\in C} \langle z^*, c\rangle$$

is Lipschitz continuous if C is bounded. Hence, if (B.3) holds only for some $W \subset Z^*$ dense, then it holds for the whole Z^*. □

Let us state and prove a simple lemma that allows us to pass from a differential inequality for the right upper Dini derivative to the corresponding distributional inequality (see also [22, Lemma A.1] and [17]).

Lemma B.4 *Let $(a, b) \subset \mathbb{R}$ be an open interval (bounded or unbounded) and let $\zeta, \eta : (a, b) \to \mathbb{R}$ be s.t. ζ is continuous in (a, b) and η is measurable and locally bounded from above in (a, b). If*

$$\frac{\mathrm{d}^+}{\mathrm{d}t} \zeta(t) \le \eta(t)$$

for every $t \in (a, b)$, then the above inequality holds also in the sense of distributions, meaning that

$$-\int_a^b \zeta(t)\varphi'(t)\,\mathrm{d}t \le \int_a^b \eta(t)\varphi(t)\,\mathrm{d}t$$

for every $\varphi \in \mathrm{C}_c^\infty(a, b)$ with $\varphi \ge 0$.

Proof Let $\varphi \in \mathrm{C}_c^\infty(a, b)$ with $\varphi \ge 0$, then there exist $a < x < y < b$ s.t. the support of φ is contained in $[x, y]$; since η is locally bounded from above, there exists a positive constant $C > 0$ s.t. $\eta(t) \le C$ for every $t \in [x, y]$. Then the function $t \mapsto \zeta(t) - Ct$ is such that

$$\frac{\mathrm{d}^+}{\mathrm{d}t} (\zeta(t) - Ct) \le 0$$

for every $t \in [x, y]$, so that it is decreasing in $[x, y]$ and hence a function of bounded variation in $[x, y]$. Its distributional derivative is hence a non positive measure T on $[x, y]$ whose absolutely continuous part (w.r.t. the 1-dimensional Lebesgue measure on $[x, y]$) coincides a.e. with the right upper Dini derivative. Then we have

$$
\begin{aligned}
-\int_a^b (\zeta(t) - Ct)\varphi'(t)\,\mathrm{d}t = T(\varphi) &= \int_a^b \frac{\mathrm{d}^+}{\mathrm{d}t}(\zeta(t) - Ct)\varphi(t)\,\mathrm{d}t + T_s(\varphi) \\
&\le \int_a^b (\eta - C)\varphi(t)\,\mathrm{d}t,
\end{aligned}
$$

where T_s is the singular part of T. This immediately gives the thesis. $\square$

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