Evolution and market behavior with endogenous investment rules*

Giulio Bottazzi Pietro Dindo

November 10, 2010

LEM and CAFED, Scuola Superiore Sant'Anna,
Piazza Martiri della Libertà 33, 56127 Pisa, Italy

Abstract

In a complete market for short-lived assets, we investigate long run wealth-driven selection on a general class of investment rules that depend on endogenously determined current and past prices. We find that market instability, leading to asset mis-pricing and informational efficiencies, is a common phenomenon and is due to two different mechanisms. First, conditioning investment decisions on asset prices implies that dominance of an investment rule on others, as measured by the relative entropy, can be different at different prevailing prices thus reducing the global selective capability of the market. Second, the feedback existing between past realized prices and current investment decisions can lead to a form of deterministic overshooting. By investigating the random dynamical system that corresponds to prices and wealths dynamics, we are able to derive general conditions for the occurrence of each type of market instability and the emergence of informational inefficiencies.

Keywords: Market selection; Informational Efficiency; Evolutionary finance; Price feedbacks; Asset pricing.
JEL Classification: D50, D80, G11, G12

*While preparing this work we have benefited from feedbacks by participants to workshops and seminars held in (order of appearance) Paris, Amsterdam, Pisa, Udine, London, Budapest, Rotterdam, and Leuven. We are particularly grateful to Mikhail Anufriev, Larry Blume (the suggestion to condition investment decisions on current prices is his), David Easley, and Cars Hommes for their comments and support. The research that has led to this work has been supported by the European Union NoE DIME. All usual disclaimers apply.

1 Introduction

The Efficient Market Hypothesis (EMH), advanced as a general interpretative and normative framework nearly forty years ago (Fama, 1970), has grown to become a widely accepted working tool for the economic profession. Rooted in the evolutionary foundations of neoclassical economics (Alchian, 1950; Friedman, 1953), the EMH is broadly based on the "as if" argument that poorly informed investors are persistently loosing wealth in favors of the better informed. If the latter is true, then, those who are poorly informed are, in the long-run, driven out of the market so that the available information about assets fundamentals is ultimately reflected in prevailing prices.

Despite its pervasive influence in economics, a general formal proof of the selective capability of asset markets and, consequently, of the ultimate convergence of asset prices toward fundamental values is still lacking. Only fairly recently scholarshiply work has started to investigate this issue. Several behavioral models (see Barberis and Thaler, 2003, and references therein) based on evidence collected from laboratory experiments and real markets, contend both the positive and normative aspects of EMH. Rational behavior does not appear as a pervasive property of trading, nor does automatically guarantee, even if appropriately implemented, better performances and higher probability to "survive" the speculative struggle. The modeling effort of these studies has been, however, limited to partial equilibrium models with exogenous price dynamics. A general equilibrium model with complete markets and an endogenous price dynamics has been firstly proposed in Blume and Easley (1992). They investigate wealth-driven market selection, and the information content of asset prices, on a class of investment rules that depend on the realization of exogenous variables such as asset dividends.

The analysis of complete markets that has stemmed from the seminal contribution of Blume and Easley (1992) can be divided in two groups of contributions. A number of works focused on investment rules not necessarily coming from utility maximization and expressed as fraction of wealth to be invested in each asset (see Evstigneev et al., 2009, for a recent survey). These rules are allowed to depend on realized dividends but not on assets' prices. Assuming all agents consume the same, a robust finding is that investing proportionally to asset expected dividends, also named the Kelly rule after Kelly (1956), is the unique globally stable rule. The result holds for both short- and long-lived assets as shown in Evstigneev et al. (2002) and Evstigneev et al. (2008) respectively. When the Kelly rule is not present in the market, for instance when a complete knowledge of the underlying dividend process is lacking, rules with the lowest relative entropy with respect to this process, or "closest" to it, are gaining all wealth in the long-run. As a result asset prices are brought as close as possible to their dividend revealing values and the market is informationally efficient. A different group of works has instead focused on selection among investment decisions explicitly coming from utility maximization, so that assets demand is not necessarily expressed as a fraction of wealth. Their main objective is to establish whether the market is able to select for agents whose beliefs, or information, are closer to the underlying dividend payment process. In case of complete markets for long-lived assets, and assuming rational expectations on realized prices, Sandroni (2000) and Blume and Easley (2006) find that the "as if" statement is correct: no matter the functional form of the utility function they maximize, agents whose belief are "closest" to the correct one are selected for in the long run, provided that they discount future consumption with the same rate.

Both groups of contributions leave some relevant questions unanswered. In particular, it is not known what happens when investment rules depend on prices and, at the same time, agents are not able to coordinate on having rational expectations. The aim of the present paper is to fill this gap. We extend the model in Blume and Easley (1992) to encompass investment rules that depend on current and past asset prices. In doing so we drop the assumption of rational expectations, and, apart assuming some technical regularity conditions, do not pose further restriction on the functional forms of investment rules. Our aim is twofold. First, we want to move closer to a formal general check of the "as if" statement, studying market selection and the ensuing asset prices behavior for a broader class of asset demands. Second, we want to better understand the functioning of markets when their role of information gatherers is directly acknowledged by traders. In fact, in a market where prices supposedly reflect fundamentals as close as possible, the use of the former as proxy of the latter is a rational behavior. Upon believing in market selection, this is the case even if one does not explicitly assume asymmetries in traders' information. If agents believe that market price reflects the best available information and use them to guide their investment decision, is market efficiency increased or decreased? Relatedly, if agents rely on exogenous information, what is the long-run effect of those strategies that are designed to trade against assets' mis-pricing?

The dependence of investment rules on current and, possibly, past prices links past and present market performances through a feedback effect in agents' demands. The effect has already been investigated in several heterogeneous agents models. The main finding is that market instability and asset mis-pricing are in general possible (see Hommes, 2006; LeBaron, 2006, for a review). However, market selection often operates according to postulated fitness indicators and not by looking at the natural measure of relative wealth (Levy et al., 2000; Farmer, 2002; Chiarella and He, 2001, are among the few exeptions). Moreover results are often derived for specific investment behaviors and in a partial equilibrium framework. Both gaps have been partially filled by our previous works, see e.g. Anufriev and Bottazzi (2010), Anufriev et al. (2006) or Anufriev and Dindo (2010), which study wealth-driven market selection on the general class of price dependent investment rules. Nevertheless, those works, being based on an essentially deterministic framework, do not tackle the information efficiency issue we are interested here.

Technically we investigate market selection and the informational role of prices by analyzing the random dynamical system that corresponds to prices and wealths dynamics. The price dependence of investment rules commands a notion of economic equilibrium compatible with the way agents form their individual demand. In particular, a requirement of consistency between agents' expectation and realized market dynamics should be introduced. This requirement, which is not necessary when agents base their investment decisions on exogenous variables, leads to the notion of "procedural consistent equilibria" (see the discussion in Anufriev and Bottazzi, 2010) which are naturally identified with the deterministic fixed points of the random dynamical system describing the market evolution. We characterize such fixed points and investigate their stochastic stability. We are able to derive general sufficient conditions ruling whether any given agent is locally dominating all others. Since our exercise can be accomplished when investment rules depend on current prices, any Constant Relative Risk Aversion (CRRA) utility maximizing behavior can be modeled.

In line with previous contributions, our analysis confirms the peculiar role played by the Kelly rule. When it is present in the market, prices asymptotically convey the correct information about the dividend process, no matter the number and type of other competing investment rules. However, when the Kelly rule is not used by any agent, the issue becomes much more complicated. In fact, the dependence of investment rules on prices brings instability and multiplicity of equilibria into the market, persistent asset mis-pricing can be observed, and prices do not longer reflect the best available information.

The presence of multiple equilibria and instability is essentially related to two causes. First, in our framework the relative entropy of a price depend investment rule with respect to the underlying dividend payment process depend on realized prices. Hence, the same is true for their average relative wealth growth rate. Given two rules, it may well happen that the first rule has a higher wealth growth rates at the prices determined by the second, while the second has a higher wealth growth rate at the prices determined by the first. A second source of instability is directly linked to the effect induced by price feedbacks. Even though a given rule has the highest relative average wealth growth at "its" prices than all other rules present in the market would have at "their" prices, it can happen that the price feedback of the given rule is too strong and acts as a destabilizing force, a form of deterministic overshooting. In both cases, market prices do not converge to the level reflecting the "best" available information but instead keeps fluctuating around it.

The outline of this paper is as follows. In Section 2 we present our model. Section 3 proposes an example which, albeit its simplicity, will hopefully help in appreciating our findings and in understanding their causes. Section 4 contains our main results, that is, existence and stability of long run market equilibria for any given set of investment rules inside the general class considered here. In Sections 5-6 we illustrate some implications of our results by discussing two specific issues. Section 5 investigates the possibility of establishing an order relation on the space of investment rules by exploiting their relative market performance. The answer will be negative. Section 6 characterizes conditions under which a generic form of learning from prices does not vanish when trading with a Kelly rule investor. Section 7 concludes. All proofs are collected in the Appendix.

2 The model

Given the set $\Sigma = \{1, \ldots, s, \ldots, S\}$ of states of the world, define the set of sequences $\Omega := \prod_{-\infty}^{+\infty} \Sigma$ with elements $\omega = (\ldots, \omega_0 \ldots, \omega_t, \ldots)$, so that $\{\omega\}_t = \omega_t \in \Sigma$ for every t, and the complete σ-algebra $\mathbb{P} = 2^\Omega$. Let ρ be a measure on $\mathbb{P}$ so that $(\Omega, \mathbb{P}, \rho)$ is a well-defined probability space. We assume that the corresponding stochastic process with realizations in $\mathbb{P}$ is ergodic, that is, there exists a unique invariant measure $\pi = (\pi_1, \ldots, \pi_S)$ on Σ such that for every finite statistics $g : \Sigma \to \mathbb{R}$ and almost all sequences ω it holds

$$\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} g(\omega_t) = \sum_{s=1}^{S} g(s) \pi_s \, .$$

Given ρ, let θ be the Bernoulli shift operator on Ω, that is, for every component t it holds $\{\theta\omega\}_t = \{\omega\}_{t+1}$. Name θ^t the composition of t operators θ, so that $\{\theta^t \omega\}_{t'} = \{\omega\}_{t'+t}$.

We study a market where, in each period t, I agents are trading $K = S$ short-lived risky assets. We denote agent i wealth in period t as ϕ_t^i, and asset k price in period t as $p_{k,t}$. Assets' dividend payoffs are paid in terms of a consumption good, the numéraire of the economy, and are random variables defined over $(\Omega, \mathbb{P}, \rho)$. We further assume that they are stationary and depend on the contemporaneous realization of the state of the world. Given the square matrix D with non-negative elements $\{D\}_{k,s} = d_k^s$, the dividend of asset k at time t is defined as $d_{k,t}(\omega) = d_k^{\omega_{t+1}}$. We assume that the matrix D is non-singular so that in each period of time the asset market is complete. Since payoffs are stationary one can also define K random variables over Σ, $d_k(s) = d_k^s$, and then use $d_{k,t}(\omega) = d_k(\omega_{t+1})$. We will use the latter notation from now onwards.

Asset demands are modeled as wealth fractions to be allocated to each asset. We denote with $\alpha^i_{k,t}$ the fraction of wealth that at period t agent i invests in asset k. Whereas previous contributions have assumed investment decisions to be also random variables over a filtration of Ω, we assume they depend on the realization of endogenous market variables, in particular current and past asset prices, as formalized by the following

Assumption 1. The fraction of wealth agent i invests on asset k at time t, $\alpha^i_{k,t}$, is determined by a time-independent function of assets' prices, also named asset k investment rule, α^i_k, so that

$$\alpha^i_{k,t} = \alpha^i_k(p_t) \quad k = 1, \dots, K\,, \tag{2.1}$$

where p_t is the vector of current and past, up to lag L, assets' prices, that is, $p_t = (\mathbf{p}_t, \mathbf{p}^1_t, \dots, \mathbf{p}^L_t)$ with $\mathbf{p}^l_t = (p^l_{1,t}, \dots, p^l_{K,t})$ and $p^l_{k,t} = p_{k,t-l}$.[1] We further assume that investment rules satisfy the following properties

P1 Each agent i invests all its wealth, or $\sum_{k=1}^K \alpha^i_k(p) = 1$;

P2 Short positions are forbidden and portfolios are maximally diversified, or $\alpha^i_k(p) > 0$ for every asset k, and agent i.

The role of Assumption 1 deserves a brief discussion. As common in this literature, we model assets' demand in terms of wealth shares allocated to each asset. We extend previous works by assuming that shares α may depend on current and past prices. **P1** implies that agents invest all their wealth in assets and, thus, do not consume. It is straightforward to extend our framework by assuming that all agents consume the same constant fraction of their wealth α^0 in every period t.[2] **P2** is sufficient to guarantee that market equilibrium prices are always well defined and positive. One could alternatively assume that *at least* one agent invests in each asset, that is, for every asset k and price vector p there exists an agent j such that $\alpha^j_k(p) > 0$. This would however introduce technical issues related to the existence of a global dynamics (see Theorem 2.1). Since the αs can be infinitesimally small, we do not consider assumption **P2** as an actual limit on the possible investment behaviors. In what follow we shall name α^i the vector valued investment rule used by agent i and $\mathcal{A}$ the set of vector valued rules complying to **P1-P2**.

Given a set of investment rules and an initial wealth distribution, market clearing and intertemporal budget constraints determine the dynamics of assets' prices and agents' wealths for all subsequent periods, that is, the time evolution of market state variable $x_t = (\phi_t, p_t) = (\phi^1_t, \dots, \phi^I_t, p_t)$. In fact, given wealths, assets' prices, and investment decisions at time t, each agent wealth at time $t+1$ is given by the scalar product of the individual assets' holding with the vector of assets' returns corresponding to the state of the world just realized or

$$\phi^i_{t+1} = \Phi^i(x_t; \omega) := \sum_{k=1}^K \frac{\alpha^i_{k,t}\phi^i_t}{p_{k,t}} d_k(\omega_{t+1})\,, \quad i = 1, \dots, I\,. \tag{2.2}$$

Given wealths and investment decisions at time time $t+1$, asset prices are computed by aggregating asset demands and imposing market clearing which, upon normalizing asset supply

[1] The compact notation for lagged prices allows l to be equal to 0, in which case trivially $\mathbf{p}^0_t = \mathbf{p}_t =$ and $p^0_{k,t} = p_{k,t}$.

[2] This amount to a rescaling of prices normalization, see also footnote 3.

to 1, leads to

$$p_{k,t+1} = \sum_{i=1}^{I} \phi_{t+1}^{i} \alpha_{k,t+1}^{i} = \sum_{i=1}^{I} \Phi^{i}(x_t;\omega) \alpha_{k,t+1}^{i} \,, \quad k = 1, \ldots, K \,. \tag{2.3}$$

If investment decision do not depend on current prices, the previous equations uniquely determine the vector of prices at time $t+1$. Conversely, when the dependence on contemporaneous prices is present in some of the αs, prices are fixed by (2.3) through a system of K implicitly defined functions. Mild conditions are sufficient to guarantee the existence of at least one vector of prices.

Theorem 2.1. *If for every agent $i = 1, \ldots, I$ it holds $\alpha^i \in \mathcal{A}$ and $\alpha^i \in \mathcal{C}^0$, then there always exists a vector $\mathbf{p}^*$ of positive prices satisfying* (2.3)*, that is, clearing the asset market.*

The uniqueness of the solution is in general not guaranteed, but smooth investment functions with a mild dependence on present prices constitute a sufficient condition.

Theorem 2.2. *If for every agent $i = 1, \ldots, I$ it holds $\alpha^i \in \mathcal{A}$ and $\alpha^i \in \mathcal{C}^1$, and if for every $k = 1, \ldots, K$ α_k^i does not depend on current prices other than the one of the same k-th asset, then the vector $\mathbf{p}^*$ is unique provided that*

$$\left| \frac{\partial \alpha_k^i}{\partial p_k} \right| < 1 \,, \quad i = 1, \ldots, I \quad k = 1, \ldots, K \,. \tag{2.4}$$

Since in the following section our analysis will be mostly local, we are not particularly bothered by the possibility that the market clearing price vector is not unique. However, when discussing the global dynamics, we shall assume that hypothesis of Theorem 2.2 are valid and, consequently, there exist K explicit global maps

$$p_{k,t+1} = f(x_t;\omega) \,, \quad k = 1, \ldots, K \,. \tag{2.5}$$

Given that assets are short-lived and agents do not consume, the total wealth in each period is given by the sum of asset dividends payed for the state of the world just realized. We can thus use total wealth to introduce a convenient normalization of prices, dividends, and individual wealths. This procedure does not change equations (2.2-2.3) upon remembering that all variables are now understood to be normalized, so that it holds[3,4]

$$\sum_{i=1}^{I} \phi_t^i = \sum_{k=1}^{K} p_{k,t} = \sum_{k=1}^{K} d_k(s) = 1 \quad s = 1, \ldots, S \quad t \in \mathbb{N}. \tag{2.6}$$

The previous normalization is, in fact, equivalent to assuming from the beginning that total wealth, given by the sum of assets' dividend for each realization of ω_t, is equal to one so that, in particular, there is no aggregate risk in the economy. This is consistent with the assumption that investment rules are not state dependent.

[3] Upon assuming that agents are consuming an equal fraction of their wealth α^0, normalization of prices and wealths leads to $\sum_{k=1}^{K} p_{k,t} = 1 - \alpha^0$ instead of $\sum_{k=1}^{K} p_{k,t} = 1$

[4] Notice that when normalizing total wealth in each period, we are implicitly changing the shape of the investment rule by changing their dependence upon prices into a dependence upon normalized prices.

Summarizing, the market evolution can be written as a system of $I + K(L+1)$ equations

$$\mathcal{F}(\omega)x_t := \left[\begin{array}{lll} \mathcal{W}(x_t;\omega) & := & \left[\begin{array}{lll} \phi^1_{t+1} & = & \Phi^1(x_t;\omega) \\ \vdots & \vdots & \vdots \\ \phi^I_{t+1} & = & \Phi^I(x_t;\omega) \end{array}\right. \\ \mathcal{P}(x_t;\omega) & := & \left[\begin{array}{lll} \mathcal{P}_1(x_t;\omega) & := & \left[\begin{array}{lll} p_{1,t+1} & = & f_1(x_t;\omega) \\ p^1_{1,t+1} & = & p_{1,t} \\ p^2_{1,t+1} & = & p^1_{1,t} \\ \vdots & \vdots & \vdots \\ p^L_{1,t+1} & = & p^{L-1}_{1,t} \end{array}\right. \\ \vdots & & \vdots \\ \mathcal{P}_K(x_t;\omega) & := & \left[\begin{array}{lll} p_{K,t+1} & = & f_K(x_t;\omega) \\ p^1_{K,t+1} & = & p_{K,t} \\ p^2_{K,t+1} & = & p^1_{K,t} \\ \vdots & \vdots & \vdots \\ p^L_{K,t+1} & = & p^{L-1}_{K,t} \end{array}\right. \end{array}\right. \end{array}\right. . \tag{2.7}$$

Due to normalizations in (2.6) and under the assumptions of Theorem 2.2, each $\mathcal{F}(\omega)$ maps the set $\mathcal{X} = \Delta^I \times (0,1)^{K(L+1)}$ in itself. The component $\mathcal{W}$ characterizes the dynamics of agents' wealth fractions, whereas $\mathcal{P}$ fixes prices using market clearing and keeps track of their past values. For any given initial state x_0, the random dynamical system representing the market dynamics is defined iterating $\mathcal{F}(\omega)$:

$$\varphi(t,\omega,x_0) = \mathcal{F}(\theta^{t-1}\omega) \circ \ldots \circ \mathcal{F}(\theta\omega) \circ \mathcal{F}(\omega)x_0. \tag{2.8}$$

Given the arbitrariness of the dividend process, population size I, memory span L, number of assets K, and investment rules αs, the analysis of the global dynamics generated by (2.8) cannot be performed in total generality. Moreover, having a multiple agent framework with heterogeneous investment behaviors, not necessarily derived from an utility maximization given preferences and expectations, we shall not apply the traditional rational expectation approach. In the present paper, instead, our interest lies in characterizing whether long-run wealth distributions where one or many agents have gained all the wealth exist and are stable.

Because of the stationary nature of the process governing the succession of states of the world and the lack of aggregate risk due to (2.6), economic equilibria are characterized by constant prices and, in accordance with Assumption 1, constant investment shares. Owing to market dynamics and wealth normalization, constant investment decisions αs and fixed long-run asset prices imply a constant wealth distribution. Hence, we are naturally lead to identify long-run market selection equilibria with the deterministic fixed points of the random market dynamics in (2.8). Deterministic fixed points are defined by the following

Definition 2.1. The state $x^* \in \mathcal{X} = (\phi^*, p^*)$ is a deterministic fixed point of the random dynamical system φ generated by the family of maps $\mathcal{F}(\omega)$ if, for almost all $\omega \in \Omega$, it holds

$$\mathcal{F}(\omega)x^* = x^*\,, \tag{2.9}$$

which implies

$$\varphi(t,\omega,x^*) = x^* \quad \text{for every} \quad t \in \mathbb{N}\,. \tag{2.10}$$

Intuitively a deterministic fixed point can correspond to a single investor possessing the entire wealth of the economy. In this case, according to (2.3), asset prices are equal to the vector of investment decisions of this investor. Alternatively, many investors could have positive wealth at equilibrium. In this case the constraints imposed by the wealth dynamics require that they all take the same investment decision at the fixed point prevailing prices (see Section 4).

In any case, not all deterministic fixed points represent interesting asymptotic states. Indeed, in order for the market dynamics to actually converge to a deterministic fixed point starting from a positive measure set of initial conditions, the point must be asymptotically stable. The definition of asymptotic stability of a deterministic fixed point is as follows

Definition 2.2. A deterministic fixed point x^* of the random dynamical system $\varphi(t, \omega, x)$ is called asymptotically stable if, for almost all $\omega \in \Omega$ and for all x in a neighborhood $U(\omega)$ of x^*, $\lim_{t\to\infty} ||\varphi(t, \omega, x) - x^*|| \to 0$.

For some equilibria we will make use of the weaker notion of stability, which will be, in our case, sufficient to guarantee that orbits do not diverge from deterministic fixed point for a positive measure set of initial conditions. The definition is as follows

Definition 2.3. A deterministic fixed point x^* of the random dynamical system $\varphi(t, \omega, x)$ is called stable if, for any neighborhood V of x^* and for almost all $\omega \in \Omega$, there exists a neighborhood $U(\omega) \subseteq V$ of x^* such that $\lim_{t\to\infty} \varphi(t, \omega, x) \in V$ for all x in $U(\omega)$.

Notice that in the previous definitions the neighborhood U might depend on the process realization ω.

When characterizing deterministic fixed points and their local stability the following terminology, describing the long-run wealth distribution, will be useful.

Definition 2.4. An agent i is said to *survive on a given trajectory* generated by the dynamics (2.8) if $\limsup_{t\to\infty} \phi^i_t > 0$ on this trajectory. Otherwise, an agent n is said to *vanish on a trajectory.* A surviving agent i is said to *dominate on a given trajectory* if she is the unique survivor on that trajectory, that is, $\liminf_{t\to\infty} \phi^i_t = 1$

Importantly, survival and dominance are defined only with respect to a given trajectory and not in general. The reason is that we are going to work exclusively with local stability conditions so that a trader may survive on a given trajectory (i.e., for certain initial conditions) but vanish on another. A similar definition is given in Blume and Easley (1992) for a stochastic system like ours and in Anufriev and Bottazzi (2010) and Anufriev and Dindo (2010) for deterministic systems.

Applying the previous definition to a deterministic fixed point, we shall say that agent i survives at x^* if her wealth share is strictly positive, $\phi^{*i} > 0$, while she vanishes if $\phi^{*i} = 0$. Such taxonomy can be applied both to a stable or unstable deterministic fixed point, but the implications are very different in the two cases. When the fixed point is stable, all trajectories starting in a neighborhood of it will stay close to it, so that a survivor in the fixed-point will also survive on all these trajectories. If, moreover, the agent is the unique survivor and the fixed point is also asymptotically stable, the agent will dominate on all trajectories starting inside a proper neighborhood. Conversely, when the fixed point is unstable, one is not able to characterize survival and dominance for trajectories starting close to it. Both vanishing and dominating agents at an unstable fixed point may survive as well as vanish on trajectories started in any one of its neighborhoods, and in absence of global results one, in general, cannot say. In the rest of the paper we shall show that the constraints on the dynamics imposed by the dividend process, the market clearing and the wealth evolution are sufficient to: first, uniquely characterize level of prices in the deterministic fixed points and describe the corresponding distributions of wealth among agents, and, second, derive general local conditions under which the long-run market dynamics converge.

3 A toy market

In this section we shall consider the simplest market dynamics where the implication of bringing prices into the investment rule can be fully appreciated. In the discussion we will make use of analytical results whose formal derivation is postponed to Section 4.

Consider an economy with two states of the world and two agents trading according to, respectively, α^1 and α^2 both in $\mathcal{A}$. Two Arrow securities are traded: security $k \in \{1, 2\}$ pays 1 if state of the world k is realized and 0 otherwise, so that the market is complete. We assume that $\alpha^1, \alpha^2 \in \mathcal{C}^1$ and start our analysis with the case in which each investment rule depends only on the last observed price. Then, according to Theorem 2.2, an unique price vector is determined at each time step. Using the notation of the previous section we fix $K = S = 2$, $I = 2$, $L = 1$. Because of wealth and prices normalizations, both agents' wealth and assets' prices add up to one in every period, so that we are left with a three dimensional random dynamical system: the wealth fraction of agent 1, the price of asset 1, and its first lag. Without loss of generality we shall assume that ω is the realization of a Bernoulli process: at every period t, $\omega_t = 1$ with probability π and $\omega_t = 2$ with probability $1 - \pi$.

Given the state of the market at time t, $x_t = (\phi_t, p_t, p^1_t = p_{t-1})$, and the diagonal structure of the dividend payoff matrix the random dynamical system representing the market dynamics can be written as the composition of the following map

$$\begin{cases} \phi_{t+1} = \begin{cases} \frac{\alpha^1(p^1_t)\phi_t}{p_t} & \text{if} \quad \omega_{t+1} = 1 \\ \\ \frac{\left(1-\alpha^1(p^1_t)\right)\phi_t}{1-p_t} & \text{if} \quad \omega_{t+1} = 2 \end{cases}, \\ \\ p_{t+1} = \alpha^1(p_t)\phi_{t+1} + \alpha^2(p_t)(1 - \phi_{t+1})\,, \\ \\ p^1_{t+1} = p_t\,. \end{cases} \tag{3.1}$$

We are interested in characterizing long-run market equilibria. Let f_π and $f_{1-\pi}$ stand for the map in (3.1) when $\omega = 1$ and $\omega = 2$, respectively. The deterministic fixed points of the system are the states x^* such that

$$x^* = f_\pi(x^*) \quad \text{and} \quad x^* = f_{1-\pi}(x^*)\,.$$

Straightforward computations show that there are three types of such points, namely

$$\begin{aligned} x^*_1 &= (\phi^* = 1, p^* = \alpha^1(p^*), p^{1*} = p^*)\,, \\ x^*_2 &= (\phi^* = 0, p^* = \alpha^2(p^*), p^{1*} = p^*)\,, \\ x^*_{1,2} &= (\phi^*, p^* = \alpha^1(p^*) = \alpha^2(p^*), p^{1*} = p^*)\,. \end{aligned}$$

Either one agent has all wealth and dominates, which occurs at x^*_1 and x^*_2, or both agents have some wealth and survive, which occurs at $x^*_{1,2}$. In both cases prices are fixed point of the

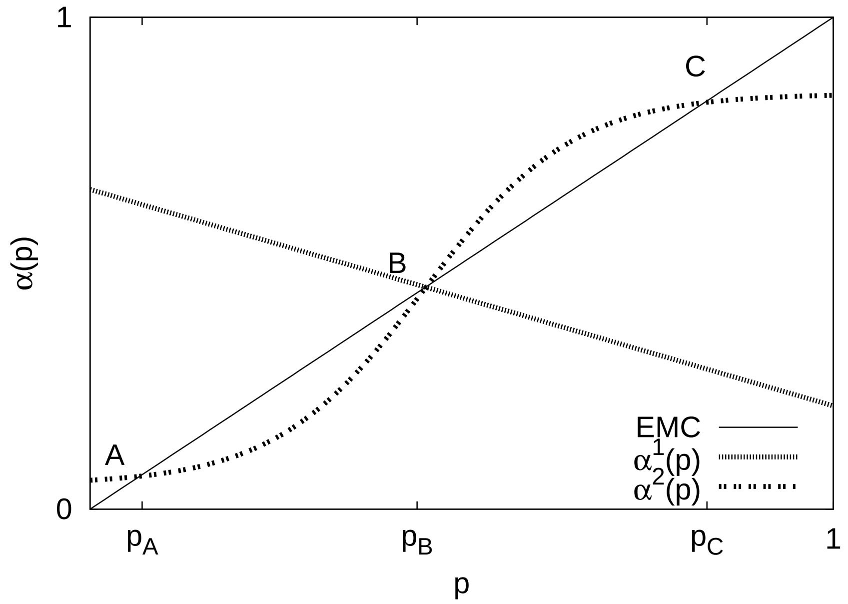

Figure 1: Investment rules $\alpha^1(p)$ and $\alpha^2(p)$ are plotted against the price of asset $k = 1$. Equilibrium prices are given by the coordinate of their intersections with the EMC, that is, points p_A, p_B and p_C.

survivor's investment rule, $p^* = \alpha^i(p^*)$, and each surviving agent receives the same earning in both states of the world.[5]

It is useful to use a plot to visualize the location of fixed points. In Fig. 1 we plot two generic investment rules as a function of the (lagged) price of the first asset p. The intersections of either investment rules (demand) with the diagonal (supply), that is, points A, B and C, are the different Walrasian equilibria corresponding to all possible deterministic fixed points of the system. Using the terminology introduced in Definition 2.4, in A and C agent 2 dominates and agent 1 vanishes. These are *single survivor* equilibria. Conversely, the existence of *multiple survivor* equilibria of the $x^*_{1,2}$ type, like the point B, in which both agents survive and neither dominates nor vanishes, requires that the first and second agent's investment rules intersect the diagonal at the same point.

No matter the shape of the investment rules, both single survivor and multiple survivors equilibria lie on the diagonal of the plot with coordinates $\alpha(p)$ and p. For analogies with previous works (Anufriev et al., 2006; Anufriev and Bottazzi, 2010; Anufriev and Dindo, 2010) we name it the "Equilibrium Market Curve" (EMC) to stress that it is the locus of all long-run market equilibria.

The stability of the deterministic fixed points of the simple system in (3.1) depends upon

[5] If the investment rules are derived from expected utility maximization, absence of aggregate risk implies that equilibrium prices correspond, no matter the shape of the utility function, to the beliefs the surviving agent assigns to the occurrence of each state.

the value of the two quantities (c.f. Theorem 4.3 and Theorem 4.5)

$$\mu(\phi,p) = \left(\left(\frac{\alpha^2(p)}{\alpha^1(p)}\right)^{\pi}\left(\frac{1-\alpha^2(p)}{1-\alpha^1(p)}\right)^{1-\pi}\right)^{2\phi-1} \qquad \phi,p\in[0,1]\,, \tag{3.2}$$

$$\lambda(\phi,p) = \phi\frac{\partial\alpha^1(p)}{\partial p}+(1-\phi)\frac{\partial\alpha^2(p)}{\partial p} \qquad \phi,p\in[0,1]\,. \tag{3.3}$$

A single survivor equilibrium x_i^* with $i=1,2$ is asymptotically stable provided that both the corresponding values $\mu(\phi^*,p^*)$ and $\lambda(\phi^*,p^*)$ have modulus smaller than one. A multiple survivor equilibrium $x_{1/2}^*$ is stable (but not asymptotically stable) provided that the corresponding $\lambda(\phi^*,p^*)$ has modulus smaller than one. Either $|\mu|$ or $|\lambda|$ being grater than one, corresponds to a different sources of instability.

Regarding μ, consider the relative entropy of the investment rule of agent i with respect to the exogenous measure on the states of the world computed at price p, defined as

$$I_\pi(\alpha^i(p)) := \pi\,\log\frac{\pi}{\alpha^i(p)}+(1-\pi)\,\log\frac{1-\pi}{1-\alpha^i(p)}\;.$$

It is immediate to see that $\log(\mu(\phi^*,p^*))$ is equal to the relative entropy of the survivor's rule minus the relative entropy of the rule of the vanishing agent, computed at equilibrium. Thus, the deterministic fixed point is stable only if the surviving agent is the one whose investment rules has, at the equilibrium prices, the lowest entropy. The intuition is that in this case the surviving agent invests, on average, better than the other agent, and, consequently, his wealth share grows at an average positive rate. The fulfillment of this condition can be directly appreciated in the EMC plot. In Fig. 2 all curves are the same as in Fig. 1 with the addition on the horizontal line π equal to the probability of occurrence of state 1. The distance between this line and α^i at a given price p is monotonically related to $I_\pi(\alpha^i(p))$. Consider the point C where agent 2 dominates and 1 vanishes, that is, $\phi^*=0$ and $p^*=p_C=\alpha^2(p_C)$. Since the distance from the π line is larger for $(p_C,\alpha^2(p_C))$ than for $(p_C,\alpha^1(p_C,))$, it is $I_\pi(\alpha^i(p))<I_\pi(\alpha^2(p))$ and this point is unstable. Conversely, in p_A the curve nearest to the π line is α^2, so that, at least according to this criterion, the point A is stable.

Concerning the second quantity, λ, it depends on the relation between past realized prices and present investment decisions. When $|\lambda(\phi^*,p^*)|>1$ price feedbacks are too strong for the dynamics to settle down, a form of deterministic overshooting which reminds of the instability of price adjustment processes. At equilibrium, only the investment rules of the surviving agents are relevant to define stability with respect to λ. Given the slope of $\alpha^2(p)$ at p_A and p_C, both A and C are stable under past prices feedback.[6] Since A is also stable when looking at the relative entropy, it represents an asymptotically stable single survivor equilibrium and a possible outcome of the long-run market dynamics. In the example of Fig. 2 it is the unique single survivor stable equilibrium, but it is not the unique long-run equilibrium. We have still to evaluate the stability with respect to past prices feedback of B, where both agents survive. Locally, the market dynamics is equivalent to the one generated by a single agent whose investment rule is the wealth weighed average of both surviving rules. Since $|\partial\alpha^1(p_B)|<1$ and $|\partial\alpha^2(p_B)|>1$ for continuity, if ϕ is large enough, than $|\lambda(\phi^*,p^*)|<1$ and the point is stable. For smaller values of ϕ, the over-reaction to price movement of α^2 destabilizes the

[6]Given **P1-P2** in Assumption 1 and upon continuity, each rule has at least one intersection with the EMC with derivative lower than one in absolute value, so if one agent is alone in the market, there exists at least one stable fixed point.

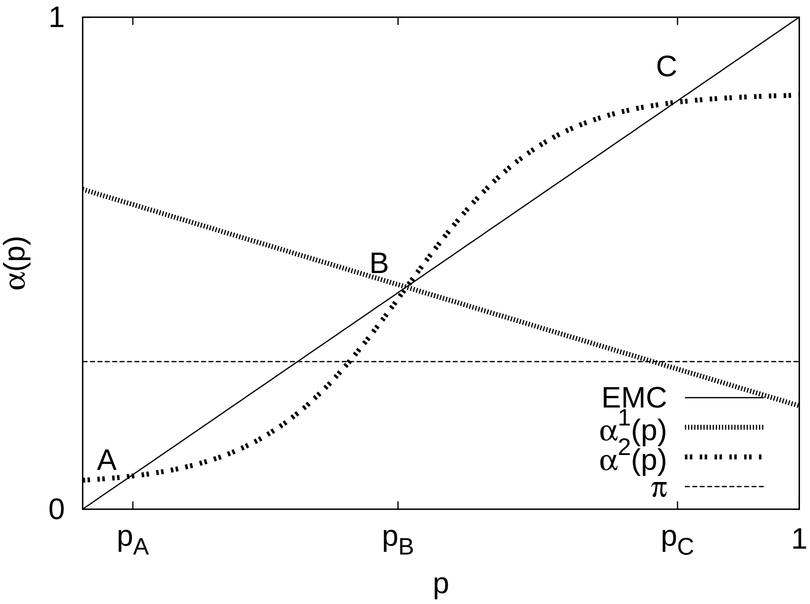

Figure 2: Local stability of a deterministic fixed point A, B, C can be appraised graphically by 1) comparing the relative distance with respect to the underlying dividend payment probability π, the closer the better, which corresponds to checking that $\mu(\phi^*, p^*)$ is smaller than one, and 2) checking that the slope of $\alpha^i(p)$ is not steeper than the EMC, which corresponds to checking $|\lambda(\phi^*, p^*)|$ is smaller than one too.

equilibrium. Notice at last that since in B investment decisions of both agents are equal, the distance of their rules in terms of relative entropy is zero and $\mu(\phi^*, p_B) = 1$. For this reason even if $|\lambda(\phi^*, p^*)| < 1$, the fixed point is not asymptotically stable. A perturbation can indeed generate a permanent change in the distribution of wealth. Prices will converge back to their equilibrium level p_B but the system will end up in a fixed point with a different distribution of wealth.

If investment rules depend only on current prices, the random dynamical system simplifies to

$$\phi_{t+1} = \begin{cases} \frac{\alpha^1(p_t)\phi_t}{p_t} & \omega_{t+1} = 1 \\ \\ \frac{(1-\alpha^1(p_t))\phi_t}{1-p_t} & \omega_{t+1} = 2 \end{cases} , \tag{3.4}$$

where $p_t(\phi_t)$ is a solution of

$$p_t = \alpha^1(p_t)\phi_t + \alpha^2(p_t)(1-\phi_t)\,. \tag{3.5}$$

As already discussed in Section 2, (3.5) can possess multiple solutions, so that the global dynamics may be ill-defined. Concerning deterministic fixed points, however, they are the same of the previous case. Moreover, as long as $\phi^*\partial_p\alpha^1(p^*) + (1-\phi^*)\partial_p\alpha^2(p^*) \neq 1$, the local dynamics around them remains well defined. Stability is now only decided by the value of $\mu(\phi, p)$: since the investment rules do not depend on past prices there is no room for the destabilizing role of price feedbacks.

Summarizing, irrespectively of the fact that price dependence is on past or present prices, the market rewards agents whose equilibrium prices are "closest" (in entropy terms) to those of the underlying assets' dividend process. Notice however that, differently from the result in Blume and Easley (1992, 2006), Sandroni (2000) or the works surveyed in Evstigneev et al. (2009), in our framework this applies only locally. It can well happen, like in the example of Fig. 2, that multiple stable equilibria do exist or, alternatively, that none of the deterministic fixed points is stable. In the latter case, Alice is doing better at Bob's prices and Bob is doing better at Alice's prices so that neither Alice nor Bob prevail and market prices fluctuates indefinitely.

4 Main results

This section is devoted to the formal investigation, in a more general case, of the possible sources of market instability appeared in the simple toy model of the previous section. For this purpose we consider a market populated by I investors trading K assets using investment rules depending on a vector of current and past L prices. We derive results about the existence and stability of deterministic fixed points, or long-run market equilibria, for market dynamics described by (2.8). In presenting our findings it is convenient to treat the case of single survivor equilibria first and move to the multiple survivors case at a later stage.

4.1 Single survivor equilibria

While Theorem 2.1 guarantees, under mild conditions, the existence of a market clearing price vector, it is silent about its uniqueness. Since all our results about long-run properties of the market are local, we are not very disturbed by this limitation. In what follows, our first step will be to characterize the single survivor deterministic fixed points. Then, we will specify the conditions under which a *local* dynamic is well-defined around them. At last we shall provide sufficient conditions for the asymptotic stability of these equilibria.

The single survivor equilibria are given by the deterministic fixed points as characterized by the following

Theorem 4.1. *Consider a market for K short-lived assets with non singular payoff matrix D, where I agents invest according to rules in $\mathcal{A}$ using L price lags. Assume agents' wealths and assets' prices evolve according to φ in (2.8). If there exists an agent $i \in \{1, \dots, I\}$ and a price vector $\mathbf{p}^*$ such that $\alpha^i_k(p^*) = p^*_k$ for every $k = 1, \dots, K$, where $p^* = (\mathbf{p}^*, \dots, \mathbf{p}^*)$, then $x^* = (\phi^*, p^*)$ is a deterministic fixed point where agent i dominates, that is, $\phi^{i*} = 1$ and $\phi^{j*} = 0$ for $j \neq i$.*

If agent i has all wealth, then prices are fixed at the intersection of the EMC, which is now an hyperplane of dimension $K-1$, with the i-th investment rule, exactly as in the example in Section 3.

Given a fixed point x^*, a well-defined local dynamics exists for a continuous differentiable dominating investment rule provided that the implicit function theorem can be applied in a neighborhood of x^*. We have the following

Theorem 4.2. *Under the hypothesis of Theorem 4.1, let x^* be a single survivor fixed point where, without loss of generality, agent I-th dominates. Assume $\alpha^I \in \mathcal{C}^1$ in a neighborhood of*

x^*. *The dynamics is locally well-defined, that is, for every $\omega \in \Omega$ there exists a neighborhood $U(\omega)$ of x^* where prices and wealths evolve according to* (2.8), *if and only if the matrix*

$$H := \begin{pmatrix} (\alpha_1^I)^{1,0} - 1 & (\alpha_1^I)^{2,0} & (\alpha_1^I)^{3,0} & \dots & (\alpha_1^I)^{K,0} \\ (\alpha_2^I)^{1,0} & (\alpha_2^I)^{2,0} - 1 & (\alpha_2^I)^{3,0} & \dots & (\alpha_3^I)^{K,0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (\alpha_K^I)^{1,0} & (\alpha_K^I)^{2,0} & (\alpha_K^I)^{3,0} & \dots & (\alpha_K^I)^{K,0} - 1 \end{pmatrix} , \tag{4.1}$$

is non-singular, where

$$(\alpha_k^i)^{h,l} := \left. \frac{\partial \alpha_k^i\,(p)}{\partial p_h^l} \right|_{x^*} , \quad i = 1, \dots, I \, , \quad l = 0, 1, \dots, L \, , \quad k, h = 1, \dots, K \, . \tag{4.2}$$

Notice that when the investment rule of agent I does not depend on current prices it holds $H = -\mathbb{I}$ and the non-singularity condition is trivially met.

Once the local dynamics in a neighborhood of a deterministic fixed point is well defined, the crucial issue is to asses whether an agent dominating (or vanishing) at that fixed point is also dominating (or vanishing) on all trajectories started close enough to it. The next theorem provides sufficient conditions for the asymptotic stability of a deterministic fixed point.

Theorem 4.3. *Under the hypothesis of Theorem 4.2, consider the fixed point $x^* = (\phi^*, p^*)$ of Theorem 4.1 where $\phi^{I*} = 1$ and $p_k^* = \alpha_k^I(p^*)$ for every $k = 1, \dots, K$, and assume that the matrix H defined in* (4.1) *is non-singular. Define the $I - 1$ quantities*

$$\mu_i := \prod_{s=1}^{K} \left(\sum_{k=1}^{K} \frac{\alpha_k^i(p^*)}{\alpha_k^I(p^*)} d_k(s) \right)^{\pi_s} , \quad i \in 1, \dots, I - 1 \, , \tag{4.3}$$

where π_s is the invariant measure probability of state s, and the polynomial in λ of LKth degree

$$P(\lambda) := \sum_{l_1=1}^{L} \dots \sum_{l_K=1}^{L} \lambda^{LK - \sum_j l_j} \sum_{\sigma} sgn(\sigma) \prod_{k=1}^{K} \left((\Delta_k^I)^{\sigma_k, l_{\sigma_k}} - \lambda \, \delta_{k,\sigma_k} \, \delta_{l_{\sigma_k},1} \, , \right) \tag{4.4}$$

where

$$(\Delta_k^I)^{h,l} := - \sum_{k'=1}^{K} \{H^{-1}\}_{k,k'} (\alpha_{k'}^I)^{h,l} \, ,$$

and $(\alpha_{k'}^I)^{h,l}$ are defined in (4.2). *The fixed point x^* is asymptotically stable provided that all μ_i and all the roots of $P(\lambda)$ have module smaller than one. Moreover, if the investment rule α_k^I depends only on asset k prices, which we name no-cross-dependence condition, $P(\lambda)$ simplifies to*

$$P(\lambda) = \prod_{k=1}^{K} \left(\lambda^L - \sum_{l=1}^{L} \lambda^{L-l} (\Delta_k^I)^{(k,l)} \right) , \tag{4.5}$$

and each $(\Delta_k^I)^{h,l}$ to

$$(\Delta_k^I)^{h,l} = \frac{(\alpha_k^I)^{h,l}}{1 - (\alpha_k^I)^{h,0}} \, .$$

Quantities μs, defined in (4.3), smaller than one in absolute value corresponds to the requirement that the relative entropy of the dominating rule with respect to the invariant measure of the dividend process is lower than the same quantity for all other competing rules. This is basically the same condition already found by Blume and Easley (1992) and all subsequent works analyzing market selection between rules depending on assets dividends, the relevant difference being that in their case the differences in relative entropies are global, while in our case they depend on prevailing prices. The second set of stability conditions pertains to the values of λs. These are the roots of a polynomial which depends on the derivatives of the dominating investment rule. The strength of price feedback in the survivor's rule becomes a separate source of market instability, independent from the relative entropy of the adopted strategy. Even though this polynomial is heavily simplified under the no cross-dependence condition, a characterization of its root is only possible when specific investment rules are given. In any case, by continuity, it holds that if investment rules are rather flat at equilibrium prices, so that their partial derivatives are close to zero, the fixed point is asymptotically stable. Indeed, as a straightforward application of Theorem 4.3 one has the following

Corollary 4.1. *Under the hypothesis of Theorem 4.3, when investment rules depend only on current prices, the asymptotic stability of x^* depends only on the value of the module of μs as defined in* (4.3).

4.2 Multiple survivors equilibria

Similar results can also be obtained for fixed points where more rules have positive wealth, or multiple survivors equilibria. They have the characteristics that all surviving agents take the same investment decision and are defined by the following

Theorem 4.4. *Consider a market for K short-lived assets with non singular payoff matrix D, where I agents invest according to a rule in $\mathcal{A}$ using L price lags. Assume agents' wealths and assets' prices evolve according to φ in* (2.8). *If there exists a price vector $\mathbf{p}^*$ and M agents, say the last M, exist such that $p_k^* = \alpha_k^m(p^*)$ for every $k = 1, \ldots, K$, and $m = I-M+1, \ldots, I$ with $p^* = (\mathbf{p}^*, \ldots, \mathbf{p}^*)$ then $x^* = (\phi^*, p^*)$ defines a manifold of deterministic fixed point where the last M agents possess all the wealth and survive, $\sum_{m=I-M+1}^{I} \phi^{m*} = 1$, and the first $I-M$ agents have zero wealth and vanish, $\phi^{j*} = 0$ for $j \leq I-M$.*

Surviving agents fix assets' prices at their common intersection with the EMC. Each common intersection define a manifold of fixed points x^* because each reallocation of wealth among surviving agents does not change equilibrium prices and is still a fixed point. We turn now to the specification of the necessary and sufficient conditions under which the dynamics in a neighborhood of a multiple survivors deterministic fixed point is well defined and to the sufficient conditions for stability. The following theorem generalizes both Theorem 4.2 and 4.3 to the present case.

Theorem 4.5. *Consider the manifold of fixed points $x^* = (\phi^*, p^*)$ of Theorem 4.4 and assume that $\alpha^i \in \mathcal{C}^1$ in a neighborhood of x^* for $i = I-M+1, \ldots, I$. Necessary and sufficient conditions for the existence of a well-defined local dynamics in a neighborhood of x^* and sufficient conditions for the stability of x^* are the same as those specified, respectively, in Theorem 4.2 and 4.3 provided that*

(i) condition (4.3) *is checked only for the $I-M$ agents with zero wealth,*

(ii) in the definition of $(\Delta_k)^{h,l}$, H, and thus $P(\lambda)$, the expression $(\alpha_k^I)^{h,l}$ is replaced by

$$\langle\alpha_k\rangle^{h,l} := \sum_{m=I-M+1}^{M} (\alpha_k^m)^{h,l}\phi^{m*}\,.$$

Intuitively, results for multiple survivors fixed points mimic those for a single survivor once the rule of the dominating agent is replaced by the weighted investment rule, with weights equal to the equilibrium wealth shares. Notice that if at a fixed point x^* all I agents take the same investment decision, all generalized eigenvalues μ will be equal to one, so that the only binding necessary condition for local stability will be given by the roots of the polynomial $P(\lambda)$, representing the strength of the "average" price feedback. This is exactly what happened in the two agents toy market encountered in the example of Section 3 and what will happen in the case considered in Section 6. Notice also that while the statement in Theorem 4.3 concerns *asymptotic* stability, the conditions of Theorem 4.5 only assure stability. This is the obvious consequence of the fact that multiple survivor fixed points are non-isolated points, but lay on a manifold. A reassignment of wealth shares among surviving agents can in fact lead the market to a new stable fixed point having the same equilibrium prices.

5 Dominance and ordering

In the remaining part of the paper we shall illustrate some implications of our results by considering two specific problems. In the present section, we define a relation on the set of rules by using the concepts of dominance and survival and show that this relation is not transitive and thus cannot be of an order type. Nevertheless, we will find that there exists a "special" rule, the Kelly rule, that cannot be dominated by any other rule. In the next section, starting from this consideration, we will examine if the survival ability of the Kelly rule is unique. In particular, we will confront the Kelly rule against a specific class of investment rules which depend on some given statistics of past prices, as in the case in which agents use the observed average past prices and its variance to forecast future assets' performances.

Let us start with the ordering problem. For each given asset market, we define the relation $\succeq$ as a subset of $\mathcal{A}\times\mathcal{A}$ by saying that (α^1,α^2) belong to the relation, or $\alpha^1 \succeq \alpha^2$, if when only the two rules are competing, for almost all initial conditions $x_0 \in \mathcal{X}$ and almost all $\omega \in \Omega$, an agent using rule α^1 either dominates or survives but never vanishes. The question is whether $\succeq$ induces an order relation on the set $\mathcal{A}$.[7]

To answer this question, consider a complete market with 2 states of the world with equal probability to occur and 2 Arrow securities. The following investment rules expressed as fraction of wealth to be invested in the first asset, whose price is denoted as p, are given:

$$\alpha^1(p) = 0.3, \quad \alpha^2(p) = \begin{cases} 0.9 & p \le 0.2 \\ 1.5-3p & 0.2 < p \le 0.3 \\ 0.6 & p > 0.3 \end{cases}, \quad \alpha^3(p) = \begin{cases} 0.2 & p \le 0.3 \\ p-0.1 & p > 0.3 \end{cases}.$$

Let us start from the case in which only rules 1 and 2 compete on the market. The market dynamics is as in Section 3 with (3.4) updating market state variables and with prices

[7]It is understood that while $\mathcal{A}$ depends only on the number of assets K and price lags L, the relation $\succeq$ depends also on the process governing the state of the worlds so that different markets have different relations.

implicitly set by (3.5). Naming ϕ the wealth fraction of strategy 1 and solving (3.5) for market prices gives:

$$p_t = 0.6 - 0.3\phi_t\,.$$

The price of asset 1 is always between 0.3 (when $\phi = 1$) and 0.6 (when $\phi = 0$). Plugging this price equation in (3.4) one obtains the 1-dimensional dynamical system describing the evolution of the market. It is straightforward to check (e.g. by plotting $\alpha^1(p)$ and $\alpha^2(p)$ on the EMC plot) that there exist two single survivor equilibria: one with $\phi^* = 1$ and $p^* = 0.3$ so that rule 1 dominates, and one with $\phi^* = 0$ and $p^* = 0.6$ so that rule 2 dominates. According to Theorem 4.3, however, only the second equilibrium is asymptotically stable, that is, rule 2 dominates on all trajectories starting in a neighborhood of $\phi = 0$. In this simple example we can also infer that, since for all possibly realized prices, $p_t \in [0.3, 0.6]$, the rule of agent 2 has a lower relative entropy than the rule of agent 1, the single survivor equilibrium $(\phi^* = 0, p^* = 0.6)$ is also globally stable. The market dynamics will converge there for almost all initial conditions and for almost all dividend process realizations. Importantly α^2 never vanishes so that $\alpha^2 \succeq \alpha^1$.

Next consider the case in which rules 1 and 3 are trading at the same time. Market clearing price as a function of rule 1 wealth, ϕ, reads

$$p_t = 0.2 + 0.1\phi_t\,,$$

and is bounded between 0.2 and 0.3. As in the previous case, only one single survivor equilibrium, the one associated with $\phi^* = 1$, is asymptotically stable. Because for all possibly realized prices the conditional entropy of rules 3 results the lowest, $(\phi^* = 1, p^* = 0.3)$ is also globally stable, implying $\alpha^1 \succeq \alpha^3$.[8]

Finally, consider the case in which rule 2 and 3 are present in the market. The dynamics is now slightly more complicated. The price of asset 1 as a function of agent 2 wealth fraction ϕ reads

$$p_t = \begin{cases} \dfrac{0.2 + 1.3\phi_t}{1 + 3\phi_t} & \phi_t \in \left[0, \dfrac{1}{4}\right] \\[2ex] 0.7 - \dfrac{0.1}{\phi_t} & \phi_t \in \left(\dfrac{1}{4}, 1\right] \end{cases}$$

which is always between 0.2 (for $\phi = 0$) and 0.6 (for $\phi = 1$). As before, only single survivor equilibria exist and it is easily checked that the fixed point is asymptotically stable when $\phi = 0$ whereas it is unstable when $\phi = 1$. In this case, however, the relative entropy of the two rules is not ordered on the entire set of possibly realized prices. As a consequence, we are not able to state global results since at different prices either rule 2 or rule 3 does better. Depending on the initial conditions, the price can converge to 0.2, and ϕ converge to 0 so that rule 3 dominates, or the price keeps fluctuating between low values close to 0.2, where rule 2 dominates and pushes it up, and high values close to 0.6, where rule 3 dominates and pushes it down. In any case it will never happen, unless for the measure zero initial condition $\phi = 1$, that rule 3 vanishes so that $\alpha^3 \succeq \alpha^2$.

[8] One can also define a stricter relation $\succ$ based on the global dominance of one strategy on the other. In this case one would have $\alpha^2 \succ \alpha^1$ and $\alpha^1 \succ \alpha^3$.

As the previous example makes clear, the relation $\succeq$ is not transitive: $\alpha^2 \succeq \alpha^1$, $\alpha^1 \succeq \alpha^3$, but it is not true that $\alpha^2 \succeq \alpha^3$. Hence, $\succeq$ is not an order relation. The same relation would be transitive, and thus could be used to order investment rules, if the latter did not depend on prices.

While, in the presence of price dependence, it is not possible to order rules according to their relative dominance and survival, there exists a rule, the Kelly rule, named after Kelly (1956), which is never vanishes. This is the content of the following

Theorem 5.1. *Consider a market for K short-lived assets with non singular payoff matrix D, where I agents invest according to rules in $\mathcal{A}$ using L price lags. Assume agents' wealths and assets' prices evolve according to φ in* (2.8)*. The rule*

$$\alpha_k^* = \sum_{s=1}^{K} \pi_s d_k(s)\,, \quad k = 1, \dots, K\,, \tag{5.1}$$

or Kelly rule, never vanishes, that is,

$$\alpha^* \succeq \alpha \quad \textit{for every} \quad \alpha \in \mathcal{A}\,.$$

Notice that to invest according to the Kelly rule, an agent must possess a perfect knowledge about the invariant measures π on the states of the world.

6 Learning from prices

We have established that the Kelly rule, not being dominated by any other rule, never vanishes. Is it the Kelly rule the unique rule having this property? The answer is negative in that there exist many other rules with this same property. Indeed one can construct many different examples by working on the local stability conditions derived in Section 4. In this section we concentrate on one such examples by considering a rule that, in using only market information given by past prices, "adapts" to any other rule and thus is never dominated, in particular not even by the Kelly rule. We first characterize the property of this price learner and then use them to appraise its survivability when competing against the Kelly rule.

Consider an investment rule $\alpha^L \in \mathcal{A}$ that depends on some statistics, like mean or variance, computed on the finite number L of past realized prices.[9] Assume further that the rule satisfies the no-cross-dependence condition and that it is consistent, that is, $\alpha_k^L(p) = p_k$, $k = 1, \dots, K$ for any constant price vector $p = (\mathbf{p}, \dots, \mathbf{p})$. If the statistics used assign equal weights to the L past prices, then all partial derivatives computed at the fixed points are equal. This implies a substantial simplification in the expression of (4.4) which in turn leads to the following

Theorem 6.1. *Consider a deterministic fixed point x^* in which only the agent using rule α^L survives. Assume that the agent investment rule does not depend on present prices, satisfies the no-cross-dependence condition, is consistent and, moreover, for every $k = 1, \dots, K$, all partial derivatives are equal, or*

$$(\alpha_k^L)^{k,l} = (\alpha_k^L)^{k,l'}\,, \quad \textit{for every} \quad l, l' = 1, \dots, L \quad k = 1, \dots, K\,. \tag{6.1}$$

[9]Several so called "technical" rule of chartist inspiration, like trend detection, ceiling or floor crossing etcetera can be considered.

Define $(\alpha_k^L)_{x^*}$ the common value of the partial derivative of investment rule k at the fixed point x^*. All the roots of polynomial $P(\lambda)$ defined in Theorem 4.3 are inside the unit circle provided that

$$(\alpha_k^L)_{x^*} \in \left(-1, \frac{1}{L}\right) . \tag{6.2}$$

The extension of the previous result to the multiple survivors case is straightforward: conditions are not on partial derivatives $(\alpha_k^L)^{k,l}$ but on convex combinations of partial derivatives of the type $\langle \alpha_k \rangle^{k,l}$. In this case the equilibrium can be stable for some mixtures of strategies and unstable for others. When this is the case, the stability condition can be re-written in terms of which wealth distributions among survivors guarantee stability.

We can now apply the previous reasoning to a market populated by a price learner and an agent using the Kelly rule, whose wealth fraction is denoted by ϕ. Consider a fixed point $x^* = (\phi^*, p^*)$ with $p^* = \alpha_k^*$ $k = 1, \ldots, K$ where both agents survive. Then, under the assumptions of Theorem 6.1, one has the following

Corollary 6.1. *Let $(\alpha_k^L)_{x^*}$ be the partial derivatives of the k-th investment rule of the price learner with respect to price. Then the fixed point $x^* = (\phi^*, p^*)$ where $p_k^* = \alpha_k^*$ for every $k = 1, \ldots, K$ is stable if*

$$\phi^* \; > \; 1 - \frac{1}{(\alpha_k^L)_{x^*} L} \quad \textit{when} \quad (\alpha_k)_{x^*} > 0 \, , \tag{6.3}$$

$$\phi^* \; > \; 1 - \frac{1}{|(\alpha_k^L)_{x^*}|} \quad \textit{when} \quad (\alpha_k)_{x^*} < 0 \, , \tag{6.4}$$

and always when $(\alpha_k)_{x^*} = 0$.

The implication of the previous result is that a price learner never vanishes when trading with an agent using the Kelly rule, in that there always exists a finite wealth fraction of the former that stabilizes the deterministic fixed point. Since it is never the case that the Kelly rule dominates a price learner, we have established that $\alpha^L \succeq \alpha^*$.[10]

7 Conclusion

We have investigated wealth-driven selection and market behavior in a complete market for short-lived assets where demands are expressed as a fraction of wealth and depend on a vector of current and past prices. We have found that market instability, leading to asset mis-pricing and informational efficiencies, is a common phenomenon which is due to two different sources, namely investment rules having too strong past prices feedbacks and relative entropy of the dominating rule being too high with respect to some other rule at the equilibrium prices it determines.

Our results cast doubts on the working of market selection, and thus on the validity of the "as if" statement, when exchange economies with uncertainty are considered. On the one hand our results imply that when an agent who has perfect knowledge regarding the underlying dividend process and exploit it at best using the Kelly rule is trading in the market, the fixed points in which she survives are the unique stable equilibria, so that prices correctly reflect,

[10] In fact, along the same lines, it is straightforward to show that $\alpha^L \succeq \alpha$ for every $\alpha \in \mathcal{A}$.

in the long run, asset's fundamental values. This is the same result found also in previous works where market selection was tested on investment rules depending on exogenous asset dividends. On the other hand, when an investor using the Kelly rule is not trading, it is not anymore the case that the market selects for the best informed trader, and informational inefficiencies due to endogenous fluctuations emerge as a generic market property.

A Appendix: Proofs

A.1 Section 2

Proof of Theorem 2.1 According to (2.3) prevailing prices $\mathbf{p}_t$ are set by the implicit equation

$$\mathbf{p}_t = \mathbf{A}(\mathbf{p}_t)$$

where $\mathbf{A}$ is the vector valued function with components $A_k = \sum_{i=1}^{I} \phi_t^i \alpha_{k,t}^i$. Due to assumptions of the theorem, $\mathbf{A}$ is a continuous function from the convex compact set $[0,1]^K$ into itself. Then the proposition follows from Brouwer's Theorem.

Proof of Theorem 2.1 Using the notation of the previous Theorem, a sufficient condition for the uniqueness of the fixed point is that $\mathbf{A}$ represents a contraction mapping, that is for each couple of prices $\mathbf{p}$ and $\mathbf{q} = \mathbf{p} + \delta\mathbf{p}$ it is

$$|\mathbf{A}(\mathbf{p}) - \mathbf{A}(\mathbf{q})| \leq |\mathbf{p} - \mathbf{q}| \ .$$

Due to the differentiability of $\mathbf{A}$, the mean value theorem implies that

$$|\mathbf{A}(\mathbf{p}) - \mathbf{A}(\mathbf{q})| = (\delta\mathbf{p})' \, Q \, \delta\mathbf{p} \ ,$$

where the matrix $Q(\mathbf{p}, \delta\mathbf{p})$ is a positive semi-definite quadratic form defined starting from the Jacobian matrix J of the function $\mathbf{A}$ as

$$Q = \int_0^1 dt_1 J'(\mathbf{p} + t_1 \, \delta\mathbf{p}) \int_0^1 dt_2 J(\mathbf{p} + t_2 \, \delta\mathbf{p}) \ ,$$

The function $\mathbf{A}$ is a contraction if for every couple $\mathbf{p}$ and $\mathbf{q}$, the matrix Q does not possess eigenvalues greater than one. This is trivially the case if the investment functions αs, and consequently the function $\mathbf{A}$, do not depend on contemporaneous prices (in this case all eigenvalues of Q are equal to zero).

If only the k-th contemporaneous price enter as a variable in the investment functions relative to asset k, the matrix Q is diagonal, with elements

$$Q_{i,i} = \left(\int_{0,1} dt_1 J_{i,i}(\mathbf{p} + t_1 \, \delta\mathbf{p}) \right)^2 \ .$$

For the triangle inequality, if $|J_{i,i}(\mathbf{p})| < 1$ for any $\mathbf{p}$, then $Q_{i,i} < 1$ and the proposition follows. □

A.2 Section 4

Proof of Theorem 4.1 The result follows from substitution of x^* in (2.7). $\mathcal{W}(x^*;\omega) = \phi^*$ holds because, for every ω, for every $i \neq I$ $\phi^{i*} = 0$ is a fixed point of $\Phi^j(\cdot,\omega)$ and, since by assumption $p^*_k = \alpha^I_k(p^*)$ for every $k = 1,\ldots,K$, $\phi^{I*} = 1$ is a fixed point of $\Phi^I(\cdot,\omega)$. $\mathcal{P}(x^*;\omega) = p^*$ holds because, regarding the current price, it is only agent I who fixes prices (all other agents have zero wealth) and $p^*_k = \alpha^I_k(p^*)$ holds by assumption for every $k = 1,\ldots,K$. Regarding lagged prices, at any fixed point they are all equal by definition. □

Proof of Theorem 4.2 After noting that prices are implicitly defined by the set of K equations in (2.3) with $\phi^I_{t+1} = 1$ and $\phi^i_{t+1} = 0$ for $i \neq I$, the result immediately follows from the implicit function theorem. □

Proof of Theorem 4.3 Consider the reduced system in $[0,1]^{I-1} \times (0,1)^{K(L+1)}$ of dimension $I - 1 + K(L+1)$ obtained by substituting $\phi^I_t = 1 - \sum_{i=1}^{I-1} \phi^i_t$. With an abuse of notation we will keep using the same names for the map f, and thus also $\mathcal{F}$, even though its definition has actually changed. In particular the definition of f given in (2.5) becomes

$$f_k(x_t;\omega) = \sum_{i=1}^{I-1} \Phi^i(x_t;\omega)(\alpha^i_{k,t+1} - \alpha^I_{k,t+1}) + \alpha^I_{k,t+1}\,, \tag{A.1}$$

for every $k = 1,\ldots,K$. $\mathcal{F}$ defined in (2.7) and x^* vary accordingly, in particular $x^* = (0,\ldots,0,p^*)$.

The Jacobian $J(\omega,x)$ of $\mathcal{F}$ can be written as

$$J(\omega,x) = \begin{pmatrix} \frac{\partial\mathcal{W}}{\partial\mathcal{W}} & \frac{\partial\mathcal{W}}{\partial\mathcal{P}} \\ \frac{\partial\mathcal{P}}{\partial\mathcal{W}} & \frac{\partial\mathcal{P}}{\partial\mathcal{P}} \end{pmatrix}\,, \tag{A.2}$$

or subdividing the part relative to price determination, with obvious notation,

$$J(\omega,x) = \begin{pmatrix} \frac{\partial\mathcal{W}}{\partial\mathcal{W}} & \frac{\partial\mathcal{W}}{\partial\mathcal{P}_1} & \cdots & \frac{\partial\mathcal{W}}{\partial\mathcal{P}_K} \\ \frac{\partial\mathcal{P}_1}{\partial\mathcal{W}} & \frac{\partial\mathcal{P}_1}{\partial\mathcal{P}_1} & \cdots & \frac{\partial\mathcal{P}_1}{\partial\mathcal{P}_K} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\mathcal{P}_K}{\partial\mathcal{W}} & \frac{\partial\mathcal{P}_K}{\partial\mathcal{P}_1} & \cdots & \frac{\partial\mathcal{P}_K}{\partial\mathcal{P}_K} \end{pmatrix}\,. \tag{A.3}$$

The element i,j of each block matrix is the partial derivative of the i-th component of the nominator with respect to the j-th component of the denominator.

In each sub-block $\partial\mathcal{W}/\partial\mathcal{P}_k$ the first column reads

$$\left\{\frac{\delta\mathcal{W}}{\delta\mathcal{P}_k}\right\}_{i,1} = \left(\sum_{k'} \frac{(\alpha^i_{k'})^{k,1}}{p_{k',t}} d_{k'}(\omega_{t+1}) - \frac{\alpha^i_{k,t}}{(p_{k,t})^2 d_k(\omega_{t+1})}\right)\phi^i_t\,, \quad i = 1,\ldots,I-1\,,$$

while for $l > 1$ it is

$$\left\{\frac{\delta\mathcal{W}}{\delta\mathcal{P}_k}\right\}_{i,l>1} = \left(\sum_{k'} \frac{(\alpha^i_{k'})^{k,l-1}}{p_{k't}} d_{k'}(\omega_{t+1})\right)\phi^i_t\,, \quad i = 1,\ldots,I-1\,, \quad L = 2,\ldots,L+1\,,$$

Upon evaluating the previous expressions at x^*, one has

$$\left\{ \frac{\partial \mathcal{W}}{\partial \mathcal{P}}\bigg|_{x^*} \right\}_{i,j} = 0 \quad \text{for all} \quad i, j\,.$$

As a result, the Jacobian matrix evaluated at x^*, $J^*(\omega) = J(\omega, x^*)$, is lower block triangular and the eigenvalues of $J^*(\omega)$ are those of the left-upper, $\partial\mathcal{W}/\partial\mathcal{W}$, and right-lower, $\partial\mathcal{P}/\partial\mathcal{P}$, blocks.

Let us start from the left-upper block. Taking the partial derivatives of wealth fractions gives

$$\left\{ \frac{\partial \mathcal{W}}{\partial \mathcal{W}} \right\}_{i,j} = \frac{\partial \Phi^i(x_t;\omega)}{\partial \phi^j_t} = \delta_{i,j} \sum_{k=1}^{K} \frac{\alpha^i_{k,t}}{p_{k,t}} d_k(\omega_{t+1}) \quad i, j = 1, \dots, I-1$$

so that the block computed in $x^* = (0, \dots, 0, p^*)$ becomes diagonal and reads

$$\frac{\partial \mathcal{W}}{\partial \mathcal{W}}\bigg|_{x^*} = \begin{pmatrix} \mu_1(\omega_{t+1}) & 0 & \dots & 0 \\ 0 & \mu_2(\omega_{t+1}) & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mu_{I-1}(\omega_{t+1}) \end{pmatrix}, \tag{A.4}$$

where, using the fact that prices are fixed by agent I's rule,

$$\mu_i(\omega_{t+1}) = \sum_{k=1}^{K} \frac{\alpha^i_k(p^*)}{\alpha^I_k(p^*)} d_k(\omega_{t+1})\,. \tag{A.5}$$

Concerning the right-lower block, $\partial\mathcal{P}/\partial\mathcal{P}$, the first row of each block $k, h = 1, \dots, K$ is computed using the implicit function theorem. At the fixed point x^* it holds that

$$\begin{aligned} \left\{ \frac{\partial \mathcal{P}_k}{\partial \mathcal{P}_h}\bigg|_{x^*} \right\}_{1,l} &= \frac{\partial f_k(x_t;\omega)}{\partial p^{l-1}_{h,t}}\bigg|_{x^*} \qquad l = 1, \dots, L+1\,, \\ &= -\sum_{k'=1}^{K} \{H^{-1}\}_{k,k'} \{M\}_{k',h_l} \end{aligned} \tag{A.6}$$

where H is the matrix defined in (4.1), non-singular by hypothesis and

$$\{M\}_{k',h_l} = \sum_{i=1}^{I-1} \left(\left\{ \frac{\partial \mathcal{W}}{\partial \mathcal{P}_h}\bigg|_{x^*} \right\}_{1,l} + \phi^i_{t+1}\big|_{x^*} \left(\left(\alpha^i_{k'}\right)^{h,l} - \left(\alpha^I_{k'}\right)^{h,l} \right) \right) + \left(\alpha^I_{k'}\right)^{h,l}\,. \tag{A.7}$$

Simplifying (A.7) and substituting it in (A.6) leads to

$$\left\{ \frac{\partial \mathcal{P}_k}{\partial \mathcal{P}_h}\bigg|_{x^*} \right\}_{1,l} = -\sum_{k'=1}^{K} \{H^{-1}\}_{k,k'} (\alpha^I_{k'})^{h,l} = (\Delta^I_k)^{h,l}\,, \quad l = 1, \dots, L \tag{A.8}$$

and $\{\partial\mathcal{P}_k/\partial\mathcal{P}_h|_{x^*}\}_{1,L+1} = 0$. The other rows are all zero but for the diagonal blocks which have a "Jordan" form, that is,

$$\left\{ \frac{\partial \mathcal{P}_k}{\partial \mathcal{P}_h} \right\}_{i>1,l} = \frac{\partial p^l_{k,t+1}(x_t;\omega)}{\partial p^{l-1}_{k,t}} = \delta_{k,h}\delta_{i+1,i}\,, \quad i = 2, \dots, L+1 \quad l = 1, \dots, L+1\,.$$

As a result

$$\left.\frac{\partial \mathcal{P}_k}{\partial \mathcal{P}_h}\right|_{x^*} = \begin{pmatrix} (\Delta_k^I)^{h,1} & (\Delta_k^I)^{h,2} & \dots & (\Delta_k^I)^{h,L} & 0 \\ \delta_{k,h} & 0 & \dots & 0 & 0 \\ 0 & \delta_{k,h} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \delta_{k,h} & 0 \end{pmatrix}, \quad k,h = 1,\dots,K\,. \tag{A.9}$$

The eigenvalues associated with the price blocks are obtained from the characteristic polynomial defined as the determinant

$$P(\lambda) = \begin{vmatrix} \frac{\partial \mathcal{P}_1}{\partial \mathcal{P}_1} - \lambda I & \dots & \frac{\partial \mathcal{P}_1}{\partial \mathcal{P}_K} & \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \mathcal{P}_K}{\partial \mathcal{P}_1} & \dots & \frac{\partial \mathcal{P}_K}{\partial \mathcal{P}_K} - \lambda I & \end{vmatrix},$$

where I stands for the $(L+1)\times(L+1)$ identity matrix. The last zero columns in each column-block is responsible for a factor λ. This generates an eigenvalue 0 of multiplicity k. Once the associated K columns, and their corresponding rows, have been removed one remains with a residual matrix of dimension KL. This matrix has K rows filled with Δs. Each other row is zero but for two elements, a 1 and a $-\lambda$. Using the Laplace formula iteratively, the final expression of the characteristic polynomial of the lower-right block becomes

$$P(\lambda) = \lambda^k \sum_{l_1=1}^{L} \dots \sum_{l_K=1}^{L} \lambda^{LK-\sum_j l_j} \begin{vmatrix} (\Delta_1^I)^{1,l_1} - \lambda\delta_{1,l_1} & (\Delta_1^I)^{2,l_2} & \dots & (\Delta_1^I)^{K,l_K} \\ (\Delta_2^I)^{1,l_1} & (\Delta_2^I)^{2,l_2} - \lambda\delta_{1,l_2} & \dots & (\Delta_2^I)^{K,l_K} \\ \vdots & \vdots & \ddots & \vdots \\ (\Delta_K^I)^{1,l_1} & (\Delta_K^I)^{2,l_2} & \dots & (\Delta_K^I)^{K,l_K} - \lambda\delta_{1,l_K} \end{vmatrix},$$

which, using the Leibniz formula for the computation of the determinant, and dropping the factor λ^k, reduces to (4.4).

Consider now the iteration, for any given T, of the stochastic linear map defined by the Jacobian computed in the fixed point

$$J^*(T,\omega) = J^*(\theta^{T-1}\omega)\dots J^*(\theta\omega)J^*(\omega)\,.$$

According to the Oseledec's multiplicative ergodic theorem (see Young, 1995, Th. 2.1.1) the eigenvalues of $J^*(T,\omega)$ can be used to compute the Lyapunov spectrum of the iterated linear map provided that the integrability condition is satisfied, that is, as long as

$$\mathrm{E}\log^+ ||J^*|| := \sum_{\omega\in\Omega} \log^+ ||J^*(\omega)||\pi(\omega) < \infty\,, \tag{A.10}$$

where $\log^+ a = \mathrm{Max}\{\log a, 0\}$. In our case, the ergodic nature of the process guarantees that, component by component,

$$\mathrm{E}\log ||\{J^*\}_{i,j}|| = \sum_{\omega\in\Omega} \log ||\{J^*(\omega)\}_{i,j}||\pi(\omega) = \sum_s \log^+ ||J^*_{i,j}(s)||\pi_s\,.$$

Due to assumptions **P1-P2** the element of J^* are finite for any realization of the process, so that (A.10) immediately follows.

Since the integrability condition is satisfied, the Lyapunov exponent of the iterated map reduce to

$$\lim_{T\to\infty} \frac{1}{T} \log |\{J^*(T,\omega)\}_{i,i}| \ .$$

Moreover, since $J^*(\omega)$ is block triangular for every ω, so it is $J^*(T,\omega)$, which can be written as

$$J^*(T,\omega) = \begin{pmatrix} \left(\frac{\partial \mathcal{W}}{\partial \mathcal{W}}\big|_{x^*}\right)^T & 0 \\ \blacksquare & \left(\frac{\partial \mathcal{P}}{\partial \mathcal{P}}\big|_{x^*}\right)^T \end{pmatrix} , \tag{A.11}$$

where

$$\begin{aligned} \blacksquare &= \sum_{t=0}^{T-1} \left(\frac{\partial \mathcal{W}}{\partial \mathcal{W}}\right)^{T-t-1} \frac{\partial \mathcal{P}}{\partial \mathcal{W}} \left(\frac{\partial \mathcal{P}}{\partial \mathcal{P}}\right)^t , \\ &= \sum_{t=0}^{T-1} \frac{\partial \mathcal{W}(\theta^T\omega)}{\partial \mathcal{W}} \cdots \frac{\partial \mathcal{W}(\theta^{t+1}\omega)}{\partial \mathcal{W}} \frac{\partial \mathcal{P}(\theta^t\omega)}{\partial \mathcal{W}} \frac{\partial \mathcal{P}(\theta^{t-1}\omega)}{\partial \mathcal{P}} \cdots \frac{\partial \mathcal{P}(\omega)}{\partial \mathcal{P}} . \end{aligned} \tag{A.12}$$

This implies that the eigenvalues of $J^*(T,\omega)$ are given by the union of the eigenvalues of the T-iteration of the 2 diagonal blocks of $J^*(\omega)$.

The left-upper block is diagonal and for any realization of the stochastic process it is

$$\mu_i(T,\omega) = \mu_i(\omega_{t+T}) \ldots \mu_i(\omega_{t+2})\mu_i(\omega_{t+1}) , \qquad i = 1 \ldots , I-1 ,$$

which, using the expression in (A.5) and the ergodic property of the process to take the limit $T \to \infty$, converges to

$$\mu_i = \prod_{s=1}^{S} \left(\sum_{k=1}^{K} \frac{\alpha_k^i(p^*)}{\alpha_k^I(p^*)} d_k(s) \right)^{\pi_s} .$$

Concerning the right-lower block, the matrix in (A.9) does not depend upon the realization of the random variable. This implies that the eigenvalues of the T-product of right-lower block are just the T power of the eigenvalue of $\partial\mathcal{P}/\partial\mathcal{P}$.

Summarizing the list of exponential of the Lyapunov exponents of the iterated linear map is

$$\lambda(T,\omega) = \{\mu_1(T,\omega), \ldots, \mu_{I-1}(T,\omega)\} \cup \left\{0, \lambda_{1,1}^T, \ldots, \lambda_{1,L}^T\right\} \ldots \left\{0, \lambda_{K,1}^T, \ldots, \lambda_{K,L}^T\right\} , \tag{A.13}$$

where the lambdas are the LK roots of (4.4). The fact that the elements of (A.13) are, in absolute value, lower than one is a sufficient condition for the stability of the iterated linear map.

Since the random dynamical system φ is $\mathcal{C}^1$ (because $\mathcal{F}$ in (2.7) is $\mathcal{C}^1$) and we proved above that the integrability condition of the Multiplicative Ergodic Theorem is satisfied, the Local Hartman-Grobman theorem (see Arnold, 1998, Th. 7.5.6) ensures that the asymptotic stability results of the stochastic linear map $J(\omega,x)$ carry over to the system φ, and the first part of the theorem is proved.

The polynomial (4.4) is heavily simplified when the investment rule of agent I in asset k depends only on current and past prices of asset k itself. In this case all off-diagonal price/price blocks (A.9) have zero entries, and the characteristic polynomial of each diagonal block $k = 1, \ldots, K$ is given by

$$P(\lambda) = \lambda \left(\lambda^L - \sum_{l=1}^{L} \lambda^{L-l} (\alpha_k^I)^{(k,l)} \right) ,$$

that is, one eigenvalue is equal to zero while the other L eigenvalues are the zeros of (4.5). □

Proof of Theorem 4.5 The proof proceeds along the same lines of that of Theorem 4.3. It is still convenient to omit the state variable ϕ_t^I by using $\phi_t^I = 1 - \sum_{i=1}^{I-1} \phi_t^i$. Consider the Jacobian $J^*(\omega)$, of $\mathcal{F}$ computed at the fixed point x^*. The components of the off-diagonal wealth/price and price/wealth blocks read

$$\left\{ \frac{\partial \mathcal{W}}{\partial \mathcal{P}_k}\Big|_{x^*} \right\}_{i,1} = \begin{cases} 0 & i = 1, \ldots, I-M \\ -\frac{\phi^{i*}}{p_k^*} d_k(\omega_{t+1}) & i = I-M+1, \ldots, I-1 \end{cases} , \tag{A.14}$$

$$\left\{ \frac{\partial \mathcal{W}}{\partial \mathcal{P}_k}\Big|_{x^*} \right\}_{i,j>1} = \begin{cases} 0 & i = 1, \ldots, I-M \\ \sum_{k'} \frac{\phi^{i*}}{p_{k'}^*} (\alpha_{k'}^i)^{k,j-1} d_{k'}(\omega_{t+1}) & i = I-M+1, \ldots, I-1 \end{cases} , \tag{A.15}$$

$$\left\{ \frac{\partial \mathcal{P}_k}{\partial \mathcal{W}}\Big|_{x^*} \right\}_{1,j} = \begin{cases} \mu_j(\omega_{t+1})(\alpha_k^j(p^*) - p_k^*) & j = 1, \ldots, I-M \\ 0 & j = I-M+1, \ldots, I-1 \end{cases} , \tag{A.16}$$

$$\left\{ \frac{\partial \mathcal{P}_k}{\partial \mathcal{W}}\Big|_{x^*} \right\}_{i>1,j} = 0 \quad j = 1, ..., I-1 , \tag{A.17}$$

for $k = 1, \ldots, K$ and where $\mu_j(\omega_{t+1})$ is defined as in (A.5). Diagonal blocks have a similar structure to that found for the single survivor case. In particular the wealth/wealth block is

$$\frac{\partial \mathcal{W}}{\partial \mathcal{W}}\Big|_{x^*} = \begin{pmatrix} \mu_1(\omega_{t+1}) & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & \mu_{I-M}(\omega_{t+1}) & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & 1 \end{pmatrix} , \tag{A.18}$$

where each $\mu_i(\omega_{t+1})$ is defined in (A.5) and the presence of ones comes from the fact that $\mu_i(\omega_{t+1}) = 1$ for all $i = I-M+1, \ldots, I-1$. Price/price blocks are obtained from (A.6) with the substitution of the derivatives of the I-th investment rule with the average of the derivative of all surviving rules, weighted with the associated equilibrium wealth shares: defining

$$\langle \alpha_k \rangle^{h,l} := \sum_{m=I-M+1}^{M} \phi^{m*} (\alpha_k^m)^{h,l} ,$$

and $\langle H\rangle$, $\langle M\rangle$, $\langle\Delta\rangle$, as in, respectively, (4.1), (A.7), (A.8) with $\langle\alpha_k\rangle^{h,l}$ in place of $(\alpha_k^I)^{h,l}$, each price/price blocks is given by

$$\left.\frac{\partial \mathcal{P}_k}{\partial \mathcal{P}_h}\right|_{x^*} = \begin{pmatrix} \langle\Delta_k\rangle^{h,1} & \langle\Delta_k\rangle^{h,2} & \dots & \langle\Delta_k\rangle^{h,L} & 0 \\ \delta_{k,j} & 0 & \dots & 0 & 0 \\ 0 & \delta_{k,j} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \delta_{k,j} & 0 \end{pmatrix}, \quad k,h = 1,\dots,K\,, \tag{A.19}$$

The resulting matrix has the structure

$$J^*(\omega) = \begin{pmatrix} W & 0 & 0 \\ 0 & I & A \\ B & 0 & P \end{pmatrix}, \tag{A.20}$$

where $\left(\begin{smallmatrix} W & 0 \\ 0 & I \end{smallmatrix}\right)$ is the wealth/wealth block (A.18), in particular W is the $(I-M)\times(I-M)$ upper diagonal block and I is the $(M-1)\times(M-1)$ identity matrix, P is the $K(L+1)\times K(L+1)$ price/price block built using (A.19), A is a $(M-1)\times K(L+1)$ matrix with elements defined by (A.14-A.15), B is a $K(L+1)\times(I_M)$ matrix with elements defined by (A.16-A.17), and 0 denotes, case by case, a matching null matrix.

It is now a trivial algebraic result that T products of (A.20) are given by

$$J^*(T,\omega) = \begin{pmatrix} W^T & 0 & 0 \\ C' & I & A' \\ B' & 0 & P^T \end{pmatrix},$$

where the exact form of the matrices A',B',C' depend on the choice of T and is not relevant for our analysis. It then follows that the determinant of $J^*(T,\omega)$ can be easily computed as the product of the determinants of its diagonal blocks W^T and P^T. As a result, sufficient conditions for stability can be derived along the same lines of the proof of Theorem 4.3, where diagonal blocks have changed from (A.4) and (A.9) to (A.18) and (A.19), respectively.

Notice that, also in the case of multiple survivors, the stochastic component enters only in the diagonal wealth/wealth block. For multiple survivors, however, the characteristic polynomial of the wealth/wealth block possesses a unit root with multiplicity $M-1$. Consequently, the fixed point is non-hyperbolic, and thus not asymptotically stable. We shall show that each fixed point $x^* = (\phi^*, p^*)$ belonging to the manifold where

$$\sum_{m=1}^{M} \phi^{(I-M+m)*} = 1$$

is nevertheless stable. For any realization ω of the Bernoulli process, the direct sum of the eigenspaces associated with each unitary eigenvalue is the linear space V_I spanned by the $M-1$ vectors $\mathbf{e}_m$, $m = I-M+1,\dots,I-1$ of the canonical base of $\mathbb{R}^{I-1+K(L+1)}$. Since the direction of each vector $\mathbf{e}_m$ corresponds to a change in the relative wealths of the m-th and I-th survivor, each small enough perturbation $v \in V_I$ away from x^* push the dynamics to a new fixed point $x'^* = x^* + v = (\phi'^*, p^*)$ where the wealth distribution ϕ'^* differs from ϕ^* for the reallocation of wealth among the M surviving agents corresponding to v^*. It is a trivial result of this last observation that the market dynamics is stable when perturbations are restricted to V_I. For the more general case notice that any perturbation h can be written as $h = h' + h^\perp$ with $h' \in V_I$, $h^\perp \in V_I^\perp$ and that the market dynamics is asymptotically stable for perturbations $h^\perp$ and stable for perturbations h'. The overall dynamics is indeed stable, but not asymptotically stable. □

B Section 5

Proof of Theorem 5.1 The result follows easily by applying the Theorems of Section 4 to a market with two agents using respectively the Kelly rule α^* defined in (5.1) and a generic rule $\alpha \in \mathcal{A}$. In fact, on the one hand, the deterministic fixed point where the agent using the Kelly rule posses all the wealth and $p^* = \alpha^*$ is either asymptotically stable, when $\alpha(p^*) \neq p^*$, or stable. On the other hand, the deterministic fixed point where the agent using α posses all wealth and p^* solves $p = \alpha(p)$ is either unstable or stable, but never asymptotically stable. Furthermore, on all other possible trajectories both agents survive.

In showing the above mentioned local stability results two are the crucial facts. First, for any given vectors $x \neq \alpha^*$ and π, both in the interior of the unit simplex Δ^K, it holds

$$\prod_{s=1}^{K}\left(\sum_{k=1}^{K}\frac{x_k}{\alpha^*_k}d_k(s)\right)^{\pi_s} \in (0,1)\,,$$

implying the needed conditions for the value of μ defined in (4.3). Second, under the Kelly rule all roots of the polynomial defined in (4.4) are zero. □

C Section 6

Proof of Theorem 6.1 Since by hypothesis the price learner rule α^L does not depend on contemporaneous prices and satisfies both the no-cross dependence condition and (6.1), the characteristic polynomial (4.5) reduces to

$$P(\lambda) = \prod_{k=1}^{K}\left(\lambda^L - (\alpha^L_k)_{x^*}\sum_{l=1}^{L}\lambda^{L-l}\right)\,,$$

Notice that $P(\lambda)$ is the product of K polynomials having all one zero root and the same form namely

$$P(x;\alpha) = x^L - \alpha\sum_{l=0}^{L-1}x^l\,.$$

The problem of determining whether the roots of $P(\lambda)$ are all inside the unit circle can thus be solved by looking at $P(x;\alpha)$.

If $\alpha = 0$ all roots are inside the unite circle. Assume that $\alpha > 0$. On the unit complex circle, $|z| = 1$, it holds

$$|z^L - P(z;\alpha)| = |\alpha\sum_{l=0}^{L-1}z^l| \leq \alpha\sum_{l=0}^{L-1}|z^l| = L\alpha\,.$$

It follows that if $\alpha < 1/L$, $|z^L - P(z;\alpha)| < 1 = |z^L|$ for $|z| = 1$. The latter inequality together with Rouchè's Theorem (see e.g. Lang, 1993) imply that the polynomial $P(z;\alpha)$ and z^L have has the same number of roots inside the unit circle. Moreover notice that if $\alpha \geq 1/L$, it holds both $P(1;\alpha) \leq 0$ and $\lim_{x\to+\infty}P(x;\alpha) = +\infty$, implying the existence of a root greater or equal to one. Provided α is positive, we have proved that $\alpha < 1/L$ is both a necessary and sufficient condition for $P(x;\alpha)$ having all the roots inside the unit circle.

Take now $\alpha < 0$. The complex polynomial $P(z;\alpha)$ can be rewritten as

$$P(z;\alpha) = \sum_{l=0}^{L} z^l - (1 - |\alpha|) \sum_{l=0}^{L-1} z^l \,.$$

and its roots are the solutions of

$$\sum_{l=0}^{L} z^l = (1 - |\alpha|) \sum_{l=0}^{L-1} z^l \,.$$

Multiplying left and right hand side by $z - 1$ (remembering we are adding the root $z = 1$) and rearranging the terms leads to

$$|z - (1 - |\alpha|)| = \frac{|\alpha|}{|z|^L} \,,$$

provided $z \neq 0$ which we can always assume since zero is never a root. Assume now $|\alpha| < 1$. If a root with modulus bigger or equal than one, but different from $z = 1$, exists, one could write

$$|\alpha| < |z - (1 - |\alpha|)| = \frac{|\alpha|}{|z|^L} \leq |\alpha| \,,$$

which is a contradiction. We have proved that $|\alpha| < 1$ is a sufficient condition for all roots being inside the unit circle. The condition is also necessary. Indeed, since the modulus of the constant in $P(z;\alpha)$, $|\alpha|$, is given by the product of the moduli of all the roots, when $|\alpha| \geq 1$ there must exist at least a root with modulus bigger or equal to 1.

Interestingly, the role of the memory parameter L is different in the case of positive and negative prices feedbacks. In general, for consistent estimators, partial derivatives depend on the number of lags considered and scale with $1/L$: the longer the agent's memory, the lower the partial derivative. Then if $(\alpha_k^L)_{x^*} < -1$, by increasing the number of past observation, that is, the memory, it is always possible to cross the bound of -1 and thus stabilize the fixed point. Conversely, if $(\alpha_k^L)_{x^*} > 1/L$, an increase in the memory of the strategy does not improve the stability of the fixed point because the bound scales with $1/L$ as well. □

Proof of Corollary 6.1 The corollary is easily proved by using results from Theorem 6.1 and upon realizing that the characteristic polynomial now depends on the convex combination of partial derivatives, that is, $\langle\alpha\rangle_k = (1 - \phi^*)(\alpha_k^L)_{x^*}$ $k = 1, \ldots, K$, rather than on $(\alpha_k)_{x^*}$ $k = 1, \ldots, K$. □

References

Alchian, A. (1950). Uncertainty, evolution, and economic theory. *The Journal of Political Economy 58*, 211–221.

Anufriev, M. and G. Bottazzi (2010). Market equilibria under procedural rationality. *Journal of Mathematical Economics*. forthcoming.

Anufriev, M., G. Bottazzi, and F. Pancotto (2006). Equilibria, stability and asymptotic dominance in a speculative market with heterogeneous agents. *Journal of Economic Dynamics and Control 30*, 1787 – 1835.

Anufriev, M. and P. Dindo (2010). Wealth-driven selection in a financial market with heterogeneous agents. *Journal of Economic Behavior and Organization 73*, 327–358.

Arnold, L. (1998). *Random Dynamical Systems.* Springer, Berlin.

Barberis, N. and R. Thaler (2003). A survey of behavioral finance. In G. Constantinides, M. Harris, and R. Stultz (Eds.), *Handbook of the Economics of Finance.* North-Holland (Handbooks in Economics Series), Amsterdam.

Blume, L. and D. Easley (1992). Evolution and market behavior. *The Journal of Economic Theory 58*, 9–40.

Blume, L. and D. Easley (2006). If you are so smart why aren't you rich? belief selection in complete and incomplete markets. *Econometrica 74*, 929–966.

Chiarella, C. and X. He (2001). Asset price and wealth dynamics under heterogeneous expectations. *Quantitative Finance 1*, 509–526.

Evstigneev, I., T. Hens, and K. Schenk-Hoppé (2002). Market selection of financial trading strategies: Global stability. *Mathematical Finance 12*, 329–339.

Evstigneev, I., T. Hens, and K. Schenk-Hoppé (2008). Globally evolutionary stable portfolio rules. *Journal of Economic Theory 140*, 197–228.

Evstigneev, I., T. Hens, and K. Schenk-Hoppé (2009). Evolutionary finance. In T. Hens and K. Schenk-Hoppé (Eds.), *Handbook of Financial Markets: Dynamics and Evolution.* North-Holland (Handbooks in Economics Series).

Fama, E. (1970). Efficient capital markets: a review of theory and empirical work. *Journal of Finance 25*, 383–422.

Farmer, J. (2002). Market force, ecology, and evolution. *Industrial and Corporate Change 11*, 895–953.

Friedman, M. (1953). *Essays in Positive Economics.* Univ. Chicago Press.

Hommes, C. (2006). Heterogeneous agent models in economics and finance. In K. Judd and L. Tesfatsion (Eds.), *Handbook of Computational Economics Vol. 2: Agent-Based Computational Economics.* North-Holland (Handbooks in Economics Series), Amsterdam.

Kelly, J. (1956). A new interpretation of information rates. *Bell System Technical Journal 35*, 917–926.

Lang, S. (1993). *Complex Analysis*. Springer, New York.

LeBaron, B. (2006). Agent-based computational finance. In K. Judd and L. Tesfatsion (Eds.), *Handbook of Computational Economics Vol. 2: Agent-Based Computational Economics*. North-Holland (Handbooks in Economics Series).

Levy, M., H. Levy, and S. Solomon (2000). *Microscopic Simulation of Financial Markets*. Academic Press, London.

Sandroni, A. (2000). Do markets favor agents able to make accurate predictions. *Econometrica 68*(6), 1303–1341.

Young, L.-S. (1995). Ergodic theory of differentiable dynamical systems. In Branner and Hjorth (Eds.), *Real and Complex Dynamics*, pp. 293–336. Kluwer Academic Publisher.