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A VERSION OF THE HOPF-LAX FORMULA IN THE HEISENBERG GROUP

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ABSTRACT

We consider Hamilton-Jacobi equations

$$u_t + H(D_\mathrm{h} u) = 0$$

in the $\mathcal{H} \times \mathbb{R}^+$, where $\mathcal{H}$ is the Heisenberg group and $D_\mathrm{h} u$ denotes the horizontal gradient of u. We establish uniqueness of bounded viscosity solutions with continuous initial data $u(p, 0) = g(p)$. When the hamiltonian H is radial, convex and superlinear the solution is given by the Hopf-Lax formula

$$u(p, t) = \inf_{q \in \mathcal{H}} \left\{ tL\left(\frac{q^{-1} \cdot p}{t}\right) + g(q) \right\},$$

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where the Lagrangian L is the horizontal Legendre transform of H lifted to $\mathcal{H}$ by requiring it to be radial with respect to the Carnot-Carathéodory metric.

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1. INTRODUCTION

The regularizing effects of solutions of Hamilton-Jacobi equations can sometimes be used as a replacement for other approximation procedures, like convolution, which are not available in the study of nonlinear elliptic equations. This approach was discovered by Lasry and Lions in Ref. [1] and used by R. Jensen in Ref. [2] to prove the comparison principle for viscosity solutions of fully nonlinear elliptic equations. See also Ref. [3] for an enlighting explanation.

When considering the possible extension of these techniques to the subelliptic case is therefore appropriate to study Hamilton-Jacobi equations in a subriemannian framework. In this article we focus on the Heisenberg group, where once we defined a suitable notion of subelliptic jet, the proof of uniqueness of solutions mirrors the Euclidean case, but the existence present difficulties arising from the lack of commutativity of the group, and more importantly, from the different metric structure. As it will be apparent in Section §3 below, the appropriate gauge is the, so called, Carnot-Carathéodory gauge instead of the more commonly used smooth gauges. The underlying geometric reason is that geodesics behave better with respect to the Carnot-Carathéodory gauge than other gauges.

While the theory of quasilinear elliptic equations in divergence form in the subelliptic setting has been extensively developed (see Refs. [4–6] and references therein), in the case of fully nonlinear subelliptic equations the theory is at a more primitive stage. Recently Bieske (Ref. [7]) considered extensions of the Jensen's maximum principle from Ref. [8] to the Heisenberg group for the special case of infinite harmonic functions and one of us (Ref. [9]) refined Bieske's ideas to prove a comparison principle for viscosity solutions of fully nonlinear elliptic equations in the Heisenberg group.

There are many articles dealing with Hopf-Lax formulas for solutions of Hamilton Jacobi equations of the form

$$\begin{cases} u_t + F(u, Du) = 0 \\ \qquad u(x, 0) = g(x) \end{cases}$$

under various hypothesis on the Hamiltonian $F(u, Du)$ and the initial datum $g(x)$. See Refs. [10–14] and references therein. However, we are not aware[1] of any Hopf-Lax type formulas when the Hamiltonian $F(x, u, Du)$ depends also on x. To illustrate our results, consider the following example in $(0, \infty) \times \mathbb{R}^3$

$$\begin{cases} u_t + \dfrac{1}{\alpha}\left[\left(\dfrac{\partial u}{\partial x} - \dfrac{y}{2}\dfrac{\partial u}{\partial z}\right)^2 + \left(\dfrac{\partial u}{\partial y} + \dfrac{x}{2}\dfrac{\partial u}{\partial z}\right)^2\right]^{\alpha/2} = 0 \\ \qquad\qquad u(x, y, z, 0) = g(x, y, z), \end{cases}$$

where $\alpha > 1$. By interpreting this equation in a subriemmanian setting, the solution $u(\xi, t)$, where $\xi = (x_1, y_1, z_1) \in \mathbb{R}^3$, is given by the Hopf-Lax formula

$$\inf\left\{\frac{t}{\beta}\left(d\left(\frac{x_1 - x}{t}, \frac{y_1 - y}{t}, \frac{z_1 - z + \frac{1}{2}(x_1 y - x y_1)}{t^2}\right)\right)^{\beta/2} + g(x, y, z)\right\}, \tag{1.1}$$

where the infimum is taken for $(x, y, z) \in \mathbb{R}^3$, the exponent $\beta = \alpha/(\alpha - 1)$ is the Hölder conjugate of α, and $d(x, y, z)$ is the Carnot gauge associated to the vector fields

$$X_1(x, y, z) = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}$$

and

$$X_2(x, y, z) = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}.$$

The gauge $d(x, y, z)$ does not have a simple closed form expression in terms of $x, y,$ and z, but its geometric significance makes the formula Eq. (1.1) very useful in the study of nonlinear subelliptic equations. See the forthcoming paper (Ref. [15]).

Our setting is the Heisenberg group $\mathcal{H}$, which is the connected and simply connected Lie group with Lie algebra $\mathfrak{h}$ spanned by the vector fields X_1, X_2 and

$$X_3(x, y, z) = \frac{\partial}{\partial z} = [X_1, X_2].$$

[1]Recently we learn of work in progress by H. Ishii and I. Capuzzo-Dolcetta who have extended and generalised some of the results in this paper.

For analysis on Carnot groups we refer to Refs. [16–20]. Endow $\mathfrak{h}$ with and inner product $\langle\cdot,\cdot\rangle$ so that $\{X_1, X_2, X_3\}$ is an orthonormal basis. The associated norm will be denoted by $\|\cdot\|$.

Using exponential coordinates in $\mathcal{H}$ we identify the vector $xX_1 + yX_2 + zX_3$ in $\mathfrak{h}$ with the point (x, y, z) in $\mathcal{H}$ (so that $\exp\colon \mathfrak{h} \to \mathcal{H}$ is the identity). The group multiplication law in $\mathcal{H}$ is given by

$$p \cdot q = \left(x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2 + \frac{1}{2}(x_1 y_2 - x_2 y_1)\right),$$

where $p = (x_1, y_1, z_1)$ and $q = (x_2, y_2, z_2)$. For a point $p = (x, y, z)$ we write $\bar{p} = (x, y, 0)$ and call points with vanishing z-coordinate horizontal points. We also denote by $\mathcal{H}_0$ the set of all horizontal points, or horizontal vectors if we think of them as members of the Lie algebra.

The Heisenberg group $\mathcal{H}$ has a family of dilations that are group homomorphisms, parameterized by $r > 0$ and given by

$$\delta_r(x, y, z) = (rx, ry, r^2 z).$$

Note that whenever s and t are positive $\delta_t \circ \delta_s = \delta_{st}$ and $\delta_s \cdot \delta_t = \delta_t \cdot \delta_s$. For negative $r < 0$ we define

$$\delta_r(p) = (\delta_{-r})^{-1}(p) = \delta_{-1/r}(p).$$

We often write $r(p)$ or even rp to denote $\delta_r(p)$ and for positive r, we write p/r to denote $\delta_{1/r}(p)$.

The Carnot-Carathéodory metric in $\mathcal{H}$ is a left-invariant metric homogeneous with respect to the dilations δ_r defined as follows. A curve $t \mapsto \gamma(t) \in \mathcal{H}$ is horizontal if its tangent vector $\gamma'(t)$ is in the two dimensional subspace generated by $\{X_1(\gamma(t)), X_2(\gamma(t))\}$. The Carnot-Carathéodory distance between the points p and q is defined as the infimum

$$d(p, q) = \inf_{\Gamma} \int_0^1 \|\gamma'(t)\|\, dt,$$

where the set Γ is the set of all horizontal curves γ such that $\gamma(0) = p$ and $\gamma(1) = q$. By Chow's theorem (see, for example Ref. [16],) any two points can be connected by a horizontal curve, which makes $d(p, q)$ a left-invariant metric on $\mathcal{H}$. A Carnot-Carathéodory ball of radius r centered at a point p_0 is given by

$$B(p_0, r) = \{p \in \mathcal{H} : d(p, p_0) < r\}.$$

A homogeneous norm for $p \in \mathcal{H}$ is defined by

$$|p| = d(0, p).$$

Note that $|\bar{p}| \leq |p|$ since the projection of a horizontal curve is horizontal and that $|\bar{p}| = \sqrt{x^2 + y^2}$ for $p = (x, y, z)$. In addition, this homogeneous norm or gauge has the following properties:

a) $|p| = |p^{-1}|$,
b) $|rp| = |r||p|$,
c) $|p| = 0$ if and only if $p = 0$,
d) $|pq| \leq |p| + |q|$ and
e) $|(x, y, z)| \sim |x| + |y| + |z|^{1/2}$. (1.2)

Property (e) is a special case of the "Ball-Box" theorem, see Ref. [16] or Ref. [19].

We shall also need a smooth gauge equivalent to the Carnot gauge just defined, called the Heisenberg gauge $p \mapsto \|p\|_H$. It is given by

$$\|p\|_H = \left(\left(x^2 + y^2 \right)^2 + z^2 \right)^{1/4}. \tag{1.3}$$

The Heisenberg gauge also satisfies properties a) through e) above (See Ref. [20]).

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2. VISCOSITY SOLUTIONS

In order to define viscosity solutions we must first identify the first order jets adapted to our framework. Motivated by the Taylor expansion (Ref. [16]) consider a differentiable function $u: \mathcal{H} \times \mathbb{R} \mapsto \mathbb{R}$ at the point (p_0, t_0). We have

$$u(p, t) = u(p_0, t_0) + \langle D_{\mathrm{h}} u(p_0, t_0), \overline{p_0^{-1} \cdot p} \rangle + u_t(p_0, t_0)(t - t_0)$$
$$+ o(|p_0^{-1} \cdot p| + |t - t_0|),$$

where $D_{\mathrm{h}} u = (X_1 u) X_1 + (X_2 u) X_2$ is the horizontal gradient of u.

Definition 1. A function $f: \mathcal{H} \mapsto \mathbb{R}$ is of class C^1 if the horizontal derivatives $X_1 f$ and $X_2 f$ are continuous.

Remark 1. Note that a function could be C^1 in this sense, but not in the usual sense. For example, we could take

$$f(x, y, z) = \|p_0^{-1} \cdot p\|_H^2,$$

that has continuous horizontal derivatives but it fails to have a continuous z-derivative at the point (x_0, y_0, z_0).

If a function u is not necessarily smooth but merely upper semicontinuous, the collection of vectors $\eta \in \mathbb{R}^3$ such that

$$u(p, t) \le u(p_0, t_0) + \langle \bar{\eta}, \overline{p_0^{-1} \cdot p} \rangle + \eta_3(t - t_0) + o\big(|p_0^{-1} \cdot p| + |t - t_0|\big) \tag{2.1}$$

is denoted by $J_u^{1,+}(p_0, t_0)$ and called the first order superjet of u at the point (p_0, t_0). Analogously we define $J_u^{1,-}(p_0, t_0)$, the first order subjet of a lower semicontinuous function u at (p_0, t_0) as the set of vectors $\gamma \in \mathbb{R}^3$ such that

$$u(p, t) \ge u(p_0, t_0) + \langle \bar{\gamma}, \overline{p_0^{-1} \cdot p} \rangle + \gamma_3(t - t_0) + o\big(|p_0^{-1} \cdot p| + |t - t_0|\big). \tag{2.2}$$

Jets can also be characterized by test functions ψ as follows.

Proposition 1. *Let u defined on a neighborhood of a point* (p_0, t_0). *Suppose that* ψ *is a* C^1 *function touching u from above at* (p_0, t_0),

$$u(p_0, t_0) = \psi(p_0, t_0)$$

and

$$u(p, t) \le \psi(p, t)$$

in a neighborhood of (p_0, t_0). *Then the vector*

$$(D_h \psi(p_0, t_0), \psi_t(p_0, t_0)) \in J_u^{1,+}(p_0, t_0).$$

Moreover, every vector

$$\eta \in J_u^{1,+}(p_0, t_0)$$

is of the form

$$(D_h \psi(p_0, t_0), \psi_t(p_0, t_0))$$

for some C^1 function ψ *that touches u from above at* (p_0, t_0).

A similar statement holds for $J_u^{1,-}(p_0, t_0)$ *replacing "touching from above" by "touching from below."*

Proof. The direct part follows easily from the Taylor expansion of ψ. To prove the converse we start from Eq. (2.1) and want to construct $\psi \in C^1$ that touches u from above at (p_0, t_0). We follow the variations introduced in Ref. [7] on the proof for the Euclidean case in Ref. [21].

Recall that $\|p\|_H$ is the Heisenberg gauge Eq. (1.3). Given $r > 0$ consider

$$g(r) = \sup\left\{ \left(u(p,t) - u(p_0,t_0) - \langle \bar{\eta}, \overline{p_0^{-1} \cdot p} \rangle - \eta_3(t - t_0) \right)^+ \right\}$$

where the supremum is taken in the set $\|p_0^{-1} \cdot p\|_H < r$ and $|t - t_0| < r$. Observe that $g(r)$ is non-decreasing and $g(r) = o(r)$ as $r \to 0^+$. Choose a non-decreasing continuous function $\tilde{g}(r)$ satisfying $\tilde{g}(r) \geq g(r)$ and still $\tilde{g}(r) = o(r)$ as $r \to 0^+$. Define

$$G(s) = \frac{1}{s} \int_s^{2s} \tilde{g}(\sigma)\, d\sigma$$

and consider

$$\psi(p,t) = u(p_0,t_0) + G(\|p_0^{-1} \cdot p\|_H) + \|p_0^{-1} \cdot p\|_H^2 + |t - t_0|^2 + \langle \bar{\eta}, \overline{p_0^{-1} \cdot p} \rangle + \eta_3(t - t_0).$$

An elementary calculation shows that $X_1\psi$ and $X_2\psi$ are continuous in a neighborhood of (p_0, t_0). Observe that ψ touches u from above at (p_0, t_0) by the construction of G.

From the fact that $D_{\mathrm{h}}(G(\|p_0^{-1} \cdot p\|_H))(p_0) = (0,0)$ we deduce

$$D_{\mathrm{h}}\psi(p_0,t_0) = \bar{\eta}$$

and

$$\psi_t(p_0,t_0) = \eta_3.$$

In the set $s \leq \min\{\|p_0^{-1} \cdot p\|_H, |t - t_0|\}$ we have $u(p,t) - \psi(p,t) + 2s^2 \leq 0$. Therefore the function $u - \psi$ has a strict maximum at (p_0, t_0). □

Definition 2. Let $F: \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. An upper semicontinuous function $u: D \times I \mapsto \mathbb{R}$, where $I \subset \mathbb{R}^+$ is an open interval and $D \subset \mathcal{H}$ is a domain, is a viscosity subsolution of the equation

$$u_t + F(D_{\mathrm{h}}u) = 0$$

if whenever $(p_0, t_0) \in D \times I$ and ψ is a C^1-test function touching u from above at (p_0, t_0), we have

$$\psi_t(p_0,t_0) + F(D_{\mathrm{h}}\psi(p_0,t_0)) \leq 0.$$

Equivalently, we can give the definition in terms of superjets. For every $\eta \in J_u^{1,+}(p_0, t_0)$ we have

$$\eta_3 + F(\bar{\eta}) \leq 0.$$

A lower semicontinuous function $v: D \times I \mapsto \mathbb{R}$ is a viscosity supersolution of the equation

$$v_t + F(D_\mathrm{h} v) = 0$$

if whenever $(p_0, t_0) \in D \times I$ and ψ is a C^1-test function touching u from below at (p_0, t_0), we have

$$\psi_t(p_0, t_0) + F(D_\mathrm{h}\psi(p_0, t_0)) \geq 0.$$

Once again, we can give an equivalent definition in terms of subjets. For every $\gamma \in J_v^{1,-}(p_0, t_0)$ we have

$$\gamma_3 + F(\bar{\gamma}) \geq 0.$$

A continuous function $u: D \times I \mapsto \mathbb{R}$ that is both a viscosity subsolution and a viscosity supersolution is called a viscosity solution.

3. THE HOPF-LAX FORMULA IN $\mathcal{H}$

In this section we extend to the Heisenberg group the Hopf-Lax formula and some of its properties from the Euclidean case when we have radial Hamiltonians. We have benefited from the techniques in Ref. [12]. Consider $H(v) = \mathfrak{f}(|v|)$, where $\mathfrak{f}$ is a convex increasing function satisfying

$$\lim_{s\to\infty} \frac{\mathfrak{f}(s)}{s} = +\infty \tag{3.1}$$

and

$$\lim_{s\to 0} \frac{\mathfrak{f}(s)}{s} = 0. \tag{3.2}$$

and $|v|^2 = a^2 + b^2$ for $v = aX_1 + bX_2$. The horizontal Legendre transform of H is defined on horizontal vectors by

$$L(v) = \sup_{w\in\mathfrak{h}_0} \{\langle v, w\rangle - H(w)\}.$$

It is easy to see that $L(v) = \phi(|v|)$ where ϕ is the (one variable) Legendre transform of $\mathfrak{f}$. Moreover ϕ has also properties Eqs. (3.1) and (3.2) (see Ref. [22]). We now lift L to the Heisenberg group by requiring that L is radial with respect to the Carnot-Carathéodory metric so that for $p \in \mathcal{H}$ we have

$$L(p) = \phi(d(0, p)).$$

Let $g:\mathcal{H}\mapsto\mathbb{R}$ be a bounded continuous function. In analogy with the classical Hopf-Lax formula we define for $t>0$ and $p\in\mathcal{H}$

$$u(p,t)=\inf_{q\in\mathcal{H}}\left\{tL\left(\frac{q^{-1}\cdot p}{t}\right)+g(q)\right\}. \tag{3.3}$$

Let us observe that the continuity and superlinearity of L, and the continuity of g show that the infimum in Eq. (3.3) is actually a minimum. Also, by taking $q=p$ we see that we always have the upper bound

$$u(p,t)\leq g(p). \tag{3.4}$$

The following semigroup property is the starting point of the theory.

Theorem 1. *For* $0<s<t$ *and all* $p\in\mathcal{H}$ *we have*

$$u(p,t)=\min_{q\in\mathcal{H}}\left\{(t-s)L\left(\frac{q^{-1}\cdot p}{t-s}\right)+u(q,s)\right\}.$$

We need a lemma:

Lemma 1. *Let* $\phi:[0,\infty)\mapsto\mathbb{R}$ *be a convex increasing function satisfying* $\phi(0)=0$ *and set*

$$L(p)=\phi(d(p,0)).$$

Then for all $\sigma,\tau\geq 0$ *such that* $\sigma+\tau=1$ *we have*

$$L((\tau p)\cdot(\sigma q))\leq\tau L(p)+\sigma L(q). \tag{3.5}$$

In particular, given any three points p, q *and* v *in* $\mathcal{H}$ *we have*

$$L(v^{-1}\cdot p)\leq\tau L\left(\frac{v^{-1}\cdot q}{\tau}\right)+\sigma L\left(\frac{q^{-1}\cdot p}{\sigma}\right). \tag{3.6}$$

Proof. Start with the triangle inequality

$$|(\tau p)\cdot(\sigma q)|\leq|\tau p|+|\sigma q|\leq\tau|p|+\sigma|q|$$

and apply the monotonicity and convexity of ϕ.

Next, given any three point p,q and v in $\mathcal{H}$ write

$$v^{-1}\cdot p=v^{-1}\cdot q\cdot q^{-1}\cdot p=\tau\left(\frac{v^{-1}\cdot q}{\tau}\right)\sigma\left(\frac{q^{-1}\cdot p}{\sigma}\right)$$

and apply Eq. (3.5) to obtain Eq. (3.6). □

PROOF OF THEOREM 1:

Proof. The proof will be divided in two parts.

Step 1: Based on the convexity property Eq. (3.6):

$$u(p,t) \leq \min_{q\in\mathcal{H}}\left\{(t-s)L\left(\frac{q^{-1}\cdot p}{t-s}\right)+u(q,s)\right\}. \tag{3.7}$$

Fix $q \in \mathcal{H}$ and choose $v \in \mathcal{H}$ such that

$$u(q,s) = sL\left(\frac{v^{-1}\cdot q}{s}\right)+g(v).$$

Apply Eq. (3.6) to p, q and v with $\sigma = s/t$ and $\tau = (t-s)/t$ to obtain:

$$tL\left(\frac{v^{-1}\cdot p}{t}\right) \leq sL\left(\frac{v^{-1}\cdot q}{s}\right)+(t-s)L\left(\frac{q^{-1}\cdot p}{t-s}\right).$$

Adding $g(v)$ to both sides and taking minima over $q \in \mathcal{H}$ we obtain the claim. □

Step 2: Based on geodesics:

$$u(p,t) \geq \min_{q\in\mathcal{H}}\left\{(t-s)L\left(\frac{q^{-1}\cdot p}{t-s}\right)+u(q,s)\right\}. \tag{3.8}$$

Given points $p, w \in \mathcal{H}$ and numbers $0 < s < t < \infty$ we want to find $q \in \mathcal{H}$ such that we have the equality

$$\frac{t-s}{t}L\left(\frac{q^{-1}\cdot p}{t-s}\right)+\frac{s}{t}L\left(\frac{w^{-1}\cdot q}{s}\right) = L\left(\frac{w^{-1}\cdot p}{t}\right). \tag{3.9}$$

For q along a minimizing geodesic from p to w we always have

$$|q^{-1}\cdot p| + |w^{-1}\cdot q| = |w^{-1}\cdot p|.$$

In addition given $\sigma, \tau \geq 0$ and satisfying $\sigma + \tau = 1$ we can always find q so that

$$|q^{-1}\cdot p| = \tau|w^{-1}\cdot p|$$

and

$$|w^{-1} \cdot q| = \sigma |w^{-1} \cdot p|.$$

Setting $\sigma = s/t$ we obtain

$$\frac{t}{t-s} \frac{|q^{-1} \cdot p|}{t} = \frac{|w^{-1} \cdot p|}{t}, \ \frac{t}{s} \frac{|w^{-1} \cdot q|}{t} = \frac{|w^{-1} \cdot p|}{t}$$

and the triple equality

$$L\left(\frac{q^{-1} \cdot p}{t-s}\right) = L\left(\frac{w^{-1} \cdot p}{t}\right) = L\left(\frac{w^{-1} \cdot q}{s}\right).$$

Therefore we found $q \in \mathcal{H}$ so that Eq. (3.9) holds. To finish the proof of the claim choose $w \in \mathcal{H}$ so that we have the equality

$$u(p,t) = tL\left(\frac{w^{-1} \cdot p}{t}\right) + g(w)$$

and add $g(w)$ to both sides of Eq. (3.9). □

If the initial datum g is a Lipschitz function then, in the Riemannian case, the function $u(p,t)$ is also Lipschitz. We do not know whether this is indeed the case in the Heisenberg group. In Theorem 2 below we prove the local Hölder continuity of $u(p,t)$ with exponent $1/2$.

Denote by $\mathrm{Lip}(g)$ the Lipschitz constant of g. Consider the constant $C(L, \mathrm{Lip}(g))$ defined as follows

$$C(L, \mathrm{Lip}(g)) = \max_{p \in \mathcal{H}} \left\{ \mathrm{Lip}(g)|p| - L(p) \right\}.$$

By setting $p = 0$ it is clear that $C(L, \mathrm{Lip}(g)) \geq 0$. Indeed, if the constant $C(L, \mathrm{Lip}(g)) = 0$, then the superlinearity of ϕ gives that $\mathrm{Lip}(g) = 0$ and therefore g is constant. Note that we have used Eq. (3.2) here. We will assume from now on that g is not constant, and so

$$C(L, \mathrm{Lip}(g)) > 0.$$

Note also that $C(L, 2\mathrm{Lip}(g))$ is strictly positive.

Choose a vector $v \in \mathcal{H}$ such that we have the equality

$$u(p,t) = tL\left(\frac{v^{-1} \cdot p}{t}\right) + g(v).$$

Since $u(p,t) \le g(p)$ we have

$$0 \ge t\,L\left(\frac{v^{-1}\cdot p}{t}\right) + g(v) - g(p) \ge t\phi\left(\frac{d(v,p)}{t}\right) - d(v,p)\mathrm{Lip}(g).$$

Dividing by $d(v,p)$ we obtain

$$0 \ge \frac{\phi(d(v,p)/t)}{d(v,p)/t} - \mathrm{Lip}(g).$$

Choose $s_0 \ge 0$ so that for $s \ge s_0$ we have

$$\frac{\phi(s)}{s} > \mathrm{Lip}(g).$$

We conclude that

$$d(v,p) \le s_0 t. \tag{3.10}$$

We are ready for the regularity properties of Eq. (3.3):

Theorem 2. *Let $u(p,t)$ be given by Eq.* (3.3). *We have*

i) *for all $p \in \mathcal{H}$ and $t > 0$*

$$|u(p,t) - g(p)| \le C(L, \mathrm{Lip}(g))\,t, \tag{3.11}$$

ii) *there exists an constant $k > 0$ independent of the initial datum g and the Lagrangian L such that for all $p, q \in \mathcal{H}$ and $t > 0$*

$$|u(p,t) - u(q,t)| \le k\,\mathrm{Lip}(g)\big(d(p,q)^{1/2} + (s_0 t)^{1/2} + (|p| + |q|)^{1/2}\big)d(p,q)^{1/2}, \tag{3.12}$$

and

iii) *given a compact set $K \subset \mathcal{H}$ here exists a constant $A = A(K,k,L, Lip(g))$ depending on $K, k, Lip(g),$ and L so that for $|t - s| < 1$ and $p \in K$*

$$|u(p,t) - u(p,s)| \le A|t - s|^{1/2}. \tag{3.13}$$

Proof. Using the Lipschitz continuity of g and the homogeneity of the Carnot metric we have

$$u(p,t) \geq g(p) + \min_{q\in\mathcal{H}}\left\{tL\left(\frac{q^{-1}\cdot p}{t}\right) - \mathrm{Lip}(g)\,d(p,q)\right\}$$
$$\geq g(p) - t\max_{v\in\mathcal{H}}\left\{-L(v) + \mathrm{Lip}(g)\,d(v,0)\right\}.$$

Together with the inequality Eq. (3.4) this implies Eq. (3.11).

To prove the Hölder continuity in the space variable, fix p and q in $\mathcal{H}$. Choose $v \in \mathcal{H}$ such that

$$u(p,t) = tL\left(\frac{v^{-1}\cdot p}{t}\right) + g(v).$$

Write the difference

$$u(q,t) - u(p,t) = \min_{w\in\mathcal{H}}\left\{tL\left(\frac{w^{-1}\cdot q}{t}\right) + g(w)\right\} - tL\left(\frac{v^{-1}\cdot p}{t}\right) - g(v).$$

Set $w = q\cdot p^{-1}\cdot v$ in the formula above and note that $d(w,q) = d(v,p)$ to get

$$u(q,t) - u(p,t) \leq g(w) - g(v) \leq \mathrm{Lip}(g)\,d(w,v). \tag{3.14}$$

To estimate $d(w,v)$ we proceed as follows. Write

$$d(w,v) = d(q\cdot p^{-1}\cdot v, v) = |v^{-1}\cdot p\cdot q^{-1}\cdot v| = |v^{-1}\cdot q\cdot p^{-1}\cdot v|. \tag{3.15}$$

Temporarily set $p\cdot q^{-1} = (a,b,c)$ and $v = (\alpha,\beta,\gamma)$. We compute

$$v^{-1}\cdot p\cdot q^{-1}\cdot v = (a, b, c + a\beta - b\alpha)$$

and use the Ball-Box estimate for the Carnot gauge to obtain

$$\begin{aligned}|v^{-1}\cdot p\cdot q^{-1}\cdot v| &\leq k\big(|a| + |b| + |c + a\beta - b\alpha|^{1/2}\big)\\ &\leq k\big(|a| + |b| + |c|^{1/2} + |a\beta - b\alpha|^{1/2}\big)\\ &\leq k\big(|p\cdot q^{-1}| + |p\cdot q^{-1}|^{1/2}|v|^{1/2}\big)\\ &\leq k\,d(p,q)^{1/2}\big(d(p,q)^{1/2} + |v|^{1/2}\big),\end{aligned}$$

where the constant k depend only on the constants in the Ball-Box property Eq. (1.2). To estimate $|v|$ we rely on Eq. (3.10) and the triangle inequality

$$|v| \leq d(v,p) + |p| \leq s_0 t + |p|. \tag{3.16}$$

Inequality Eq. (3.12) follows from Eq. (3.14), Eqs. (3.15) and (3.16).

To establish the Hölder continuity in time note that from Eq. (3.3) it follows that for $0 < s \leq t$ we have

$$u(p,t) - u(p,s) \leq 0.$$

Start with the semigroup property from Theorem 1 and add $-u(p,s)$ to both sides. We obtain

$$0 \geq u(p,t) - u(p,s) = \min_{q \in \mathcal{H}} \left\{ (t-s) L\left(\frac{q^{-1} \cdot p}{t-s} \right) + u(q,s) - u(p,s) \right\}.$$

We estimate the right hand side using Eq. (3.12):

$$\begin{aligned} 0 &\geq u(p,t) - u(p,s) \\ &\geq \min_{q \in \mathcal{H}} \left\{ (t-s) L\left(\frac{q^{-1} \cdot p}{t-s} \right) \right. \\ &\quad \left. - k \operatorname{Lip}(g)\, d(p,q)^{1/2} \left[d(p,q)^{1/2} + (s_0 s)^{1/2} + (|p| + |q|)^{1/2} \right] \right\}. \end{aligned}$$

Set $q = p \cdot \delta_{t-s}(v^{-1})$ for a variable $v \in \mathcal{H}$ and $h = t - s > 0$ in the above formula. Then we may write

$$v = \delta_{1/h}(q^{-1} \cdot p)$$

so that $d(p,q) = h|v|$. We obtain

$$\begin{aligned} 0 &\geq u(p,t) - u(p,s) \\ &\geq \min_{v \in \mathcal{H}} \left\{ hL(v) - k \operatorname{Lip}(g)(h|v|)^{1/2} \right. \\ &\quad \left. \times \left[(h|v|)^{1/2} + (s_0 s)^{1/2} + (|p| + |p \cdot \delta_h(v^{-1})|)^{1/2} \right] \right\} \\ &= - \max_{v \in \mathcal{H}} \left\{ - hL(v) + k \operatorname{Lip}(g)(h|v|)^{1/2} \right. \\ &\quad \left. \times \left[(h|v|)^{1/2} + (s_0 s)^{1/2} + (|p| + |p \cdot \delta_h(v^{-1})|)^{1/2} \right] \right\} \end{aligned}$$

Factor $h^{1/2}$ and using the fact that $h \leq 1$ we get

$$\begin{aligned} 0 &\geq u(p,t) - u(p,s) \\ &\geq -h^{1/2} \max_{v \in \mathcal{H}} \left\{ - h^{1/2} L(v) + k \operatorname{Lip}(g)(|v|)^{1/2} \right. \\ &\quad \left. \times \left[(|v|)^{1/2} + (s_0 s)^{1/2} + (2|p| + |v|)^{1/2} \right] \right\} \end{aligned}$$

It remains to observe that the expression inside the maximum is uniformly bounded since L is positive and we control v by Eq. (3.10) and p varies in a compact set. □

Our main result is the following:

Theorem 3. *Let* $H(v) = \tilde{\mathfrak{f}}(|v|)$ where $v \in \mathcal{H}_0$ *and* $\tilde{\mathfrak{f}}\colon [0, \infty) \mapsto [0, \infty)$ *is a convex increasing function satisfying Eqs.* (3.1) *and* (3.2). *Let* $g\colon \mathcal{H} \mapsto \mathbb{R}$ *be a bounded and Lipschitz continuous function. The problem*

$$\begin{cases} u_t + H(D_{\mathrm{h}} u) = 0 & \text{in } \mathcal{H} \times (0, T) \\ u(p, 0) = g(p) & \text{in } \mathcal{H} \times \{0\} \end{cases} \tag{3.17}$$

has a unique viscosity solution given by the Hopf-Lax formula

$$u(p, t) = \inf_{q \in \mathcal{H}} \left\{ tL\left(\frac{q^{-1} \cdot p}{t}\right) + g(q) \right\}, \tag{3.18}$$

where $L(p) = \phi(d(0, p))$ *and* ϕ *is the Legendre transform of* $\tilde{\mathfrak{f}}$.

Proof. Let us observe that by Theorem 2 the function given by Eq. (3.18) is bounded and continuous. The uniqueness statement is a particular case of Theorem 4 below. All we need to do is to show that Eq. (3.18) is indeed a viscosity solution of Eq. (3.17).

Let ψ be a C^1 test function touching u from above at the point (p_0, t_0). We want to prove that

$$\psi_t(p_0, t_0) + H(D_{\mathrm{h}} \psi(p_0, t_0)) \leq 0.$$

For $t < t_0$ the semigroup property of Theorem 1 gives

$$u(p_0, t_0) = \inf_{p \in \mathcal{H}} \left\{ (t_0 - t) L\left(\frac{p^{-1} \cdot p_0}{t_0 - t}\right) + u(p, t) \right\}$$

and therefore we have

$$u(p_0, t_0) - u(p, t) \leq (t_0 - t) L\left(\frac{p^{-1} \cdot p_0}{t_0 - t}\right).$$

Since $\psi(p_0, t_0) - \psi(p, t) \leq u(p_0, t_0) - u(p, t)$ we obtain

$$\psi(p_0, t_0) - \psi(p, t) \leq (t_0 - t) L\left(\frac{p^{-1} \cdot p_0}{t_0 - t}\right).$$

Use the Taylor development of ψ at the point (p_0, t_0) to obtain

$$\psi_t(p_0, t_0)h + h\langle D_{\rm h}\psi(p_0, t_0), \overline{q^{-1}}\rangle + o(h|q| + h) \leq hL(q^{-1}),$$

where we have set $h = t_0 - t$ and q is such that we have

$$p = p_0 \cdot hq.$$

Dividing by h and letting $h \to 0$ we have

$$\psi_t(p_0, t_0) + \langle D_{\rm h}\psi(p_0, t_0), \overline{q^{-1}}\rangle \leq L(q^{-1})$$

for every $q \in \mathcal{H}$. Since this holds for every $q \in \mathcal{H}$ we can write

$$\psi_t(p_0, t_0) + \max_{q\in\mathcal{H}}\{\langle D_{\rm h}\psi(p_0, t_0), \bar{q}\rangle - L(q)\} \leq 0.$$

We now identify the term within the maximum using the fact that $L(\bar{q}) \leq L(q)$

$$\begin{aligned}\max_{q\in\mathcal{H}}\{\langle D_{\rm h}\psi(p_0, t_0), \bar{q}\rangle - L(q)\} &= \max_{q\in\mathcal{H}}\{\langle D_{\rm h}\psi(p_0, t_0), \bar{q}\rangle - L(\bar{q})\} \\ &= H(D_{\rm h}\psi(p_0, t_0)),\end{aligned}$$

We conclude

$$\psi_t(p_0, t_0) + H(D_{\rm h}\psi(p_0, t_0)) \leq 0.$$

Suppose now that ψ is a C^1 test function touching u from below at the point (p_0, t_0). We want to prove that

$$\psi_t(p_0, t_0) + H(D_{\rm h}\psi(p_0, t_0)) \geq 0.$$

Suppose that this is not true. We can find $\delta > 0$ so that for (p, t) near (p_0, t_0) we have

$$\psi_t(p, t) + H(D_{\rm h}\psi(p, t)) \leq -\delta < 0. \tag{3.19}$$

Let $t < t_0$ and set $h = t_0 - t$. From the semigroup property of Theorem 1 we obtain the existence of $p_t \in \mathcal{H}$ such that

$$u(p_0, t_0) = hL\left(\frac{p_t^{-1} \cdot p_0}{h}\right) + u(p_t, t). \tag{3.20}$$

Denote by q_t the point defined by the relation $p_t^{-1} \cdot p_0 = hq_t$. Note that $d = h|q_t|$ where $d = d(p_t, p_0)$. Let $s \to \gamma(s)$ be a minimizing geodesic from

p_t to p_0 such that $\gamma(0) = p_t$ and $\gamma(d) = p_0$. Consider the curve $s \to \Gamma(s)$ in $\mathcal{H} \times \mathbb{R}$ given by

$$\Gamma(s) = \left(\gamma(s), t + \left(\frac{h}{d}\right)s\right).$$

Note that since $s \to \gamma(s)$ is a horizontal curve we have

$$\frac{d}{ds}(\psi(\Gamma(s))) = \langle D_{\mathrm{h}}\psi(\Gamma(s)), \gamma'(s)\rangle + \psi_t(\Gamma(s))\frac{h}{d}\cdot$$

Estimate the difference

$$\begin{aligned}
\psi(p_0, t_0) - \psi(p_t, t) &= \int_0^d \frac{d}{ds}(\psi(\Gamma(s)))\, ds \\
&= \int_0^d \langle D_{\mathrm{h}}\psi(\Gamma(s)), \gamma'(s)\rangle + \psi_t(\Gamma(s))\frac{h}{d}\, ds \\
&= \int_0^d \left[\left\langle D_{\mathrm{h}}\psi(\Gamma(s)), \frac{d}{h}\gamma'(s)\right\rangle + \psi_t(\Gamma(s))\right]\frac{h}{d}\, ds \\
&\le \left(L\left(\frac{d}{h}\gamma'(s)\right) - \delta\right)h
\end{aligned}$$

where we have used Eq. (3.19). Observe that from the superlinearity of ϕ we can see that $p_t \to p_0$ when $t \to t_0$. Use now Eq. (3.20) and the fact ψ touches u from below at (p_0, t_0) to obtain

$$hL(q_t) \le -\delta h + hL\left(\frac{d}{h}\gamma'(s)\right).$$

Cancel h and set $v = (d/h)\gamma'(s)$. Note that $|v| \le |q_t|$ and use the monotonicity of L to obtain

$$L(v) \le -\delta + L(v)$$

which is clearly impossible. □

4. A UNIQUENESS RESULT

In this section we adapt the uniqueness proof of Ref. [23] to the subelliptic case. Quite possibly a more general theorem can be established using the refinements in Ref. [24], but the theorem below is enough for our purposes in the previous section

Theorem 4. *Let* $F:\mathbb{R}^2 \mapsto \mathbb{R}$ *be a continuous Hamiltonian. For* $g:\mathcal{H}\mapsto\mathbb{R}$ *bounded and continuous the problem*

$$\begin{cases} u_t + F(D_h u) = 0 & \text{in } \mathcal{H}\times(0,T) \\ \quad u(p,0) = g(p) & \text{in } \mathcal{H}\times\{0\} \end{cases} \tag{4.1}$$

has at most one bounded viscosity solution u satisfying

$$\omega_u(t) = \sup_{p\in\mathcal{H}}\{|u(p,t)-u(p,0)|\} \quad \to \quad 0$$

as $t\to 0$.

Proof. The proof of this theorem is based on the proof of Theorem 4.1 in Ref. [23] We will only indicate the changes needed to accommodate our subelliptic setting. Suppose that we have two solutions u_1 and u_2

$$\sup_{\mathcal{H}\times[0,T]} (u_1(p,t) - u_2(p,t)) = \theta > 0.$$

We will show that this strict inequality leads to a contradiction.

Let $M=\max\{\|u_1\|_\infty, \|u_2\|_\infty\}$ and $\lambda>0$. Consider a function $\phi_0\in C^\infty(\mathbb{R})$ such that $\phi_0(0)=1$ and $\phi_0(t)=0$ for $t>1$. For $\epsilon>0$ define the function

$$\beta_\epsilon(p,t) = \phi_0\left(\left(\frac{x}{\epsilon}\right)^4 + \left(\frac{y}{\epsilon}\right)^4 + \left(\frac{t}{\epsilon}\right)^4\right),$$

where $p=(x,y,z)\in\mathcal{H}$ and $t\geq 0$. Observe that β_ϵ is a smooth nonnegative function bounded above by 1 and $\beta_\epsilon(0,0)=1$. It is clear that if $x^4+y^4+z^4>\epsilon^4$ then $\beta_\epsilon(p,t)=0$. For our purposes we wish to observe the property

$$D_z(\beta_\epsilon(p,t)) = 0. \tag{4.2}$$

We define

$$\Phi(p,q,t,s) = u_1(p,t) - u_2(q,s) - \lambda(t+s) + (5M+2\lambda T)\beta_\epsilon(q^{-1}\cdot p, t-s).$$

For a given $\delta>0$ we can find a point (p_0,q_0,t_0,s_0) so that

$$\Phi(p_0,q_0,t_0,s_0) > \sup\Phi - \delta.$$

The same argument as in Theorem 4.1 in Ref. [23] shows that if λ, δ and ϵ are sufficiently small, then we can find $\mu>0$ so that

$$t_0 > \mu \quad \text{ and } s_0 > \mu$$

and μ is independent of λ, δ and ϵ.

Next, consider a function $\phi_1 \in C^\infty(\mathbb{R})$ such that $\phi_1(0) = 1$ and $\phi_1(t) = 0$ for $t > \mu/2$. Define the bump function centered at (p_0, q_0, t_0, s_0)

$$\zeta(p, q, t, s) = \phi_1\big(\|p_0^{-1} \cdot p\|_H^4 + \|q_0^{-1} \cdot q\|_H^4 + (t - t_0)^2 + (s - s_0)^2\big).$$

If $\|p_0^{-1} \cdot p\|_H^4 + \|q_0^{-1} \cdot q\|_H^4 + (t - t_0)^2 + (s - s_0)^2 > \mu^2/4$ then $\zeta(p, q, t, s) = 0$. Define now

$$\Psi(p, q, t, s) = \Phi(p, q, t, s) + 2\delta\zeta(p, q, t, s).$$

The function Ψ attains its maximum at a point (p_1, q_1, t_1, s_1) satisfying

$$\|p_0^{-1} \cdot p_1\|_H^4 + \|q_0^{-1} \cdot q_1\|_H^4 + (t_1 - t_0)^2 + (s_1 - s_0)^2 \leq \mu^2/4.$$

In particular we have $t_1 \geq \mu/2$ and $s_1 \geq \mu/2$.

With these modifications, following the argument of Theorem 4.1 in Ref. [23] and using Eq. (4.2) to simplify the calculation of the horizontal gradient we arrive at a contradiction by letting $\delta \to 0$. □

Remark 2. One could also consider existence results for Eq. (4.1) for general Hamiltonians. In the Euclidean case (Refs. [10,25,26]) when the Hamiltonian depends on x, existence is based on control theory and differential games techniques.

Remark 3. For the case of general Carnot groups the definition of viscosity solutions, the characterization of jets, and the semigroup property all hold with similar proofs as in the Heisenberg case. However, we use specific properties of the Heisenberg group in the proof of Theorem 2 ii) and iii).

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