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Categorical Aspects of Algebraic Quantum Field Theory

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Alla mia famiglia

Abstract

The aim of this work is to introduce and investigate 3 distinct problems in Algebraic Quantum Field Theory whose solution relies on (higher) multicategorical methods. More precisely, we will discuss a model-independent comparison between Algebraic Quantum Field Theory and Factorization Algebras, a 2-categorical notion of algebraic quantum field theory more sensitive to global aspects of gauge theories than previous approaches and a "smoothness axiom" for algebraic quantum field theories encoding "smooth responses of observable algebras" to "smooth variations of spacetimes".

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CONTENTS

INTRODUCTION

An algebraic quantum field theory (AQFT) is, roughly speaking, a law associating associative and unital ($*$-)*algebras* of observables to spacetimes, satisfying some physically motivated axioms, most notably *Einstein causality*, i.e. the axiom imposing observables with no causal relationship to commute with each other ([HK63, BFV03]).

Due to its rigour and expressiveness, this approach to axiomatize quantum field theories on Lorentzian spacetimes has been widely studied in the last 50 years, see e.g. [BDFY15] for a review of the recent successes of AQFT. As can be inferred by this book, AQFT is a very broad field that can be approached from various viewpoints, e.g. from analytical (operator algebras), geometrical and algebraic perspectives. Our work contributes to the latter picture, more precisely the long-standing tradition of categorical approaches.

In recent years various categorical techniques have been explored, leading to relevant advancements in AQFT. For example, the use of multicategories has largely improved the study of the relationships between AQFTs (see [BS19b, BSW21]), the main reason for this success being the encoding of Einstein causality as a *structure* and not as a *property*.

More recently, AQFT has witnessed huge developments toward the inclusion of gauge theoretic models. This was started by Fredenhagen and Rejzner by introducing chain complex-valued AQFTs ([FR12, FR13]) through the Batalin-Vilkovisky formalism. Later, this framework was formalized by Benini, Schenkel and Woike in [BS19a] and [BSW19b], motivating the use of higher categorical methods to study AQFTs.

The aim of this thesis is to discuss our humble contributions to this field ([BPS19, BPS20, BPSW21]) and, leveraging on multicategorical and higher categorical techniques, to show the effectiveness of Category Theory, not just as a unifying language, but also as an active source of new ideas.

More precisely, the common thread of this work will be the use of higher (multi)categorical techniques to approach 3 rather different questions that motivated our studies.

The first question concerns, broadly speaking, the role of Category Theory as a unifying language. In particular, we will discuss the relationship between *Factorization Algebras* (FA) and Algebraic Quantum Field Theory (AQFT). A comparison at the level of examples between these axiomatizations of quantum field theory was studied by Gwilliam and Rejzner in [GR17] and our main contribution will be considering a categorical and model-independent one. Recall that Factorization Algebras are an axiomatization of quantum field theory due to Costello and Gwilliam (see [CG17]) that takes a slightly different perspective than AQFT, namely, a *prefactorization algebra* (PFA) assigns *vector spaces* of observables to open subsets of a topological manifold M and comes endowed with *factorization products*, i.e. with laws on how to multiply observables coming from disjoint spacetimes. We will see (Theorem 2.4.1) that, provided we assume some natural conditions on both sides, algebraic quantum field theories and prefactorization algebras are intimately related, or, in categorical terms, form the collections of objects of equivalent categories.

The second question concerns the use of higher categorical techniques to study gauge theories. As mentioned earlier, higher categorical tools have already been used in AQFT in the context of chain complex-valued AQFTs. The problem with these methods, though, is that they are perturbative and detect only infinitesimal aspects of gauge theories. Our aim is to study alternative higher categorical approaches, more sensitive to global features.

In particular, we will develop a 2-categorical analogue of AQFTs, namely 2-algebraic quantum field theories (2AQFTs), that assign $\mathbb{K}$-*linear locally presentable* categories (with $\mathbb{K}$ a field of characteristic 0) of observables to spacetime regions. The $\mathbb{K}$-linear categories assigned by a 2AQFT should be interpreted as a quantization of the quasi-coherent sheaf category of the phase space of a physical system ([Toe14]). This becomes in particular relevant when the phase space is a stack, as it is the case in gauge theory, due to the fact that quasi-coherent sheaf categories carry more information than function algebras (see e.g. [Lur04] for a reconstruction theorem of geometric stacks from their quasi-coherent sheaf categories).

The third question concerns the use of (higher) Topos Theory to endow algebraic quantum field theories with a further axiom, which we call *smoothness axiom*. Such axiom should encode the idea that the "algebras of observables shall respond smoothly to smooth variations of spacetimes". Let us try to be a bit clearer. Suppose we are given a Lorentzian manifold and we decide to "vary smoothly" the coefficients of its metric tensor, therefore obtaining a family $\{M_s | s \in S\}$ of spacetimes "depending smoothly" on the parameter $s \in S$. We would expect the algebras of observables $\mathfrak{A}(M_s)$ to "vary smoothly" accordingly, since we presume a "small change of spacetime" to induce a "small change of observable algebras", but no axiom in AQFT encodes this property. To tackle this inadequacy in full generality is an irksome problem which will probably require the efforts of an heterogeneous community of mathematicians (see Subsection 5.2.3). To make a first step toward fulfilling this lacuna and set a possible framework for discussions, we will focus on 1-dimensional quantum field theories. In particular, we will use *stacks*, a 2-categorical analogue of sheaves, to introduce smooth refinements of a suitable category of spacetimes and of the category of associative and unital $*$-algebras, therefore defining smooth algebraic quantum field theories as stack morphisms between such stacks.

Let us conclude this introduction by outlining the content of the remainder of this thesis:

(a) In Chapter 1 we briefly recall the main tools needed to understand the content of this work. From a categorical perspective we will need to talk about multicategories and involutive categories (see Subsections 1.1.1 and 1.1.2). While the former are a straightforward generalization of categories in which morphisms have multiple inputs, the latter represent the correct environment where to study $*$-structures. The other concepts we will need to recall come mainly from the world of AQFT. In particular, in Subsection 1.2.1 we introduce the category **Loc** of globally hyperbolic Lorentzian manifolds while in Subsections 1.2.2 and 1.2.3 we present the core objects of this thesis: (ordinary) algebraic quantum field theories. We will initially define an algebraic quantum field theory to be a functor $\mathbf{Loc} \to \mathbf{Alg}_{\mathbb{K}}$, where $\mathbf{Alg}_{\mathbb{K}}$ denotes the category of associative and unital algebras, satisfying the Einstein causality axiom (i.e. causally independent observables must commute), while we will subsequentially opt for a more generic and flexible definition replacing the category of spacetimes $\mathbf{Loc}$ with any *orthogonal* category $\mathbf{Sp}^{\perp}$ (an orthogonal category is a category endowed with a suitable relation between morphisms with the same target) and replacing the category $\mathbf{Alg}_{\mathbb{K}}$ with any category of algebras $\mathbf{Alg}(\mathbf{C})$ in a generic symmetric monoidal (in general also closed and bicomplete) category $(\mathbf{C}, \otimes, I)$. Furthermore, we will introduce the AQFT multicategory $\mathcal{O}_{\mathbf{Sp}^{\perp}}$ and we will see that algebraic quantum field theories can be conveniently interpreted in terms of multicategory-morphisms (*multifunctors*) $\mathcal{O}_{\mathbf{Sp}^{\perp}} \to \mathbf{C}$.

(b) In Chapter 2 we will approach the first of the questions mentioned earlier, i.e. what is the relationship between Factorization Algebras and Algebraic Quantum Field Theory? Recall that a *prefactorization algebra* is a law that associates to any spacetime M a *vector space* of observables $\mathfrak{F}(M)$ and to each tuple of disjoint $\mathbf{Loc}$-morphisms $(f_1 : M_1 \to N, \dots, f_n : M_n \to N)$ (i.e. morphisms whose images are mutually disjoint) a *factorization product* $\mathfrak{F}(\underline{f}) : \mathfrak{F}(M_1) \otimes \cdots \otimes \mathfrak{F}(M_n) \to \mathfrak{F}(N)$.

In order to obtain a meaningful comparison we will restrict our attention to algebraic quantum field theories $\mathfrak{A} : \mathbf{Loc} \to \mathbf{Alg}(\mathbf{C})$, i.e. to quantum field theories defined on globally hyperbolic Lorentzian manifolds, and we will endow prefactorization algebras and algebraic quantum field theories with two further axioms, namely *Cauchy constancy* and *additivity* (see [Few13],[FV12]). We will obtain functorial comparisons between the categories formed by these objects, and we will prove that the category of additive Cauchy constant algebraic quantum field theories is equivalent to the category of *time-orderable* additive Cauchy constant prefactorization algebras, where time-orderable prefactorization algebras are prefactorization algebras that admit factorization products just for tuples of $\mathbf{Loc}$-morphisms that are *time-orderable* in some appropriate sense.

(c) In Chapter 3 we begin by introducing multicategorical analogues of 2-categories, pseudo-functors, pseudo-natural transformations and modifications, namely *2-multicategories*, *pseudo-multifunctors*, *pseudo-multinatural transformations* and *multimodifications* and we proceed by proving that algebraic quantum field theories can be equivalently interpreted as $\mathbf{Alg}(\mathbf{C})$-valued prefactorization algebras, a fact that we will leverage to introduce 2-algebraic quantum field theories as pseudo-multifunctors $\mathcal{P}_{\mathbf{Sp}^{\perp}} \to \mathbf{Pr}_{\mathbb{K}}$, where $\mathcal{P}_{\mathbf{Sp}^{\perp}}$ denotes the *prefactorization multicategory* associated to the orthogonal multicategory $\mathbf{Sp}^{\perp}$ and $\mathbf{Pr}_{\mathbb{K}}$ is the 2-multicategory of locally presentable $\mathbb{K}$-linear categories. In this context, we will investigate the properties of 2-algebraic quantum field theories and define a *gauging construction* that will enable us to obtain simple toy-models of non-truncated 2-algebraic quantum field theories, i.e. 2-algebraic quantum field theories that do not arise naturally (in some appropriate sense) from an ordinary algebraic quantum field theory. Moreover, we will discuss local-to-global properties of our constructions and a categorification of Fredenhagen's universal algebra, which we call *Fredenhagen's universal category*.

(d) In Chapter 4 we discuss smooth refinements of the categories **Loc** and $^*\mathbf{Alg}_{\mathbb{C}}$ (of associative and unital $*$-algebras) using *stacks* (a 2-categorical analogue of sheaves) on the category **Man** of manifolds and smooth maps. We will then introduce smooth 1-dimensional algebraic quantum field theories as stack morphism and, more importantly, we will add a further level of smoothness to our setting by defining a stack $\mathbf{AQFT}_1^\infty$ that will allow us to define concepts such as "smooth curves of smooth 1-dimensional algebraic quantum field theories". Furthermore, we will introduce the *smooth automorphism group* of a smooth algebraic quantum field theory and we will give examples of the constructions introduced. More precisely, we will discuss smooth refinements of the canonical (anti-)commutation relation functors and define suitable concepts of Green operators for *vertical differential operators*, which we will leverage to obtain a smooth analogue of the 1-dimensional massive scalar field with smoothly varying mass parameter and an example of a $U(1)$-equivariant smooth 1-dimensional AQFT, namely a smooth counterpart of the 1-dimensional massless Dirac field together with its (global) $U(1)$-symmetry.

1

PRELIMINARIES

An Algebraic Quantum Field Theory (AQFT) on curved spacetimes is a functor $\mathbf{Loc} \to {}^*\mathbf{Alg}_{\mathbb{C}}$ from the category of globally hyperbolic Lorentzian manifolds to the category of $\mathbb{C}$-algebras with $*$-involutions satisfying some physically motivated axioms ([BFV03]). In more recent years this picture has been generalized in many different directions, from considering broader kinds of spaces and symmetric monoidal target categories, to the use of multicategories for a more comprehensive study of the categorical aspects. The aim of this chapter is to recall and explain some of these advancements, in particular those that will be leveraged throughout the thesis. More specifically, in Section 1.1 we recall the fundamentals of the theory of multicategories, such as the definition of *multicategory*, *multifunctor* and *multinatural transformation* and slightly more advanced topics such as *multicategorical left Kan extensions* and the *Boardman-Vogt tensor product* ([EM09, Wei07]). Moreover, we introduce the *prefactorization multicategory* ([BPS19]) and a natural categorical environment where to study $*$-involutions, namely *involutive categories* ([BSW19a, Jac12]).

In Section 1.2, we begin exploring the world of Algebraic Quantum Field Theory (AQFT) by recalling some basic definitions and results from Lorentzian Geometry ([BS06, BGP07, BDH13]), by introducing the category **Loc** and by listing the Brunetti-Fredenhagen-Verch axioms while giving some physical motivation ([BDFY15]). We will proceed by defining categorically what an Algebraic Quantum Field Theory valued in a generic symmetric monoidal category is ([BSW21]) and by introducing the multicategorical formulation of **AQFT**s ([BSW21]), showing that AQFTs can be conveniently interpreted in terms of multifunctors (see Subsection 1.2.3).

For the sake of clarity we would like to point out that the only original result of this section is Theorem 1.2.16.

1.1 MULTICATEGORIES AND INVOLUTIVE CATEGORIES

1.1.1 *Multicategories*

Symmetric multicategories (or *operads*, see [EM09, Wei07, Yau16]) are a generalization of categories obtained by considering morphisms with more than one input (*operations*). In the spirit of Category Theory where the key focus is not on the objects themselves, but on the relationships among those and their higher relationships, we will introduce a multicategorical analogue of functors and natural transformations, called *multifunctors* and *multinatural transformations*. To continue the analogy we will see how multicategories, multifunctors and multinatural transformations form a 2-category **MULT** in which the 2-category **CAT** of categories, functors and natural transformations is embedded. Furthermore we will give some examples of multicategories and introduce multicategorical left Kan extensions and the Boardman-Vogt tensor product ([EM09, Wei07]).

Definition 1.1.1. A **Set**-valued *symmetric multicategory* $\mathcal{O}$ consists of the following data ([EM09, Wei07, Yau16]):

(a) A collection of objects (or *colours*) $\mathcal{O}_0$.

(b) For all $t \in \mathcal{O}_0$, $n \geq 0$, $\underline{c} := (c_1, \ldots, c_n) \in \mathcal{O}_0^n$, a set of *n-operations* $\mathcal{O}\binom{t}{\underline{c}}$.

(c) For all $t \in \mathcal{O}_0$, $n \geq 1$, $\underline{a} \in \mathcal{O}_0^n$, $m_i \geq 0$, $\underline{b}_i \in \mathcal{O}_0^{m_i}$, with $i = 1, \ldots, n$, a *composition* map $\gamma : \mathcal{O}\binom{t}{\underline{a}} \times \prod_{i=1}^n \mathcal{O}\binom{a_i}{\underline{b}_i} \to \mathcal{O}\binom{t}{\underline{\underline{b}}}$, where

$$\underline{\underline{b}} = (b_{11}, \ldots, b_{1m_1}, \ldots, b_{n1}, \ldots, b_{nm_n}).$$

We write compactly $\phi\, \underline{\psi} := \gamma(\phi, (\psi_1, \ldots, \psi_n))$ for the composition of operations.

(d) For every $t \in \mathcal{O}_0$ a function $\mathbb{1}_t : \{*\} \to \mathcal{O}\binom{t}{t}$, where $\{*\}$ is the one-object set. We also write $\mathbb{1}_t \in \mathcal{O}\binom{t}{t}$ for the corresponding identity.

(e) A right action of the permutation group on n letters Σ_n on the collection of all n-operations

$$\mathcal{O}(\sigma) : \mathcal{O}\binom{t}{\underline{c}} \to \mathcal{O}\binom{t}{\underline{c}\sigma},$$

where $\underline{c}\sigma := (c_{\sigma(1)}, \ldots, c_{\sigma(n)})$. We write $\phi \cdot \sigma := \mathcal{O}(\sigma)(\phi)$ for the permutation action (notice the dot) while we use the symbol $\underline{\psi}\sigma$ to denote the permutation $(\psi_{\sigma(1)}, \ldots, \psi_{\sigma(n)})$ of an n-tuple of operations $(\psi_1, \ldots, \psi_n)$.

These data have to satisfy the following axioms:

(a) *Associativity*: for all $t \in \mathcal{O}_0$, $n \geq 1$, $\underline{a} \in \mathcal{O}_0^n$, $\phi \in \mathcal{O}\binom{t}{\underline{a}}$, $m_i \geq 1$, $\underline{b}_i \in \mathcal{O}_0^{m_i}$, $\psi_i \in \mathcal{O}\binom{a_i}{\underline{b}_i}$, $s_{j_i} \geq 0$, $\underline{c}_{j_i} \in \mathcal{O}_0^{s_{j_i}}$, $\rho_{j_i} \in \mathcal{O}\binom{b_{j_i}}{\underline{c}_{j_i}}$ with $i = 1, \ldots, n$ and $j_i = 0, \ldots, m_i$, the following equation holds:

$$(\phi\, \underline{\psi})\, \underline{\underline{\rho}} = \phi\, (\underline{\psi}\, \underline{\underline{\rho}}) \tag{1.1.1}$$

Therefore, we can unbiasedly write $\phi\, \underline{\psi}\, \underline{\underline{\rho}}$.

(b) *Unitality*: for all $t \in \mathcal{O}_0$, $n \geq 0$ $\underline{c} \in \mathcal{O}_0^n$, $m \geq 1$, $\underline{b} \in \mathcal{O}_0^m$ the following equations hold :

$$\mathbb{1}_t\, \phi = \phi \tag{1.1.2}$$
$$\psi \mathbb{1}_{\underline{b}} = \psi \tag{1.1.3}$$

where $\phi \in \mathcal{O}\binom{t}{\underline{c}}$, $\psi \in \mathcal{O}\binom{t}{\underline{b}}$ and $\mathbb{1}_{\underline{b}} = (\mathbb{1}_{b_1}, \ldots, \mathbb{1}_{b_m})$.

(c) *Equivariance*: for all $t \in \mathcal{O}_0$, $n \geq 1$, $\underline{a} \in \mathcal{O}_0^n$, $\phi \in \mathcal{O}\binom{t}{\underline{a}}$, $m_i \geq 0$, $\underline{b}_i \in \mathcal{O}_0^{m_i}$, $\psi_i \in \mathcal{O}\binom{a_i}{\underline{b}_i}$, $\sigma \in \Sigma_n$, $\sigma_i \in \Sigma_{m_i}$ with $i = 1, \ldots, n$, the following equations hold:

$$\phi\underline{\psi} \cdot \sigma\langle m_1, \ldots, m_n\rangle = (\phi \cdot \sigma)(\underline{\psi}\sigma\langle m_1, \ldots, m_n\rangle) \tag{1.1.4}$$

$$(\phi\underline{\psi}) \cdot (\sigma_1 \oplus \cdots \oplus \sigma_n)) = \phi\,(\psi_1 \cdot \sigma_1, \ldots, \psi_n \cdot \sigma_n) \tag{1.1.5}$$

where $\sigma\langle m_1, \ldots, m_n\rangle$ is the block permutation on $m_1 + \cdots + m_n$ elements obtained from σ and $\sigma_1 \oplus \cdots \oplus \sigma_n$ is the permutation on $m_1 + \cdots + m_n$ elements obtained from the σ_is.

Examples of multicategories abound. In particular, every category is a multicategory.

Example 1.1.2. Let $\mathbf{C}$ be a category. The associated multicategory $\mathcal{C}$ is given by the following data:

(a) The collection of objects is $\mathcal{C}_0 = \mathbf{Ob}(\mathbf{C})$, where $\mathbf{Ob}(\mathbf{C})$ is the underlying collection of objects of $\mathbf{C}$.

(b) For every $c, t \in \mathcal{C}_0$, sets $\mathcal{C}\binom{t}{c} := \mathbf{C}(c, t)$, where $\mathbf{C}(c, t)$ is the set of morphisms $c \to t$ in the category $\mathbf{C}$ ($\mathbf{C}$-morphisms). For every tuple of objects $\underline{c} \in \mathcal{C}_0^n$ with $n \neq 1$ and for every object $t \in \mathcal{C}_0$, sets $\mathcal{C}\binom{t}{\underline{c}} := \emptyset$.

(c) The composition maps for 1-operations are obtained from the composition maps of the underlying category $\mathbf{C}$ and are trivial for other arities.

(d) For every $t \in \mathcal{C}_0$ the unit element is $\mathbb{1}_t := \mathrm{id}_t$, where id_t is the identity $\mathbf{C}$-endomorphism of t.

(e) The permutation actions are trivial.

The choice of setting $\mathcal{C}\binom{t}{\emptyset} = \emptyset$ for every $t \in \mathcal{C}_0$ may appear arbitrary and the reader might notice that we could have just as well imposed $\mathcal{C}\binom{t}{\emptyset} = \{*\}$. The reason behind it is that, as we will see later, the former choice gives rise to an adjunction $\mathbf{CAT} \leftrightarrows \mathbf{MULT}$, while the latter does not. $\triangledown$

One other interesting source of examples is the multicategories arising from symmetric monoidal categories. In fact, we will see that for each symmetric monoidal category $\mathbf{C}$ there exists an associated multicategory $\mathbf{C}$. Notice that we are using the same notations to denote a symmetric monoidal category and its associated multicategory since the context should always make clear which kind of object we are referring to.

Example 1.1.3. Let $(\mathbf{C}, \otimes, I)$ be a symmetric monoidal category, not necessarily strict. The associated multicategory $\mathbf{C}$ is given by the following data:

(a) The collection of objects is $\mathbf{C}_0 = \mathbf{Ob}(\mathbf{C})$.

(b) For every $t \in \mathbf{C}_0$, $n \geq 0$, $\underline{c} \in \mathbf{C}_0^n$ the sets of operations are

$$\mathbf{C}\binom{t}{\underline{c}} = \mathbf{C}(((\ldots(((c_1 \otimes c_2) \otimes c_3) \ldots) \otimes c_n), t).$$

We will see that, since the tensor product is associative (up to invertible associators), the order of the parenthesis is not relevant and every choice of an ordering will lead to the same multicategory up to isomorphism. Notice that we use the convention $\otimes_{i=1}^n c_n = I$ when $n = 0$.

(c) The composition maps are obtained from those of the monoidal category.

(d) For every $t \in \mathbf{C}_0$ the unit $\mathbb{1}_t = \mathrm{id}_t$.

(e) The right action of Σ_n on the collection of all n-operations is given by the symmetric braiding of $(\mathbf{C}, \otimes, I)$. In particular, let $n \neq 0$, $\underline{c} \in \mathbf{C}_0^n$ and $f \in \mathbf{C}\binom{t}{\underline{c}}$. The action of σ on f, which defines an operation $f \cdot \sigma \in \mathbf{C}\binom{t}{\underline{c}\sigma}$, is given by $f \cdot \sigma = f \circ \mathrm{flip}_\sigma$ where flip_σ is obtained from the symmetric braiding.

Two symmetric monoidal categories that will be of particular relevance throughout the thesis are the symmetric monoidal category $(\mathbf{Vec}_\mathbb{K}, \otimes_\mathbb{K}, \mathbb{K}, \tau)$ of $\mathbb{K}$-vector spaces (with its standard symmetric monoidal structure), where $\mathbb{K}$ will be always a field of characteristic 0, and the symmetric monoidal category $(\mathbf{Alg}_\mathbb{K}, \otimes, \mathbb{K}, \tau)$ of associative and unital algebras over the field $\mathbb{K}$ (concretely, the tensor product of algebras A, $B \in \mathbf{Alg}_\mathbb{K}$ is given by the tensor product algebra $A \otimes B$, i.e. the algebra whose multiplication is given by $(a \otimes b)\,(a' \otimes b') := (a\,a') \otimes (b\,b')$ and the unit element is $1_A \otimes 1_B \in A \otimes B$, the monoidal unit is $\mathbb{K} \in \mathbf{Alg}_\mathbb{K}$ and the symmetric braiding is given by $\tau : A \otimes B \to B \otimes A\,,\ a \otimes b \mapsto b \otimes a$) (notice that we will always consider *unital* algebras). $\triangledown$

Examples 1.1.2 and 1.1.3 provide a lot of instances of multicategories, but they do not exhaust them all. In particular, we will introduce multicategories, such as the associative multicategory As and the multicategory $\mathcal{P}_{\mathbf{C}^\perp}$ associated to an orthogonal category $\mathbf{C}^\perp$, that do not arise in this fashion.

Example 1.1.4. The commutative multicategory Com is given by the following data:

(a) The collection of objects $\mathsf{Com}_0 := \{*\}$, where $\{*\}$ is the set with one-element.

(b) For all $n \geq 0$, the sets $\mathsf{Com}(n) := \{*_n\}$. In particular there is exactly one n-operation for each $n \geq 0$.

The units, the composition maps and the permutation actions are defined in the only way possible. $\triangledown$

Example 1.1.5. The associative multicategory As is defined by the following data:

(a) The collection of objects $\mathsf{As}_0 := \{*\}$, where $\{*\}$ is the one-element set.

(b) For all $n \geq 0$, the sets $\mathsf{As}(n) := \Sigma_n$, where Σ_n is the permutation group on n elements. In particular, for $n = 0$ we impose $\mathsf{As}(0) := \{*_0\}$.

(c) For all $n \geq 1$ and $m_i \geq 0$, for $i = 1, \ldots, n$, the composition map $\gamma : \mathsf{As}(n) \times \prod_{i=1}^{n} \mathsf{As}(m_i) \to \mathsf{As}(m_1 + \cdots + m_n)$,

$$\sigma(\sigma_1, \ldots, \sigma_n) \; := \; \sigma\langle m_{\sigma^{-1}(1)}, \ldots, m_{\sigma^{-1}(n)}\rangle \; (\sigma_1 \oplus \cdots \oplus \sigma_m) \tag{1.1.6}$$

for every $\sigma \in \mathsf{As}(n)$, $\sigma_i \in \mathsf{As}(m_i)$, with $i = 1, \ldots, n$, i.e. the group multiplication in $\Sigma_{m_1+\cdots+m_n}$ of the block permutation $\sigma\langle m_{\sigma^{-1}(1)}, \ldots, m_{\sigma^{-1}(n)}\rangle$ induced by σ and the sum permutation $\sigma_1 \oplus \cdots \oplus \sigma_n$ induced by the σ_is.

(d) The function $\mathbb{1}_* : \{*\} \to \mathsf{As}(1)$ picking out the identity permutation $\mathbb{1}_* := e$.

(e) The right action of Σ_n on the collection of all n-operations

$$\mathsf{As}(\sigma) : \mathsf{As}(n) \to \mathsf{As}(n),$$

given by $\sigma' \cdot \sigma = \sigma'\sigma$ where $\sigma' \in \mathsf{As}(n)$ and $\sigma'\sigma$ is the group product of σ' and σ in $\mathsf{As}(n)$.

$\triangledown$

Orthogonal categories are essential to generalize AQFTs and *Factorization Algebras* (FA) to generic classes of spaces endowed with a suitable notion of orthogonality.

Definition 1.1.6 ([BSW21]). An *orthogonal category* $\mathbf{D}^{\perp}$ is a pair $\mathbf{D}^{\perp} := (\mathbf{D}, \perp)$ consisting of a category $\mathbf{D}$ and a subcollection $\perp \subseteq \mathrm{Mor}\mathbf{D}\,{}_{\mathrm{t}}\!\times_{\mathrm{t}} \mathrm{Mor}\mathbf{D}$ of the collection of pairs of morphisms with a common target (called *orthogonality relation*), such that the following conditions hold true:

1. *Symmetry:* If $(f_1, f_2) \in \perp$, then $(f_2, f_1) \in \perp$.
2. $\circ$*-Stability:* If $(f_1, f_2) \in \perp$, then $(g\, f_1\, h_1, g\, f_2\, h_2) \in \perp$, for all composable $\mathbf{D}$-morphisms g, h_1 and h_2.

We denote orthogonal pairs $(f_1, f_2) \in \perp$ also by $f_1 \perp f_2$.

An *orthogonal functor* $F : \mathbf{D}^{\perp} \to \mathbf{E}^{\perp}$ is a functor $F : \mathbf{D} \to \mathbf{E}$ such that $F(f_1) \perp_{\mathbf{E}} F(f_2)$ for all $f_1 \perp_{\mathbf{D}} f_2$.

We denote by **OrthCAT** the category of orthogonal categories and orthogonal functors.

Notice that given an orthogonal category $\mathbf{E}^{\perp} = (\mathbf{E}, \perp_{\mathbf{E}})$, a category $\mathbf{D}$ and a functor $F : \mathbf{D} \to \mathbf{E}$, we can endow $\mathbf{D}$ with the *pullback orthogonality relation* $\perp_{\mathbf{D}} := F^*(\perp_{\mathbf{E}})$ given by: $f \perp_{\mathbf{D}} g \iff F(f) \perp_{\mathbf{E}} F(g)$. In particular, $F : \mathbf{D}^{\perp} \to \mathbf{E}^{\perp}$ is an orthogonal functor.

Example 1.1.7. Let $\mathbf{Open}(M)$ be the category of non-empty open subsets $U \subseteq M$ of a manifold M with morphisms $U \to V$ given by subset inclusions $U \subseteq V \subseteq M$. We introduce an orthogonality relation $\perp_d$ by declaring two morphisms $U_1, U_2 \subseteq V \subseteq M$ to be orthogonal if and only if $U_1 \cap U_2 = \emptyset$. We denote this orthogonal category by $\mathbf{Open}(M)^{\perp_d}$. $\triangledown$

Example 1.1.8. Let M be a manifold and consider the orthogonal category $\mathbf{Disk}(M)^{\perp_d} = (\mathbf{Disk}(M), \perp_d)$ where $\mathbf{Disk}(M) \subseteq \mathbf{Open}(M)$ is the full subcategory of $\mathbf{Open}(M)$ (see Example 1.1.7) consisting of the open non-empty subsets $U \subseteq M$ such that U is a Cartesian space, i.e. $U \cong \mathbb{R}^m$ for $m = \dim(M)$, and where $\perp_d$ is the pullback orthogonality relation induced by the inclusion functor $J : \mathbf{Disk}(M) \to \mathbf{Open}(M)^{\perp_d}$ (see Definition 1.1.6). ▽

Definition 1.1.9. The *prefactorization multicategory* $\mathcal{P}_{\mathbf{D}^\perp}$ associated to an orthogonal category $\mathbf{D}^\perp$ is the multicategory defined by the following data (see [BPS19]):

(a) The collection of objects is $\mathcal{P}_{\mathbf{D}^\perp_0} = \mathbf{Ob}(\mathbf{D})$, where $\mathbf{Ob}(\mathbf{D})$ is the collection of objects underlying the orthogonal category $\mathbf{D}^\perp$.

(b) For all $n \geq 0$, $t \in \mathcal{P}_{\mathbf{D}^\perp_0}$, $\underline{d} \in \mathcal{P}^n_{\mathbf{D}^\perp_0}$, the sets

$$\mathcal{P}_{\mathbf{D}^\perp}\binom{t}{\underline{d}} := \left\{ \underline{f} := (f_1, \ldots, f_n) \in \prod_{i=1}^n \mathbf{D}(d_i, t) \ : \ f_i \perp f_j \text{ for all } i \neq j \right\} \quad , \tag{1.1.7}$$

For the empty tuple $\underline{d} = \emptyset$, we set $\mathcal{P}_{\mathbf{D}^\perp}\binom{t}{\emptyset} := \{*_t\}$, where $\{*_t\}$ is a set with one element.

(c) For all $n \geq 1$, $t \in \mathcal{P}_{\mathbf{D}^\perp_0}$, $\underline{a} \in \mathcal{P}^n_{\mathbf{D}^\perp_0}$, $m_i \geq 0$, $\underline{b}_i \in \mathcal{P}^{m_i}_{\mathbf{D}^\perp_0}$, for $i = 1, \ldots, n$, the composition maps $\gamma : \mathcal{P}_{\mathbf{D}^\perp}\binom{t}{\underline{a}} \times \prod_{i=1}^n \mathcal{P}_{\mathbf{D}^\perp}\binom{a_i}{\underline{b}_i} \to \mathcal{P}_{\mathbf{D}^\perp}\binom{t}{\underline{\underline{b}}}$ are obtained from the compositions in the underlying orthogonal category $\mathbf{D}^\perp$:

$$\gamma\big(\underline{f}, (\underline{g}_1, \ldots, \underline{g}_n)\big) := \underline{f}\,\underline{\underline{g}} := \big(f_1\, g_{11}, \ldots, f_1\, g_{1m_1}, \ldots, f_n\, g_{n1}, \ldots, f_n\, g_{nm_n}\big) \quad . \tag{1.1.8}$$

(d) For every $t \in \mathcal{P}_{\mathbf{D}^\perp_0}$, the functions $\mathbb{1}_t : \{*\} \to \mathcal{P}_{\mathbf{C}^\perp}\binom{t}{t}$ picking out the identity map id_t.

(e) The permutation actions $\mathcal{P}_{\mathbf{D}^\perp}(\sigma) : \mathcal{P}_{\mathbf{D}^\perp}\binom{t}{\underline{d}} \to \mathcal{P}_{\mathbf{D}^\perp}\binom{t}{\underline{d}\sigma}$ are given by

$$\mathcal{P}_{\mathbf{D}^\perp}(\sigma)(\underline{f}) := \underline{f} \cdot \sigma := \underline{f}\sigma := (f_{\sigma(1)}, \ldots, f_{\sigma(n)}) \quad . \tag{1.1.9}$$

Remark 1.1.10. While we introduced prefactorization multicategories associated just to orthogonal categories $\mathbf{D}^\perp$, i.e. categories $\mathbf{D}$ equipped with a *binary* relation on the set of morphisms with the same target, there are more general prefactorization multicategories that are constructed out of n-ary relations on the sets of morphisms with the same target for all n (we will see the example of the time-orderable prefactorization multicategory $\mathcal{P}_{t\mathbf{Loc}}$ in Chapter 2). △

Defining multifunctors, a generalization of functors in the context of multicategories, is pretty straightforward and relies on the *adagio* that, as a map of categories is structure preserving, i.e. preserves compositions and units, a map between multicategories should be structure preserving as well, i.e. compatible with compositions, units and permutation actions. In particular, leveraging on Example 1.1.2, we will see that functors are special examples of multifunctors.

Definition 1.1.11. Let $\mathcal{O}$ and $\mathcal{P}$ be symmetric multicategories. A *multifunctor* $F : \mathcal{O} \to \mathcal{P}$ is given by the following data:

(a) A map on the underlying collections of objects $F_0 : \mathcal{O}_0 \to \mathcal{P}_0$.

(b) For all $t \in \mathcal{O}_0$, $n \geq 0$ and $\underline{c} \in \mathcal{O}_0^n$, functions $F^t_{\underline{c}} : \mathcal{O}\binom{t}{\underline{c}} \to \mathcal{P}\binom{Ft}{F\underline{c}}$, where $F\underline{c} = (Fc_1, \dots, Fc_n)$.

Notice: We will drop the superscripts and subscripts when clear from the context.

Satisfying the following axioms:

(a) *Preservation of compositions*: for all $t \in \mathcal{O}_0$, $n \geq 1$, $\underline{a} \in \mathcal{O}_0^n$, $\phi \in \mathcal{O}\binom{t}{\underline{a}}$, $m_i \geq 0$, $\underline{b}_i \in \mathcal{O}_0^{m_i}$, $\psi_i \in \mathcal{O}\binom{a_i}{\underline{b}_i}$, with $i = 1, \dots, n$, the following equation holds:

$$F(\phi\, \underline{\psi}) = F(\phi)\, F(\underline{\psi}) \tag{1.1.10}$$

where $F(\underline{\psi}) = (F\psi_1, \dots, F\psi_n)$.

(b) *Preservation of units*: for all $t \in \mathcal{O}_0$ the following equation holds:

$$F(\mathbb{1}_t) = \mathbb{1}_{F(t)} \tag{1.1.11}$$

(c) *Preservation of permutations*: for all $t \in \mathcal{O}_0$, $n \geq 0$, $\underline{c} \in \mathcal{O}_0^n$, $\phi \in \mathcal{O}\binom{t}{\underline{c}}$, $\sigma \in \Sigma_n$, the following equation holds:

$$F(\phi \cdot \sigma) = F(\phi) \cdot \sigma \tag{1.1.12}$$

The collection of multicategories and multifunctors form a *bicomplete* ([EM09])) category **MULT** in which the 1-category **CAT** of categories and functors naturally embeds fully faithfully.

Example 1.1.12. Let **C**, **D** be categories, $\mathcal{C}$, $\mathcal{D}$ their associated multicategories (see Example 1.1.2) and let $F : \mathbf{C} \to \mathbf{D}$ be a functor. There is an obvious multifunctor $F : \mathcal{C} \to \mathcal{D}$ which coincides with the functor F on objects and 1-operations.

▽

Remark 1.1.13. Notice that every multicategory $\mathcal{O}$ gives rise to a category $\pi_0(\mathcal{O})$ by restriction to 1-operations. In particular, the category $\pi_0(\mathcal{O})$ is given by the following data:

(a) The collection of objects $\mathbf{Ob}(\pi_0(\mathcal{O})) := \mathcal{O}_0$.

(b) For all objects $c, c' \in \mathbf{Ob}(\pi_0(\mathcal{O}))$, the sets $\pi_0(\mathcal{O})(c, c') := \mathcal{O}\binom{c'}{c}$.

(c) For every object $c \in \mathbf{Ob}(\pi_0(\mathcal{O}))$, the identity element $\mathrm{id}_c := \mathbb{1}_c$.

(d) Categorical composition is given by multicategorical composition.

Notice further that, given multicategories $\mathcal{O}$ and $\mathcal{P}$, for every multifunctor $F : \mathcal{O} \to \mathcal{P}$ there exists an associated functor $\pi_0(F) : \pi_0(\mathcal{O}) \to \pi_0(\mathcal{P})$ obtained by restriction of F to 1-operations.

It can be checked that these assignments are functorial and together with the functor in Example 1.1.12 form the data of a 1-adjunction $\hookrightarrow: \mathbf{CAT} \leftrightarrows \mathbf{MULT} : \pi_0$. △

Example 1.1.14. Let $\mathbf{D}^\perp = (\mathbf{D}, \perp_\mathbf{D})$ and $\mathbf{E}^\perp = (\mathbf{E}, \perp_\mathbf{E})$ be orthogonal categories and let $j : \mathbf{D}^\perp \to \mathbf{E}^\perp$ be an orthogonal functor. Then, there is an obvious multifunctor $J : \mathcal{P}_{\mathbf{D}^\perp} \to \mathcal{P}_{\mathbf{E}^\perp}$ (notice the abuse of notations), where $\mathcal{P}_{\mathbf{D}^\perp}$ and $\mathcal{P}_{\mathbf{E}^\perp}$ denote the prefactorization multicategories associated to $\mathbf{D}^\perp$ and $\mathbf{E}^\perp$ respectively (see Definition 1.1.9). It is given by the following data:

(a) $J_0 : \mathcal{P}_{\mathbf{D}^\perp_0} \to \mathcal{P}_{\mathbf{E}^\perp_0}$ associates to any object $d \in \mathcal{P}_{\mathbf{D}^\perp_0}$ the object $j(d) \in \mathcal{P}_{\mathbf{E}^\perp_0}$

(b) For every $t \in \mathcal{P}_{\mathbf{D}^\perp_0}$, $n \geq 0$, $\underline{d} \in \mathcal{P}^n_{\mathbf{D}^\perp_0}$, J assigns to each $\underline{f} \in \mathcal{P}_{\mathbf{D}^\perp}\binom{t}{\underline{d}}$ the tuple $J(\underline{f}) = (j(f_1), \dots, j(f_n))$ (which is a well-defined n-operation in $\mathcal{P}_{\mathbf{E}^\perp}$ since j is an orthogonal functor).

▽

Example 1.1.15. Let $(\mathbf{C}, \otimes_\mathbf{C}, I_\mathbf{C})$ and $(\mathbf{D}, \otimes_\mathbf{D}, I_\mathbf{D})$ be (not necessarily strict) symmetric monoidal categories and $(F, F_1, F_2) : \mathbf{C} \to \mathbf{D}$ be a symmetric Lax-monoidal functor with $F_1 : I_\mathbf{D} \to F(I_\mathbf{C})$ the *unit-laxator* and $F_2 : F(c) \otimes_\mathbf{D} F(c') \to F(c \otimes_\mathbf{C} c')$ the *monoidal product-laxator natural transformation*. The following data defines a multifunctor $F : \mathbf{C} \to \mathbf{D}$, where $\mathbf{C}$ and $\mathbf{D}$ are the multicategories associated to the symmetric monoidal categories $\mathbf{C}$ and $\mathbf{D}$ (see Example 1.1.3):

(a) The function F_0 on the collection of objects is given by the underlying map on objects of F.

(b) For every $t \in \mathbf{C}_0$, $n \geq 1$, $\underline{c} \in \mathbf{C}_0^n$, $\phi \in \mathbf{C}\binom{t}{\underline{c}}$ the operation $Fc_1 \otimes_\mathbf{D} \cdots \otimes_\mathbf{D} Fc_n \to F_t$ defined by $F(\phi) \circ F_2^{n-1}$. For every $t \in \mathbf{C}_0$, $\phi \in \mathbf{C}\binom{t}{\emptyset} := \mathbf{C}(I_\mathbf{C}, t)$ the operation $I_\mathbf{D} \to F(t)$ defined by $F(\phi) \circ F_1$.

▽

Remark 1.1.16. The assignments in Examples 1.1.3, 1.1.15 define the data of a functor $\mathbf{SMCAT^{Lax}} \to \mathbf{MULT}$, where $\mathbf{SMCAT^{Lax}}$ is the category of symmetric monoidal categories and Lax-monoidal functors ([EM09, Wei07]), which restricts to a functor $\mathbf{SMCAT} \to \mathbf{MULT}$, where we denote by $\mathbf{SMCAT}$ the category of symmetric monoidal categories and strong monoidal functors. Notice, that we will denote by $\mathbf{SMCat}$ the category of *small* symmetric monoidal categories and strong monoidal functors.

Notice that there exists a functor $(-)^\otimes : \mathbf{MULT} \to \mathbf{SMCAT}$, left adjoint to $\mathbf{SMCAT} \to \mathbf{MULT}$, called the *monoidal envelope* functor ([Hor17]). This functor assigns to each multicategory $\mathcal{O}$ the (strict) symmetric monoidal category $\mathcal{O}^\otimes$ given by the following data:

(a) The collection of objects is $\mathbf{Ob}(\mathcal{O}^{\otimes}) := \coprod_{n\in\mathbb{N}} \mathcal{O}_0^n$, i.e. the collection of all tuples with elements in $\mathcal{O}_0$ (or equivalently the set underlying the free monoid on the elements of $\mathcal{O}_0$).

(b) For all $n \geq 0$, $m \geq 0$, $\underline{c} \in \mathbf{Ob}(\mathcal{O}^{\otimes})$ of length n, $\underline{d} \in \mathbf{Ob}(\mathcal{O}^{\otimes})$ of length m, the sets

$$\mathcal{O}^{\otimes}(\underline{c}, \underline{d}) := \coprod_{\alpha:n\to m} \prod_{0\leq i\leq m} \mathcal{O}\big(\begin{smallmatrix} d_i \\ \underline{c}_{\alpha,i} \end{smallmatrix}\big),$$

where the coproduct is taken over all functions $\alpha : n \to m$ (in this context n and m denote the sets $\{1, \ldots, n\}$ and $\{1, \ldots, m\}$ respectively) and $\underline{c}_{\alpha,i}$ is the (possibly empty) sub-tuple of $\underline{c}$ containing only the c_j's satisfying $\alpha(j) = i$. We will denote morphisms by $(\alpha, \underline{\phi}) := (\alpha, \phi_1, \ldots, \phi_m)$

(c) The composition maps associate to any $(\alpha, \underline{\phi}) : \underline{b} \to \underline{a}$ and $(\beta, \underline{\psi}) : \underline{a} \to \underline{t}$ in $\mathcal{O}^{\otimes}$ the morphism $(\beta, \underline{\psi}) \circ (\alpha, \underline{\phi}) := (\beta\alpha, \underline{\gamma}) : \underline{b} \to \underline{t}$, where $\beta\alpha$ is the usual composition of maps of sets and $\underline{\gamma} := (\gamma_1, \ldots, \gamma_\ell)$ is the tuple of operations $\gamma_k := \psi_k \, \underline{\phi}_{\beta,k} \in \mathcal{O}\big(\begin{smallmatrix} t_k \\ \underline{b}_{\beta\alpha,k} \end{smallmatrix}\big)$ determined by multicategorical composition, for $k = 1, \ldots, \ell$, where $\underline{\phi}_{\beta,k}$ is the sub-tuple of $\underline{\phi} = (\phi_1, \ldots, \phi_m)$ containing only the ϕ_j's satisfying $\beta(j) = k$.

(d) For every $\underline{c} = (c_1, \ldots, c_n) \in \mathcal{O}^{\otimes}$ the identities $\mathrm{id}_{\underline{c}} := (\mathrm{id}, (\mathbb{1}_{c_1}, \ldots, \mathbb{1}_{c_n})) : \underline{c} \to \underline{c}$.

(e) The monoidal product $\mathcal{O}^{\otimes} \times \mathcal{O}^{\otimes} \to \mathcal{O}^{\otimes}$ is obtained from the free monoid functor $\mathbf{Set} \to \mathbf{Set}$ (i.e. concatenation of tuples $\underline{c} \otimes \underline{c}' := (\underline{c}, \underline{c}')$), the symmetric braiding is obtained leveraging the permutation actions of the multicategory $\mathcal{O}$, while the monoidal unit is the empty tuple $\emptyset$.

To each multifunctor $F : \mathcal{O} \to \mathcal{P}$ it assigns the (strict) monoidal functor $F^{\otimes} : \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ given by the following data:

(a) The map $F^{\otimes} : \mathbf{Ob}(\mathcal{O}^{\otimes}) \to \mathbf{Ob}(\mathcal{P}^{\otimes})$ that assigns, for all $n \geq 0$, to each object $\underline{c} \in \mathbf{Ob}(\mathcal{O}^{\otimes})$ of length n the object $F^{\otimes}\underline{c} := (Fc_1, \ldots, Fc_n)$.

(b) For all $n \geq 0$, $m \geq 0$, $\underline{c} \in \mathbf{Ob}(\mathcal{O}^{\otimes})$ of length n, $\underline{d} \in \mathbf{Ob}(\mathcal{O}^{\otimes})$ of length m, functions

$$\mathcal{O}^{\otimes}(\underline{c}, \underline{d}) \to \mathcal{P}^{\otimes}(F^{\otimes}\underline{c}, F^{\otimes}\underline{d}),$$

sending $(\alpha, \phi_1, \ldots, \phi_m) \in \mathcal{O}^{\otimes}(\underline{c}, \underline{d})$ to $(\alpha, F\phi_1, \ldots, F\phi_m) \in \mathcal{P}^{\otimes}(F^{\otimes}\underline{c}, F^{\otimes}\underline{d})$.

△

After defining multicategories and multifunctors it is natural to define what corresponds to natural transformation in the multicategorical setting, i.e. *multinatural transformations*.

Definition 1.1.17. Let $\mathcal{O}$, $\mathcal{P}$ be multicategories and $F, G : \mathcal{O} \to \mathcal{P}$ multifunctors. A *multinatural transformation* $\zeta : F \to G$ is given by the following data:

(a) Functions $\zeta_t : \{*\} \to \mathcal{P}\big(\begin{smallmatrix} Gt \\ Ft \end{smallmatrix}\big)$, for each $t \in \mathcal{O}_0$. We also write $\zeta_t \in \mathcal{P}\big(\begin{smallmatrix} Gt \\ Ft \end{smallmatrix}\big)$ for the corresponding 1-operation.

Satisfying the following axiom:

(a) *Naturality*: for each $t \in \mathcal{O}_0$ and $\phi \in \mathcal{O}\binom{t}{\underline{c}}$ the following equation holds:

$$G(\phi)\zeta_{\underline{c}} = \zeta_t F(\phi) \tag{1.1.13}$$

where $\zeta_{\underline{c}} = (\zeta_{c_1}, \ldots, \zeta_{c_n})$.

Example 1.1.18. Let $\mathbf{C}$, $\mathbf{D}$ be categories, $\mathcal{C}$, $\mathcal{D}$ their associated multicategories (see Example 1.1.2), $F, G : \mathbf{C} \to \mathbf{D}$ functors, $F, G : \mathcal{C} \to \mathcal{D}$ their associated multifunctors (see Example 1.1.12) and $\zeta : F \to G$ a natural transformation. It is easy to verify that the collection of maps ζ_t with $t \in \mathcal{C}_0$ defines a multinatural transformation $\zeta : F \to G$. $\triangledown$

Theorem 1.1.19 ([EM09])**.** *Let $\mathcal{O}$, $\mathcal{P}$ be multicategories. The collection of multifunctors $\mathcal{O} \to \mathcal{P}$ and multinatural transformations between those form the data of a category $[\mathcal{O}, \mathcal{P}]$.*

Moreover, the collection of all multicategories, multifunctors and multinatural transformations forms a 2-category **MULT** *of which* **CAT** *is a fully faithful 2-subcategory (the embedding* **CAT** $\hookrightarrow$ **MULT** *is obtained from Examples 1.1.2, 1.1.12 and 1.1.18).*

In Remark 1.1.13 we noticed there exists a 1-adjunction $\hookrightarrow$: **CAT** $\leftrightarrows$ **MULT** : π_0 where **CAT** and **MULT** are considered as 1-categories, **CAT** $\hookrightarrow$ **MULT** is fully faithful and π_0 : **MULT** $\to$ **CAT** is given by truncation to 1-operations. Is not difficult to see that this adjunction lifts to an adjunction between 2-categories, i.e. a *biadjunction*. In fact, let $F, G : \mathcal{O} \to \mathcal{P}$ be multifunctors and let $\zeta : F \to G$ be a multinatural transformation. The collection of morphisms ζ_t for every $t \in \mathbf{Ob}(\pi_0(\mathcal{O}))$ forms the data of a natural transformation $\pi_0(\zeta) : \pi_0(F) \to \pi_0(G)$. In particular, the following result can be proven:

Theorem 1.1.20. *The adjunction of 1-categories $\hookrightarrow$:* **CAT** $\leftrightarrows$ **MULT** : π_0 *in Remark 1.1.13 lifts to an adjunction of* **CAT***-enriched categories $\hookrightarrow$:* **CAT** $\leftrightarrows$ **MULT** : π_0.

Remark 1.1.21. We have seen in Theorem 1.1.19 that given multicategories $\mathcal{O}$ and $\mathcal{P}$ we can form a category $\mathbf{Alg}_{\mathcal{O}}(\mathcal{P}) := [\mathcal{O}, \mathcal{P}]$ called the *category of algebras over $\mathcal{O}$ with values in $\mathcal{P}$* or the *category of $\mathcal{P}$-valued $\mathcal{O}$-algebras*. In this thesis, we will mostly deal with the case $\mathcal{P} = \mathbf{C}$ where $\mathbf{C}$ is the multicategory associated to a symmetric monoidal category $(\mathbf{C}, \otimes, I)$ (see Example 1.1.3). By definition an algebra over the multicategory $\mathcal{O}$ with values in $\mathbf{C}$ is a multifunctor from $\mathcal{O}$ to $\mathbf{C}$. Concretely, such a multifunctor assigns:

(a) To every object c of $\mathcal{O}_0$ an object $A(c) \in \mathbf{C}_0$.

(b) To every operation $\phi \in \mathcal{O}\binom{t}{\underline{c}}$ a morphism

$$A(\phi) : A(\underline{c}) \to A(t),$$

where $\underline{c} = (c_1, \ldots, c_n)$ and $A(\underline{c}) = A(c_1) \otimes \cdots \otimes A(c_n)$.

It is then clear that an algebra morphism, i.e. a multinatural transformation, is just an operation preserving map.

In particular, if $\mathcal{O} = \mathsf{As}$ (see Example 1.1.5), the category $\mathbf{Alg}_{\mathsf{As}}(\mathbf{C})$ is (isomorphic to) the category $\mathbf{Mon}(\mathbf{C})$ of monoids in $\mathbf{C}$. Moreover, when $\mathbf{C}$ is the symmetric monoidal category $\mathbf{Vec}_{\mathbb{K}}$ of vector spaces over $\mathbb{K}$ (see Example 1.1.2), $\mathbf{Alg}_{\mathsf{As}}(\mathbf{Vec}_{\mathbb{K}})$ specializes to the category $\mathbf{Alg}_{\mathbb{K}}$ of associative and unital algebras over the field $\mathbb{K}$.

Notice furthermore that, when $\mathcal{O} = \mathsf{Com}$, $\mathbf{Alg}_{\mathsf{Com}}(\mathbf{C})$ (see Example 1.1.4) is the category of commutative monoids in $\mathbf{C}$.

To see that the last statements hold is an exercise and can be intuitively understood by noticing that the associative and commutative multicategories count respectively in how many essentially different ways the elements of an associative or commutative algebra can be multiplied. $\triangle$

To shed some light on Remark 1.1.21, in particular to understand why $\mathbf{Alg}_{\mathsf{As}}(\mathbf{Vec}_{\mathbb{K}}) \cong \mathbf{Alg}_{\mathbb{K}}$, we recall briefly how to generate a multicategory from a collection of operations and relations.

The relevant adjunction which comes into play is between $\mathbf{MULT}$ and the category $\mathbf{ColO}$ of *collections of operations* ([Wei07]). A *collection of operations* can be thought of as the rough initial data we would like to build a multicategory from: some objects and operations. More precisely, a *collection of operations* X on a set of inputs X_0 consists of a family of sets $X\binom{t}{\underline{c}}$, for every $t \in X_0$, for every $n \geq 0$ and for every $\underline{c} = (c_1, \dots, c_n) \in X_0^n$. A morphism of collections $F : (X_0, X) \to (X_0', X')$ consists of a function $F_0 : X_0 \to X_0'$ and a function $F : X\binom{t}{\underline{c}} \to X'\binom{F_0(t)}{F_0(\underline{c})}$ for every $t \in X_0$, for every $n \geq 0$ and for every $\underline{c} \in X_0^n$. We call the category of collections of operations and their morphisms $\mathbf{ColO}$.

It is clear that there exists a forgetful functor $U : \mathbf{MULT} \to \mathbf{ColO}$, which sends a multicategory to its underlying objects and operations. What is more interesting is that U admits a left adjoint.

Theorem 1.1.22 ([Wei07])**.** *The forgetful functor $U : \mathbf{MULT} \to \mathbf{ColO}$ has a left adjoint* $\mathrm{Free} : \mathbf{ColO} \to \mathbf{MULT}$.

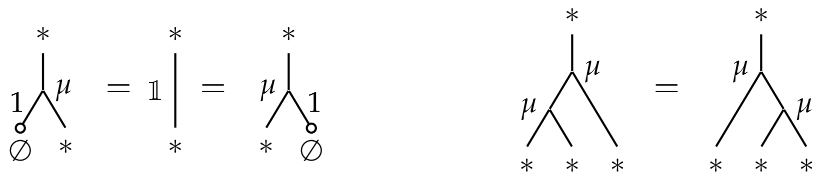
Theorem 1.1.22 provides a practical tool to build a multicategory just by specifying a set of objects and a collection of generators G. In order to implement relations on the generators it is then enough to build a collection R, a family of morphisms $F_i : R \to U\mathrm{Free}(G)$ representing the relations, and consider the coequalizer of the family of maps $\overline{F}_i : \mathrm{Free}(R) \to \mathrm{Free}(G)$ in $\mathbf{MULT}$, where $\overline{F}_i$ is the map associated to F_i under the adjunction for every i.

Example 1.1.23. As an exercise let us try to generate the multicategory As from Example 1.1.5 using the aforementioned adjunction.

Suppose we are given a symmetric monoidal category $(\mathbf{C}, \otimes, I)$ and we want to generate a multicategory As such that $\mathbf{Alg}_{\mathsf{As}}(\mathbf{C})$ can be identified with the category of associative and unital algebras in $\mathbf{C}$. Recalling from Remark 1.1.21 what an algebra with values in $\mathbf{C}$ is and keeping in mind the previous discussion, we realize that the (a multicategory isomorphic to the) associative multicategory As can be obtained considering a collection of objects consisting of a single element $\{*\}$ and requiring the generators

(1.1.14)

representing the unit and the multiplication, to satisfy the relations



(1.1.15)

$\triangledown$

Theorem 1.1.24 ([EM09])**.** *Let $(\mathbf{C}, \otimes, I)$ be a bicomplete symmetric monoidal category and let $\mathcal{O}$ be a multicategory. The category $\mathbf{Alg}_{\mathcal{O}}(\mathbf{C})$ is bicomplete.*

One of the multicategorical techniques we will use more often is called *multicategorical left Kan extension*, an analogue of the categorical left Kan extension in the multicategorical setting.

Definition 1.1.25. Let $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$ be categories, let $\phi : \mathbf{C} \to \mathbf{D}$ be a functor and denote by ϕ^* the pullback functor $[\mathbf{D}, \mathbf{E}] \to [\mathbf{C}, \mathbf{E}]$ given on functors by $\phi^*(H) = H \circ \phi$. If ϕ^* admits a left adjoint $\phi_! : [\mathbf{C}, \mathbf{E}] \to [\mathbf{D}, \mathbf{E}]$ we will say that $\phi_!$ is the *categorical left Kan extension* along ϕ.

Since most of the times the symmetric monoidal categories of interest will be bicomplete and closed, we will assume from now on they are, if not stated otherwise.

The following Theorem provides left Kan extensions for multicategorical algebras along multifunctors $\phi : \mathcal{O} \to \mathcal{P}$:

Theorem 1.1.26 (Multicategorical left Kan extension)**.** *Let $\mathcal{O}$, $\mathcal{P}$ be multicategories, $\phi : \mathcal{O} \to \mathcal{P}$ a multifunctor, $(\mathbf{C}, \otimes, I)$ a symmetric monoidal category (closed and bicomplete) and let $\phi^* : [\mathcal{P}, \mathbf{C}] \to [\mathcal{O}, \mathbf{C}]$ be the pullback functor defined on multifunctors $H \in [\mathcal{P}, \mathbf{C}]$ by $H \to H \circ \phi$, where $\mathbf{C}$ is the multicategory associated to $\mathbf{C}$ (see Example 1.1.3). There exists an adjunction*

$$\phi_! \, : \, \mathbf{Alg}_{\mathcal{O}}(\mathbf{C}) \rightleftarrows \mathbf{Alg}_{\mathcal{P}}(\mathbf{C}) \, : \, \phi^* \quad . \tag{1.1.16}$$

In particular, we will say that $\phi_!$ is the multicategorical left Kan extension along ϕ.

It can be shown ([Hor17]) that $\phi_!$ can be obtained as an ordinary left Kan extension along the functor $\phi^\otimes : \mathcal{O}^\otimes \to \mathcal{P}^\otimes$ (see Remark 1.1.16). Given an algebra $\mathfrak{A} : \mathcal{O} \to \mathbf{C}$, the following coend describes the value of $\phi_!\mathfrak{A}$ on an object $p \in \mathcal{P}_0$:

$$\phi_!\mathfrak{A}(p) = \int^{\underline{q} \in \mathcal{O}^\otimes} \mathcal{P}^\otimes\binom{p}{\phi^\otimes \underline{q}} \otimes \mathfrak{A}(\underline{q}) \tag{1.1.17}$$

where $n \geq 0$ and the tensor product of $\mathcal{P}^{\otimes}\binom{p}{\phi^{\otimes}\underline{q}}$ and $\mathfrak{A}(\underline{q}) = \bigotimes_{i=1}^{n} \mathfrak{A}(q_i)$ is given by **Set***-tensoring. In fact, since the category* **C** *is bicomplete, the tensor product of a set X with an element $c \in$* **C** *is defined by $X \otimes c := \coprod_{x \in X} c$.*

Remark 1.1.27. The coend in Equation (1.1.17) can alternatively be obtained as the following colimit:

$$\phi_!(\mathfrak{A})(p) := \operatorname{colim}\left(\phi^{\otimes}/(p) \xrightarrow{\text{forget}} \mathcal{O}^{\otimes} \xrightarrow{\underline{\mathfrak{A}}} \mathbf{C} \right) \tag{1.1.18}$$

where $(p) \in \mathbf{Ob}(\mathcal{P}^{\otimes})$ is the length 1-tuple consisting of the object $p \in \mathcal{P}_0$, $\phi^{\otimes}/(p)$ is the slice category of the functor $\phi^{\otimes} : \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$, forget : $\phi^{\otimes}/(p) \to \mathcal{O}^{\otimes}$ is the forgetful functor from the slice category to $\mathcal{O}^{\otimes}$ and $\underline{\mathfrak{A}} : \mathcal{O}^{\otimes} \to \mathbf{C}$ is the functor obtained by universal property of the monoidal envelope, i.e. the functor obtained by post-composing $\mathfrak{A}^{\otimes}$ with the counit ϵ at **C** of the adjunction **MULT** $\leftrightarrows$ **SMCAT** from Remark 1.1.16 △

The last ingredient we want to recall from the general theory of multicategories is the *Boardman-Vogt tensor product*. The Boardman-Vogt tensor product is a monoidal product on the category of multicategories and has some very nice properties. For instance, the Boardman-Vogt tensor product admits a right adjoint, the internal-Hom of **MULT**, therefore, it preserves colimits.

Definition 1.1.28 (Boardman-Vogt tensor product [EM09, Wei07])**.** Let $\mathcal{O}$ and $\mathcal{P}$ be multicategories. The *Boardman-Vogt tensor product* $\mathcal{O} \otimes_{BV} \mathcal{P}$ is given in terms of the following generators:

(a) The collection of objects underlying $\mathcal{O} \otimes_{BV} \mathcal{P}$ is $\mathcal{O}_0 \times \mathcal{P}_0$.

(b) For every $q \in \mathcal{O}_0$, $n \geq 0$, $\underline{q} \in \mathcal{O}_0^n$, $p \in \mathcal{P}_0$ there are generators $\phi \otimes p \in \mathcal{O} \otimes_{BV} \mathcal{P}\binom{(q,p)}{((q_1,p),\dots,(q_n,p))}$ for every $\phi \in \mathcal{O}\binom{q}{\underline{q}}$.

(c) For every $p \in \mathcal{P}_0$, $n \geq 0$, $\underline{p} \in \mathcal{P}_0^n$, $q \in \mathcal{O}_0$ there are generators $q \otimes \psi \in \mathcal{O} \otimes_{BV} \mathcal{P}\binom{(q,p)}{((q,p_1),\dots,(q,p_n))}$ for every $\psi \in \mathcal{P}\binom{p}{\underline{p}}$.

Subject to the following relations:

(a) For every $q \in \mathcal{O}_0$, $n \geq 1$, $\underline{q} \in \mathcal{O}_0^n$, $m_i \geq 0$, $\underline{q}_i \in \mathcal{O}_0^{m_i}$, $p \in \mathcal{P}_0$ we impose

$$(\phi \otimes p)((\phi_1 \otimes p), \dots, (\phi_n \otimes p)) = (\phi\underline{\phi}) \otimes p. \tag{1.1.19}$$

where $\phi \in \mathcal{O}\binom{q}{\underline{q}}$ and $\phi_i \in \mathcal{O}\binom{q_i}{\underline{q}_i}$ for $i = 1, \dots, n$.

(b) For every $p \in \mathcal{P}_0$, $n \geq 1$, $\underline{p} \in \mathcal{P}_0^n$, $m_i \geq 0$, $\underline{p}_i \in \mathcal{P}_0^{m_i}$, $q \in \mathcal{O}_0$ we impose

$$(q \otimes \psi)((q \otimes \psi_1), \dots, (q \otimes \psi_n)) = q \otimes (\psi\underline{\psi}) \tag{1.1.20}$$

where $\psi \in \mathcal{P}\binom{p}{\underline{p}}$ and $\psi_i \in \mathcal{P}\binom{p_i}{\underline{p}_i}$ for $i = 1, \dots, n$.

(c) For every $q \in \mathcal{O}_0$ and $p \in \mathcal{P}_0$ we impose

$$\mathbb{1}_q \otimes p = \mathbb{1}_{(p,q)} = q \otimes \mathbb{1}_p. \tag{1.1.21}$$

(d) For every $q \in \mathcal{O}_0$, $n \geq 0$, $\underline{q} \in \mathcal{O}_0^n$, $p \in \mathcal{P}_0$ we impose

$$(\phi \otimes p) \cdot \sigma = (\phi \cdot \sigma) \otimes p \tag{1.1.22}$$

where $\phi \in \mathcal{O}\binom{q}{\underline{q}}$ and $\sigma \in \Sigma_n$.

(e) For every $p \in \mathcal{P}_0$, $n \geq 0$, $\underline{p} \in \mathcal{P}_0^n$, $q \in \mathcal{O}_0$ we impose

$$(q \otimes \psi) \cdot \sigma = q \otimes (\psi \cdot \sigma) \tag{1.1.23}$$

where $\psi \in \mathcal{P}\binom{p}{\underline{p}}$ and $\sigma \in \Sigma_n$.

(f) For every $q \in \mathcal{O}_0$, $n \geq 0$, $\underline{q} \in \mathcal{O}_0^n$, $p \in \mathcal{P}_0$, $m \geq 0$, $\underline{p} \in \mathcal{P}_0^m$ we impose

$$(\phi \otimes p)((q_1 \otimes \psi), \ldots, (q_n \otimes \psi)) = ((q \otimes \psi)((\phi \otimes p_1), \ldots, (\phi \otimes p_m))) \cdot \sigma \tag{1.1.24}$$

where $\phi \in \mathcal{O}\binom{q}{\underline{q}}$, $\psi \in \mathcal{P}\binom{p}{\underline{p}}$ and σ is the permutation that changes

$$((q_1, p_1), \ldots, (q_n, p_1), \ldots, (q_1, p_m), \ldots, (q_n, p_m))$$

into

$$((q_1, p_1), \ldots, (q_1, p_m), \ldots, (q_n, p_1), \ldots, (q_n, p_m)).$$

1.1.2 *Involutive categories*

An involution on an object c of a category $\mathbf{C}$ is normally understood to be a $\mathbf{C}$-morphism $j : c \to c$ that squares to the identity, i.e. $j \circ j = \mathrm{id}_c$. The problem with this notion in the context of quantum theory is that the associative and unital $\mathbb{C}$-algebra of observables $\mathfrak{A}$ of a quantum system comes endowed with a map $* : \mathfrak{A} \to \mathfrak{A}$ that although squaring to the identity is not a $\mathbf{Vec}_{\mathbb{C}}$-morphism, but what is called a $\mathbb{C}$-*anti-linear* map.

Motivated by the need to fill this gap Jacobs developed the theory of *involutive categories* ([Jac12]), which was extended for the needs of quantum field theory, with a multicategorical flavour, in [BSW19a].

In what follows, we present *(symmetric monoidal) involutive categories*, $*$*-objects* and order-reversing $*$-monoids setting the foundations to endow the categories $\mathbf{AQFT}(\mathbf{Sp}^{\perp})$ (see Subsection 1.2.3) and the stack ${}^*\mathbf{Alg}^{\infty}_{\mathbb{C}}$ (see Subsection 4.2.2) with involutive structures.

Definition 1.1.29. An *involutive category* is a triple $(\mathbf{C}, J, j)$, where $\mathbf{C}$ is a category, $J : \mathbf{C} \to \mathbf{C}$ an endofunctor and $j : \mathrm{Id}_{\mathbf{C}} \to J^2$ is a natural isomorphism satisfying:

$$jJ = Jj : J \longrightarrow J^3 \quad . \tag{1.1.25}$$

Example 1.1.30. Let $V \in \mathbf{Vec}_{\mathbb{C}}$ be a complex vector space. Its *complex conjugate* vector space $\overline{V}$ is the vector space that has the same underlying set of elements and additive group structure of V but where the scalar multiplication $\cdot : \mathbb{C} \times \overline{V} \to \overline{V}$ is defined by $c \cdot \bar{v} = \overline{\bar{c}v}$, where $\bar{c}$ denotes the complex conjugate of $c \in \mathbb{C}$.

Let $\overline{(-)} : \mathbf{Vec}_{\mathbb{C}} \to \mathbf{Vec}_{\mathbb{C}}$ be the endofunctor that assigns to every vector space V its complex conjugate vector space $\overline{V}$ and that assigns to a linear map $f : V \to W$ the induced linear map $\overline{f} : \overline{V} \to \overline{W}$ that has the same action of f. It can be shown that $(\mathbf{Vec}_{\mathbb{C}}, \overline{(-)}, \mathrm{id}_{\mathrm{Id}_{\mathbf{Vec}_{\mathbb{C}}}})$, where $\mathrm{id}_{\mathrm{Id}_{\mathbf{Vec}_{\mathbb{C}}}} : \mathrm{Id}_{\mathbf{Vec}_{\mathbb{C}}} \to \mathrm{Id}_{\mathbf{Vec}_{\mathbb{C}}}$ is the identity natural transformation, is an involutive category. ▽

Example 1.1.31. Let $\mathbf{Alg}_{\mathbb{C}} \cong \mathbf{Mon}(\mathbf{Vec}_{\mathbb{C}})$ be the category of $\mathbb{C}$-algebras. The triple $(\mathbf{Alg}_{\mathbb{C}}, \overline{(-)}, \mathrm{id})$, where $\overline{(-)} : \mathbf{Alg}_{\mathbb{C}} \to \mathbf{Alg}_{\mathbb{C}}$ is the functor that assigns:

(a) to every algebra $(\mathfrak{A}, \mu_{\mathfrak{A}}, \eta_{\mathfrak{A}}) \in \mathbf{Alg}_{\mathbb{C}}$ the algebra $(\overline{\mathfrak{A}}, \mu_{\overline{\mathfrak{A}}}, \eta_{\overline{\mathfrak{A}}})$, where $\overline{\mathfrak{A}}$ is the complex conjugate vector space of $\mathfrak{A}$ (Example 1.1.30), $\mu_{\overline{\mathfrak{A}}}$ is the linear morphism $\overline{\mathfrak{A}} \otimes \overline{\mathfrak{A}} \to \overline{\mathfrak{A} \otimes \mathfrak{A}} \to \overline{\mathfrak{A}}$ obtained by post-composing the canonical morphism $\overline{\mathfrak{A}} \otimes \overline{\mathfrak{A}} \to \overline{\mathfrak{A} \otimes \mathfrak{A}}$ with $\overline{\mu_{\mathfrak{A}}^{\mathrm{op}}} : \overline{\mathfrak{A} \otimes \mathfrak{A}} \to \overline{\mathfrak{A}}$, the complex conjugate of $\mu_{\mathfrak{A}}^{\mathrm{op}} : \mathfrak{A} \otimes \mathfrak{A} \to \mathfrak{A}$, and $\eta_{\overline{\mathfrak{A}}} = \overline{\eta_{\mathfrak{A}}} \circ * : \mathbb{C} \to \overline{\mathbb{C}} \to \overline{\mathfrak{A}}$ is obtained by post-composing the complex conjugation morphism $* : \mathbb{C} \to \overline{\mathbb{C}}$ with $\overline{\eta_{\mathfrak{A}}} : \overline{\mathbb{C}} \to \overline{\mathfrak{A}}$, the complex conjugate of $\eta_{\mathfrak{A}}$ (notice that the $*$-involutions of algebras are order-reversing, i.e. $(ab)^* = b^* a^*$).

(b) to every algebra morphism $f : \mathfrak{A}_1 \to \mathfrak{A}_2$ its complex conjugate $\overline{f} : \overline{\mathfrak{A}_1} \to \overline{\mathfrak{A}_2}$.

and where $\mathrm{id} : \mathrm{Id}_{\mathbf{C}} \to \overline{\overline{(-)}} = \mathrm{Id}_{\mathbf{C}}$ denotes the identity natural transformation, is an involutive category. ▽

Definition 1.1.32. Let $(\mathbf{C}, J, j)$ be an involutive category. A *$*$-object* in $\mathbf{C}$ is a couple $(c, *_c)$ where c is an object belonging to $\mathbf{Ob}(\mathbf{C})$ and $*_c : c \to Jc$ is a $\mathbf{C}$-morphism satisfying:

$$(J *_c) \circ *_c \; = \; j_c \tag{1.1.26}$$

where j_c is the component at c of the natural transformation j.

A *$*$-morphism* $f : (c, *_c) \to (c', *_{c'})$ is a $\mathbf{C}$-morphism $f : c \to c'$ satisfying:

$$\begin{tikzcd} c \arrow[r, "f"] \arrow[d, "*_c"'] & c' \arrow[d, "*_{c'}"] \\ Jc \arrow[r, "Jf"'] & Jc' \end{tikzcd} \tag{1.1.27}$$

The collection of $*$-objects and $*$-morphisms, with the natural composition and identities obtained from $\mathbf{C}$, form a category ${}^*\mathbf{Obj}(\mathbf{C})$.

Remark 1.1.33. We will see in Subsection 1.2.2 that *the category of $*$-algebras* ${}^*\mathbf{Alg}_{\mathbb{C}} \cong {}^*\mathbf{Obj}(\mathbf{Alg}_{\mathbb{C}})$ (see Definition 1.1.32 and Example 1.1.31), naturally arises in quantum mechanics. △

Definition 1.1.34. An *involutive symmetric monoidal category* is a triple$(\mathbf{C}, J, j)$, where $\mathbf{C} := (\mathbf{C}, \otimes, I)$ is a symmetric monoidal category, $J := (J, J_1, J_2) : \mathbf{C} \to \mathbf{C}$ is a symmetric monoidal endofunctor (see Example 1.1.3 for notations) and $j : \mathrm{Id}_{\mathbf{C}} \to J^2$ is a symmetric monoidal natural transformation satisfying equation (1.1.26).

Example 1.1.35. Let $(\mathbf{Vec}_{\mathbb{C}}, \overline{(-)}, \mathrm{id}_{\mathrm{Id}_{\mathbf{Vec}_{\mathbb{C}}}})$ be the involutive category from Example 1.1.30. It is easy to see that the functor $\overline{(-)}$ can be promoted to a symmetric monoidal functor $(\mathbf{Vec}_{\mathbb{C}}, \otimes, \mathbb{C}) \to (\mathbf{Vec}_{\mathbb{C}}, \otimes, \mathbb{C})$, where $\otimes : \mathbf{Vec}_{\mathbb{C}} \times \mathbf{Vec}_{\mathbb{C}} \to \mathbf{Vec}_{\mathbb{C}}$ is the usual tensor product of complex vector spaces. In particular, the unit-laxator $\overline{(-)}_1 : \mathbb{C} \to \overline{\mathbb{C}}$ is given by complex conjugation and the monoidal product-laxator $\overline{(-)}_2 : \overline{V} \otimes \overline{W} \to \overline{V \otimes W}$ is given by the canonical map. ▽

Example 1.1.36. From Examples 1.1.35 and 1.1.31 and the natural symmetric monoidal structure of $\mathbf{Alg}_{\mathbb{C}}$ it can be shown that $(\mathbf{Alg}_{\mathbb{C}}, \overline{(-)}, \mathrm{id}_{\mathrm{Id}_{\mathbf{Alg}_{\mathbb{C}}}})$ is a symmetric monoidal involutive category. ▽

Definition 1.1.37. Let $(\mathbf{C}, J, j)$ and $(\mathbf{C}', J', j')$ be symmetric monoidal involutive categories. An *involutive symmetric monoidal functor* $(F, \nu) : (\mathbf{C}, J, j) \to (\mathbf{C}', J', j')$ consists of a symmetric monoidal functor $F : \mathbf{C} \to \mathbf{C}'$ and a symmetric monoidal natural transformation $\nu : FJ \to J'F$ satisfying

$$\begin{tikzcd}
F \arrow[rr, equal] \arrow[d, "Fj"'] & & F \arrow[d, "j'F"] \\
FJ^2 \arrow[r, "\nu J"'] & J'FJ \arrow[r, "J'\nu"'] & J'^2F
\end{tikzcd} \tag{1.1.28}$$

An *involutive symmetric monoidal natural transformation* $\zeta : (F, \nu) \to (G, \chi)$ between involutive symmetric monoidal functors $(F, \nu), (G, \chi) : (\mathbf{C}, J, j) \to (\mathbf{C}', J', j')$ is a symmetric monoidal natural transformation $\zeta : F \to G$ satisfying the following axiom:

$$\begin{tikzcd}
FJ \arrow[r, "\zeta J"] \arrow[d, "\nu"'] & GJ \arrow[d, "\chi"] \\
J'F \arrow[r, "J'\zeta"'] & J'G
\end{tikzcd} \tag{1.1.29}$$

Definition 1.1.38. We denote by **ISMCAT** the 2-category consisting of involutive symmetric monoidal categories, involutive symmetric monoidal functors and involutive symmetric monoidal natural transformations.

In the literature an associative and unital $*$-algebra over the field of complex numbers $\mathbb{C}$ (or an order-reversing $*$-monoid in $\mathbf{Vec}_{\mathbb{C}}$) $(A, \mu_A, \eta_A, *_A)$ is a monoid (A, μ_A, η_A) in the category $\mathbf{Vec}_{\mathbb{C}}$ of complex vector spaces, where μ_A and η_A denote respectively the product and the unit of A, together with a linear map $*_A : A \to \overline{A}$ satisfying the following axioms:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "*"] \arrow[d, "\eta"'] & \overline{\mathbb{C}} \arrow[d, "\overline{\eta}"] \\
A \arrow[r, "*_A"'] & \overline{A}
\end{tikzcd} \tag{1.1.30}$$

$$\begin{tikzcd}
A \arrow[r, "*_A"] \arrow[dr, "\cong"'] & \overline{A} \arrow[d, "\overline{*_A}"] \\
 & \overline{\overline{A}}
\end{tikzcd} \tag{1.1.31}$$

$$
\begin{tikzcd}
A \otimes A \arrow[rr, "\mu_A"] \arrow[d, "*_A \otimes *_A"'] & & A \arrow[d, "*_A"] \\
\overline{A} \otimes \overline{A} \arrow[r, "\cong"'] & \overline{A \otimes A} \arrow[r, "\overline{\mu_A^{\mathrm{op}}}"'] & \overline{A}
\end{tikzcd}
\tag{1.1.32}
$$

This set of axioms can be generalized to any involutive symmetric monoidal category $(\mathbf{C}, J, j)$ and we call a tuple $(c, \mu_c, \eta_c, *_c)$ satisfying such generalization an *order-reversing $*$-monoid*. In particular:

Proposition 1.1.39 ([BSW19a]). *There exists a 2-functor $*$-$\mathbf{Mon}_{\mathrm{rev}} : \mathbf{ISMCAT} \to \mathbf{CAT}$ defined by the following data:*

(a) It sends each involutive symmetric monoidal category $\mathbf{C} = (\mathbf{C}, J, j)$ to the category $*$-$\mathbf{Mon}_{\mathrm{rev}}(\mathbf{C})$ whose objects are order-reversing $*$-monoids in $\mathbf{C}$ and whose morphisms $f : (c, \mu_c, \eta_c, *_c) \to (c', \mu_{c'}, \eta_{c'}, *_{c'})$ are $\mathbf{C}$-morphisms $f : c \to c'$ preserving the monoid structure and the $*$-structure.

(b) It sends an involutive symmetric monoidal functor

$$(F, \nu) : (\mathbf{C}, J, j) \to (\mathbf{C}', J', j')$$

to the obvious functor $*$-$\mathbf{Mon}_{\mathrm{rev}}(F, \nu) : *$-$\mathbf{Mon}_{\mathrm{rev}}(\mathbf{C}) \to *$-$\mathbf{Mon}_{\mathrm{rev}}(\mathbf{C}')$ and an involutive symmetric monoidal natural transformation $\eta : (F, \nu) \to (G, \chi)$ to the obvious natural transformation $*$-$\mathbf{Mon}_{\mathrm{rev}}(\eta) : *$-$\mathbf{Mon}_{\mathrm{rev}}(F, \nu) \to *$-$\mathbf{Mon}_{\mathrm{rev}}(G, \chi)$.

In particular, the category ${}^*\mathbf{Alg}_{\mathbb{C}}$ from Remark 1.1.33 can be equivalently interpreted as the category $*$-$\mathbf{Mon}_{\mathrm{rev}}(\mathbf{Vec}_{\mathbb{C}})$ of order-reversing $*$-monoids in $\mathbf{Vec}_{\mathbb{C}}$.

1.2 AQFT

1.2.1 *The category* **Loc**

In this section we introduce a category of well-behaved (free from future/past directed closed causal curves) spacetimes, namely that of globally hyperbolic Lorentzian manifolds **Loc**. In particular, we argue that these spacetimes represent a good setting for studying initial value problems.

We begin by listing some basic definitions from Lorentzian geometry ([ONe83, BG12]).

A *Lorentzian manifold* is a manifold M together with a metric g of signature $(- + \cdots +)$. The Lorentzian nature of the metric g enables us talking about the *causal structure* of the manifold: we say that a non-zero tangent vector $v \in T_xM$ at a point $x \in M$ is *time-like* if $g(v, v) < 0$, *light-like* if $g(v, v) = 0$, *space-like* if $g(v, v) > 0$ and *causal* if it is either time-like or light-like. Analogously, we will say that a smooth curve $\gamma : I \to M$ is *time-like/light-like/space-like/causal* if all its tangent vectors are time-like/light-like/space-like/causal.

We say that M is *time-orientable* if there exists a smooth section $\mathfrak{t} \in \Gamma^\infty(TM)$ of its tangent bundle that is everywhere time-like. To avoid being notationally heavy, we will not display the metric g and time-orientation $\mathfrak{t}$ of a time-oriented Lorentzian manifold if not strictly necessary. A non-zero causal vector $v \in T_xM$ in the tangent space to a time-oriented Lorentzian manifold M at $x \in M$ is *future directed* if $g(\mathfrak{t}_x, v) < 0$ and *past-directed* if $g(\mathfrak{t}_x, v) > 0$. Analogously, we will say that a smooth curve $\gamma : I \to M$ is *future/past directed* if all the vectors tangent to γ are future/past directed.

We denote with $I^+(x)$ and $I^-(x)$ respectively the *chronological future* and the *chronological past* of $x \in M$, i.e. the set of points that can be reached from x via future or past directed time-like curves respectively. Given $S \subseteq M$ we define $I^+(S) := \cup\{I^+(x)|x \in S\}$ and $I^-(S) := \cup\{I^-(x)|x \in S\}$. Similarly, we denote with $J^\pm(x)$ the *causal future/past* of the point $x \in M$, i.e. the set of points that can be reached from x via future/past directed causal curves and given $S \subseteq M$ we define $J^+(S) := \cup\{J^+(x)|x \in S\}$ and $J^-(S) := \cup\{J^-(x)|x \in S\}$.

Notice that none of the definitions given so far excludes pathological (at least from a physical perspective) aspects, such as closed past/future directed causal curves, from entering the picture.

Example 1.2.1 (Gödel spacetime). The Gödel spacetime is the manifold $M = \mathbb{R}^4$ with line-element that in standard coordinates (t, x, y, z) reads as

$$\mathrm{d}s^2 = -(\mathrm{d}t + e^{2ky}dx)^2 + \mathrm{d}y^2 + \frac{e^{4ky}}{2}\mathrm{d}x^2 + \mathrm{d}z^2$$

After introducing the following change of coordinates

$$e^{2ky} = \cosh(2kr) + \sinh(2kr)\cos\varphi, \quad \sqrt{2}kxe^{2ky} = \sinh(2kr)\sin\varphi,$$
$$\frac{kt}{\sqrt{2}} = \frac{kt'}{\sqrt{2}} - \frac{\varphi}{2} + \arctan\left(e^{-2kr}\tan\frac{\varphi}{2}\right),$$

where $|k(t-t')| < \frac{\pi}{\sqrt{2}}$, $r \in [0, \infty)$ and $\varphi \in [0, 2\pi)$, the line-element in the coordinates (t', z, ϕ, r) reads:

$$ds^2 = -dt'^2 + dr^2 + dz^2 - \frac{\sqrt{8}}{k}\sinh^2(kr)d\varphi dt' + \frac{1}{k^2}\left(\sinh^2(kr) - \sinh^4(kr)\right)d\varphi^2.$$

It is possible to check that any curve with t' and z fixed, and r larger then or equal to $r_0 = (1/k)\ln(1+\sqrt{2})$, is closed causal.

$\triangledown$

The solution to the problem of closed causal curves relies on the notion of *causal convexity*:

Definition 1.2.2. Let M be a time-oriented Lorentzian manifold. We say that $S \subseteq M$ is *causally convex* if $J^+(S) \cap J^-(S) \subseteq S$.

It is clear from Definition 1.2.2 that asking every point $x \in M$ to have arbitrarily small causally convex open neighbourhoods is sufficient to avoid our manifold showing pathologies like future/past directed closed causal curves. There is another issue though: we will be interested in spacetimes that allow a well-posed initial value problem for hyperbolic differential operators and so far none on the conditions imposed to the manifold help us with that. The idea is that we want the time-oriented manifold M to have codimension 1 subspaces on which we can assign initial data. The solution to this issue is contained in the following definition:

Definition 1.2.3. A Lorentzian manifold M is called *globally hyperbolic* if it admits a *Cauchy surface* Σ, i.e. there exists a set $\Sigma \subseteq M$ such that every inextensible time-like curve meets Σ exactly once.

Although Definition 1.2.3 looks quite obscure in the sense that it seems to suggest we are going to assign initial data just from a distinguished (for what reason?) hypersurface, it turns out that it is the right condition to consider:

Theorem 1.2.4 ([BS06]). *Let M be a time-oriented Lorentzian manifold. Then the following are equivalent:*

(a) M is globally hyperbolic.

(b) M is isometric to $\mathbb{R} \times \Sigma$ with line element $ds^2 = -\beta dt^2 + h_t$, where $\beta \in C^\infty(M)$ is strictly positive and h_t is a smooth Riemannian metric on Σ depending smoothly on t. Furthermore, each $\{t\} \times \Sigma$ is a smooth Cauchy hypersurface in M.

Remark 1.2.5. Theorem 1.2.4 not only implies that a globally hyperbolic Lorentzian manifold admits infinitely many Cauchy hypersurfaces; it also implies that we can choose them to be smooth, providing a good framework for initial value problems. Moreover, it can be proven that every globally hyperbolic Lorentzian manifold M is *strongly causal*, i.e. admits arbitrary small causally convex open neighbourhoods for every point $x \in M$. In particular, there are no closed future/past directed causal curves in M. △

There is one last notion from Lorentzian geometry we need, i.e. *causal disjointness*. The idea is that two subsets S, $S' \subseteq M$ of a Lorentzian manifold M are causally disjoint if they have no causal relationship, i.e. there are no future/past directed causal curves connecting S and S', or equivalently:

Definition 1.2.6. Let M be a Lorentzian manifold and let S, $S' \subseteq M$. We say that S and S' are *causally disjoint* if $(J^+(S) \cup J^-(S)) \cap S' = \emptyset$.

We are now ready to define the central object of this section, the category **Loc**:

Definition 1.2.7. We denote by **Loc** the category whose objects are all the connected oriented and time-oriented globally hyperbolic Lorentzian manifolds M (we will call these objects *spacetimes*) and whose morphisms are all orientation and time-orientation preserving isometric embeddings $f : M \to N$ with causally convex and open image $f(M) \subseteq N$.

We denote by $\mathbf{Loc}_m$ the full subcategory of **Loc** consisting of the oriented and time-oriented globally hyperbolic Lorentzian manifolds M of dimension $\dim(M) = m$.

Remark 1.2.8. Since we will need it in Chapter 4 we give an explicit description of the category $\mathbf{Loc}_1$ of 1-dimensional globally hyperbolic Lorentzian manifolds (which we will often refer to as the "category of 1-dimensional spacetimes" for brevity).

(a) *Objects*: A 1-dimensional spacetime (time interval) is a pair (I, e) where $I \subseteq \mathbb{R}$ is an open interval and $e \in \Omega^1(I)$ is a non-degenerate 1-form encoding metric $g = e \otimes e$ and orientation of I.

(b) *Morphisms*: A $\mathbf{Loc}_1$-morphism $f : (I, e) \to (I', e')$ is an open embedding $f : I \to I'$ preserving the 1-forms, i.e. $f^*(e') = e$, where $f^*(e')$ denotes the pullback 1-form along f of e'.

$\triangle$

To conclude this section, we introduce some nomenclature that will be pivotal throughout the thesis.

Definition 1.2.9. (a) Let M, $N \in \mathbf{Loc}$ and let $f : M \to N$ be a $\mathbf{Loc}$-morphism. We say that f is a *Cauchy morphism* if $f(M)$ contains a Cauchy hypersurface of N.

(b) Let M_1, M_2, $N \in \mathbf{Loc}$ and let $f_i : M_i \to N$ be a $\mathbf{Loc}$-morphism for $i = 1, 2$. We say that f_1 and f_2 are *causally disjoint* if $f(M_1)$ and $f(M_2)$ are causally disjoint subsets of N.

(c) Let M_1, M_2, $N \in \mathbf{Loc}$ and let $f_i : M_i \to N$ be a $\mathbf{Loc}$-morphism for $i = 1, 2$. We say that f_1 and f_2 are *disjoint morphisms* if $f(M_1)$ and $f(M_2)$ are disjoint subsets of N. More generally, we will say that a tuple of $\mathbf{Loc}$-morphisms $\underline{f} = (f_1 : M_1 \to N, \dots, f_n : M_n \to N)$ is *disjoint* if the morphisms $f_1, \dots, f_n$ are mutually disjoint.

Remark 1.2.10. Notice that Definition 1.2.9 suggests two orthogonality relations (see Definition 1.1.6) on the category $\mathbf{Loc}$. The first is given by the collection of pairs with common target $\perp_c \subseteq \mathrm{Mor}\mathbf{Loc}\,{}_{\mathrm{t}}{\times}_{\mathrm{t}}\,\mathrm{Mor}\mathbf{Loc}$ that are causally disjoint. We denote the prefactorization multicategory (see Example 1.1.9) associated to the orthogonal category $(\mathbf{Loc}, \perp_c)$ with the symbol $\mathcal{P}_{\mathbf{Loc}^{\perp_c}}$. The second is given by the collection of pairs with common target $\perp_d \subseteq \mathrm{Mor}\mathbf{Loc}\,{}_{\mathrm{t}}{\times}_{\mathrm{t}}\,\mathrm{Mor}\mathbf{Loc}$ that are disjoint. We denote the prefactorization multicategory associated to the orthogonal category $(\mathbf{Loc}, \perp_d)$ with the symbol $\mathcal{P}_{\mathbf{Loc}^{\perp_d}}$. $\triangle$

1.2.2 *The Brunetti-Fredenhagen-Verch axioms*

In the last 50 years the study of quantum field theories on backgrounds different from Minkowski spacetime (*curved backgrounds*) has seen renovated efforts, most notably in the field of Cosmology. In this picture Algebraic Quantum Field Theory is an attempt to axiomatize the Heisenberg picture of quantum mechanics i.e. an attempt to encode the dependence of observables on spacetimes. The Brunetti-Fredenhagen-Verch axioms ([BFV03]) formalize this idea. In particular, an Algebraic Quantum Field Theory (AQFT) is a map $\mathfrak{A}$ satisfying the following physically motivated axioms:

Axiom (Algebraic structure of observables). The map $\mathfrak{A}$ assigns to every spacetime $M \in \mathbf{Loc}$ a complex algebra $\mathfrak{A}(M) \in \mathbf{Alg}_{\mathbb{C}}$ endowed with an involution $(-)^* : \mathfrak{A}(M) \to \mathfrak{A}(M)$, i.e. $\mathfrak{A}(M) \in {}^*\mathbf{Alg}_{\mathbb{C}}$ (Example 1.1.33).

The axiom states that the algebra of observables $\mathfrak{A}(M)$ of a quantum field theory is complex linear (which is not surprising to ask since reducing to the 1-dimensional case, i.e. quantum mechanics, the algebra of linear operators on a Hilbert space is a complex vector space) and comes endowed with an involution $* : \mathfrak{A}(M) \to \mathfrak{A}(M)$ of algebras which permits distinguishing the physical and non-physical observables.

In particular, a *physical observable* in the algebra $\mathfrak{A}(M)$ is an element $a \in \mathfrak{A}(M)$ such that $a^* = a$ (considering the example of a finite complex Hilbert space $\mathbb{C}^n$ and the involution on the linear maps $\mathbb{C}^n \to \mathbb{C}^n$ given by the conjugate transpose should make clear why this makes sense).

Axiom (Functoriality). The map $\mathfrak{A}$ assigns to every **Loc**-morphism $f : M \to N$ a map of $*$-algebras $\mathfrak{A}(f) : \mathfrak{A}(M) \to \mathfrak{A}(N)$. This assignment must respect identities and compositions. In particular, an Algebraic Quantum Field Theory is a functor $\mathfrak{A} : \mathbf{Loc} \to {}^*\mathbf{Alg}_{\mathbb{C}}$.

The Functoriality Axiom tells us that pushing forward observables along spacetime embeddings is compatible with compositions and identities.

The axiom we want to introduce next appears in the literature with different names such as *axiom of causal locality* or *Einstein causality axiom* and relies on the notion of $\perp_c$-commutativity for a functor $\mathfrak{A} : \mathbf{Loc} \to {}^*\mathbf{Alg}_{\mathbb{C}}$ (see Remark 1.2.10). Before describing the axiom we introduce the following definition, which generalizes $\perp_c$-commutativity to more generic orthogonal categories of spaces, since we will need it in what follows.

Definition 1.2.11. Let $\mathbf{Sp}^\perp = (\mathbf{Sp}, \perp)$ be an orthogonal category, let $(\mathbf{C}, \otimes, I)$ be a symmetric monoidal category and let $\mathbf{Alg}(\mathbf{C}) := \mathbf{Mon}(\mathbf{C})$ denote the category of monoids in $\mathbf{C}$.

A functor $\mathfrak{A} : \mathbf{Sp} \to \mathbf{Alg}(\mathbf{C})$ is called $\perp$-*commutative* if for every s_1, s_2, $s \in \mathbf{Sp}$, and for every $f_i : s_i \to s$, with $i = 1, 2$, such that $f_1 \perp f_2$ the diagram

$$\begin{tikzcd}
\mathfrak{A}(s_1) \otimes \mathfrak{A}(s_2) \arrow[r, "\mathfrak{A}(f_1)\otimes\mathfrak{A}(f_2)"] \arrow[d, "\mathfrak{A}(f_1)\otimes\mathfrak{A}(f_2)"'] & \mathfrak{A}(s) \otimes \mathfrak{A}(s) \arrow[d, "\mu_s^{\mathrm{op}}"] \\
\mathfrak{A}(s) \otimes \mathfrak{A}(s) \arrow[r, "\mu_s"'] & \mathfrak{A}(s)
\end{tikzcd} \tag{1.2.1}$$

in $\mathbf{C}$ commutes. Here μ_s denotes the multiplication on $\mathfrak{A}(s)$ and $\mu_s^{\mathrm{op}} := \mu_s \tau$ the opposite multiplication on $\mathfrak{A}(s)$, with τ the symmetric braiding of $\mathbf{C}$.

Axiom (Einstein causality). Given causally disjoint **Loc**-morphisms $f_1 : M_1 \to N$, $f_2 : M_2 \to N$ the functor $\mathfrak{A}$ is such that

$$\mu_M(\mathfrak{A}(f_1)a_1, \mathfrak{A}(f_2)a_2) = \mu_M(\mathfrak{A}(f_2)(a_2), \mathfrak{A}(f_1)(a_1))$$

where μ_M is the multiplication map of the associative algebra $\mathfrak{A}(M)$ and $a_i \in \mathfrak{A}(M_i)$ for $i = 1, 2$. In particular, $\mathfrak{A} : \mathbf{Loc} \to {}^*\mathbf{Alg}_{\mathbb{C}}$ is $\perp_c$-commutative (see Remark 1.2.10).

The Einstein causality axiom expresses the fact that observables coming from causally disjoint regions, i.e. that have no causal relationship, commute with each other. In particular, the axiom encodes the physical principle that information cannot travel faster than light.

There is one more optional axiom which is dynamical in nature and may or may not be considered. In particular, when enforced, the axiom implies that for each spacetime M the algebra $\mathfrak{A}(M)$ is determined by the value of $\mathfrak{A}$ on any causally convex open subset N of M containing a Cauchy hypersurface of M:

Axiom (Time-slice). For every Cauchy morphism $f : M \to N \in \operatorname{Mor}\mathbf{Loc}$ (see Definition 1.2.9), $\mathfrak{A}(f)$ is an isomorphism in ${}^*\mathbf{Alg}_{\mathbb{C}}$.

1.2.3 $\mathbf{C}$-valued AQFTs and the multicategorical perspective

In the previous section we have defined AQFTs to be functors $\mathfrak{A} : \mathbf{Loc} \to {}^*\mathbf{Alg}_{\mathbb{C}}$ satisfying the Einstein causality axiom (and optionally the time-slice axiom). Although this approach is quite general we notice the following insufficiencies:

1. There exist approaches to quantum field theory where the orthogonal category of globally hyperbolic Lorentzian spacetimes is replaced by intervals in the circle $\mathbb{S}^1$ or by Riemannian manifolds ([Sch99, Kaw15]).

2. More modern approaches to AQFT consider AQFTs valued in categories of monoids other than $\mathbf{Alg}_{\mathbb{C}}$, such as the category $\mathbf{Alg}(\mathbf{Ch}_{\mathbb{K}})$ of algebras in chain complexes ([FR12, BSW19b]) or, more generally, the category $\mathbf{Alg}(\mathbf{C})$ of algebras in a symmetric monoidal category $(\mathbf{C}, \otimes, I)$, see e.g. [BSS15, Yau19, BSW21].

These considerations motivate the following more general definition ([BSW21]):

Definition 1.2.12. Let $\mathbf{Sp}^{\perp} = (\mathbf{Sp}, \perp)$ be an orthogonal category and let $(\mathbf{C}, \otimes, I)$ be a bicomplete closed symmetric monoidal category.

A $\mathbf{C}$*-valued Algebraic Quantum Field Theory* on $\mathbf{Sp}^{\perp}$ is a $\perp$-commutative functor (see Definition 1.2.12) $\mathfrak{A} : \mathbf{Sp} \to \mathbf{Alg}(\mathbf{C})$.

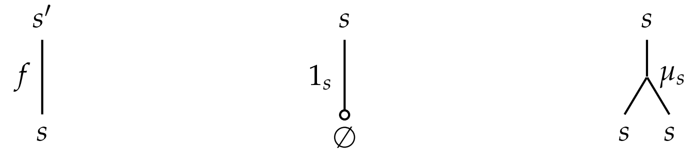
The collection of $\perp$-commutative functors $\mathfrak{A} : \mathbf{Sp} \to \mathbf{Alg}(\mathbf{C})$ and natural transformations between those form the category $\mathbf{AQFT}(\mathbf{Sp}^{\perp}) \subseteq [\mathbf{Sp}, \mathbf{Alg}(\mathbf{C})]$ of $\mathbf{C}$*-valued Algebraic Quantum Field Theories on* $\mathbf{Sp}^{\perp}$.

Remark 1.2.13. Notice that in Definition 1.2.12 we dropped the $*$-structure. The reason for this choice is twofold. Firstly, there are axiomatizations of quantum field theories, such as *Factorization Algebras*, where the sets of observables are not normally endowed with a $*$-algebra structure. Secondly, when suitable and possible, we will use the theory of involutive categories to add involutions to the picture. Notice for example that it is possible to endow the category $\mathbf{AQFT}(\mathbf{Sp}^{\perp})$ of $\mathbf{Vec}_{\mathbb{C}}$-valued algebraic quantum field theories with an involutive structure, $\overline{(-)} : \mathbf{AQFT}(\mathbf{Sp}^{\perp}) \to \mathbf{AQFT}(\mathbf{Sp}^{\perp})$. Concretely, the involution assigns to every $\mathfrak{A} \in \mathbf{AQFT}(\mathbf{Sp}^{\perp})$ the functor $\overline{\mathfrak{A}} : \mathbf{Sp} \to \mathbf{Alg}_{\mathbb{C}}$ that assigns to $s \in \mathbf{Sp}$ the algebra $(\overline{\mathfrak{A}}(s), \mu_{\overline{\mathfrak{A}}}, \eta_{\overline{\mathfrak{A}}})$ (see Example 1.1.31). It is easy to confirm that the category ${}^*\mathbf{AQFT}(\mathbf{Sp}^{\perp})$ of $*$-objects (Remark 1.1.33) agrees with the one from the literature [BFV03, BDFY15].

△

Although being rather straightforward, Definition 1.2.12 is not ideal to work with. In fact, $\perp$-commutativity is a property that a functor may or may not satisfy. Moreover, it is not clear what the properties of the category $\mathbf{AQFT}(\mathbf{Sp}^{\perp})$ are. Is it complete? Is it cocomplete? Some basic questions to which is difficult to give an immediate answer. Luckily, we will be able to interpret AQFTs in terms of multicategories and provide solutions to these questions. More precisely, we specify in the following example generators and relations (mimicking the axioms of an AQFT), similarly to what we have done for the associative multicategory As in Example 1.1.23, to obtain a multicategory $\mathcal{O}_{\mathbf{Sp}^{\perp}}$ whose algebras are precisely the AQFTs.

Example 1.2.14 (AQFT multicategory)**.** Let $\mathbf{Sp}^{\perp} = (\mathbf{Sp}, \perp)$ be an orthogonal category. As we mentioned in the previous discussion the goal of this example is to introduce generators and relations mimicking the axioms of an AQFT in order to obtain a multicategory $\mathcal{O}_{\mathbf{Sp}^{\perp}}$ whose category of algebras will be (equivalent to) that of AQFTs. Taking inspiration from Example 1.1.23 and looking at Definition 1.2.12 it is clear that we need as generators the morphisms of the category $\mathbf{Sp}$ as well as a multiplication and a unit:



(1.2.2)

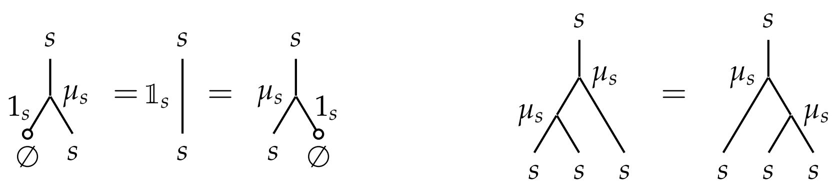
for every $s, s' \in \mathbf{Ob}(\mathbf{Sp})$ and $(f : s \to s') \in \operatorname{Mor}\mathbf{Sp}$.

After having determined the generators, we need to impose some relations, beginning with functoriality:

(1.2.3)

for every $s \in \mathbf{Ob}(\mathbf{Sp})$ and every pair of composable morphisms $f : s' \to s'', g : s \to s' \in \operatorname{Mor}\mathbf{Sp}$.

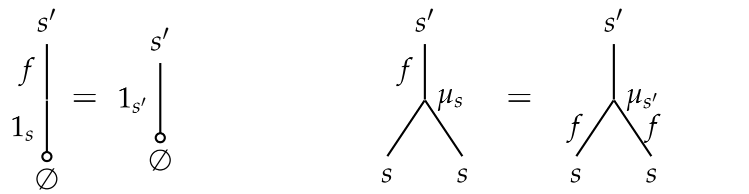
Next, as in the case of the associative multicategory we need to impose relations encoding the associativity and unitality of the algebras (see Example 1.1.23):



(1.2.4)

for every $s \in \mathbf{Ob}(\mathbf{Sp})$.

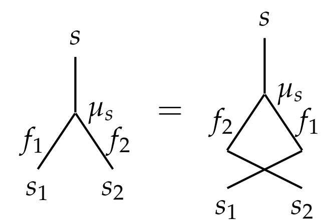
Moreover, we need to enforce a compatibility relation between the morphisms (push forwards) and the algebraic structure:



(1.2.5)

for every $f : s \to s' \in \operatorname{Mor}\mathbf{Sp}$.

Finally, we need a relation forcing the commutativity of pairs of observables coming from push forwards along orthogonal embeddings:



(1.2.6)

for every couple of orthogonal $\mathbf{Sp}$-morphisms $f_1 \perp f_2$.

The multicategory $\mathcal{O}_{\mathbf{Sp}^\perp}$ obtained from this set of generators and relations is called the *AQFT multicategory*. ▽

Although the description in Example 1.2.14 of the multicategory $\mathcal{O}_{\mathbf{Sp}^\perp}$ in terms of generators and relations is perfectly legit, it will be sometimes better to consider alternative (isomorphic) descriptions:

Theorem 1.2.15 ([BSW21]). *Let $\mathbf{Sp}^\perp = (\mathbf{Sp}, \perp)$ be an orthogonal category. The AQFT multicategory $\mathcal{O}_{\mathbf{Sp}^\perp} \in \mathbf{MULT}$ from Example 1.2.14 is (isomorphic to) the multicategory specified by the following data:*

(a) The collection of objects $\mathcal{O}_{\mathbf{Sp}_0^\perp} := \mathbf{Ob}(\mathbf{Sp})$;

(b) For every $s \in \mathcal{O}_{\mathbf{Sp}_0^\perp}$, $n \geq 0$, $\underline{s} = (s_1, \ldots, s_n) \in \mathcal{O}^n_{\mathbf{Sp}_0^\perp}$, the sets

$$\mathcal{O}_{\mathbf{Sp}^\perp}\big(\begin{smallmatrix} s \\ \underline{s} \end{smallmatrix}\big) = \Big(\Sigma_n \times \prod_{i=1}^n \mathbf{Sp}(s_i, s)\Big) \Big/ \sim_\perp \quad , \tag{1.2.7}$$

where $\mathbf{Sp}(s_i, s)$ are Hom-sets, and the equivalence relation is defined as follows: $(\sigma, \underline{f}) \sim_\perp (\sigma', \underline{f}')$ if and only if $\underline{f} = \underline{f}'$ and the right permutation $\sigma\sigma'^{-1} : \underline{f}\sigma^{-1} \to \underline{f}\sigma'^{-1}$ is generated by transpositions of adjacent orthogonal pairs.

(c) For all $t \in \mathcal{O}_{\mathbf{Sp}_0^\perp}$, $n \geq 1$, $\underline{a} \in \mathcal{O}^n_{\mathbf{Sp}_0^\perp}$, $m_i \geq 0$, $\underline{b}_i \in \mathcal{O}_0^{m_i}$, for $i = 1, \ldots, n$, the composition map $\gamma : \mathcal{O}_{\mathbf{Sp}^\perp}\big(\begin{smallmatrix} t \\ \underline{a} \end{smallmatrix}\big) \times \prod_{i=1}^n \mathcal{O}_{\mathbf{Sp}^\perp}\big(\begin{smallmatrix} a_i \\ \underline{b}_i \end{smallmatrix}\big) \to \mathcal{O}_{\mathbf{Sp}^\perp}\big(\begin{smallmatrix} t \\ \underline{b} \end{smallmatrix}\big)$:

$$\gamma\big([\sigma, \underline{f}], ([\sigma_1, \underline{g}_1], \ldots, [\sigma_n, \underline{g}_n])\big) = \big[\sigma(\sigma_1, \ldots, \sigma_n), \underline{f}(\underline{g}_1, \ldots, \underline{g}_n)\big] \quad ,$$

where $\sigma(\sigma_1, \ldots, \sigma_n) = \sigma\langle m_{\sigma^{-1}(1)}, \ldots, m_{\sigma^{-1}(n)}\rangle \ (\sigma_1 \oplus \cdots \oplus \sigma_n)$ is the product of the block permutation induced by σ and the sum permutation induced by the σ_i, and $\underline{f}(\underline{g}_1, \ldots, \underline{g}_n) = \big(f_1\, g_{11}, \ldots, f_1\, g_{1m_1}, \ldots, f_n\, g_{n1}, \ldots, f_n\, g_{nm_n}\big)$ is given by composition in the category $\mathbf{Sp}$.

(d) Units $[e, \mathrm{id}_s] \in \mathcal{O}_{\mathbf{Sp}^\perp}\big(\begin{smallmatrix} s \\ s \end{smallmatrix}\big)$, where $e \in \Sigma_1$ is the identity permutation and $\mathrm{id}_s : s \to s$ the identity morphism in $\mathbf{Sp}$.

(e) The permutation actions are $\mathcal{O}_{\mathbf{Sp}^\perp}(\sigma')([\sigma, \underline{f}]) = [\sigma\sigma', \underline{f}\sigma']$.

There is a third alternative (isomorphic) description of the $\mathcal{O}_{\mathbf{Sp}^\perp}$ multicategory in terms of the prefactorization algebra multicategory $\mathcal{P}_{\mathbf{Sp}^\perp}$ (Example 1.1.9) associated to $\mathbf{Sp}^\perp$ and the associative multicategory As (Example 1.1.5) using the Boardman-Vogt tensor product (Definition 1.1.28):

Theorem 1.2.16. *The multicategory $\mathcal{O}_{\mathbf{Sp}^\perp}$ is equivalent to the multicategory $\mathcal{P}_{\mathbf{Sp}^\perp} \otimes_{BV} \mathsf{As}$ ([BPSW21]).*

Before proving Theorem 1.2.16, we want to discuss what it means for two multicategories to be *equivalent*:

Remark 1.2.17. As in any 2-category, there is a notion of equivalence in the 2-category **MULT** of multicategories, i.e. two multicategories $\mathcal{O}$ and $\mathcal{P}$ are *equivalent* if there exist multifunctors $F : \mathcal{O} \to \mathcal{P}$ and $G : \mathcal{P} \to \mathcal{O}$ such that $G \circ F \cong \mathrm{Id}_{\mathcal{O}}$ and $F \circ G \cong \mathrm{Id}_{\mathcal{P}}$. Due to the analogy with categories it does not come as a surprise that $\mathcal{O}$ and $\mathcal{P}$ are equivalent multicategories if and only if there exists a *fully faithful essentially surjective* multifunctor $F : \mathcal{O} \to \mathcal{P}$, where essential surjectivity means that the functor $\pi_0(F)$ (see Remark 1.1.13) is essentially surjective and fully faithfulness means that the maps $F : \mathcal{O}\binom{t}{\underline{c}} \to \mathcal{P}\binom{t}{\underline{c}}$ are bijective for all $\underline{c}$ and t. △

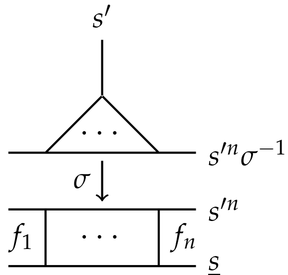
Proof of Theorem 1.2.16. To prove the result we introduce a fully faithful multifunctor $F : \mathcal{O}_{\mathbf{Sp}^\perp} \to \mathcal{P}_{\mathbf{Sp}^\perp} \otimes_{BV} \mathsf{As}$ given by the following data:

(a) It assigns to an object $s \in \mathcal{O}_{\mathbf{Sp}_0^\perp}$, the object $(s, *)$, where $* \in \mathsf{As}_0$ is the only element of As_0.

(b) For all $s \in \mathcal{O}_{\mathbf{Sp}_0^\perp}$, $n \geq 0$, $\underline{s} = (s_1, \ldots, s_n) \in \mathcal{O}^n_{\mathbf{Sp}_0^\perp}$ it assigns to an n-operation $[\sigma, \underline{f}] \in \mathcal{O}_{\mathbf{Sp}^\perp}\binom{s}{\underline{s}}$ the morphism $(s \otimes \sigma)((f_1 \otimes *), \ldots, (f_n \otimes *))$ (see Definition 1.1.28).

Being bijective on the underlying collections of objects, this functor is clearly essentially surjective. To see that it is also well defined and fully faithful it is convenient to think about the source and target multicategories in terms of generators and relations (see Examples 1.1.23, 1.1.9, 1.2.14). To check that the functor is well-defined, i.e. that the relations satisfied by the generators of $\mathcal{O}_{\mathbf{Sp}^\perp}$ are preserved by the functor, it suffices to notice the following: the relations in equation (1.2.4) are preserved because they are satisfied by the associative multicategory, the relations in equation (1.2.5) are preserved because of equation (1.1.24), while relation (1.2.6) can be seen to be preserved after direct inspection by using the permutation action defined by the BV-tensor product (see Definition 1.1.28). Fully faithfulness is a consequence of the fact that from the descriptions of $\mathcal{O}_{\mathbf{Sp}^\perp}$, $\mathcal{P}_{\mathbf{Sp}^\perp}$ and As in terms of generators and relations it emerges that the images of the generators of $\mathcal{O}_{\mathbf{Sp}^\perp}$ are in 1-to-1 correspondence with the generators of Boardman-Vogt tensor product of $\mathcal{P}_{\mathbf{Sp}^\perp}$ and As. □

Notice that the multicategory $\mathcal{O}_{\mathbf{Sp}^\perp}$ has a natural graphical interpretation. We may graphically visualize an element $[\sigma, \underline{f}] \in \mathcal{O}_{\mathbf{Sp}^\perp}\binom{s'}{\underline{s}}$ by



(1.2.8)

This picture should be read from bottom to top and it should be understood as the following ([BSW21]) :

(a) Apply the morphisms $\underline{f}$ to observables on $\underline{s} = (s_1, \dots, s_n)$;

(b) Permute the resulting observables on s'^n.

(c) Multiply the resulting observables on $s'^n\sigma^{-1}$ according to the order in which they appear.

As we mentioned earlier we can use the alternative descriptions of $\mathcal{O}_{\mathbf{Sp}^\perp}$ introduced in this section to obtain equivalent representations of the category $\mathbf{AQFT}(\mathbf{Sp}^\perp)$. More precisely:

Theorem 1.2.18. *Let $\mathbf{Sp}^\perp$ be an orthogonal category and let $\mathbf{C}$ be a bicomplete closed symmetric monoidal category. The category $\mathbf{AQFT}(\mathbf{Sp}^\perp)$ of $\mathbf{C}$-valued AQFTs on $\mathbf{Sp}^\perp$ is equivalent to the category $\mathbf{Alg}_{\mathcal{O}_{\mathbf{Sp}^\perp}}(\mathbf{C})$ of $\mathbf{C}$-valued $\mathcal{O}_{\mathbf{Sp}^\perp}$-algebras. Therefore, the category $\mathbf{AQFT}(\mathbf{Sp}^\perp)$ is bicomplete (Theorem 1.1.24).*

Remark 1.2.19. We will not discuss, even though it would be feasible (see e.g. [BSW21]), generalizations of the time-slice axiom (see Subsection 1.2.2) to algebraic quantum field theories defined on orthogonal categories other than $\mathbf{Loc}^{\perp_c}$. It is also straightforward to generalize the time-slice axiom to $\mathbf{C}$-valued algebraic quantum field theories on $\mathbf{Loc}^{\perp_c}$ for generic symmetric monoidal categories $(\mathbf{C}, \otimes, I)$. More precisely, we say that an algebraic quantum field theory $\mathfrak{A} : \mathbf{Loc} \to \mathbf{Alg}(\mathbf{C})$ is *Cauchy constant* if $\mathfrak{A}(f)$ is an $\mathbf{Alg}(\mathbf{C})$-isomorphism whenever f is a Cauchy morphism (see Definition 1.2.9). We denote by $\mathbf{AQFT}^c(\mathbf{Loc}^{\perp_c}) \subseteq \mathbf{AQFT}(\mathbf{Loc}^{\perp_c})$ the full subcategory of Cauchy constant AQFTs. △

We conclude this section by briefly showing an example of how multicategorical left Kan extension (see Theorem 1.1.26) can be used to extend algebraic quantum field theories defined on an orthogonal subcategory $\mathbf{Sp}^\perp_{nice} \subseteq \mathbf{Sp}^\perp$ of "nice" spacetimes to the whole orthogonal category $\mathbf{Sp}^\perp$.

Example 1.2.20. Let $\mathbf{Loc}_\diamond^{\perp_c} \subseteq \mathbf{Loc}^{\perp_c}$ be the orthogonal category of *diamond* globally hyperbolic Lorentzian manifolds, i.e the full subcategory of $\mathbf{Loc}$ whose objects are the manifolds $M \in \mathbf{Ob}(\mathbf{Loc})$ diffeomorphic to $\mathbb{R}^m$ for some $m \geq 0$, endowed with pullback orthogonality relation $\perp_c$ induced by the inclusion functor $j : \mathbf{Loc}_\diamond \to \mathbf{Loc}$ (see Definition 1.1.6) and denote by $J : \mathcal{O}_{\mathbf{Loc}_\diamond^{\perp_c}} \to \mathcal{O}_{\mathbf{Loc}^{\perp_c}}$ the obvious multifunctor induced by $j : \mathbf{Loc}_\diamond \to \mathbf{Loc}$ (see Example 1.1.14). The multicategorical left Kan extension $J_!$ along $J : \mathcal{O}_{\mathbf{Loc}_\diamond^{\perp_c}} \to \mathcal{O}_{\mathbf{Loc}^{\perp_c}}$ is a functor that takes an $\mathfrak{A} \in \mathbf{AQFT}(\mathbf{Loc}_\diamond^{\perp_c})$ defined on the category of diamond locally hyperbolic Lorentzian manifolds and extends it to an algebraic quantum field theory $J_!(\mathfrak{A}) \in \mathbf{AQFT}(\mathbf{Loc}^{\perp_c})$ defined on all spacetimes $M \in \mathbf{Ob}(\mathbf{Loc})$. The algebra $J_!(\mathfrak{A})(M)$ is a multicategorical refinement of the ordinary categorical left Kan extension leading to *Fredenhagen's universal algebra* [Fre90, Fre93, FRS92]. We refer to [BSW21] for simple criteria on M under which the multicategorical and traditional universal algebras coincide. $\triangledown$

2

AQFT VS PFA

The aim of this chapter is to compare from a model-independent perspective two axiomatizations of Quantum Field Theory on globally hyperbolic Lorentzian manifolds, namely the AQFT (see Definition 1.2.12) and the Factorization Algebra (FA) approach ([CG17]), drawing framework and results from our paper [BPS19].

As we have seen in Subsection 1.2.3, given a bicomplete closed symmetric monoidal category $\mathbf{C}$, an Algebraic Quantum Field Theory is a functor $\mathfrak{A} : \mathbf{Loc} \to \mathbf{Alg}(\mathbf{C})$ associating to each $M \in \mathbf{Loc}$ an *associative and unital algebra* $(\mathfrak{A}(M), \mu_M, \eta_M)$, where μ_M denotes the multiplication and η_M the unit map, satisfying some axioms, most notably the Einstein causality axiom, which states that observables coming from causally disjoint regions commute, or equivalently, that $\mathfrak{A}$ is $\perp_c$-commutative (see Remark 1.2.10 and Definition 1.2.12). A *prefactorization algebra*, instead, is a multifunctor $\mathfrak{F} : \mathcal{P}_{\mathbf{Loc}^{\perp_d}} \to \mathbf{C}$ (see Remark 1.2.10) that associates to each tuple of pairwise disjoint $\mathbf{Loc}$-morphisms (see Definition 1.2.9) $\underline{f} = (f_1 : M_1 \to N, \dots, f_n : M_n \to N)$ a *factorization product* $\mathfrak{F}(M_1) \otimes \cdots \otimes \mathfrak{F}(M_n) \to \mathfrak{F}(N)$ ([CG17]). In particular, in the factorization algebra approach, manifolds *do not* come equipped with a multiplication of observables $\mu_M : \mathfrak{F}(M) \otimes \mathfrak{F}(M) \to \mathfrak{F}(M)$ since there is no factorization product associated to the tuple $(\mathrm{id}_M : M \to M, \mathrm{id}_M : M \to M)$, being not disjoint.

Although these axiomatizations look (and are) different, we will see that they coincide (are isomorphic) when required to satisfy some further natural axioms, namely the *Cauchy constancy* and the *additivity* axioms. To be more precise, our main result states that there exists an equivalence between the category $\mathbf{AQFT}^{\mathrm{add},c}$ of *additive Cauchy constant* algebraic quantum field theories on $\mathbf{Loc}$ and the category $\mathbf{tPFA}^{\mathrm{add},c}$ of *additive Cauchy constant time-orderable* prefactorization algebras on $\mathbf{Loc}$. We will see that our result admits an interpretation in terms of multicategories, suitable for potential generalizations to higher quantum field theories such as gauge theories. In order to prove our main Theorem we structure the chapter as follows:

In Section 2.1 we introduce *additivity* and *Cauchy constancy* for Factorization Algebras and AQFTs. Broadly speaking, a prefactorization algebra $\mathfrak{F}$ (or an AQFT $\mathfrak{A}$) is additive if it is such that for each spacetime M the observables $\mathfrak{F}(M)$ (or $\mathfrak{A}(M)$) are generated by those coming from the *relatively compact* and *causally convex* open subsets of M. Cauchy constancy was defined in Remark 1.2.19 just in the case of AQFTs, but the concept extends straightforwardly to Factorization Algebras. More precisely, we say that a prefactorization algebra $\mathfrak{F}$ is Cauchy constant if $\mathfrak{F}(f)$ is an isomorphism for every Cauchy morphism f (see Definition 1.2.9).

In Section 2.2 we set the ground for our comparison Theorem by introducing a functor $\mathbb{A} : \mathbf{PFA}^{\mathrm{add},c} \to \mathbf{AQFT}^{\mathrm{add},c}$ sending an additive and Cauchy constant prefactorization algebra $\mathfrak{F}$ to an additive and Cauchy constant algebraic quantum field theory $\mathbb{A}[\mathfrak{F}]$. In particular, we endow each spacetime M with a multiplication $\mu_M : \mathfrak{F}(M) \otimes \mathfrak{F}(M) \to \mathfrak{F}(M)$ defined by taking any couple of disjoint Cauchy inclusions $\iota^M_{\underline{U}} = (\iota^M_{U_+} : U_+ \to M, \iota^M_{U_-} : U_- \to M)$ such that there exist a Cauchy surface Σ in M for which $U_\pm \subseteq I^\pm(\Sigma)$ and composing the associated *iso*morphism $\mathfrak{F}(M) \otimes \mathfrak{F}(M) \to \mathfrak{F}(U_+) \otimes \mathfrak{F}(U_-)$ with the factorization product $\mathfrak{F}(U_+) \otimes \mathfrak{F}(U_-) \to \mathfrak{F}(M)$.

In Section 2.3 we introduce the concept of *time-orderability* for tuples of **Loc**-morphisms $\underline{f} = (f_1 : M_1 \to N, \dots, f_n : M_n \to N)$ and define a functor $\mathbb{F} : \mathbf{AQFT} \to \mathbf{tPFA}$ sending each algebraic quantum field theory $\mathfrak{A}$ to a *time-orderable prefactorization algebra* $\mathbb{F}[\mathfrak{A}]$, i.e. a prefactorization algebra that has factorization products just for time-orderable tuples of **Loc**-morphisms. The need to restrict our attention to time-orderable tuples is based on the fact that, given an algebraic quantum field theory $\mathfrak{A}$, it is not possible (at least for us) to associate an equivariant factorization product to each tuple of disjoint **Loc**-morphisms $\underline{f} = (f_1 : M_1 \to N, \dots, f_n : M_n \to N)$, since there is *a priori* no canonical ordering of observables (see Section 2.3). Moreover, we prove that $\mathbb{F}$ restricts to a functor $\mathbb{F} : \mathbf{AQFT}^{\mathrm{add},c} \to \mathbf{tPFA}^{\mathrm{add},c}$.

In Section 2.4 we show that the functor $\mathbb{A} : \mathbf{PFA}^{\mathrm{add},c} \to \mathbf{AQFT}^{\mathrm{add},c}$ of Section 2.2 can be restricted to a functor $\mathbb{A} : \mathbf{tPFA}^{\mathrm{add},c} \to \mathbf{AQFT}^{\mathrm{add},c}$, which is inverse to the functor $\mathbb{F} : \mathbf{AQFT}^{\mathrm{add},c} \to \mathbf{tPFA}^{\mathrm{add},c}$ introduced in Section 2.3, proving in particular that $\mathbf{AQFT}^{\mathrm{add},c} \cong \mathbf{tPFA}^{\mathrm{add},c}$. To conclude we will use this isomorphism to endow the category $\mathbf{tPFA}^{\mathrm{add},c}$ with an involutive structure and we will apply our comparison result to the example of the free Klein-Gordon field $\mathfrak{A}_{\mathrm{KG}} \in \mathbf{AQFT}^{\mathrm{add},c}$.

Notice that [BPS19] is not the first effort toward obtaining such a comparison Theorem but is significantly more general than the previous discussion in [GR17]: we study a comparison in a model-independent framework and we study in detail the unitality, associativity and Einstein causality of the multiplications $\mu_M : \mathfrak{F}(M) \otimes \mathfrak{F}(M) \to \mathfrak{F}(M)$.

2.1 ADDITIVE AND CAUCHY CONSTANT PFAS AND AQFTS

A prefactorization algebra is a multifunctor $\mathcal{P}_{\mathbf{Sp}^\perp} \to \mathbf{C}$, where $\mathbf{Sp}^\perp = (\mathbf{Sp}, \perp)$ is usually an orthogonal category of topological spaces or Riemannian manifolds and $\mathbf{C}$ is a bicomplete closed monoidal symmetric category ([CG17]). In order to consider a context in which studying both Algebraic Quantum Field Theories and Factorization Algebras makes sense we will fix $\mathbf{Sp}$ to be the category **Loc** of globally hyperbolic Lorentzian manifolds (see Subsection 1.2.1). In what follows we will use interchangeably multicategorical or more explicit descriptions of the objects in play both for practical convenience as for the sake of clarity. In particular, we will not ascribe to a definitive picture but we will use this plurality of interpretations to make the concepts introduced along the way, hopefully, more transparent.

Remark 2.1.1. Form now on, when referring to the objects of a category $\mathbf{D}$, we will not always enforce the notation $d \in \mathbf{Ob}(\mathbf{D})$ and prefer the slimmer $d \in \mathbf{D}$. △

Definition 2.1.2. A *prefactorization algebra* $\mathfrak{F}$ on **Loc** with values in **C** is a multifunctor $\mathfrak{F} : \mathcal{P}_{\mathbf{Loc}^{\perp_d}} \to \mathbf{C}$, or more explicitly, a law given by the following data (see Remark 1.1.21):

(a) For each $M \in \mathbf{Loc}$, an object $\mathfrak{F}(M) \in \mathbf{C}$.

(b) For each tuple $\underline{f} = (f_1, \dots, f_n) : \underline{M} \to N$ of pairwise disjoint morphisms, a **C**-morphism $\mathfrak{F}(\underline{f}) : \bigotimes_{i=1}^n \mathfrak{F}(M_i) \to \mathfrak{F}(N)$ (called *factorization product*).

satisfying the following axioms:

(a) For every $n \geq 0$, $m_i \geq 0$, for every tuple of pairwise disjoint **Loc**-morphisms $\underline{f} = (f_1, \dots, f_n) : \underline{M} \to N$ and for every tuple of pairwise disjoint **Loc**-morphisms $\underline{g}_i = (g_{i1}, \dots, g_{im_i}) : \underline{L}_i \to M_i$ with $i = 1, \dots, n$, the diagram

$$\begin{tikzcd} \bigotimes\limits_{i=1}^{n} \bigotimes\limits_{j=1}^{m_i} \mathfrak{F}(L_{ij}) \arrow[r, "\otimes_i \mathfrak{F}(\underline{g}_i)"] \arrow[dr, "\mathfrak{F}(\underline{f}(\underline{g}_1, \dots, \underline{g}_n))"'] & \bigotimes\limits_{i=1}^{n} \mathfrak{F}(M_i) \arrow[d, "\mathfrak{F}(\underline{f})"] \\ & \mathfrak{F}(N) \end{tikzcd} \tag{2.1.1}$$

in **C** commutes, where $\underline{f}(\underline{g}_1, \dots, \underline{g}_n) := (f_1\, g_{11}, \dots, f_n\, g_{nm_n}) : (\underline{L}_1, \dots, \underline{L}_n) \to N$ is given by composition in **Loc**.

(b) For every $M \in \mathbf{Loc}$, $\mathfrak{F}(\mathrm{id}_M) = \mathrm{id}_{\mathfrak{F}(M)} : \mathfrak{F}(M) \to \mathfrak{F}(M)$.

(c) For every tuple of pairwise disjoint **Loc**-morphisms $\underline{f} = (f_1, \dots, f_n) : \underline{M} \to N$ and every permutation $\sigma \in \Sigma_n$, the diagram

$$\begin{tikzcd} \bigotimes\limits_{i=1}^{n} \mathfrak{F}(M_i) \arrow[r, "\mathfrak{F}(\underline{f})"] \arrow[d, "\text{permute}"'] & \mathfrak{F}(N) \\ \bigotimes\limits_{i=1}^{n} \mathfrak{F}(M_{\sigma(i)}) \arrow[ur, "\mathfrak{F}(\underline{f}\sigma)"'] & \end{tikzcd} \tag{2.1.2}$$

in **C** commutes, where $\underline{f}\sigma := (f_{\sigma(1)}, \dots, f_{\sigma(n)}) : \underline{M}\sigma \to N$ is given by right permutation.

A similar concept of prefactorization algebra on **Loc** appeared in [GR17].

Remark 2.1.3. We know from Remark 1.1.21 that the collection of multifunctors $\mathcal{P}_{\mathbf{Loc}^{\perp_d}} \to \mathbf{C}$ and multinatural transformations between those form a category $\mathbf{PFA} := \mathbf{Alg}_{\mathcal{P}_{\mathbf{Loc}^{\perp_d}}}(\mathbf{C})$. In the spirit of Definition 2.1.2, we want to give an explicit description of what a morphism of prefactorization algebras is. A morphism $\zeta : \mathfrak{F} \to \mathfrak{G}$ of prefactorization algebras is a family $\zeta_M : \mathfrak{F}(M) \to \mathfrak{G}(M)$ of **C**-morphisms, for all

$M \in \mathbf{Loc}$, that is compatible with the factorization products, i.e. for all $\underline{f} : \underline{M} \to N$ the following diagram (in $\mathbf{C}$) commutes:

$$\begin{tikzcd} \bigotimes_{i=1}^{n} \mathfrak{F}(M_i) \arrow[r, "\mathfrak{F}(\underline{f})"] \arrow[d, "\otimes_i \zeta_{M_i}"'] & \mathfrak{F}(N) \arrow[d, "\zeta_N"] \\ \bigotimes_{i=1}^{n} \mathfrak{G}(M_i) \arrow[r, "\mathfrak{G}(\underline{f})"'] & \mathfrak{G}(N) \end{tikzcd} \tag{2.1.3}$$

△

A factorization algebra is a prefactorization algebra that satisfies codescent for *Weiss covers* ([CG17]).

Definition 2.1.4. Let M be a topological space. We say that a collection of open subsets $\mathfrak{U}$ of M is a *Weiss cover* if for any finite set S of points of M there exist an open $U \in \mathfrak{U}$ covering S.

A prefactorization algebra $\mathfrak{F}$ is a *factorization algebra* if it is satisfies the codescent condition for all Weiss covers, i.e. if it is a cosheaf with respect to Weiss covers ([CG17]). More precisely, $\mathfrak{F}$ is a factorization algebra if for every spacetime $M \in \mathbf{Loc}$ and every Weiss cover $\{U_\alpha : \alpha \in A\}$ of M the canonical diagram

$$\coprod_{\substack{U,V \in \{U_\alpha : \alpha \in A\} \\ U \cap V \neq \emptyset}} \mathfrak{F}(U \cap V) \rightrightarrows \coprod_{U \in \mathbf{RC}_M} \mathfrak{F}(U) \longrightarrow \mathfrak{F}(M)$$

is a coequalizer in $\mathbf{C}$.

In our discussion we will consider prefactorization algebras satisfying a weaker property than codescent, namely *additivity*, and we will see that every factorization algebra is an additive prefactorization algebra. Additivity is defined considering a particular kind of Weiss cover of a globally hyperbolic Lorentzian manifold M, i.e. the one given by its relatively compact and causally convex open subsets.

Lemma 2.1.5. *Let $M \in \mathbf{Loc}$. We denote by $\mathbf{RC}_M$ the subcategory of $\mathbf{Loc}$ whose objects are the relatively compact and causally convex open subsets U of M (with the induced metric, orientation and time-orientation) and whose morphisms are given by subset inclusions $\iota_U^V : U \to V$. The set of objects $\mathbf{Ob}(\mathbf{RC}_M)$ of $\mathbf{RC}_M$ is directed. In particular, $\mathbf{Ob}(\mathbf{RC}_M)$ is a Weiss cover of M.*

Proof. Let $U_1, U_2 \in \mathbf{RC}_M$. We shall construct $U \in \mathbf{RC}_M$ such that $U_i \subseteq U$, for $i = 1, 2$. Since $K := \overline{U_1} \cup \overline{U_2}$ is compact, there exists a Cauchy surface Σ of M such that $K \subseteq I_M^-(\Sigma)$. We set $S := J_M^+(K) \cap J_M^-(\Sigma)$ and observe that this is a compact subset of M by [BGP07, Corollary A.5.4]. Using also [BGP07, Lemma A.5.12], it follows that $U := I_M^+(K) \cap I_M^-(S)$ belongs to $\mathbf{RC}_M$. By construction, U contains both U_1 and U_2. □

For each spacetime $M \in \mathbf{Loc}$ the category $\mathbf{RC}_M$ can be endowed with the pullback orthogonality relation $\perp_d$ induced by the inclusion $\mathbf{RC}_M \subseteq \mathbf{Loc}$ (see Definition 1.1.6). We can therefore consider the obvious embedding of multicategories

$\iota : \mathcal{P}_{\mathbf{RC}_M^{\perp_d}} \to \mathcal{P}_{\mathbf{Loc}^{\perp_d}}$ and use (1.1.16) to restrict any prefactorization algebra $\mathfrak{F} \in \mathbf{PFA}$ to a multifunctor $\mathfrak{F}_{|M} := \iota^*(\mathfrak{F}) \in \mathbf{Alg}_{\mathcal{P}_{\mathbf{RC}_M^{\perp_d}}}(\mathbf{C})$.

Definition 2.1.6. Let $\mathfrak{F}$ be a prefactorization algebra on $\mathbf{Loc}$. We say that $\mathfrak{F}$ is *additive* if for every spacetime $M \in \mathbf{Loc}$ the natural map

$$\operatorname{colim}\Big(\pi_0(\mathfrak{F}|_M) : \mathbf{RC}_M \to \mathbf{C}\Big) \longrightarrow \mathfrak{F}(M) \tag{2.1.4}$$

is a $\mathbf{C}$-isomorphism, where π_0 is the functor $\mathbf{MULT} \to \mathbf{CAT}$ from Remark 1.1.13. We denote by $\mathbf{PFA}^{\mathrm{add}} \subseteq \mathbf{PFA}$ the full subcategory of additive prefactorization algebras.

Remark 2.1.7. In this chapter we will only deal with $\mathbf{C}$-valued algebraic quantum field theories on the orthogonal category $\mathbf{Loc}^{\perp_c}$. Therefore, we will abuse notations writing $\mathbf{AQFT} := \mathbf{AQFT}(\mathbf{Loc}^{\perp_c})$ for its entire duration. △

Analogously, additivity can be defined for AQFTs.

Definition 2.1.8. Let $\mathfrak{A} \in \mathbf{AQFT}$ be an algebraic quantum field theory on $\mathbf{Loc}^{\perp_c}$ (see Definition 1.2.12). We say that $\mathfrak{A}$ is *additive* if for every spacetime $M \in \mathbf{Loc}$ the natural map

$$\operatorname{colim}\Big(\mathfrak{A}|_M : \mathbf{RC}_M \to \mathbf{Alg}(\mathbf{C})\Big) \longrightarrow \mathfrak{A}(M) \tag{2.1.5}$$

is an $\mathbf{Alg}(\mathbf{C})$-isomorphism. We denote by $\mathbf{AQFT}^{\mathrm{add}} \subseteq \mathbf{AQFT}$ the full subcategory of additive algebraic quantum field theories and we denote by $\mathbf{AQFT}^{\mathrm{add},c}$ the full subcategory of additive Cauchy constant algebraic quantum field theories (see Remark 1.2.19).

Remark 2.1.9. Notice that since $\mathbf{Ob}(\mathbf{RC}_M)$ is a directed set (see Lemma 2.1.5), the colimit in Definition 2.1.8

$$\operatorname{colim}\Big(\mathfrak{A}|_M : \mathbf{RC}_M \to \mathbf{Alg}(\mathbf{C})\Big) \tag{2.1.6}$$

can be computed in the underlying category $\mathbf{C}$, see e.g. [Fre17, Proposition 1.3.6].

Therefore, to check additivity of an algebraic quantum field theory $\mathfrak{A} \in \mathbf{AQFT}$, it is enough to consider the underlying functor $\mathfrak{A} : \mathbf{Loc} \to \mathbf{C}$ to the category $\mathbf{C}$, i.e. forgetting the algebra structures, and verify that the natural map $\operatorname{colim}\big(\mathfrak{A}|_M : \mathbf{RC}_M \to \mathbf{C}\big) \to \mathfrak{A}(M)$ is an isomorphism in $\mathbf{C}$.

As we mentioned in the introduction to this chapter, we can think of an additive prefactorization algebra (additive algebraic quantum field theory) to be a prefactorization algebra $\mathfrak{F}$ (an algebraic quantum field theory $\mathfrak{A}$) such that, for every spacetime $M \in \mathbf{Loc}$, the observables of $\mathfrak{F}(M)$ ($\mathfrak{A}(M)$) are generated by those of the relatively compact and causally convex open subsets of M. △

Proposition 2.1.10. *Every factorization algebra $\mathfrak{F}$ on* $\mathbf{Loc}$ *is an additive prefactorization algebra.*

Proof. Suppose that $\mathfrak{F}$ is a factorization algebra (see Definition 2.1.4), i.e. it satisfies a cosheaf condition with respect to all Weiss covers of every $M \in \mathbf{Loc}$. For every $M \in \mathbf{Loc}$, the cover defined by $\mathbf{RC}_M$ is a Weiss cover (see Lemma 2.1.5). The property of being a factorization algebra then implies that the canonical diagram

$$\coprod_{\substack{U,V \in \mathbf{RC}_M \\ U\cap V \neq \varnothing}} \mathfrak{F}(U\cap V) \rightrightarrows \coprod_{U \in \mathbf{RC}_M} \mathfrak{F}(U) \longrightarrow \mathfrak{F}(M) \tag{2.1.7}$$

is a coequalizer in $\mathbf{C}$. Our claim then follows by observing that the cocones of (2.1.4) are canonically identified with the cocones of (2.1.7). Indeed, any cocone $\{\alpha_U : \mathfrak{F}(U) \to Z\}$ of (2.1.4) defines a cocone of (2.1.7) because $U \cap V \in \mathbf{RC}_M$ (whenever nonempty) and hence the diagram

$$\begin{tikzcd} & \mathfrak{F}(U) \arrow[dr, "\alpha_U"] & \\ \mathfrak{F}(U\cap V) \arrow[ur, "\mathfrak{F}(\iota^U_{U\cap V})"] \arrow[rr, "\alpha_{U\cap V}"] \arrow[dr, "\mathfrak{F}(\iota^V_{U\cap V})"'] & & Z \\ & \mathfrak{F}(V) \arrow[ur, "\alpha_V"'] & \end{tikzcd} \tag{2.1.8a}$$

in $\mathbf{C}$ commutes. Vice versa, any cocone $\{\alpha_U : \mathfrak{F}(U) \to Z\}$ of (2.1.7) defines a cocone of (2.1.4) because $U \cap V = U$, for all $U \subseteq V$, and hence the diagram

$$\begin{tikzcd} & \mathfrak{F}(U) \arrow[dr, "\alpha_U"] \arrow[dd, "\mathfrak{F}(\iota^V_U)"] & \\ \mathfrak{F}(U\cap V) \arrow[ur, equal] \arrow[dr, "\mathfrak{F}(\iota^V_{U\cap V})"'] & & Z \\ & \mathfrak{F}(V) \arrow[ur, "\alpha_V"'] & \end{tikzcd} \tag{2.1.8b}$$

in $\mathbf{C}$ commutes. □

To conclude this section we recall a *time-slice* axiom for prefactorization algebras (see Remark 1.2.19 for a comparison).

Definition 2.1.11. Let $\mathfrak{F}$ be a prefactorization algebra on $\mathbf{Loc}$. We say that $\mathfrak{F}$ is *Cauchy constant* if $\mathfrak{F}(f)$ is a $\mathbf{C}$-isomorphism for every Cauchy morphism $f \in \mathrm{Mor}\mathbf{Loc}$. We denote by $\mathbf{PFA}^c \subseteq \mathbf{PFA}$ the full subcategory of Cauchy constant prefactorization algebras on $\mathbf{Loc}$ and we denote by $\mathbf{PFA}^{\mathrm{add},c} \subseteq \mathbf{PFA}$ the full subcategory of Cauchy constant and additive prefactorization algebras on $\mathbf{Loc}$.

2.2 FROM PFA TO AQFT

The aim of this section is to define a functor $\mathbb{A} : \mathbf{PFA}^{\mathrm{add},c} \to \mathbf{AQFT}^{\mathrm{add},c}$.

To achieve this goal we will follow three steps and dedicate a whole subsection to each of those. More precisely, in Subsection 2.2.1 we show that for every $\mathfrak{F} \in \mathbf{PFA}^c$ and for every $M \in \mathbf{Loc}$, the object $\mathfrak{F}(M)$ can be endowed with an associative and unital algebra structure (notice that we are not asking $\mathfrak{F}$ to be additive so far). In Subsection 2.2.2 we rely on additivity and Cauchy constancy to show that for every $\mathfrak{F} \in \mathbf{PFA}^{\mathrm{add},c}$ the algebras defined in the first step are compatible with the $\mathbf{C}$-morphisms $\mathfrak{F}(f)$ for every $f \in \mathrm{Mor}\mathbf{Loc}$, i.e. we define a functor $\mathbb{A} : \mathbf{PFA}^{\mathrm{add},c} \to [\mathbf{Loc}, \mathbf{Alg}(\mathbf{C})]$. In Subsection 2.2.3 we carefully check that the image of $\mathbb{A}$ is actually contained in the category $\mathbf{AQFT}^{\mathrm{add},c}$ by proving that $\mathbb{A}[\mathfrak{F}]$ satisfies the Einstein causality axiom for every $\mathfrak{F} \in \mathbf{PFA}^{\mathrm{add},c}$.

2.2.1 *Algebraic structure*

In this subsection we will assume every prefactorization algebra $\mathfrak{F}$ to be Cauchy constant.

We begin the first step toward the construction of the aforementioned functor $\mathbb{A} : \mathbf{PFA}^{\mathrm{add},c} \to \mathbf{AQFT}^{\mathrm{add},c}$ by recalling the intuition behind the statement that for each spacetime $M \in \mathbf{Loc}$ the object $\mathfrak{F}(M) \in \mathbf{C}$ carries an algebra structure. Suppose there exist causally convex open subsets $U_+, U_- \subseteq M$ and a Cauchy hypersurface $\Sigma \subseteq M$ such that U_+ and U_- are contained in the chronological future and in the chronological past of Σ respectively, i.e. $U_\pm \subseteq I^\pm(\Sigma)$, and that the inclusions $\iota^M_{U_\pm} : U_\pm \xrightarrow{c} M$ are Cauchy morphisms. We can then compose the factorization product $\mathfrak{F}(\iota^M_{\underline{U}})$ associated to the tuple of disjoint morphisms $\iota^M_{\underline{U}} = (\iota^M_{U_+}, \iota^M_{U_-})$ with the inverse of the $\mathbf{C}$-isomorphism $\mathfrak{F}(\iota_{U_+}) \otimes \mathfrak{F}(\iota_{U_-})$ (notice that we are using Cauchy constancy) to obtain the map:

$$\begin{tikzcd}
\mathfrak{F}(M) \otimes \mathfrak{F}(M) \arrow[rr, "\mu_M"] & & \mathfrak{F}(M) \\
& \mathfrak{F}(U_+) \otimes \mathfrak{F}(U_-) \arrow[ul, "\mathfrak{F}(\iota^M_{U_+}) \otimes \mathfrak{F}(\iota^M_{U_-})", "\cong"'] \arrow[ur, "\mathfrak{F}(\iota^M_{\underline{U}})"'] &
\end{tikzcd} \tag{2.2.1}$$

It is not clear, however, whether different choices of U_+ and U_- lead to the same multiplication μ_M. We claim that this is actually the case and to prove it we proceed in the following way:

1. We collect all possible choices in a category $\mathbf{P}_M$ (see Definition 2.2.1).
2. We prove that $\mathbf{P}_M$ is a non-empty and connected category (see Lemma 2.2.2).
3. We prove that any two objects in $\mathbf{P}_M$ connected by an arrow lead to the same multiplication μ_M of equation (2.2.1), proving the claim (see Corollary 2.2.3).

Definition 2.2.1. Let $M \in \mathbf{Loc}$. We denote by $\mathbf{P}_M$ the category given by the following data:

(a) The collection of objects consists of tuples of pairwise disjoint morphisms $\iota^M_{\underline{U}} = (\iota^M_{U_+}, \iota^M_{U_-}) : \underline{U} \to M$ where U_+, U_- are causally convex open subsets of M such that there exists a Cauchy hypersurface Σ of M with $U_\pm \subseteq I^\pm_M(\Sigma)$ and such that $\iota^M_{U_\pm} : U_\pm \xrightarrow{c} M$ are Cauchy morphisms.

(b) A a unique morphism $(\iota^M_{\underline{U}} : \underline{U} \to M) \to (\iota^M_{\underline{V}} : \underline{V} \to M)$ if and only if $U_\pm \subseteq V_\pm$.

Lemma 2.2.2. *For every $M \in \mathbf{Loc}$, the category $\mathbf{P}_M$ is non-empty and connected.*

Proof. Non-empty: Choose any Cauchy surface Σ of M and define $\Sigma_\pm := I^\pm_M(\Sigma)$. Then $\iota^M_{\underline{\Sigma}} = (\iota^M_{\Sigma_+}, \iota^M_{\Sigma_-}) : \underline{\Sigma} \to M$ defines an object in $\mathbf{P}_M$.

Connected: We have to prove that there exists a zig-zag of morphisms in $\mathbf{P}_M$ between every pair of objects $\iota^M_{\underline{U}} : \underline{U} \to M$ and $\iota^M_{\underline{V}} : \underline{V} \to M$. For every object $\iota^M_{\underline{U}} : \underline{U} \to M$ in $\mathbf{P}_M$, there exists by hypothesis a Cauchy surface Σ of M such that $U_\pm \subseteq \Sigma_\pm := I^\pm_M(\Sigma)$. Hence, there exists a morphism $(\iota^M_{\underline{U}} : \underline{U} \to M) \to (\iota^M_{\underline{\Sigma}} : \underline{\Sigma} \to M)$. As a consequence, our original problem reduces to finding a zig-zag of morphisms in $\mathbf{P}_M$ between $\iota^M_{\underline{\Sigma}} : \underline{\Sigma} \to M$ and $\iota^M_{\underline{\Sigma}'} : \underline{\Sigma}' \to M$, for any two Cauchy surfaces Σ, Σ' of M. To exhibit such a zig-zag, let us introduce $\widetilde{U}_+ := \Sigma_+ \cap \Sigma'_+$ and $\widetilde{U}_- := \Sigma_- \cap \Sigma'_-$. If we could prove that $\iota^M_{\widetilde{U}_\pm} : \widetilde{U}_\pm \xrightarrow{c} M$ are Cauchy morphisms, then

$$\big(\iota^M_{\underline{\Sigma}} : \underline{\Sigma} \to M\big) \longleftarrow \big(\iota^M_{\underline{\widetilde{U}}} : \underline{\widetilde{U}} \to M\big) \longrightarrow \big(\iota^M_{\underline{\Sigma}'} : \underline{\Sigma}' \to M\big) \tag{2.2.2}$$

would provide a zig-zag that proves connectedness of $\mathbf{P}_M$.

It remains to show that $\widetilde{U}_+ = \Sigma_+ \cap \Sigma'_+ = I^+_M(\Sigma) \cap I^+_M(\Sigma') \subseteq M$ contains a Cauchy surface of M. (A similar argument shows that $\widetilde{U}_- \subseteq M$ also contains a Cauchy surface of M.) Because Σ, Σ' are by hypothesis Cauchy surfaces of M, there exists a Cauchy surface $\Sigma_1 \subset I^+_M(\Sigma)$ of M in the future of Σ and a Cauchy surface $\Sigma'_1 \subset I^+_M(\Sigma')$ of M in the future of Σ'. We define the subset

$$\widetilde{\Sigma} := \big(\Sigma_1 \cap J^+_M(\Sigma'_1)\big) \cup \big(J^+_M(\Sigma_1) \cap \Sigma'_1\big) \subset \widetilde{U}_+ \subseteq M \tag{2.2.3}$$

and claim that $\widetilde{\Sigma}$ is a Cauchy surface of M. To prove the last statement, consider any inextensible time-like curve $\gamma : I \to M$, which we may assume without loss of generality to be future directed. (If γ would be past directed, then change the orientation of the interval I.) Because Σ_1 and Σ'_1 are Cauchy surfaces of M, there exist *unique* $t, t' \in I$ such that $\gamma(t) \in \Sigma_1$ and $\gamma(t') \in \Sigma'_1$. If $t \geq t'$, then $\gamma(t) \in \Sigma_1 \cap J^+_M(\Sigma'_1) \subseteq \widetilde{\Sigma}$, and if $t' \geq t$, then $\gamma(t') \in J^+_M(\Sigma_1) \cap \Sigma'_1 \subseteq \widetilde{\Sigma}$. Hence, γ meets $\widetilde{\Sigma} \subset M$ at least once. Multiple intersections are excluded by the definition of $\widetilde{\Sigma}$ in (2.2.3) and the fact that both Σ_1 and Σ'_1 are Cauchy surfaces of M. □

Corollary 2.2.3. *For every $M \in \mathbf{Loc}$, the multiplication map μ_M in* (2.2.1) *does not depend on the choice of object $\iota^M_{\underline{U}} : \underline{U} \to M$ in $\mathbf{P}_M$.*

Proof. By Lemma 2.2.2, it is sufficient to prove that $\iota^M_{\underline{U}} : \underline{U} \to M$ and $\iota^M_{\underline{V}} : \underline{V} \to M$ define the same multiplication if $U_+ \subseteq V_+$ and $U_- \subseteq V_-$. This is a consequence of the commutative diagram

$$\begin{tikzcd}
 & \mathfrak{F}(V_+) \otimes \mathfrak{F}(V_-) \arrow[dl, "\mathfrak{F}(\iota^M_{V_+}) \otimes \mathfrak{F}(\iota^M_{V_-})"', "\cong"] \arrow[dr, "\mathfrak{F}(\iota^M_{\underline{V}})"] & \\
\mathfrak{F}(M) \otimes \mathfrak{F}(M) & & \mathfrak{F}(M) \\
 & \mathfrak{F}(U_+) \otimes \mathfrak{F}(U_-) \arrow[ul, "\mathfrak{F}(\iota^M_{U_+}) \otimes \mathfrak{F}(\iota^M_{U_-})", "\cong"'] \arrow[ur, "\mathfrak{F}(\iota^M_{\underline{U}})"'] \arrow[uu, "\mathfrak{F}(\iota^{V_+}_{U_+}) \otimes \mathfrak{F}(\iota^{V_-}_{U_-})"] &
\end{tikzcd} \tag{2.2.4}$$

where one also uses the composition properties (2.1.1) of prefactorization algebras. □

Equation (2.2.1) and Corollary 2.2.3 endow, for all Cauchy constant prefactorization algebra $\mathfrak{F}$ and all $M \in \mathbf{Loc}$, the object $\mathfrak{F}(M)$ with a multiplication μ_M. To obtain an associative and unital algebra structure on $\mathfrak{F}(M)$ though, we have to define a unit $\eta_M : I \to \mathfrak{F}(M)$. The natural choice for such a map is the distinguished object $\eta_M = \mathfrak{F}(*_M)$, where $*_M$ is the 0-operation in $\mathcal{P}_{\mathbf{Loc}^{\perp_d}}\binom{M}{\emptyset}$.

Proposition 2.2.4. *Let $\mathfrak{F} \in \mathbf{PFA}^c$ and let $M \in \mathbf{Loc}$. The object $\mathfrak{F}(M) \in \mathbf{C}$ can be endowed with an associative an unital algebra structure $(\mathfrak{F}(M), \mu_M, \eta_M)$ where $\mu_M : \mathfrak{F}(M) \otimes \mathfrak{F}(M) \to \mathfrak{F}(M)$ is given by* (2.2.1) *and the unit $\eta_M : I \to \mathfrak{F}(M)$, where I denotes the unit of the monoidal category $\mathbf{C}$, is given by evaluating $\mathfrak{F}$ on the unique 0-operation $\emptyset \to M \in \mathcal{P}_{\mathbf{Loc}^{\perp_d}}\binom{M}{\emptyset}$.*

Proof. To prove that the multiplication μ_M is associative, we consider two Cauchy surfaces Σ_0, Σ_1 of M such that $\Sigma_1 \subset I^+_M(\Sigma_0)$, i.e. Σ_1 is in the future of Σ_0. Using the independence result from Corollary 2.2.3 and the composition properties of prefactorization algebras, one easily confirms that $\mu_M\,(\mathrm{id} \otimes \mu_M)$ is the upper path and $\mu_M\,(\mu_M \otimes \mathrm{id})$ the lower path from $\mathfrak{F}(M)^{\otimes 3}$ to $\mathfrak{F}(M)$ in the commutative diagram

$$\begin{tikzcd}
\mathfrak{F}(\Sigma_{1+}) \otimes \mathfrak{F}(\Sigma_{1-} \cap \Sigma_{0+}) \otimes \mathfrak{F}(\Sigma_{0-}) \arrow[r, "\mathrm{id}\otimes\mathfrak{F}(\iota^{\Sigma_{1-}}_{\Sigma_{1-}\cap\Sigma_{0+}},\iota^{\Sigma_{1-}}_{\Sigma_{0-}})"] \arrow[d, "\cong", "\mathfrak{F}(\iota^M_{\Sigma_{1+}})\otimes\mathfrak{F}(\iota^M_{\Sigma_{1-}\cap\Sigma_{0+}})\otimes\mathfrak{F}(\iota^M_{\Sigma_{0-}})"'] & \mathfrak{F}(\Sigma_{1+}) \otimes \mathfrak{F}(\Sigma_{1-}) \arrow[d, "\mathfrak{F}(\iota^M_{\Sigma_{1+}},\iota^M_{\Sigma_{1-}})"] \\
\mathfrak{F}(M) \otimes \mathfrak{F}(M) \otimes \mathfrak{F}(M) & \mathfrak{F}(M) \\
\mathfrak{F}(\Sigma_{1+}) \otimes \mathfrak{F}(\Sigma_{1-} \cap \Sigma_{0+}) \otimes \mathfrak{F}(\Sigma_{0-}) \arrow[r, "\mathfrak{F}(\iota^{\Sigma_{0+}}_{\Sigma_{1+}},\iota^{\Sigma_{0+}}_{\Sigma_{1-}\cap\Sigma_{0+}})\otimes\mathrm{id}"'] \arrow[u, "\cong"', "\mathfrak{F}(\iota^M_{\Sigma_{1+}})\otimes\mathfrak{F}(\iota^M_{\Sigma_{1-}\cap\Sigma_{0+}})\otimes\mathfrak{F}(\iota^M_{\Sigma_{0-}})"] & \mathfrak{F}(\Sigma_{0+}) \otimes \mathfrak{F}(\Sigma_{0-}) \arrow[u, "\mathfrak{F}(\iota^M_{\Sigma_{0+}},\iota^M_{\Sigma_{0-}})"']
\end{tikzcd} \tag{2.2.5}$$

where as before we denote by $\Sigma_\pm := I^\pm_M(\Sigma) \subseteq M$ the chronological future/past of a Cauchy surface Σ of M. Unitality of the product follows immediately from the fact that there exists a unique morphism $\emptyset \to M$ for each $M \in \mathbf{Loc}$ and the composition properties (2.1.1) of prefactorization algebras.

$\square$

2.2.2 *Preservation of algebraic structures*

In this subsection we prove that for all $f \in \mathrm{Mor}\mathbf{Loc}$ and $\mathfrak{F} \in \mathbf{PFA}^{\mathrm{add},c}$ the $\mathbf{C}$-morphisms $\mathfrak{F}(f)$ are $\mathbf{Alg}(\mathbf{C})$-morphisms with respect to the algebraic structures defined by Proposition 2.2.4. In particular, we prove the existence of a functor $\mathbb{A} : \mathbf{PFA}^{\mathrm{add},c} \to [\mathbf{Loc}, \mathbf{Alg}(\mathbf{C})]$. We will proceed in the following way:

(a) We show that given a Cauchy constant prefactorization algebra $\mathfrak{F}$, a $\mathbf{Loc}$-morphism $f : M \to N$ with relatively compact image $f(M) \subseteq N$ is an algebra morphism when $\mathfrak{F}(M)$ and $\mathfrak{F}(N)$ are endowed with the algebra structure defined in Proposition 2.2.4.

(b) Next, we show that, whenever $\mathfrak{F}$ is also additive, $\mathfrak{F}(f)$ is an algebra morphism for every $\mathbf{Loc}$-morphism f. To do so we leverage item (a), the fact that in a bicomplete closed symmetric monoidal category $(\mathbf{C}, \otimes, I)$ the monoidal product $\otimes$ commutes with colimits and that $\mathbf{Ob}(\mathbf{RC}_M)$ is a directed set for every $M \in \mathbf{Loc}$.

(c) Finally, we prove the existence of the aforementioned functor

$$\mathbb{A} : \mathbf{PFA}^{\mathrm{add},c} \to [\mathbf{Loc}, \mathbf{Alg}(\mathbf{C})] \quad .$$

It is important to notice that while additivity did not play any role in Subsection 2.2.1 it is of fundamental importance in this subsection.

Lemma 2.2.5. *Let $\mathfrak{F} \in \mathbf{PFA}^c$ be a Cauchy constant (and* not *necessarily additive) prefactorization algebra and let $f : M \to N$ be a* **Loc***-morphism with relatively compact image $f(M) \subseteq N$. Then, endowing $\mathfrak{F}(M)$ and $\mathfrak{F}(N)$ with the algebraic structure from Proposition 2.2.4, makes $\mathfrak{F}(f) : \mathfrak{F}(M) \to \mathfrak{F}(N)$ an $\mathbf{Alg}(\mathbf{C})$-morphism, i.e. $\mu_N\, (\mathfrak{F}(f) \otimes \mathfrak{F}(f)) = \mathfrak{F}(f)\, \mu_M$ and $\eta_N = \mathfrak{F}(f)\, \eta_M$.*

Proof. The units are clearly preserved for every **Loc**-morphism $f : M \to N$ because composing the unique empty tuple $\varnothing \to M$ with $f : M \to N$ yields the unique empty tuple $\varnothing \to N$.

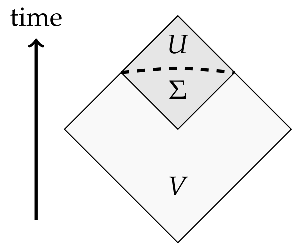
Let us focus now on the multiplications. Because $f(M) \subseteq N$ is by hypothesis relatively compact, its closure $\overline{f(M)} \subseteq N$ is compact. Let us take any Cauchy surface Σ of M and note that $\overline{f(\Sigma)} \subseteq N$ is a compact subset. Using further that $f(M) \subseteq N$ is causally convex and that the causality relation induced by time-like curves is open (cf. [ONe83, Lemma 14.3]), it follows that $\overline{f(\Sigma)} \subseteq N$ is achronal, i.e. every time-like curve in N meets this subset at most once. By [BS06, Theorem 3.8], there exists a Cauchy surface $\widetilde{\Sigma}$ of N such that $f(\Sigma) \subseteq \widetilde{\Sigma}$.

Using the Cauchy surfaces constructed above, we can define the multiplication μ_M in terms of $\Sigma_\pm := I^\pm_M(\Sigma)$ and the multiplication μ_N in terms of $\widetilde{\Sigma}_\pm := I^\pm_N(\widetilde{\Sigma})$, cf. (2.2.1). By construction, $f : M \to N$ restricts to **Loc**-morphisms $f^{\widetilde{\Sigma}_\pm}_{\Sigma_\pm} : \Sigma_\pm \to \widetilde{\Sigma}_\pm$. Our claim that $\mathfrak{F}(f) : \mathfrak{F}(M) \to \mathfrak{F}(N)$ preserves the multiplications then follows by observing that the diagram

$$\begin{tikzcd}
\mathfrak{F}(M) \otimes \mathfrak{F}(M) \arrow[d, "\mathfrak{F}(f)\otimes\mathfrak{F}(f)"'] & \mathfrak{F}(\Sigma_+) \otimes \mathfrak{F}(\Sigma_-) \arrow[l, "\mathfrak{F}(\iota^M_{\Sigma_+})\otimes\mathfrak{F}(\iota^M_{\Sigma_-})"', "\cong"] \arrow[r, "\mathfrak{F}(\iota^M_{\Sigma})"] \arrow[d, "\mathfrak{F}(f^{\widetilde{\Sigma}_+}_{\Sigma_+})\otimes\mathfrak{F}(f^{\widetilde{\Sigma}_-}_{\Sigma_-})"] & \mathfrak{F}(M) \arrow[d, "\mathfrak{F}(f)"] \\
\mathfrak{F}(N) \otimes \mathfrak{F}(N) & \mathfrak{F}(\widetilde{\Sigma}_+) \otimes \mathfrak{F}(\widetilde{\Sigma}_-) \arrow[l, "\cong", "\mathfrak{F}(\iota^N_{\widetilde{\Sigma}_+})\otimes\mathfrak{F}(\iota^N_{\widetilde{\Sigma}_-})"'] \arrow[r, "\mathfrak{F}(\iota^N_{\widetilde{\Sigma}})"'] & \mathfrak{F}(N)
\end{tikzcd} \tag{2.2.6}$$

commutes. □

Remark 2.2.6. In the proof of Lemma 2.2.5 we relied on the fact that given a spacetime M, a Cauchy surface Σ of M and a **Loc**-morphism $f : M \to N$ such that the image $f(M)$ is a relatively compact subset of N, there exists a Cauchy hypersurface $\widetilde{\Sigma} \subseteq N$ such that $f(\Sigma) \subseteq \widetilde{\Sigma}$. If $f(M) \subseteq N$ fails to be relatively compact such a $\widetilde{\Sigma}$ might not exist as the example of the inclusion $\iota^V_U : U \to V$ of the following diamond regions in 2-dimensional Minkowski spacetime demonstrates (notice that U is not relatively compact in V):



(2.2.7)

The set $f(\Sigma) \subseteq V$ is clearly not contained in any Cauchy hypersurface $\widetilde{\Sigma}$ of V. Therefore, given a Cauchy constant prefactorization algebra $\mathfrak{F}$ it might be the case that $\mathfrak{F}(\iota_U^V)$ is not an algebra morphism with respect to the algebra structures on $\mathfrak{F}(U)$ and $\mathfrak{F}(V)$ defined by Proposition 2.2.4. As promised, we will see that this insufficiency is overcome by considering prefactorization algebras that are both additive and Cauchy constant. $\triangle$

Proposition 2.2.7. *Let $\mathfrak{F} \in \mathbf{PFA}^{\mathrm{add},c}$ be an additive Cauchy constant prefactorization algebra on* **Loc**, *let $f : M \to N$ be a* **Loc***-morphism and endow $\mathfrak{F}(M)$ and $\mathfrak{F}(N)$ with the algebra structure from Proposition 2.2.4. Then, the* **C***-morphism $\mathfrak{F}(f) : \mathfrak{F}(M) \to \mathfrak{F}(N)$ is an* $\mathbf{Alg}(\mathbf{C})$*-morphism.*

Proof. We already observed in the proof of Lemma 2.2.5 that $\mathfrak{F}(f)$ preserves the units.

For the multiplications we have to prove that $\mu_N\,(\mathfrak{F}(f) \otimes \mathfrak{F}(f)) = \mathfrak{F}(f)\,\mu_M$ as **C**-morphisms from $\mathfrak{F}(M) \otimes \mathfrak{F}(M)$ to $\mathfrak{F}(N)$. Because $\mathfrak{F}$ is by hypothesis additive (cf. Definition 2.1.6) and the monoidal product $\otimes$ in a cocomplete closed symmetric monoidal category preserves colimits in both entries, it follows that

$$\mathfrak{F}(M) \otimes \mathfrak{F}(M) \;\cong\; \mathrm{colim}_{U,V \in \mathbf{RC}_M}\big(\mathfrak{F}(U) \otimes \mathfrak{F}(V)\big) \;\cong\; \mathrm{colim}_{U \in \mathbf{RC}_M}\big(\mathfrak{F}(U) \otimes \mathfrak{F}(U)\big) \quad , \tag{2.2.8}$$

where in the last step we also used that $\mathbf{RC}_M$ is directed by Lemma 2.1.5. For every $U \in \mathbf{RC}_M$, consider the diagram

$$\begin{tikzcd}
\mathfrak{F}(U)\otimes\mathfrak{F}(U) \arrow[r, "\mu_U"] \arrow[dr, "\mathfrak{F}(\iota_U^M)\otimes\mathfrak{F}(\iota_U^M)"] \arrow[ddr, "\mathfrak{F}(f_U)\otimes\mathfrak{F}(f_U)"'] & \mathfrak{F}(U) \arrow[dr, "\mathfrak{F}(\iota_U^M)"] \arrow[ddr, "\mathfrak{F}(f_U)"'] & \\
& \mathfrak{F}(M)\otimes\mathfrak{F}(M) \arrow[r, "\mu_M"] \arrow[d, "\mathfrak{F}(f)\otimes\mathfrak{F}(f)"] & \mathfrak{F}(M) \arrow[d, "\mathfrak{F}(f)"] \\
& \mathfrak{F}(N)\otimes\mathfrak{F}(N) \arrow[r, "\mu_N"'] & \mathfrak{F}(N)
\end{tikzcd} \tag{2.2.9}$$

where $f_U : U \to N$ denotes the restriction of $f : M \to N$ to $U \subseteq M$. The top and bottom squares of this diagram commute because of Lemma 2.2.5 and the fact that both $U \subseteq M$ and $f(U) \subseteq N$ are relatively compact subsets. The two triangles commute by direct inspection. By universality of the colimit in (2.2.8), this implies that the front square in (2.2.9) commutes, proving our claim. $\square$

Corollary 2.2.8. *Every Cauchy constant additive prefactorization algebra $\mathfrak{F} \in \mathbf{PFA}^{\mathrm{add},c}$ defines a functor $\mathbb{A}[\mathfrak{F}] : \mathbf{Loc} \to \mathbf{Alg}(\mathbf{C})$ assigning to each $M \in \mathbf{Loc}$ the algebra*

$\mathbb{A}[\mathfrak{F}](M) := (\mathfrak{F}(M), \mu_M, \eta_M)$ and to each **Loc***-morphisms $f : M \to N$ the algebra map $\mathbb{A}[\mathfrak{F}](f) := \mathfrak{F}(f)$. The assignment $\mathfrak{F} \mapsto \mathbb{A}[\mathfrak{F}]$ canonically extends to a functor $\mathbb{A} : \mathbf{PFA}^{\mathrm{add},c} \to [\mathbf{Loc}, \mathbf{Alg}(\mathbf{C})]$.*

Proof. It remains to prove that every morphism $\zeta : \mathfrak{F} \to \mathfrak{G}$ in $\mathbf{PFA}^{\mathrm{add},c}$ defines a natural transformation $\mathbb{A}[\zeta] : \mathbb{A}[\mathfrak{F}] \to \mathbb{A}[\mathfrak{G}]$ between $\mathbf{Alg}(\mathbf{C})$-valued functors on **Loc**, i.e. that all components $\zeta_M : \mathfrak{F}(M) \to \mathfrak{G}(M)$ preserve the multiplications and units. For the units this is immediate, while for the multiplications it follows from the fact that the diagram

$$\begin{tikzcd}
\mathfrak{F}(M) \otimes \mathfrak{F}(M) \arrow[d, "\zeta_M \otimes \zeta_M"'] & \mathfrak{F}(U_+) \otimes \mathfrak{F}(U_-) \arrow[l, "\mathfrak{F}(\iota^M_{U_+}) \otimes \mathfrak{F}(\iota^M_{U_-})"', "\cong"] \arrow[r, "\mathfrak{F}(\iota^M_{\underline{U}})"] \arrow[d, "\zeta_{U_+} \otimes \zeta_{U_-}"'] & \mathfrak{F}(M) \arrow[d, "\zeta_M"] \\
\mathfrak{G}(M) \otimes \mathfrak{G}(M) & \mathfrak{G}(U_+) \otimes \mathfrak{G}(U_-) \arrow[l, "\cong", "\mathfrak{G}(\iota^M_{U_+}) \otimes \mathfrak{G}(\iota^M_{U_-})"'] \arrow[r, "\mathfrak{G}(\iota^M_{\underline{U}})"'] & \mathfrak{G}(M)
\end{tikzcd} \tag{2.2.10}$$

commutes by the compatibility properties (2.1.3) of prefactorization algebra morphisms. $\square$

2.2.3 *Einstein causality*

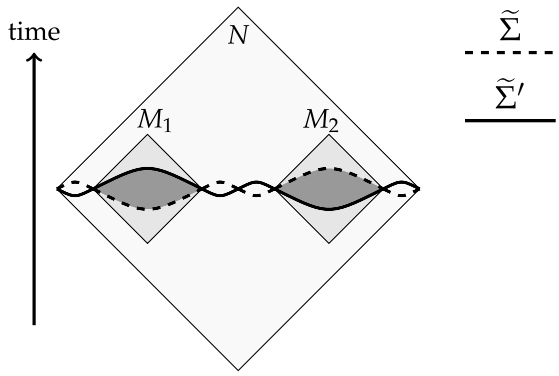
The aim of this section is to show that the functor $\mathbb{A} : \mathbf{PFA}^{\mathrm{add},c} \to [\mathbf{Loc}, \mathbf{Alg}(\mathbf{C})]$ factors through the category $\mathbf{AQFT}^{\mathrm{add},c}$, i.e. defines a functor $\mathbb{A} : \mathbf{PFA}^{\mathrm{add},c} \to \mathbf{AQFT}^{\mathrm{add},c}$. In order to prove this result we proceed similarly to Subsection 2.2.2, by first restricting our attention to **Loc**-morphisms with relatively compact image and then obtaining the general case by leveraging the fact that the monoidal product of a bicomplete closed symmetric monoidal category commutes with colimits and that $\mathbf{Ob}(\mathbf{RC}_M)$ is a directed set for every $M \in \mathbf{Loc}$.

Lemma 2.2.9. *Let $\mathfrak{F} \in \mathbf{PFA}^c$ be a Cauchy constant (and* not *necessarily additive) prefactorization algebra and let $(f_1 : M_1 \to N) \perp_c (f_2 : M_2 \to N)$ be a causally disjoint pair of* **Loc***-morphisms with relatively compact images $f_1(M_1), f_2(M_2) \subseteq N$. Then $\mu_N^{\mathrm{op}}(\mathfrak{F}(f_1) \otimes \mathfrak{F}(f_2)) = \mu_N(\mathfrak{F}(f_1) \otimes \mathfrak{F}(f_2))$, where $\mu_N^{(\mathrm{op})}$ denotes the (opposite) multiplication on $\mathfrak{F}(N)$ from Proposition 2.2.4.*

Proof. In order to compare the two morphisms $\mu_N(\mathfrak{F}(f_1) \otimes \mathfrak{F}(f_2))$ and $\mu_N^{\mathrm{op}}(\mathfrak{F}(f_1) \otimes \mathfrak{F}(f_2))$ from $\mathfrak{F}(M_1) \otimes \mathfrak{F}(M_2)$ to $\mathfrak{F}(N)$, we introduce convenient ways to compute these composites. Let us choose arbitrary Cauchy surfaces Σ_1 of M_1 and Σ_2 of M_2. As in the proof of Lemma 2.2.5, we deduce that $\overline{f_1(\Sigma_1)}, \overline{f_2(\Sigma_2)} \subseteq N$ are achronal compact subsets. Causal disjointness of the pair $f_1 \perp_c f_2$ entails achronality of the union $\overline{f_1(\Sigma_1)} \cup \overline{f_2(\Sigma_2)} \subseteq N$. By [BS06, Theorem 3.8], there exists a Cauchy surface $\widetilde{\Sigma}$ of N that contains the union $\overline{f_1(\Sigma_1)} \cup \overline{f_2(\Sigma_2)} \subseteq \widetilde{\Sigma}$. Similarly, choosing any Cauchy surface $\Sigma_1' \subset I^+_{M_1}(\Sigma_1)$ of M_1 that lies in the future of Σ_1 and any Cauchy surface $\Sigma_2' \subset I^-_{M_2}(\Sigma_2)$ of M_2 that lies in the past of Σ_2, there exists a Cauchy surface $\widetilde{\Sigma}'$ of N that contains the union $\overline{f_1(\Sigma_1')} \cup \overline{f_2(\Sigma_2')} \subseteq \widetilde{\Sigma}'$. Let us introduce

$$U_1 := I^+_{M_1}(\Sigma_1) \cap I^-_{M_1}(\Sigma_1') \subseteq M_1 \quad , \qquad U_2 := I^+_{M_2}(\Sigma_2') \cap I^-_{M_2}(\Sigma_2) \subseteq M_2 \quad , \tag{2.2.11}$$

and also consider $\widetilde{\Sigma}_\pm := I_N^\pm(\widetilde{\Sigma}) \subseteq N$ and $\widetilde{\Sigma}'_\pm := I_N^\pm(\widetilde{\Sigma}') \subseteq N$. By construction, $\iota_{U_i}^{M_i} : U_i \xrightarrow{c} M_i$, for $i = 1, 2$, and $\iota^N_{\widetilde{\Sigma}^{(\prime)}_\pm} : \widetilde{\Sigma}^{(\prime)}_\pm \xrightarrow{c} N$ are Cauchy morphisms. The following picture illustrates in dark gray the chosen subsets $U_1 \subseteq M_1$ and $U_2 \subseteq M_2$:



(2.2.12)

With these preparations, we can compute $\mu_N\left(\mathfrak{F}(f_1) \otimes \mathfrak{F}(f_2)\right)$ by

$$\begin{tikzcd}
\mathfrak{F}(M_1) \otimes \mathfrak{F}(M_2) \arrow[r, "\mathfrak{F}(f_1)\otimes\mathfrak{F}(f_2)"] & \mathfrak{F}(N) \otimes \mathfrak{F}(N) \arrow[r, "\mu_N"] & \mathfrak{F}(N) \\
\mathfrak{F}(U_1) \otimes \mathfrak{F}(U_2) \arrow[u, "\mathfrak{F}(\iota^{M_1}_{U_1})\otimes\mathfrak{F}(\iota^{M_2}_{U_2})", "\cong"'] \arrow[r, "\mathfrak{F}\left((f_1)^{\widetilde{\Sigma}_+}_{U_1}\right)\otimes\mathfrak{F}\left((f_2)^{\widetilde{\Sigma}_-}_{U_2}\right)"'] & \mathfrak{F}(\widetilde{\Sigma}_+) \otimes \mathfrak{F}(\widetilde{\Sigma}_-) \arrow[u, "\mathfrak{F}(\iota^N_{\widetilde{\Sigma}_+})\otimes\mathfrak{F}(\iota^N_{\widetilde{\Sigma}_-})", "\cong"'] \arrow[ur, "\mathfrak{F}(\iota^N_{\underline{\widetilde{\Sigma}}})"'] &
\end{tikzcd} \tag{2.2.13}$$

where $(f_1)^{\widetilde{\Sigma}_+}_{U_1} : U_1 \to \widetilde{\Sigma}_+$ denotes the restriction of $f_1 : M_1 \to N$ to $U_1 \subseteq M_1$, and analogously for $(f_2)^{\widetilde{\Sigma}_-}_{U_2}$. Similarly, $\mu_N^{\mathrm{op}}\left(\mathfrak{F}(f_1) \otimes \mathfrak{F}(f_2)\right)$ can be computed by

$$\begin{tikzcd}
\mathfrak{F}(M_1) \otimes \mathfrak{F}(M_2) \arrow[r, "\mathfrak{F}(f_1)\otimes\mathfrak{F}(f_2)"] \arrow[d, "\mathrm{flip}"'] & \mathfrak{F}(N) \otimes \mathfrak{F}(N) \arrow[r, "\mu_N^{\mathrm{op}}"] \arrow[d, "\mathrm{flip}"'] & \mathfrak{F}(N) \\
\mathfrak{F}(M_2) \otimes \mathfrak{F}(M_1) \arrow[r, "\mathfrak{F}(f_2)\otimes\mathfrak{F}(f_1)"] & \mathfrak{F}(N) \otimes \mathfrak{F}(N) \arrow[ur, "\mu_N"] & \\
\mathfrak{F}(U_2) \otimes \mathfrak{F}(U_1) \arrow[u, "\mathfrak{F}(\iota^{M_2}_{U_2})\otimes\mathfrak{F}(\iota^{M_1}_{U_1})", "\cong"'] \arrow[r, "\mathfrak{F}\left((f_2)^{\widetilde{\Sigma}'_+}_{U_2}\right)\otimes\mathfrak{F}\left((f_1)^{\widetilde{\Sigma}'_-}_{U_1}\right)"'] & \mathfrak{F}(\widetilde{\Sigma}'_+) \otimes \mathfrak{F}(\widetilde{\Sigma}'_-) \arrow[u, "\mathfrak{F}(\iota^N_{\widetilde{\Sigma}'_+})\otimes\mathfrak{F}(\iota^N_{\widetilde{\Sigma}'_-})", "\cong"'] \arrow[uur, "\mathfrak{F}(\iota^N_{\underline{\widetilde{\Sigma}'}})"'] &
\end{tikzcd} \tag{2.2.14}$$

The claim follows from the equivariance property (2.1.2) of prefactorization algebras:

$$\begin{tikzcd}
\mathfrak{F}(U_1) \otimes \mathfrak{F}(U_2) \arrow[r, "\mathfrak{F}(\iota^N_{\underline{U}})"] \arrow[d, "\mathrm{flip}"', "\cong"] & \mathfrak{F}(N) \\
\mathfrak{F}(U_2) \otimes \mathfrak{F}(U_1) \arrow[ur, "\mathfrak{F}(\iota^N_{\underline{U}}\cdot\tau)"'] &
\end{tikzcd} \tag{2.2.15}$$

where τ is the permutation that flips 1 and 2.

$\square$

Proposition 2.2.10. *Let $\mathfrak{F} \in \mathbf{PFA}^{\mathrm{add},c}$ and let $(f_1 : M_1 \to N) \perp_c (f_2 : M_2 \to N)$ be a pair of causally disjoint* **Loc***-morphisms. Then $\mu_N^{\mathrm{op}} \left(\mathfrak{F}(f_1) \otimes \mathfrak{F}(f_2)\right) = \mu_N \left(\mathfrak{F}(f_1) \otimes \mathfrak{F}(f_2)\right)$, where $\mu_N^{(\mathrm{op})}$ denotes the (opposite) multiplication on $\mathfrak{F}(N)$ from Proposition 2.2.4.*

Proof. Because $\mathfrak{F}$ is by hypothesis additive (cf. Definition 2.1.6) and the monoidal product $\otimes$ in a cocomplete closed symmetric monoidal category preserves colimits in both entries, it follows that

$$\mathfrak{F}(M_1) \otimes \mathfrak{F}(M_2) \;\cong\; \mathrm{colim}_{(U_1,U_2) \in \mathbf{RC}_{M_1} \times \mathbf{RC}_{M_2}} \big(\mathfrak{F}(U_1) \otimes \mathfrak{F}(U_2)\big) \quad . \tag{2.2.16}$$

For every $(U_1, U_2) \in \mathbf{RC}_{M_1} \times \mathbf{RC}_{M_2}$, consider the diagram

$$\begin{tikzcd}
\mathfrak{F}(U_1) \otimes \mathfrak{F}(U_2) \arrow[dr, "\mathfrak{F}(\iota_{U_1}^{M_1}) \otimes \mathfrak{F}(\iota_{U_2}^{M_2})"] \arrow[drr, bend left, "\mathfrak{F}\left((f_1)_{U_1}\right) \otimes \mathfrak{F}\left((f_2)_{U_2}\right)"] \arrow[ddr, bend right, "\mathfrak{F}\left((f_1)_{U_1}\right) \otimes \mathfrak{F}\left((f_2)_{U_2}\right)"'] & & \\
& \mathfrak{F}(M_1) \otimes \mathfrak{F}(M_2) \arrow[r, "\mathfrak{F}(f_1) \otimes \mathfrak{F}(f_2)"] \arrow[d, "\mathfrak{F}(f_1) \otimes \mathfrak{F}(f_2)"'] & \mathfrak{F}(N) \otimes \mathfrak{F}(N) \arrow[d, "\mu_N^{\mathrm{op}}"] \\
& \mathfrak{F}(N) \otimes \mathfrak{F}(N) \arrow[r, "\mu_N"'] & \mathfrak{F}(N)
\end{tikzcd} \tag{2.2.17}$$

where $(f_i)_{U_i} : U_i \to N$ denotes the restriction of $f_i : M_i \to N$ to $U_i \subseteq M_i$, for $i = 1, 2$. The two triangles coincide and commute by direct inspection. Furthermore, for every $(U_1, U_2) \in \mathbf{RC}_{M_1} \times \mathbf{RC}_{M_2}$, the outer square commutes as a consequence of Lemma 2.2.9 applied to the causally disjoint pair $(f_1)_{U_1} \perp_c (f_2)_{U_2}$, whose images $f_1(U_1), f_2(U_2) \subseteq N$ are relatively compact subsets. Hence, by universality of the colimit in (2.2.16), the inner square commutes as well, which is our claim. □

We can finally prove that $\mathbb{A} : \mathbf{PFA}^{\mathrm{add},c} \to [\mathbf{Loc}, \mathbf{Alg}(\mathbf{C})]$ factors through $\mathbf{AQFT}^{\mathrm{add},c}$.

Theorem 2.2.11. *Every Cauchy constant additive prefactorization algebra $\mathfrak{F} \in \mathbf{PFA}^{\mathrm{add},c}$ defines a Cauchy constant additive algebraic quantum field theory $\mathbb{A}[\mathfrak{F}] \in \mathbf{AQFT}^{\mathrm{add},c}$. Hence, the functor $\mathbb{A} : \mathbf{PFA}^{\mathrm{add},c} \to [\mathbf{Loc}, \mathbf{Alg}(\mathbf{C})]$ from Corollary 2.2.8 factors through the full subcategory $\mathbf{AQFT}^{\mathrm{add},c} \subseteq [\mathbf{Loc}, \mathbf{Alg}(\mathbf{C})]$.*

Proof. Proposition 2.2.10 implies that the functor $\mathbb{A}[\mathfrak{F}] : \mathbf{Loc} \to \mathbf{Alg}$ defined in Corollary 2.2.8 is an algebraic quantum field theory, i.e. it satisfies the Einstein causality axiom (see Definition 1.2.12). Because $\mathfrak{F}$ is by hypothesis Cauchy constant, it follows that $\mathbb{A}[\mathfrak{F}]$ is Cauchy constant too. Because the underlying functors $\mathbb{A}[\mathfrak{F}]|_M = \mathfrak{F}|_M : \mathbf{RC}_M \to \mathbf{C}$ to the category $\mathbf{C}$ coincide, additivity of $\mathfrak{F} \in \mathbf{PFA}^{\mathrm{add},c}$ and Remark 2.1.9 immediately imply additivity of $\mathbb{A}[\mathfrak{F}]$. Hence, $\mathbb{A}[\mathfrak{F}] \in \mathbf{AQFT}^{\mathrm{add},c}$. □

2.3 FROM AQFT TO TPFA

The goal of this section is to build a functor $\mathbb{F} : \mathbf{AQFT}^{\mathrm{add},c} \to \mathbf{tPFA}^{\mathrm{add},c}$ associating to any (additive Cauchy constant) algebraic quantum field theory $\mathfrak{A}$ on $\mathbf{Loc}^{\perp_c}$ a *time-orderable* (additive Cauchy constant) prefactorization algebra $\mathbb{F}[\mathfrak{A}]$ on $\mathbf{Loc}$. The reason for introducing a new type of prefactorization algebras, namely *time-orderable* prefactorization algebras, is the following: Suppose we are given the task to construct an additive Cauchy constant prefactorization algebra $\mathbb{F}[\mathfrak{A}]$ out of an additive Cauchy constant algebraic quantum field theory $\mathfrak{A} \in \mathbf{AQFT}^{\mathrm{add},c}$. The first step one might want to take is trying to define the factorization products $\mathfrak{F}(\underline{f})$ for every tuple $\underline{f} = (f_1 : M_1 \to N, \dots, f_n : M_N \to N)$ of disjoint $\mathbf{Loc}$-morphisms. To make things easier we focus on the case $n = 2$, i.e. the case in which we have a couple $(f_1 : M_1 \to N, f_2 : M_2 \to N)$ of disjoint $\mathbf{Loc}$-morphisms. A reasonable attempt is trying to define $\mathbb{F}[\mathfrak{A}](\underline{f}) : \mathfrak{A}(M_1) \otimes \mathfrak{A}(M_2) \to \mathfrak{A}(N)$ via the following diagram

$$\begin{tikzcd}
\mathfrak{A}(M_1) \otimes \mathfrak{A}(M_2) \arrow[rr, "\mathbb{F}[\mathfrak{A}](\underline{f})"] \arrow[dr, "\mathfrak{A}(f_1)\otimes\mathfrak{A}(f_2)"'] & & \mathfrak{A}(N) \\
& \mathfrak{A}(N) \otimes \mathfrak{A}(N) \arrow[ur, "\mu_N"'] &
\end{tikzcd} \tag{2.3.1}$$

Unluckily, this simple approach does not work, in fact, the equivariance axiom for prefactorization algebras (see Equation (2.1.2)) requires us to verify that $\mathbb{F}[\mathfrak{A}](\underline{f}) = \mathbb{F}[\mathfrak{A}](\underline{f} \cdot \tau) \circ \mathrm{flip}$, i.e. that the following equation holds $\mu_N(\mathfrak{A}(f_1) \otimes \mathfrak{A}(f_2)) = \mu_N^{\mathrm{op}}(\mathfrak{A}(f_1) \otimes \mathfrak{A}(f_2))$ for every pair of *disjoint* morphisms and this is not true in general unless f_1 and f_2 are *causally disjoint*. To deal with this insufficiency, we introduce a new sort of prefactorization algebras called *time-orderable* in which factorization products exist just for couples of disjoint $\mathbf{Loc}$-morphisms $(f_1 : M_1 \to N, f_2 : M_2 \to N)$ where $f(M_1)$ is in some sense "later" than $f(M_2)$.

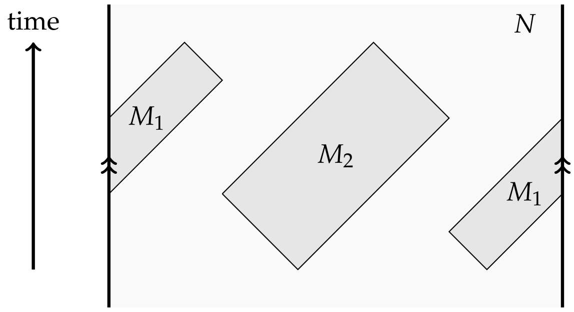
Definition 2.3.1. Let $n \geq 0$, $f_i : M_i \to N \in \mathrm{Mor}\mathbf{Loc}$ for every $i = 1, \dots, n$. We say that:

(a) $\underline{f} = (f_1, \dots, f_n)$ is *time-ordered* if $J_N^+(f_i(M_i)) \cap f_j(M_j) = \emptyset$ for all $1 \leq i < j \leq n$, where $J_N^+(f(M_i))$ denotes the causal future in N of $f(M_i)$ for every i.

(b) $\underline{f}$ is *time-orderable* if there exists a permutation $\rho \in \Sigma_n$ such that $\underline{f}\rho = (f_{\rho(1)}, \dots, f_{\rho(n)})$ is time-ordered. We call such a ρ a *time-ordering* permutation.

By convention all 0-tuples $\emptyset \to N$ and all 1-tuples $f : M \to N$ are time-ordered.

Remark 2.3.2. Time-orderability is a condition that a tuple of disjoint $\mathbf{Loc}$-morphisms $\underline{f}$ may or may not satisfy. To see that this is the case, consider for example the inclusion of the following causally convex open subsets M_1 and M_2 into the Lorentzian cylinder N:



(2.3.2)

In this picture the left and right boundaries are identified as indicated, thereby producing the Lorentzian cylinder $N = (\mathbb{R} \times \mathbb{S}^1, g = -\mathrm{d}t^2 + \mathrm{d}\phi^2, \mathfrak{t} = \frac{\partial}{\partial t})$. It is clear that the tuple $\iota^N_{\underline{M}} = (\iota^N_{M_1}, \iota^N_{M_2})$ is not time-orderable. △

Definition 2.3.3. The following data defines a multicategory $\mathcal{P}_{t\mathbf{Loc}}$, called the *time-orderable prefactorization multicategory*:

(a) The collection $\mathcal{P}_{t\mathbf{Loc}_0}$ consists of the objects of **Loc**, i.e. $\mathcal{P}_{t\mathbf{Loc}_0} := \mathbf{Ob}(\mathbf{Loc})$.

(b) For every $N \in \mathcal{P}_{t\mathbf{Loc}_0}$, $n \geq 0$, $\underline{M} \in \mathcal{P}^n_{t\mathbf{Loc}_0}$, the sets $\mathcal{P}_{t\mathbf{Loc}}\binom{N}{\underline{M}} = \{\underline{f} = (f_1 : M_1 \to N, \dots, f_n : M_n \to N) \,:\, \underline{f}$ is a time-orderable tuple of **Loc**-morphisms$\}$.

The identities, composition maps and permutation actions are obtained restricting those of the multicategory $\mathcal{P}_{\mathbf{Loc}^{\perp_d}}$ to time-orderable tuples.

To check that Definition 2.3.3 satisfies the axioms of a multicategory it is sufficient to prove the following Lemma:

Lemma 2.3.4. *(a) Let $\underline{f} = (f_1 : M_1 \to N, \dots, f_n : M_n \to N)$ be a time-orderable tuple, ρ_0 a time-ordering permutation for $\underline{f}$ and $\sigma \in \Sigma_n$. Then $\sigma^{-1}\rho \in \Sigma_n$ is a time-ordering permutation for $\underline{f}\sigma = (f_{\sigma(1)}, \dots, f_{\sigma(n)}) : \underline{M}\sigma \to N$.*

(b) Let $\underline{f} = (f_1 : M_1 \to N, \dots, f_n : M_n \to N)$ be a time-orderable tuple, ρ_0 a time-ordering permutation for $\underline{f}$, $\underline{g}_i = (g_{i1}, \dots, g_{im_i}) : \underline{L}_i \to M_i$ be a time-orderable tuple with time-ordering permutation ρ_i for every $i = 1, \dots, n$. Then, the permutation

$$\rho_0\langle m_1, \dots, m_n\rangle \, (\rho_{\rho_0(1)} \oplus \dots \oplus \rho_{\rho_0(n)}) \in \Sigma_{m_1 + \dots + m_n} \quad , \tag{2.3.3}$$

where $\rho_0\langle m_1, \dots, m_n\rangle$ denotes the block permutation corresponding to ρ_0 and $\rho_{\rho_0(1)} \oplus \dots \oplus \rho_{\rho_0(n)}$ the sum permutation of the $\rho_{\rho_0(i)}$, is a time-ordering permutation for

$$\underline{f}(\underline{g}_1, \dots, \underline{g}_n) := (f_1\, g_{11}, \dots, f_n\, g_{nm_n}) \,:\, (\underline{L}_1, \dots, \underline{L}_n) \longrightarrow N \quad . \tag{2.3.4}$$

(c) Let $\underline{f} = (f_1 : M_1 \to N, \dots, f_n : M_n \to N)$ be a time-orderable tuple and let ρ_0 and ρ_0' be time-ordering permutations for $\underline{f}$. Then, the right permutation $\rho_0^{-1}\rho_0' : \underline{f}\rho_0 \to \underline{f}\rho_0'$ is generated by transpositions of adjacent causally disjoint pairs of morphisms.

Proof. (a): Trivial.
(b): Since

$$\underline{f}(\underline{g}_1,\dots,\underline{g}_n)\,\rho_0\langle m_1,\dots,m_n\rangle\,(\rho_{\rho_0(1)}\oplus\dots\oplus\rho_{\rho_0(n)}) = (\underline{f}\rho_0)(\underline{g}_{\rho_0(1)}\rho_{\rho_0(1)},\dots,\underline{g}_{\rho_0(n)}\rho_{\rho_0(n)}) \quad , \tag{2.3.5}$$

it is sufficient to prove that the composition of time-ordered tuples is time-ordered. Therefore, assuming that $\underline{f}$ and $\underline{g}_i$, for $i=1,\dots,n$, are time-ordered, we have to show that $(f_1\,g_{11},\dots,f_n\,g_{nm_n})$ is time-ordered, i.e. $J_N^+(f_ig_{ii'}(L_{ii'}))\cap f_jg_{jj'}(L_{jj'})=\emptyset$ for the following two cases: Case 1 is $i<j$ and arbitrary $i'=1,\dots,m_i$ and $j'=1,\dots,m_j$. Case 2 is $i=j$ and $j<j'$. Case 1 follows immediately from the hypothesis that $\underline{f}$ is time-ordered, i.e. $J_N^+(f_i(M_i))\cap f_j(M_j)=\emptyset$ for all $i<j$. For case 2 we use that $\underline{g}_i$ is time-ordered, i.e. $J_{M_i}^+(g_{ii'}(L_{ii'}))\cap g_{ij'}(L_{ij'})=\emptyset$ for all $j<j'$, and hence by the properties of **Loc**-morphisms

$$J_N^+(f_ig_{ii'}(L_{ii'}))\cap f_ig_{ij'}(L_{ij'}) = f_i\Big(J_{M_i}^+(g_{ii'}(L_{ii'}))\cap g_{ij'}(L_{ij'})\Big)=\emptyset \quad . \tag{2.3.6}$$

This proves that $(f_1\,g_{11},\dots,f_n\,g_{nm_n})$ is time-ordered.
(c): Suppose that $\rho_0^{-1}\rho_0' : \underline{f}\rho_0\to\underline{f}\rho_0'$ reverses the time-ordering between f_k and f_ℓ, i.e. $\rho_0(i)=k=\rho_0'(i')$ and $\rho_0(j)=\ell=\rho_0'(j')$ with $i<j$ and $j'<i'$ or vice versa with $j<i$ and $i'<j'$. Let us consider the case $i<j$ and $j'<i'$, the other one being similar. By hypothesis, we have that $J_N^+(f_{\rho_0(i)}(M_{\rho_0(i)}))\cap f_{\rho_0(j)}(M_{\rho_0(j)})=\emptyset$ and $J_N^+(f_{\rho_0'(j')}(M_{\rho_0'(j')}))\cap f_{\rho_0'(i')}(M_{\rho_0'(i')})=\emptyset$, which is equivalent to $f_k\perp_c f_\ell$ being causally disjoint. Summing up, this proves that every pair (f_k,f_ℓ) of morphisms whose time-ordering is reversed by $\rho_0^{-1}\rho_0'$ is causally disjoint $f_k\perp_c f_\ell$.

To conclude the proof, let us recall that every permutation $\sigma : (h_1,\dots,h_n)\to(h_{\sigma(1)},\dots,h_{\sigma(n)})$ admits a (not necessarily unique) factorization into adjacent transpositions that flip only elements whose order is reversed by σ. (One way to obtain such a factorization is as follows: Start from $(h_1,\dots,h_n)$ and move by adjacent transpositions the element $h_{\sigma(1)}$ to the leftmost position. Then move by adjacent transpositions the element $h_{\sigma(2)}$ to the second leftmost position, and so on.) This implies that we obtain a factorization $\rho_0^{-1}\rho_0'=\tau_1\cdots\tau_N : \underline{f}\rho_0\to\underline{f}\rho_0'$, where each $\tau_l : \underline{f}\rho_0\tau_1\cdots\tau_{l-1}\to\underline{f}\rho\tau_1\cdots\tau_l$ transposes two adjacent **Loc**-morphisms whose time-ordering is reversed by $\rho_0^{-1}\rho_0'$. Our result in the previous paragraph then implies that each τ_l is a transposition of adjacent causally disjoint pairs of morphisms, which completes our proof. □

Definition 2.3.5. We denote by **tPFA** the category $\mathbf{Alg}_{\mathcal{P}_{t\mathbf{Loc}}}(\mathbf{C})$ of **C**-valued algebras on $\mathcal{P}_{t\mathbf{Loc}}$ (see Remark 1.1.21) and call its objects *time-orderable prefactorization algebras*. Like we did in the case of Definition 2.1.2 we give a more explicit description of a generic time-orderable prefactorization algebra $\mathfrak{F}$. In particular, a time-orderable prefactorization algebra consists of the following data:

(a) For each $M\in\mathbf{Loc}$, an object $\mathfrak{F}(M)\in\mathbf{C}$.

(b) For each tuple of time-orderable **Loc**-morphisms $\underline{f}=(f_1,\dots,f_n):\underline{M}\to N$, a **C**-morphism $\mathfrak{F}(\underline{f}):\bigotimes_{i=1}^n\mathfrak{F}(M_i)\to\mathfrak{F}(N)$ (called *time-ordered product*).

Satisfying the analogues of the prefactorization algebra axioms from Definition 2.1.2 for time-orderable tuples.

A morphism $\zeta : \mathfrak{F} \to \mathfrak{G}$ of time-orderable prefactorization algebras is a family $\zeta_M : \mathfrak{F}(M) \to \mathfrak{G}(M)$ of $\mathbf{C}$-morphisms, for all $M \in \mathbf{Loc}$, that is compatible with the time-ordered products as in Definition 2.1.2.

In analogy to what we have done for algebraic quantum field theories and prefactorization algebras we introduce the categories $\mathbf{tPFA}^{\mathrm{add}}$, $\mathbf{tPFA}^{c}$, $\mathbf{tPFA}^{\mathrm{add},c}$ of additive, Cauchy constant and additive Cauchy constant time-orderable prefactorization algebras.

Remark 2.3.6. There are two multifunctors which play a central role in the discussion that follows. The first is the multifunctor $\psi : \mathcal{P}_{t\mathbf{Loc}} \to \mathcal{P}_{\mathbf{Loc}^{\perp_d}}$, which is the identity on objects and the inclusion on the sets of n-operations. This functor is clearly faithful and essentially surjective, but not full (see Example 2.3.2). As we have seen in Theorem 1.1.26, ψ defines an associated pullback functor $\psi^* : \mathbf{PFA} := \mathbf{Alg}_{\mathcal{P}_{\mathbf{Loc}^{\perp_d}}}(\mathbf{C}) \to \mathbf{tPFA} := \mathbf{Alg}_{\mathcal{P}_{t\mathbf{Loc}}}(\mathbf{C})$ which is faithful and preserves both additivity and Cauchy constancy. The role of this multifunctor is to take a prefactorization algebra $\mathfrak{F} \in \mathbf{PFA}$ and give back a time-orderable prefactorization algebra $\psi^*\mathfrak{F} \in \mathbf{tPFA}$ obtained by restriction to tuples of time-orderable morphisms.

The second is the multifunctor $\phi : \mathcal{P}_{t\mathbf{Loc}} \to \mathcal{O}_{\mathbf{Loc}^{\perp_c}}$ defined by sending each object to itself and by sending each time-orderable tuple $\underline{f} \in \mathcal{P}_{t\mathbf{Loc}}$ with time-ordering permutation ρ to the couple $(\rho^{-1}, \underline{f}) \in \mathcal{O}_{\mathbf{Loc}^{\perp_c}}$ (the appearance of the inverse of ρ is justified by the physical interpretation of the multicategory $\mathcal{O}_{\mathbf{Loc}^{\perp_c}}$, see equation (1.2.8)). To see that this functor is well defined is a consequence of Lemma 2.3.4, in particular of the fact that given time-ordering permutations ρ, ρ' of $\underline{f}$, the permutation $\rho^{-1}\rho' : \underline{f}\rho \to \underline{f}\rho'$ is generated by transpositions of causally disjoint morphisms, i.e. $(\rho^{-1}, \underline{f})$ and $(\rho'^{-1}, \underline{f})$ define the same n-operations in $\mathcal{O}_{\mathbf{Loc}^{\perp_c}}$ (see Theorem 1.2.15). Theorem 1.1.26 shows there are two multifunctors associated to $\phi : \mathcal{P}_{t\mathbf{Loc}} \to \mathcal{O}_{\mathbf{Loc}^{\perp_c}}$, namely its pullback $\phi^* : \mathbf{AQFT} \to \mathbf{tPFA}$ and the multicategorical left Kan extension $\phi_! : \mathbf{tPFA} \to \mathbf{AQFT}$ along ϕ. It turns out that $\phi^* : \mathbf{AQFT} \to \mathbf{tPFA}$ is the functor $\mathbb{F} : \mathbf{AQFT} \to \mathbf{tPFA}$ we are interested in. More explicitly, for any algebraic quantum field theory $\mathfrak{A}$, the time-orderable prefactorization algebra $\mathbb{F}[\mathfrak{A}]$ associates to a tuple of disjoint $\mathbf{Loc}$-morphisms $\underline{f} = (f_1 : M_1 \to N, \dots, f_n : M_n \to N)$ with time-ordering permutation ρ the following time-ordered product:

$$\begin{tikzcd}
\bigotimes_{i=1}^{n} \mathfrak{A}(M_i) \arrow[r, "{\mathbb{F}[\mathfrak{A}](\underline{f})}"] \arrow[d, "\text{permute}"'] & \mathfrak{A}(N) \\
\bigotimes_{i=1}^{n} \mathfrak{A}(M_{\rho(i)}) \arrow[r, "\otimes_i \mathfrak{A}(f_{\rho(i)})"'] & \mathfrak{A}(N)^{\otimes n} \arrow[u, "\mu_N^{(n)}"']
\end{tikzcd} \tag{2.3.7}$$

The product $\mathbb{F}[\mathfrak{A}](\underline{f})$ can be understood as the following procedure:

(a) Consider observables $a_1 \in \mathfrak{A}(M_1), \dots, a_n \in \mathfrak{A}(M_n)$.

(b) Swap the observables according to the permutation ρ and push them forward to N.

(c) Multiply the observables using the multiplication μ_N in N according to the order in which they appear after the last step.

The case $n = 2$ is particularly illuminating:

(a) If f_1 and f_2 are causally disjoint (in particular time-ordered) then $\phi : \mathcal{P}_{t\mathbf{Loc}} \to \mathcal{O}_{\mathbf{Loc}^{\perp_c}}$ sends $\underline{f}$ to $(e, \underline{f}) = (\tau, \underline{f})$, where τ is the permutation swapping 1 and 2. In particular, Einstein causality shows that there is no ambiguity in using equation (2.3.7) to define $\mathbb{F}[\mathfrak{A}](\underline{f})$.

(b) If f_1 and f_2 are time-ordered, then the product of observables $a_1 \in \mathfrak{A}(M_1)$ and $a_2 \in \mathfrak{A}(M_2)$ is given by $a_1 a_2 \in \mathfrak{A}(N)$, otherwise it is given by $a_2 a_1 \in \mathfrak{A}(N)$, where $a_1 a_2$ and $a_2 a_1$ denote the two possible products of the push-forwards in N of a_1 and a_2.

$\triangle$

We summarize the construction of $\mathbb{F} : \mathbf{AQFT} \to \mathbf{tPFA}$ just discussed in the following Theorem:

Theorem 2.3.7. *Let $\mathfrak{A} \in \mathbf{AQFT}$ be an algebraic quantum field theory. Then the following data defines a time-orderable prefactorization algebra $\mathbb{F}[\mathfrak{A}] \in \mathbf{tPFA} := \mathbf{Alg}_{\mathcal{P}_{t\mathbf{Loc}}}(\mathbf{C})$:*

(a) $\mathbb{F}[\mathfrak{A}]$ assigns to each each $M \in \mathcal{P}_{t\mathbf{Loc}_0}$ the object $\mathfrak{A}(M)$, where $\mathfrak{A}(M)$ is considered as an object of $\mathbf{C}$ via the forgetful functor $\mathbf{Alg}(\mathbf{C}) \to \mathbf{C}$;

(b) $\mathbb{F}[\mathfrak{A}]$ assigns to each time-orderable tuple of pairwise disjoint morphisms $\underline{f} = (f_1, \dots, f_n) : \underline{M} \to N$, the time-ordered product $\mathbb{F}[\mathfrak{A}](\underline{f}) : \bigotimes_{i=1}^{n} \mathbb{F}[\mathfrak{A}](M_i) \to \mathbb{F}[\mathfrak{A}](N)$ defined by equation (2.3.7), and assigns the unit $\eta_N : I \to \mathbb{F}[\mathfrak{A}](N)$ of $\mathfrak{A}(N)$ to the only morphism in $\mathcal{P}_{t\mathbf{Loc}}\binom{N}{\varnothing}$ for every $N \in \mathcal{P}_{t\mathbf{Loc}_0}$.

The assignment $\mathfrak{A} \mapsto \mathbb{F}[\mathfrak{A}]$ canonically extends to a functor $\mathbb{F} : \mathbf{AQFT} \to \mathbf{tPFA}$.

We can finally prove that the functor $\mathbb{F}$ from Theorem 2.3.7 restricts to functors $\mathbb{F} : \mathbf{AQFT}^{\mathrm{add}} \to \mathbf{tPFA}^{\mathrm{add}}$, $\mathbb{F} : \mathbf{AQFT}^{c} \to \mathbf{tPFA}^{c}$ and $\mathbb{F} : \mathbf{AQFT}^{\mathrm{add},c} \to \mathbf{tPFA}^{\mathrm{add},c}$.

Proposition 2.3.8. *Let $\mathfrak{A} \in \mathbf{AQFT}$. Then $\mathfrak{A}$ is additive (respectively Cauchy constant) if and only if $\mathbb{F}[\mathfrak{A}] \in \mathbf{tPFA}$ is additive (respectively Cauchy constant). In particular, the functor $\mathbb{F} : \mathbf{AQFT} \to \mathbf{tPFA}$ from Theorem 2.3.7 restricts to the categories $\mathbf{AQFT}^{\mathrm{add}}$, $\mathbf{AQFT}^{c}$ and $\mathbf{AQFT}^{\mathrm{add},c}$ as $\mathbb{F} : \mathbf{AQFT}^{\mathrm{add}} \to \mathbf{tPFA}^{\mathrm{add}}$, $\mathbb{F} : \mathbf{AQFT}^{c} \to \mathbf{tPFA}^{c}$ and $\mathbb{F} : \mathbf{AQFT}^{\mathrm{add},c} \to \mathbf{tPFA}^{\mathrm{add},c}$ respectively.*

Proof. Let us recall that, by our construction, the underlying functors $\mathbb{F}[\mathfrak{A}] = \mathfrak{A} : \mathbf{Loc} \to \mathbf{C}$ to the category $\mathbf{C}$ coincide. It is then a consequence of Remark 2.1.9 that $\mathbb{F}[\mathfrak{A}]$ is additive if and only if $\mathfrak{A}$ is additive. Furthermore, because the forgetful functor $\mathbf{Alg}(\mathbf{C}) \to \mathbf{C}$ preserves and detects isomorphisms, it follows that $\mathbb{F}[\mathfrak{A}]$ is Cauchy constant if and only if $\mathfrak{A}$ is Cauchy constant. $\square$

2.4 EQUIVALENCE THEOREM

The aim of this section is to introduce an equivalence between the category of additive Cauchy constant time-orderable prefactorization algebras on **Loc** and the category of additive Cauchy constant algebraic quantum field theories on **Loc** by leveraging the functors $\mathbb{A} : \mathbf{PFA}^{\mathrm{add},c} \to \mathbf{AQFT}^{\mathrm{add},c}$ and $\mathbb{F} : \mathbf{AQFT}^{\mathrm{add},c} \to \mathbf{tPFA}^{\mathrm{add},c}$ in Sections 2.1 and 2.3. In particular, we consider the factorization of $\mathbb{A} : \mathbf{PFA}^{\mathrm{add},c} \to \mathbf{AQFT}^{\mathrm{add},c}$ through the category $\mathbf{tPFA}^{\mathrm{add},c}$ and denote it by $\mathbb{A} : \mathbf{tPFA}^{\mathrm{add},c} \to \mathbf{AQFT}^{\mathrm{add},c}$. Moreover, using this equivalence, we discuss how to introduce $*$-involutions for additive Cauchy constant time-orderable prefactorization algebras. To conclude, we apply our equivalence Theorem to the example of the free Klein-Gordon field to recover the results from [GR17].

Theorem 2.4.1. *The functors $\mathbb{A} : \mathbf{tPFA}^{\mathrm{add},c} \to \mathbf{AQFT}^{\mathrm{add},c}$ (see Section 2.1 and the previous discussion) and $\mathbb{F} : \mathbf{AQFT}^{\mathrm{add},c} \to \mathbf{tPFA}^{\mathrm{add},c}$ (see Section 2.3) are inverse to each other. In particular, the categories $\mathbf{AQFT}^{\mathrm{add},c}$ and $\mathbf{tPFA}^{\mathrm{add},c}$ are isomorphic*

Proof. The only non-trivial check to confirm that $\mathbb{A} \circ \mathbb{F} = \mathrm{id}_{\mathbf{AQFT}^{\mathrm{add},c}}$ amounts to show that, for every $\mathfrak{A} \in \mathbf{AQFT}^{\mathrm{add},c}$, the multiplications on $\mathbb{A}[\mathbb{F}[\mathfrak{A}]](M)$ and on $\mathfrak{A}(M)$ coincide, for all $M \in \mathbf{Loc}$. By (2.2.1) and (2.3.7), the multiplication on $\mathbb{A}[\mathbb{F}[\mathfrak{A}]](M)$ is given by

$$\mathfrak{A}(M)^{\otimes 2} \xleftarrow[\cong]{\mathfrak{A}(\iota^M_{U_+})\otimes\mathfrak{A}(\iota^M_{U_-})} \mathfrak{A}(U_+) \otimes \mathfrak{A}(U_-) \xrightarrow{\mathfrak{A}(\iota^M_{U_+})\otimes\mathfrak{A}(\iota^M_{U_-})} \mathfrak{A}(M)^{\otimes 2} \xrightarrow{\mu_M} \mathfrak{A}(M) \quad , \tag{2.4.1}$$

where $\iota^M_{\underline{U}} = (\iota^M_{U_+}, \iota^M_{U_-}) : \underline{U} \to M$ is any object of $\mathbf{P}_M$. This clearly coincides with the original multiplication μ_M on $\mathfrak{A}(M)$.

Conversely, to show that $\mathbb{F} \circ \mathbb{A} = \mathrm{id}_{\mathbf{tPFA}^{\mathrm{add},c}}$, we have to confirm that the time-ordered products of $\mathbb{F}[\mathbb{A}[\mathfrak{F}]] \in \mathbf{tPFA}^{\mathrm{add},c}$ coincide with the original time-ordered products of $\mathfrak{F} \in \mathbf{tPFA}^{\mathrm{add},c}$. In arity $n = 0$ and $n = 1$ this is obvious. For $n \geq 2$, this is more complicated and requires some preparations. Using equivariance under permutation actions, it is sufficient to compare the time-ordered products for *time-ordered* (in contrast to time-orderable) tuples $\underline{f} = (f_1, \dots, f_n) : \underline{M} \to N$. Because of additivity, we can further restrict to the case where $\underline{f} : \underline{M} \to N$ has relatively compact images, i.e. $f_i(M_i) \subseteq N$ is relatively compact, for all $i = 1, \dots, n$. We shall now show that, due to Cauchy constancy, we can further restrict our attention to time-ordered tuples $\underline{h} = (h_1, \dots, h_n) : \underline{L} \to N$ with relatively compact images for which there exists a Cauchy surface Σ of N such that

$$h_1(L_1), \dots, h_{n-1}(L_{n-1}) \subseteq \Sigma_+ := I^+_N(\Sigma) \subseteq N \qquad \text{and} \qquad h_n(L_n) \subseteq \Sigma_- := I^-_N(\Sigma) \subseteq N \quad . \tag{2.4.2}$$

Indeed, given any time-ordered tuple $\underline{f} : \underline{M} \to N$ with relatively compact images, we shall prove below that there exists a family of Cauchy morphisms $g_i : L_i \xrightarrow{c} M_i$, for $i = 1, \dots, n$, such that $\underline{h} := \underline{f}(g_1, \dots, g_n) = (f_1\, g_1, \dots, f_n\, g_n) : \underline{L} \to N$ admits a Cauchy surface Σ that satisfies (2.4.2). Cauchy constancy and the fact that the time-ordered products of $\mathbb{F}[\mathbb{A}[\mathfrak{F}]]$ and $\mathfrak{F}$ agree in arity $n = 1$ then implies that $\mathbb{F}[\mathbb{A}[\mathfrak{F}]](\underline{f}) = \mathfrak{F}(\underline{f})$

if and only if $\mathbb{F}[\mathbb{A}[\mathfrak{F}]](\underline{h}) = \mathfrak{F}(\underline{h})$. To exhibit such a family of Cauchy morphisms for $\underline{f} : \underline{M} \to N$, let us choose Cauchy surfaces Σ_i of M_i, for $i = 1, \dots, n$, and define $L_i := I^+_{M_i}(\Sigma_i)$, for $i = 1, \dots, n-1$, and $L_n := I^-_{M_n}(\Sigma_n)$. Let us further define $g_i := \iota^{M_i}_{L_i} : L_i \xrightarrow{c} M_i$ by subset inclusion, for $i = 1, \dots, n$. A Cauchy surface Σ of N is constructed by extending via [BS06, Theorem 3.8] the compact and achronal subset

$$\widetilde{\Sigma} := \bigcup_{i=1}^{n} \Big(\overline{f_i(\Sigma_i)} \Big\backslash I^+_N \Big(\bigcup_{j=i+1}^{n} \overline{f_j(\Sigma_j)} \Big) \Big) \subseteq N \quad . \tag{2.4.3}$$

By direct inspection one observes that Σ fulfils (2.4.2).

Using (2.4.2), we obtain a factorization

$$\underline{h} = \iota^N_{\underline{\Sigma}} \big((h_1^{\Sigma_+}, \dots, h_{n-1}^{\Sigma_+}), h_n^{\Sigma_-} \big) \quad , \tag{2.4.4}$$

where on the right-hand side we regard $h_i^{\Sigma_+} : L_i \to \Sigma_+$ as morphisms to Σ_+, for $i = 1, \dots, n-1$, and $h_n^{\Sigma_-} : L_n \to \Sigma_-$ as a morphism to Σ_-. Iterating this construction, we observe that it is sufficient to prove that $\mathbb{F}[\mathbb{A}[\mathfrak{F}]](\iota^N_{\underline{\Sigma}}) = \mathfrak{F}(\iota^N_{\underline{\Sigma}})$, for all $\iota^N_{\underline{\Sigma}} = (\iota^N_{\Sigma_+}, \iota^N_{\Sigma_-}) : \underline{\Sigma} \to N$, where $N \in \mathbf{Loc}$ and the Cauchy surface Σ of N is arbitrary. Using (2.3.7) and (2.2.1), we obtain that $\mathbb{F}[\mathbb{A}[\mathfrak{F}]](\iota^N_{\underline{\Sigma}}) : \mathfrak{F}(\Sigma_+) \otimes \mathfrak{F}(\Sigma_-) \to \mathfrak{F}(N)$ is given by

$$\mathfrak{F}(\Sigma_+) \otimes \mathfrak{F}(\Sigma_-) \xrightarrow{\mathfrak{F}(\iota^N_{\Sigma_+}) \otimes \mathfrak{F}(\iota^N_{\Sigma_-})} \mathfrak{F}(N) \otimes \mathfrak{F}(N) \xleftarrow[\cong]{\mathfrak{F}(\iota^N_{\Sigma_+}) \otimes \mathfrak{F}(\iota^N_{\Sigma_-})} \mathfrak{F}(\Sigma_+) \otimes \mathfrak{F}(\Sigma_-) \xrightarrow{\mathfrak{F}(\iota^N_{\underline{\Sigma}})} \mathfrak{F}(N) \quad , \tag{2.4.5}$$

which clearly coincides with the original time-ordered product $\mathfrak{F}(\iota^N_{\underline{\Sigma}}) : \mathfrak{F}(\Sigma_+) \otimes \mathfrak{F}(\Sigma_-) \to \mathfrak{F}(N)$. This concludes our proof. □

Remark 2.4.2. We mentioned in Remark 2.3.6 that the functor $\phi : \mathcal{P}_{t\mathbf{Loc}} \to \mathcal{O}_{\mathbf{Loc}^{\perp_c}}$ induces two functors, namely the pullback $\phi^* = \mathbb{F} : \mathbf{AQFT} \to \mathbf{tPFA}$ and the multicategorical left Kan extension $\phi_! : \mathbf{tPFA} \to \mathbf{AQFT}$ along ϕ. It is clear then, after restricting to the appropriate domains and codomains, that the functor $\phi_!$ and $\mathbb{A}$ coincide (are equivalent), being adjoint to the same functor $\mathbb{F}$.

Note, that in order to obtain our results, we could have avoided taking into account the multicategorical perspective, but we expect it to be crucial for higher-categorical generalizations of our equivalence Theorem, i.e. to situations in which the target monoidal category $\mathbf{C}$ is replaced by some higher category or model category. These generalizations are needed to describe quantum gauge theories in terms of factorization algebras and algebraic quantum field theories. Moreover, it can be noticed how removing the burden of sticking to a particular perspective, i.e. the freedom of passing from a categorical to a multicategorical perspective and vice-versa, helps clarifying and explaining the results. △

2.4.1 *Transfer of $*$-involutions*

In Remark 1.2.13 we observed that the category of $\mathbf{Vec}_{\mathbb{C}}$-valued algebraic quantum field theories naturally carries the structure of an involutive category $(\mathbf{AQFT}, \overline{(-)}, \mathrm{id})$.

It is not difficult to see that the involutive structure $\overline{(-)} : \mathbf{AQFT} \to \mathbf{AQFT}$ restricts to an involutive structure functor $\overline{(-)} : \mathbf{AQFT}^{\mathrm{add},c} \to \mathbf{AQFT}^{\mathrm{add},c}$. In particular, we can use the isomorphism in Theorem 2.4.1 to obtain a transferred involutive structure $\overline{(-)} : \mathbf{tPFA}^{\mathrm{add},c} \to \mathbf{tPFA}^{\mathrm{add},c}$ and hence an involutive category $(\mathbf{tPFA}, \overline{(-)}, \mathrm{id})$. More explicitly the functor $\overline{(-)} : \mathbf{tPFA}^{\mathrm{add},c} \to \mathbf{tPFA}^{\mathrm{add},c}$ assigns to any $\mathfrak{F} \in \mathbf{tPFA}^{\mathrm{add},c}$ its conjugate $\overline{\mathfrak{F}}$ given by $\mathbb{F}[\overline{\mathbb{A}[\mathfrak{F}]}]$. We can then consider the category ${}^{*}\mathbf{tPFA}^{\mathrm{add},c}$ of $*$-objects in $\mathbf{tPFA}^{\mathrm{add},c}$ (see Definition 1.1.32).

Notice that the definition of the involutive structure $\overline{(-)} : \mathbf{tPFA}^{\mathrm{add},c} \to \mathbf{tPFA}^{\mathrm{add},c}$ relies on additivity and Cauchy constancy. In particular, the transfer of the $*$-structure is not generalizable to *all* time-orderable prefactorization algebras. The reason for this insufficiency lies on the fact that Cauchy constancy is needed to define the conjugate factorization products. More specifically, let $M \in \mathbf{Loc}$ and let Σ_+ and Σ_- be respectively the chronological past and the chronological future of a Cauchy hypersurface $\Sigma \subseteq M$. The complex conjugate $\overline{\mathfrak{F}}(\iota^M_{(\Sigma_+,\Sigma_-)}) : \overline{\mathfrak{F}}(\Sigma_+) \otimes \overline{\mathfrak{F}}(\Sigma_-) \to \overline{\mathfrak{F}}(M)$, where $\iota^M_{\Sigma_\pm} : \Sigma_\pm \to M$ denotes the inclusion of $\Sigma_\pm$ in M, is given by the following diagram:

$$
\begin{tikzcd}
\overline{\mathfrak{F}(\Sigma_+)} \otimes \overline{\mathfrak{F}(\Sigma_-)} \arrow[rr, "\overline{\mathfrak{F}}(\iota^M_{(\Sigma_+,\Sigma_-)})"] \arrow[d, equal] & & \overline{\mathfrak{F}(M)} \\
\overline{\mathfrak{F}(\Sigma_+) \otimes \mathfrak{F}(\Sigma_-)} \arrow[r, "\overline{\mathfrak{F}(\iota^M_{\Sigma_+}) \otimes \mathfrak{F}(\iota^M_{\Sigma_-})}"'] & \overline{\mathfrak{F}(M) \otimes \mathfrak{F}(M)} & \overline{\mathfrak{F}(\Sigma_-) \otimes \mathfrak{F}(\Sigma_+)} \arrow[l, "\cong", "\overline{\mathfrak{F}(\iota^M_{\Sigma_-}) \otimes \mathfrak{F}(\iota^M_{\Sigma_+})}"'] \arrow[u, "\overline{\mathfrak{F}(\iota^M_{(\Sigma_-,\Sigma_+)})}"']
\end{tikzcd}
\tag{2.4.6}
$$

Notice that the rightmost arrow in the second line uses explicitly Cauchy constancy, therefore, it is unclear, at least to us, whether it is possible to get rid of this issue and define an involutive structure on $\mathbf{tPFA}$.

2.4.2 *Free Klein-Gordon*

In this subsection we recall the description of the Free Klein-Gordon field $\mathfrak{A}_{\mathrm{KG}} \in \mathbf{AQFT}^{\mathrm{add},c}$ in terms of Algebraic Quantum Field Theory and use the isomorphism in Theorem 2.4.1 to obtain the corresponding description $\mathbb{F}[\mathfrak{A}_{\mathrm{KG}}] \in \mathbf{tPFA}^{\mathrm{add},c}$ in terms of Factorization Algebras.

Example 2.4.3. Let $M \in \mathbf{Loc}$ be a spacetime and let P_M denote the Klein-Gordon operator defined by $-\Box_M + m^2 : C^\infty(M) \to C^\infty(M)$, where $\Box_M$ is the d'Alembert operator and $m^2 \geq 0$ is a mass parameter and let $G^\pm_M : C^\infty_c(M) \to C^\infty(M)$ denote the uniquely defined retarded/advanced Green's operator associated to P_M, where the subscript 'c' denotes compactly supported functions. Then, the real vector space of linear observables is defined by the following cokernel:

$$
C^\infty_c(M) \xrightarrow{P_M} C^\infty_c(M) \longrightarrow \mathcal{V}(M) := C^\infty_c(M)/P_M(C^\infty_c(M)) \quad . \tag{2.4.7}
$$

Since $C^\infty_c : \mathbf{Loc} \to \mathbf{Vec}_{\mathbb{R}}$ is a cosheaf with respect to (causally convex) open covers, $P : C^\infty_c \to C^\infty_c$ is a natural transformation and cosheaves cokernels can be computed point-wise, it follows that $\mathcal{V} : \mathbf{Loc} \to \mathbf{Vec}_{\mathbb{R}}$ is a cosheaf too. The associative and unital algebra $\mathfrak{A}_{\mathrm{KG}}(M) := \big(\mathrm{Sym}_{\mathbb{C}}(\mathcal{V}(M)), \star_M, \eta_M\big) \in \mathbf{Alg}_{\mathbb{C}}$ that the algebraic quantum field theory version of the Free Klein-Gordon field assigns to M is a deformation of the complexified symmetric commutative algebra $\mathrm{Sym}_{\mathbb{C}}(\mathcal{V}(M)) \in \mathbf{Alg}_{\mathsf{Com}}(\mathbf{Vec}_{\mathbb{C}})$ (see Remark 1.1.21) obtained in the following way:

We define a differential $\mathrm{d} : \mathrm{Sym}_{\mathbb{C}}(\mathcal{V}(M)) \to \mathrm{Sym}_{\mathbb{C}}(\mathcal{V}(M)) \otimes \mathcal{V}(M)$ by setting on monomials

$$\mathrm{d}\big(\varphi_1 \cdots \varphi_n\big) := \sum_{i=1}^{n} \varphi_1 \cdots \overset{i}{\vee} \cdots \varphi_n \otimes \varphi_i \quad , \tag{2.4.8}$$

where $\overset{i}{\vee}$ means omission of φ_i. Using the causal propagator $G_M := G_M^+ - G_M^- : \mathcal{V}(M) \to \ker P_M$ and the integration map $\int_M : \mathcal{V}(M) \otimes \ker P_M \to \mathbb{R}$, $\varphi \otimes \Phi \mapsto \int_M \varphi\, \Phi\, \mathrm{vol}_M$, we define the bi-differential operator

$$\begin{tikzcd}
\mathrm{Sym}_{\mathbb{C}}(\mathcal{V}(M))^{\otimes 2} \arrow[r, "\langle G_M, \mathrm{d}\otimes\mathrm{d}\rangle"] \arrow[d, "(\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(\mathrm{d}\otimes\mathrm{d})"'] & \mathrm{Sym}_{\mathbb{C}}(\mathcal{V}(M))^{\otimes 2} \\
\mathrm{Sym}_{\mathbb{C}}(\mathcal{V}(M))^{\otimes 2} \otimes \mathcal{V}(M) \otimes \mathcal{V}(M) \arrow[r, "\mathrm{id}\otimes\mathrm{id}\otimes G_M"'] & \mathrm{Sym}_{\mathbb{C}}(\mathcal{V}(M))^{\otimes 2} \otimes \mathcal{V}(M) \otimes \ker P_M \arrow[u, "\mathrm{id}\otimes\int_M"']
\end{tikzcd} \tag{2.4.9}$$

where we recall that τ is the symmetric braiding on $\mathbf{Vec}_{\mathbb{C}}$, i.e. the flip map. The $\star$-product $\star_M : \mathrm{Sym}_{\mathbb{C}}(\mathcal{V}(M))^{\otimes 2} \to \mathrm{Sym}_{\mathbb{C}}(\mathcal{V}(M))$ is defined by composing

$$\mathrm{Sym}_{\mathbb{C}}(\mathcal{V}(M))^{\otimes 2} \xrightarrow{\exp\left(\frac{i}{2}\langle G_M, \mathrm{d}\otimes\mathrm{d}\rangle\right)} \mathrm{Sym}_{\mathbb{C}}(\mathcal{V}(M))^{\otimes 2} \xrightarrow{\cdot_M} \mathrm{Sym}_{\mathbb{C}}(\mathcal{V}(M)) \quad , \tag{2.4.10}$$

where $\cdot_M$ denotes the commutative product on $\mathrm{Sym}_{\mathbb{C}}(\mathcal{V}(M))$. (The exponential series converges because it terminates for polynomials.) The unit η_M is the unit of $\mathrm{Sym}_{\mathbb{C}}(\mathcal{V}(M))$, for all $M \in \mathbf{Loc}$.

In particular, $\mathfrak{A}_{\mathrm{KG}} \in \mathbf{AQFT}^{\mathrm{add},c}$, where the additivity follows from the fact that $\mathcal{V} : \mathbf{Loc} \to \mathbf{Vec}_{\mathbb{R}}$ is a cosheaf. $\triangledown$

To understand the time-orderable prefactorization algebra $\mathbb{F}[\mathfrak{A}_{\mathrm{KG}}]$ obtained from $\mathfrak{A}_{\mathrm{KG}}$ by applying the isomorphism from Theorem 2.4.1 it is instructive to compute explicitly the factorization products $\mathbb{F}[\mathfrak{A}_{\mathrm{KG}}](\underline{f})$, where $\underline{f} = (f_1 : M_1 \to N, f_2 : M_2 \to N)$ is a time-orderable couple of $\mathbf{Loc}$-morphisms (see Definition 2.3.1). We can distinguish two cases:

(a) If $\underline{f}$ is time-ordered, we obtain from (2.3.7), (2.4.10) and the support properties of $G_N^{\pm}$ that

$$\mathfrak{F}_{\mathrm{KG}}(\underline{f}) = \cdot_N \circ \exp\left(\tfrac{i}{2}\langle G_N^+, \mathrm{d}\otimes\mathrm{d}\rangle\right) \circ \big(\mathfrak{A}_{\mathrm{KG}}(f_1) \otimes \mathfrak{A}_{\mathrm{KG}}(f_2)\big). \tag{2.4.11a}$$

(b) If $\underline{f} = (f_1, f_2) : \underline{M} \to N$ is anti-time-ordered, i.e. $J_N^+(f_2(M_2)) \cap f_1(M_1) = \emptyset$, we obtain

$$\mathfrak{F}_{\mathrm{KG}}(\underline{f}) = \cdot_N \circ \exp\left(\tfrac{i}{2}\langle G_N^-, \mathrm{d}\otimes\mathrm{d}\rangle\right) \circ \big(\mathfrak{A}_{\mathrm{KG}}(f_1) \otimes \mathfrak{A}_{\mathrm{KG}}(f_2)\big). \tag{2.4.11b}$$

In particular, using once more the support properties of $G_N^{\pm}$, we obtain a combined formula for the two cases:

$$\mathfrak{F}_{\mathrm{KG}}(\underline{f}) = \cdot_N \circ \exp\big(i\,\langle G_N^{\mathrm{D}}, \mathrm{d}\otimes\mathrm{d}\rangle\big) \circ \big(\mathfrak{A}_{\mathrm{KG}}(f_1)\otimes\mathfrak{A}_{\mathrm{KG}}(f_2)\big) \quad (\underline{f}\text{ time-orderable}) \quad , \tag{2.4.12}$$

where $G_N^{\mathrm{D}} := \frac{1}{2}(G_N^+ + G_N^-)$ is the so-called Dirac propagator. In perturbative algebraic quantum field theory (see e.g. [FR12]), the products $\cdot_{\mathcal{T}_N} := \cdot_N \circ \exp\big(i\,\langle G_N^{\mathrm{D}}, \mathrm{d}\otimes\mathrm{d}\rangle\big)$ are called time-ordered products, henceforth, the additive Cauchy constant time-orderable prefactorization algebra $\mathbb{F}[\mathfrak{A}_{\mathrm{KG}}]$ associated to the free Klein-Gordon additive Cauchy constant algebraic quantum field theory $\mathfrak{A}_{\mathrm{KG}}$ encodes the time-ordered products obtained via the Dirac propagator. In particular, this result agrees with [GR17].

3

CATEGORIFICATION OF AQFTS

The aim of this chapter is to introduce a 2-categorical version of algebraic quantum field theories drawing framework and results from our paper [BPSW21]. As we have seen in the previous chapters, an algebraic quantum field theory is, broadly speaking, a functor that associates *associative* and *unital algebras* of observables to spacetimes, satisfying some physically motivated axioms, e.g. *Einstein causality*. Although this perspective has been proven fruitful it is not sufficient to capture the higher categorical features that appear in gauge theories. Such insufficiency has been already widely noticed and attempts to deal with it have been explored (see e.g. [FR12], [FR13] and [BSW19b] for an approach that considers homotopy-coherent algebraic quantum field theories with values in differential graded algebras). Our claim is that the 2-categorical approach we propose is more sensitive to global aspects of quantum gauge theories than previous approaches (see [FR12, FR13, BSW19b]).

We will prove (see Section 3.1) that any $\mathbf{Vec}_{\mathbb{K}}$-valued algebraic quantum field theory can be interpreted as an $\mathbf{Alg}_{\mathbb{K}}$-valued prefactorization algebra, i.e. as a multifunctor $\mathfrak{A} : \mathcal{P}_{\mathbf{Sp}^{\perp}} \to \mathbf{Alg}_{\mathbb{K}}$. Hence, we will leverage this fact to define 2-algebraic quantum field theories as *pseudo-multifunctors* $\mathcal{P}_{\mathbf{Sp}^{\perp}} \to \mathbf{Pr}_{\mathbb{K}}$, where $\mathbf{Sp}^{\perp} = (\mathbf{Sp}, \perp)$ is an orthogonal category, $\mathcal{P}_{\mathbf{Sp}^{\perp}}$ is the prefactorization algebra multicategory (see Definition 1.1.9) on $\mathbf{Sp}^{\perp}$ and $\mathbf{Pr}_{\mathbb{K}}$ is the 2-multicategory of *locally presentable* $\mathbb{K}$*-linear categories* (Section 3.2). Saying it otherwise, a 2-algebraic quantum field theory is a 2-categorical multifunctor that assigns a locally presentable $\mathbb{K}$-linear category to each spacetime $s \in \mathbf{Sp}$. To understand why we want our 2-algebraic quantum field theories to assign locally presentable $\mathbb{K}$-linear categories to spacetimes $s \in \mathbf{Sp}$ it is worth recalling why an *ordinary* (i.e. 1-categorical) algebraic quantum field theory assigns associative and unital $\mathbb{K}$-algebras. The algebra $\mathfrak{A}$ should be interpreted as a quantization of the commutative algebra $\mathcal{O}(X)$ of functions on the phase space X of a physical system. If X is "nice", i.e. it is an affine scheme, then passing to its function algebra does not lose any information, hence it is justified to quantize X by deforming its function algebra $\mathcal{O}(X)$ to a noncommutative algebra $\mathfrak{A}$. However, many important examples of phase spaces that feature in physics are *not* of this "nice" kind. For instance, if the phase space X is a *stack* (for example the classifying stack $\mathbf{B}G$ of a group G), as happens in a gauge theory, it is generically not true that X is faithfully encoded by its function algebra $\mathcal{O}(X)$, which, in this case, is a differential graded algebra (see [Toe14, BSW19b]). For instance, if $X := \mathbf{B}\mathbb{Z}_2$ the dg-algebra $\mathcal{O}(\mathbf{B}\mathbb{Z}_2)$ coincides with the dg-algebra $C^\bullet(\mathbb{Z}_2, \mathbb{K})$ of group cochains with values in the trivial $\mathbb{Z}_2$-representation $\mathbb{K}$. Hence, since for all $n \neq 0$ all the cohomology groups $H^n(\mathbb{Z}_2, \mathbb{K}) = 0$ are trivial and since $H^0(\mathbb{Z}_2, \mathbb{K}) = \mathbb{K}$, we obtain that $\mathcal{O}(\mathbf{B}\mathbb{Z}_2) \simeq \mathbb{K} = \mathcal{O}(\{*\})$. Therefore, the function algebra loses all the information regarding the group $\mathbb{Z}_2$.

A possible solution to this insufficiency is obtained by considering, in place of the function algebra $\mathcal{O}(X)$, the *category* $\mathbf{QCoh}(X)$ of quasi-coherent sheaves over X. In fact, it can be proven that $\mathbf{QCoh}(X)$, which should be thought as a well behaved analogue of the category of vector bundles on a ringed space, fully faithfully encodes the space X. More precisely a theorem of Lurie (see [Lur04]) states that a geometric stack X can be reconstructed from its category $\mathbf{QCoh}(X)$ of quasi-coherent sheaves. For example, if X is an affine scheme, $\mathbf{QCoh}(X) \cong \mathbf{Mod}_{\mathcal{O}(X)}$, hence, the algebra $\mathcal{O}(X)$ can be recovered as the endomorphism algebra $\mathrm{End}(\mathcal{O}(X))$ of the free rank 1-module $\mathcal{O}(X)$. When X is not affine, things get more interesting. For instance, if $X = \mathbf{B}G$, we obtain that $\mathbf{QCoh}(X)$ is equivalent to the *symmetric monoidal category* $\mathbf{Rep}_{\mathbb{K}}(G)$ of $\mathbb{K}$-linear representations of G, which is, in general, much richer than the dg-algebra $\mathcal{O}(\mathbf{B}G)$.

The question we have to answer then is: what does it mean to quantize $\mathbf{QCoh}(X)$? When X is an affine scheme $\mathbf{QCoh}(X) \cong \mathbf{Mod}_{\mathcal{O}(X)}$, where $\mathbf{Mod}_{\mathcal{O}(X)}$ is the *symmetric monoidal category* of right $\mathcal{O}(X)$-modules. Therefore, given an algebra $\mathfrak{A}$ that quantizes $\mathcal{O}(X)$, a natural choice for the quantization of $\mathbf{QCoh}(X)$ is the category $\mathbf{Mod}_{\mathfrak{A}}$ of right $\mathfrak{A}$-modules. Since $\mathfrak{A}$ is in general non-commutative, the category $\mathbf{Mod}_{\mathfrak{A}}$ can not be equipped with the relative tensor product $\otimes_A$, however, it is a pointed (i.e. endowed with a distinguished object, namely the rank 1-module $\mathfrak{A}$) locally presentable (i.e. "obtained from small generators" and relations) $\mathbb{K}$-linear category. This suggests that the quantization of $\mathbf{QCoh}(X)$ should be a pointed locally presentable $\mathbb{K}$-linear category.

Let us outline in more detail the content of this chapter.

In Section 3.1 we present the relevant background on **Cat**-*enriched multicategories* (or 2-multicategories), *pseudo-multifunctors*, *pseudo-multinatural transformations* and *multi-modifications*, i.e. multicategorical analogues of 2-categories, pseudo-functors, pseudo-multifunctors and modifications.

In Section 3.2 we begin by proving that *ordinary* (i.e. 1-categorical) algebraic quantum field theories can alternatively be described as $\mathbf{Alg}_{\mathbb{K}}$-valued prefactorization algebras and we proceed, leveraging this fact, by defining 2-*algebraic quantum field theories* as pseudo-multifunctors $\mathcal{P}_{\mathbf{Sp}^{\perp}} \to \mathbf{Pr}_{\mathbb{K}}$, where $\mathbf{Pr}_{\mathbb{K}}$ is the 2-multicategory of locally presentable $\mathbb{K}$-linear categories.

In Section 3.3 we describe a fully faithful *inclusion* pseudo-functor ι : $\mathbf{AQFT}(\mathbf{Sp}^{\perp}) \to \mathbf{2AQFT}(\mathbf{Sp}^{\perp})$ and a *truncation* 2-functor $\pi : \mathbf{2AQFT}(\mathbf{Sp}^{\perp}) \to \mathbf{AQFT}(\mathbf{Sp}^{\perp})$, where $\mathbf{2AQFT}(\mathbf{Sp}^{\perp})$ denotes the category of 2-algebraic quantum field theories over the orthogonal category $\mathbf{Sp}^{\perp} = (\mathbf{Sp}, \perp)$. More precisely, the inclusion pseudo-functor takes an ordinary algebraic quantum field theory $\mathfrak{A}$ and produces a 2-algebraic quantum field theory $\iota(\mathfrak{A})$ that assigns to every spacetime $s \in \mathbf{Sp}$ the locally presentable $\mathbb{K}$-linear category $\mathbf{Mod}_{\mathfrak{A}(s)}$. The truncation 2-functor takes a 2-algebraic quantum field theory $\mathfrak{A} \in \mathbf{2AQFT}(\mathbf{Sp}^{\perp})$ and produces an ordinary algebraic quantum field theory $\pi(\mathfrak{A}) \in \mathbf{AQFT}(\mathbf{Sp}^{\perp})$ assigning to every spacetime $s \in \mathbf{Sp}$ the associative and unital $\mathbb{K}$-algebra $\mathrm{End}(a_s)$, where a_s is the *pointing* of $\mathfrak{A}(s)$, i.e. the object that is picked out by the co-continuous $\mathbb{K}$-linear functor $\mathfrak{A}(*_s) : \mathbf{Mod}_{\mathbb{K}} \to \mathfrak{A}(s)$, image of the only 0-operation $*_s \in \mathcal{P}_{\mathbf{Sp}^\perp}\binom{s}{\varnothing}$ (any co-continuous $\mathbb{K}$-linear functor $F : \mathbf{Mod}_{\mathbb{K}} \to A$, where A is a locally presentable $\mathbb{K}$-linear category, is fully determined by $F(\mathbb{K})$, the image of the rank 1 free module $\mathbb{K} \in \mathbf{Mod}_{\mathbb{K}}$). Moreover, we prove that $\iota \dashv \pi$ forms a *biadjoint* pair of morphisms that exhibits $\mathbf{AQFT}(\mathbf{Sp}^\perp)$ as a *coreflective full* 2*-subcategory* of $\mathbf{2AQFT}(\mathbf{Sp}^\perp)$. This means that applying the inclusion and the truncation functor to an ordinary algebraic quantum field theory $\mathfrak{A} \in \mathbf{AQFT}(\mathbf{Sp}^\perp)$ in succession one obtains an ordinary algebraic quantum field theory equivalent to $\mathfrak{A}$ itself (which is easily seen by considering the following chain of equivalences $\pi(\iota(\mathfrak{A}(s))) \cong \pi(\mathbf{Mod}_{\mathfrak{A}(s)}) \cong \mathrm{End}(\mathfrak{A}(s)) \cong \mathfrak{A}(s)$). Therefore, we conclude that ordinary algebraic quantum field theories can be equivalently studied in the 2-category of 2-algebraic quantum field theories. Finally, we introduce a notion of "truncatedness" with the following meaning: we say that a 2-algebraic quantum field theory is *truncated* if it is equivalent to an algebraic quantum field theory of the form $\iota(\mathfrak{A})$, where $\mathfrak{A}$ is an ordinary algebraic quantum field theory.

In Section 3.4 we introduce a *gauging construction*, i.e. a functor associating to any ordinary algebraic quantum field theory $\mathfrak{A} \in \mathbf{AQFT}(\mathbf{Sp}^\perp)$ endowed with a G-action $\rho : G \to \mathrm{Aut}(\mathfrak{A})$, a 2-algebraic quantum field theory $\mathfrak{A}^G$ defined by sending each $s \in \mathbf{Sp}$ to the locally presentable $\mathbb{K}$-linear category $G\text{-}\mathbf{Mod}_{\mathfrak{A}(s)}$ of G-equivariant right $\mathfrak{A}(s)$-modules (i.e. the category whose objects are $\mathbb{K}$-linear representations $V \in \mathbf{Mod}_{\mathbb{K}}$ of G endowed with a G-equivariant right $\mathfrak{A}(s)$-action $V \otimes \mathfrak{A}(s) \to V$). We will see that $\mathfrak{A}^G$ can be physically interpreted as a local gauging of $\mathfrak{A}$ with respect to G and we call it the *categorified orbifold construction* of $(\mathfrak{A}, \rho)$. Finally, we show that this construction provides toy-models for non-truncated 2-algebraic quantum field theories and use Hopf-Galois theory to determine when categorified orbifold constructions are truncated.

In Section 3.5 we introduce a categorification of Fredenhagen's universal algebra (see Example 1.2.20), called *Fredenhagen's universal category*, obtained by noticing (see Theorem 1.1.26) that any orthogonal functor $J : \mathbf{D}^\perp \to \mathbf{E}^\perp$ gives rise to an extension/restriction *biadjunction* $J_! : \mathbf{2AQFT}(\mathbf{D}^\perp) \cong \mathbf{Alg}_{\mathcal{P}_{\mathbf{D}^\perp}}(\mathbf{Pr}_K) \leftrightarrows \mathbf{Alg}_{\mathcal{P}_{\mathbf{E}^\perp}}(\mathbf{Pr}_{\mathbb{K}}) \cong \mathbf{2AQFT}(\mathbf{E}^\perp) : J$. To conclude, we use this construction to study extensions to the entire circle $\mathbb{S}^1$ of 2-algebraic quantum field theories defined in proper connected open subsets $I \subset \mathbb{S}^1$.

3.1 **Cat**-ENRICHED MULTICATEGORIES

In this section we briefly recall the theory of **Cat**-enriched multicategories (or 2-multicategories) by introducing *pseudo-multifunctors*, *pseudo-multinatural transformations* and *multimodifications* ([BPSW21]), multicategorical analogues of pseudo-functors, pseudo-natural transformations and modifications (see e.g. [Lei04, SP09, Lac10]). It is worth noticing that our approach is slightly different from the one considered by Corner and Gurski (see [CG13]) since we allow pseudo-multifunctors to preserve permutation actions only up to isomorphism. The reason for this difference is that our quantum field theoretic examples come with non-trivial coherences for permutations.

Remark 3.1.1. Generalizing multicategories, multifunctors and multinatural transformations to the 2-categorical setting can be done in several ways, depending on the degree of strictness one wants the coherence axioms to satisfy. For our purposes it is sufficient to consider **Cat**-enriched (or 2-strict multicategories), the strictest form of 2-categorical multicategory, pseudo-multifunctors and pseudo-multinatural transformations, i.e. 2-categorical multifunctors and multinatural transformations whose coherence axioms are determined up to isomorphism. △

Definition 3.1.2. A **Cat**-enriched symmetric multicategory $\mathcal{O}$ consists of the following data:

(a) A collection of objects (or *colours*) $\mathcal{O}_0$.

(b) For all $t \in \mathcal{O}_0$, $n \geq 0$, $\underline{c} := (c_1, \ldots, c_n) \in \mathcal{O}_0^n$, a category $\mathcal{O}\binom{t}{\underline{c}}$ whose objects are called 1-*morphisms* (or n-operations, if one wants to specify the arities of 1-morphisms) and are denoted with symbols like ϕ, ψ and whose maps are called 2-*morphisms* and are denoted with symbols like α, β.

(c) For all $t \in \mathcal{O}_0$, $n \geq 1$, $\underline{a} \in \mathcal{O}_0^n$, $m_i \geq 0$, $\underline{b}_i \in \mathcal{O}_0^{m_i}$, with $i = 1, \ldots, n$, *composition* functors $\gamma : \mathcal{O}\binom{t}{\underline{a}} \times \prod_{i=1}^n \mathcal{O}\binom{a_i}{\underline{b}_i} \to \mathcal{O}\binom{t}{\underline{\underline{b}}}$, where

$$\underline{\underline{b}} = (b_{11}, \ldots, b_{1m_1}, \ldots, b_{n1} \ldots, b_{nm_n}).$$

We write compactly $\phi\,\underline{\psi} := \gamma(\phi, (\psi_1, \ldots, \psi_n))$ for the composition of 1-morphisms and $\alpha * \underline{\beta} = \gamma(\alpha, (\beta_1, \ldots, \beta_n))$ for the composition of 2-morphisms. We denote by Id the identity 2-morphisms.

(d) For every $t \in \mathcal{O}_0$ functors $\mathbb{1}_t : \mathbf{1} \to \mathcal{O}\binom{t}{t}$, where $\mathbf{1}$ is the category with one object. We also write $\mathbb{1}_t \in \mathcal{O}\binom{t}{t}$ for the corresponding identity 1-operations.

(e) A right action of the permutation group on n letters Σ_n on the collection of categories of n-operations for all n

$$\mathcal{O}(\sigma) : \mathcal{O}\binom{t}{\underline{c}} \to \mathcal{O}\binom{t}{\underline{c}\sigma},$$

where $\underline{c}\sigma := (c_{\sigma(1)}, \ldots, c_{\sigma(n)})$ and $\mathcal{O}(\sigma)$ is a functor for every $\sigma \in \Sigma_n$. We write $\phi \cdot \sigma := \mathcal{O}(\sigma)(\phi)$ and $\alpha \cdot \sigma := \mathcal{O}(\sigma)(\alpha)$ for the permutation action (notice the dot) while we use the symbol $\underline{\psi}\sigma$ to denote the permutation $(\psi_{\sigma(1)}, \ldots, \psi_{\sigma(n)})$ of an n-tuple of 1-morphisms $(\psi_1, \ldots, \psi_n)$.

Satisfying analogues of the Associativity, Unitality and the Equivariance axiom (see Definition 1.1.1)

Definition 3.1.3. Let $\mathcal{O}$ and $\mathcal{P}$ be **Cat**-enriched symmetric multicategories. A *pseudo-multifunctor* $F : \mathcal{O} \to \mathcal{P}$ is given by the following data:

(a) A map on the underlying collections of objects $F_0 : \mathcal{O}_0 \to \mathcal{P}_0$.

(b) For all $t \in \mathcal{O}_0$, $n \geq 0$ and $\underline{c} \in \mathcal{O}_0^n$, functors $F_{\underline{c}}^t : \mathcal{O}\binom{t}{\underline{c}} \to \mathcal{P}\binom{Ft}{F\underline{c}}$, where $F\underline{c} = (Fc_1, \ldots, Fc_n)$.

Note: We will drop the apexes and subscripts when clear from the context.

(c) For all $t \in \mathcal{O}_0$, $n \geq 1$, $\underline{a} \in \mathcal{O}_0^n$, $m_i \geq 0$, $\underline{b}_i \in \mathcal{O}_0^{m_i}$, with $i = 1, \dots, n$, natural isomorphisms

$$\begin{tikzcd}
\mathcal{O}\binom{t}{\underline{a}} \times \prod_{i=1}^{n} \mathcal{O}\binom{a_i}{\underline{b}_i} \arrow[r, "F \times \prod_i F"] \arrow[d, "\gamma^{\mathcal{O}}"'] & \mathcal{P}\binom{Ft}{F\underline{a}} \times \prod_{i=1}^{n} \mathcal{P}\binom{Fa_i}{F\underline{b}_i} \arrow[d, "\gamma^{\mathcal{P}}"] \arrow[dl, Rightarrow, "F^2"'] \\
\mathcal{O}\binom{t}{\underline{\underline{b}}} \arrow[r, "F"'] & \mathcal{P}\binom{Ft}{F\underline{\underline{b}}}
\end{tikzcd} \tag{3.1.1}$$

(d) For all $t \in \mathcal{O}_0$, natural isomorphisms

$$\begin{tikzcd}
\mathbf{1} \arrow[d, "\mathbb{1}^{\mathcal{O}}"'] \arrow[dr, "\mathbb{1}^{\mathcal{P}}"] \arrow[dr, phantom, "\Swarrow F^0"', near start] & \\
\mathcal{O}\binom{t}{t} \arrow[r, "F"'] & \mathcal{P}\binom{Ft}{Ft}
\end{tikzcd} \tag{3.1.2}$$

(e) For all $t \in \mathcal{O}_0$, $n \geq 0$, $\underline{c} := (c_1, \dots, c_n) \in \mathcal{O}_0^n$, $\sigma \in \Sigma_n$, natural isomorphisms

$$\begin{tikzcd}
\mathcal{O}\binom{t}{\underline{c}} \arrow[r, "F"] \arrow[d, "\mathcal{O}(\sigma)"'] & \mathcal{P}\binom{Ft}{F\underline{c}} \arrow[d, "\mathcal{P}(\sigma)"] \arrow[dl, Rightarrow, "F^{\sigma}"'] \\
\mathcal{O}\binom{t}{\underline{c}\sigma} \arrow[r, "F"'] & \mathcal{P}\binom{Ft}{F\underline{c}\sigma}
\end{tikzcd} \tag{3.1.3}$$

Satisfying the following axioms:

(a) *Preservation of compositions*:

$$\begin{tikzcd}
(F\phi)\,(F\underline{\psi})\,(F\underline{\underline{\rho}}) \arrow[r, "F^2 * \prod \mathrm{Id}"] \arrow[d, "\mathrm{Id} * \prod F^2"'] & F(\phi\,\underline{\psi})\,(F\underline{\underline{\rho}}) \arrow[d, "F^2"] \\
(F\phi)\,F(\underline{\psi}\,\underline{\underline{\rho}}) \arrow[r, "F^2"'] & F(\phi\,\underline{\psi}\,\underline{\underline{\rho}})
\end{tikzcd} \tag{3.1.4}$$

(b) *Preservation of units*:

$$\begin{tikzcd}
\mathbb{1}^{\mathcal{P}}\,(F\phi) \arrow[d, "F^0 * \mathrm{Id}"'] \arrow[dr, "\mathrm{Id}"] & \\
(F\mathbb{1}^{\mathcal{O}})\,(F\phi) \arrow[r, "F^2"'] & F(\mathbb{1}^{\mathcal{O}}\,\phi)
\end{tikzcd}
\qquad
\begin{tikzcd}
(F\phi)\prod \mathbb{1}^{\mathcal{P}} \arrow[d, "\mathrm{Id} * \prod F^0"'] \arrow[dr, "\mathrm{Id}"] & \\
(F\phi)\prod F\mathbb{1}^{\mathcal{O}} \arrow[r, "F^2"'] & F(\phi \prod \mathbb{1}^{\mathcal{O}})
\end{tikzcd} \tag{3.1.5}$$

(c) *Preservation of permutations*:

$$
\begin{tikzcd}
((F\phi)\cdot\sigma)\cdot\sigma' \arrow[r, "F^{\sigma}\cdot\sigma'"] \arrow[d, "\mathrm{Id}"'] & (F(\phi\cdot\sigma))\cdot\sigma' \arrow[d, "F^{\sigma'}"] \\
(F\phi)\cdot(\sigma\sigma') \arrow[r, "F^{\sigma\sigma'}"'] & F(\phi\cdot(\sigma\sigma'))
\end{tikzcd}
\qquad
\begin{tikzcd}
F\phi \arrow[d, "\mathrm{Id}"'] \arrow[dr, "\mathrm{Id}"] & \\
(F\phi)\cdot e \arrow[r, "F^{e}"'] & F(\phi\cdot e)
\end{tikzcd}
\tag{3.1.6}
$$

$$
\begin{tikzcd}
((F\phi)\,(F\underline{\psi}))\cdot\sigma\langle m_1,\dots,m_n\rangle \arrow[rr, "F^2\cdot\sigma\langle m_1,\dots,m_n\rangle"] \arrow[d, "\mathrm{Id}"'] & & (F(\phi\,\underline{\psi}))\cdot\sigma\langle m_1,\dots,m_n\rangle \arrow[d, "F^{\sigma\langle k_1,\dots,k_n\rangle}"] \\
((F\phi)\cdot\sigma)\,(F\underline{\psi}\sigma) \arrow[d, "F^{\sigma}*\prod\mathrm{Id}"'] & & F\big((\phi\,\underline{\psi})\cdot\sigma\langle m_1,\dots,m_n\rangle\big) \arrow[d, "\mathrm{Id}"] \\
F(\phi\cdot\sigma)\,(F\underline{\psi}\sigma) \arrow[rr, "F^2"'] & & F\big((\phi\cdot\sigma)\,(\underline{\psi}\sigma)\big)
\end{tikzcd}
\tag{3.1.7}
$$

$$
\begin{tikzcd}
((F\phi)\,(F\underline{\psi}))\cdot(\sigma_1\oplus\cdots\oplus\sigma_n) \arrow[rr, "F^2\cdot(\sigma_1\oplus\cdots\oplus\sigma_n)"] \arrow[d, "\mathrm{Id}"'] & & (F(\phi\,\underline{\psi}))\cdot(\sigma_1\oplus\cdots\oplus\sigma_n) \arrow[d, "F^{\sigma_1\oplus\cdots\oplus\sigma_n}"] \\
(F\phi)\,\big((F\underline{\psi})\cdot(\sigma_1\oplus\cdots\oplus\sigma_n)\big) \arrow[d, "\mathrm{Id}*\prod F^{\sigma_i}"'] & & F\big((\phi\,\underline{\psi})\cdot(\sigma_1\oplus\cdots\oplus\sigma_n)\big) \arrow[d, "\mathrm{Id}"] \\
(F\phi)\,\big(F(\underline{\psi}\cdot(\sigma_1\oplus\cdots\oplus\sigma_n))\big) \arrow[rr, "F^2"'] & & F\big(\phi\,(\underline{\psi}\cdot(\sigma_1\oplus\cdots\oplus\sigma_n))\big)
\end{tikzcd}
\tag{3.1.8}
$$

Example 3.1.4. There is an obvious functor $\iota : \mathbf{Set} \to \mathbf{Cat}$ that takes a set S and produces a category $\iota(S)$ whose underlying collection of objects is S and where the morphisms are identities. Therefore, any **Set**-enriched multicategory $\mathcal{O}$ produces a **Cat**-enriched multicategory by "upgrading" the sets of morphisms $\mathcal{O}\binom{t}{\underline{c}} \in \mathbf{Ob}(\mathbf{Set})$ to categories $\iota(\mathcal{O}\binom{t}{\underline{c}}) \in \mathbf{Ob}(\mathbf{Cat})$. $\triangledown$

Definition 3.1.5. Let $\mathcal{O}, \mathcal{P}$ be **Cat**-enriched symmetric multicategories and $F, G : \mathcal{O} \to \mathcal{P}$ pseudo-multifunctors. A *pseudo-multinatural transformation* $\zeta : F \to G$ is given by the following data:

(a) Functors $\zeta_t : \mathbf{1} \to \mathcal{P}\binom{Gt}{Ft}$, for each $t \in \mathcal{O}_0$. We also write $\zeta_t \in \mathcal{P}\binom{Gt}{Ft}$ for the corresponding 1-operation.

(b) For all $t \in \mathcal{O}_0$, $n \geq 0$, $\underline{c} \in \mathcal{O}_0^n$, natural isomorphisms

$$\begin{tikzcd}
\mathcal{O}\binom{t}{\underline{c}} \times \prod_{i=1}^{n} \mathbf{1} \arrow[rr, "G \times \prod_i \zeta_{c_i}"] \arrow[d, "\cong"'] & & \mathcal{P}\binom{Gt}{G\underline{c}} \times \prod_{i=1}^{n} \mathcal{P}\binom{Gc_i}{Fc_i} \arrow[dd, "\gamma^{\mathcal{P}}"] \arrow[ddll, Rightarrow, "\zeta^{\bullet}"'] \\
\mathbf{1} \times \mathcal{O}\binom{t}{\underline{c}} \arrow[d, "\zeta_t \times F"'] & & \\
\mathcal{P}\binom{Gt}{Ft} \times \mathcal{P}\binom{Ft}{F\underline{c}} \arrow[rr, "\gamma^{\mathcal{P}}"'] & & \mathcal{P}\binom{Gt}{F\underline{c}}
\end{tikzcd} \tag{3.1.9}$$

Satisfying the following axioms:

(a) *Naturality*:

$$\begin{tikzcd}
(G\phi)\,(G\underline{\psi})\,\prod \zeta_{b_{ij}} \arrow[r, "\mathrm{Id} * \prod \zeta^{\bullet}"] \arrow[d, "G^2 * \prod \mathrm{Id}"'] & (G\phi)\,\prod \zeta_{a_i}\,(F\underline{\psi}) \arrow[r, "\zeta^{\bullet} * \prod \mathrm{Id}"] & \zeta_t\,(F\phi)\,(F\underline{\psi}) \arrow[d, "\mathrm{Id} * F^2"] \\
G(\phi\,\underline{\psi})\,\prod \zeta_{b_{ij}} \arrow[rr, "\zeta^{\bullet}"'] & & \zeta_t\,F(\phi\,\underline{\psi})
\end{tikzcd} \tag{3.1.10a}$$

$$\begin{tikzcd}
\mathbb{1}^{\mathcal{P}}\,\zeta_t \arrow[r, "G^0 * \mathrm{Id}"] \arrow[d, "\mathrm{Id}"'] & (G\mathbb{1}^{\mathcal{O}})\,\zeta_t \arrow[d, "\zeta^{\bullet}"] \\
\zeta_t\,\mathbb{1}^{\mathcal{P}} \arrow[r, "\mathrm{Id} * F^0"'] & \zeta_t\,(F\mathbb{1}^{\mathcal{O}})
\end{tikzcd} \tag{3.1.10b}$$

$$\begin{tikzcd}
((G\phi)\,\prod \zeta_{c_i}) \cdot \sigma \arrow[r, "\zeta^{\bullet} \cdot \sigma"] \arrow[d, "\mathrm{Id}"'] & (\zeta_t\,(F\phi)) \cdot \sigma \arrow[d, "\mathrm{Id}"] \\
((G\phi) \cdot \sigma)\,\prod \zeta_{c_{\sigma(i)}} \arrow[d, "G^{\sigma} * \prod \mathrm{Id}"'] & \zeta_t\,((F\phi) \cdot \sigma) \arrow[d, "\mathrm{Id} * F^{\sigma}"] \\
G(\phi \cdot \sigma)\,\prod \zeta_{c_{\sigma(i)}} \arrow[r, "\zeta^{\bullet}"'] & \zeta_t\,F(\phi \cdot \sigma)
\end{tikzcd} \tag{3.1.10c}$$

where $\zeta_{\underline{c}} = (\zeta_{c_1}, \ldots, \zeta_{c_n})$.

The last definition we introduce is a multicategorical analogue of modifications, namely *multimodifications*.

Definition 3.1.6. Let $\mathcal{O}$ and $\mathcal{P}$ be **Cat**-enriched symmetric multicategories, $F, G : \mathcal{O} \to \mathcal{P}$ pseudo-multifunctors and $\zeta, \kappa : F \Rightarrow G$ pseudo-natural transformations. A *multimodification* $\Gamma : \zeta \Rrightarrow \kappa$ consists of the following data:

(a) For all $t \in \mathcal{O}_0$, natural transformations

$$\begin{tikzcd} \mathbf{1} \arrow[r, bend left, "\zeta_c"] \arrow[r, bend right, "\kappa_c"'] \arrow[r, phantom, "\Downarrow \Gamma_c"] & \mathcal{P}\binom{Gc}{Fc} \end{tikzcd} \tag{3.1.11}$$

Satisfying the following axiom:

$$\begin{tikzcd} (G\phi) \prod \zeta_{c_i} \arrow[r, "\mathrm{Id} * \prod \Gamma_{c_i}"] \arrow[d, "\zeta^\bullet"'] & (G\phi) \prod \kappa_{c_i} \arrow[d, "\kappa^\bullet"] \\ \zeta_t (F\phi) \arrow[r, "\Gamma_t * \mathrm{Id}"'] & \kappa_t (F\phi) \end{tikzcd} \tag{3.1.12}$$

Remark 3.1.7. It can be shown that **Cat**-enriched multicategories, pseudo-multifunctors, pseudo-natural transformations and multimodifications form a tricategory, where compositions are straightforward multicategorical analogues of the ones for the tricategory of bicategories (see [SP09, Appendix A.1] for a brief review or [GPS95] for a more thorough treatment). Therefore, similarly to Remark 1.1.21, we can introduce, for all **Cat**-enriched multicategories $\mathcal{O}$ and $\mathcal{P}$, the 2-category $\mathbf{Alg}_{\mathcal{O}}(\mathcal{P}) := [\mathcal{O}, \mathcal{P}]$ consisting of pseudo-multifunctors, pseudo-multinatural transformations and multimodifications, called the 2-*category of $\mathcal{P}$-valued $\mathcal{O}$-algebras*. If $\mathcal{O}$ and $\mathcal{P}$ are **Set**-enriched multicategories (see Definition 1.1.1) the 2-category $\mathbf{Alg}_{\mathcal{O}}(\mathcal{P})$ coincides with the 1-category $\mathbf{Alg}_{\mathcal{O}}(\mathcal{P})$ from Remark 1.1.21 considered as a 2-category (notice that every 1-category can be considered as a 2-category by adding identity 2-morphisms to it). Moreover, given pseudo-multifunctors $F : \mathcal{O} \to \mathcal{O}'$ and $G : \mathcal{P} \to \mathcal{P}'$ we define pseudo-functors

$$F^* : [\mathcal{O}', \mathcal{P}] \longrightarrow [\mathcal{O}, \mathcal{P}] \quad , \qquad G_* : [\mathcal{O}, \mathcal{P}] \longrightarrow [\mathcal{O}, \mathcal{P}'] \quad , \tag{3.1.13}$$

called the *pullback* and *pushforward* respectively (see Definition 1.1.25 for comparison). △

Remark 3.1.8. The reader not acquainted with 2-categories, pseudo-functors, pseudo-natural transformations and modifications might still be wondering what they are. The definitions of these concepts are easily obtained from those of **Cat**-enriched multicategory, pseudo-multifunctor, pseudo-multinatural transformation and multimodification by considering only operations of arity 1.

More precisely:

(a) *A* **Cat**-*enriched category* is a **Cat**-enriched multicategory in which only operations of arity 1 appear.

(b) Given **Cat**-enriched categories **D** and **E**, *a pseudo-functor* $X : \mathbf{D} \to \mathbf{E}$ is a pseudo-multifunctor $X : \mathbf{D} \to \mathbf{E}$.

(c) Given $X, Y : \mathbf{D} \to \mathbf{E}$ pseudo-functors, a *pseudo-natural transformation* (or 1-*morphism*) $F : X \to Y$ is a pseudo-multinatural transformation $F : X \to Y$.

(d) Given pseudo-natural transformations (1-morphisms) $F, G : X \to Y$ a *modification* (or 2-*morphism*) $\eta : F \to G$ is a multimodification $\eta : F \to G$.

$\triangle$

Example 3.1.9. Analogously to Example 3.1.4, any category $\mathbf{D}$ produces a $\mathbf{Cat}$-enriched category $\mathbf{D}$ (notice the abuse of notations). $\triangledown$

3.2 DEFINITION OF 2AQFTS

The aim of this section is to introduce a 2-categorical analogue of algebraic quantum field theories. Broadly speaking, while an algebraic quantum field theory associates associative and unital $\mathbb{K}$-algebras of observables to spacetimes, a 2-algebraic quantum field theory associates locally presentable $\mathbb{K}$-linear categories to spacetimes.

In order to prepare the ground for the aforementioned categorification it is important to notice that the category $\mathbf{AQFT}(\mathbf{Sp}^{\perp})$ of $\mathbf{Vec}_{\mathbb{K}}$-valued algebraic quantum field theories over $\mathbf{Sp}^{\perp} = (\mathbf{Sp}, \perp)$ (see Definition 1.2.12) is equivalent to the 1-category $\mathbf{Alg}_{\mathcal{P}_{\mathbf{Sp}^{\perp}}}(\mathbf{Alg}_{\mathbb{K}})$ of $\mathbf{Alg}_{\mathbb{K}}$-valued $\mathcal{P}_{\mathbf{Sp}^{\perp}}$-algebras, where $\mathbf{Alg}_{\mathbb{K}}$ is the multicategory associated to the symmetric monoidal category $\mathbf{Alg}_{\mathbb{K}}$ of associative and unital $\mathbb{K}$-algebras (see Example 1.1.3). This is easily seen by leveraging the fact that $\mathrm{Hom}_{\mathbf{MULT}}(\mathsf{As}, \mathbf{Vec}_{\mathbb{K}})$ and $\mathbf{Alg}_{\mathbb{K}}$ are equivalent multicategories ($\mathrm{Hom}_{\mathbf{MULT}}$ denotes the internal-Hom of $\mathbf{MULT}$, right adjoint to the Boardman-Vogt tensor product) and by using the following chain of equivalences: $\mathbf{AQFT}(\mathbf{Sp}^{\perp}) \cong [\mathcal{O}_{\mathbf{Sp}^{\perp}}, \mathbf{Vec}_{\mathbb{K}}] \cong [\mathcal{P}_{\mathbf{Sp}^{\perp}} \otimes_{BV} \mathsf{As}, \mathbf{Vec}_{\mathbb{K}}] \cong [\mathcal{P}_{\mathbf{Sp}^{\perp}}, \mathrm{Hom}_{\mathbf{MULT}}(\mathsf{As}, \mathbf{Vec}_{\mathbb{K}})] \cong [\mathcal{P}_{\mathbf{Sp}^{\perp}}, \mathbf{Alg}_{\mathbb{K}}]$ where the first equivalence is given by Theorem 1.2.15, the second by Theorem 1.2.16, the third by the BV-tensor product/ internal-Hom adjunction and the fourth is consequence of $\mathrm{Hom}_{\mathbf{MULT}}(\mathsf{As}, \mathbf{Vec}_{\mathbb{K}}) \cong \mathbf{Alg}_{\mathbb{K}}$.

Theorem 3.2.1. *Let $\mathbf{Sp}^{\perp} = (\mathbf{Sp}, \perp)$ be an orthogonal category. Then, the category $\mathbf{AQFT}(\mathbf{Sp}^{\perp})$ of $\mathbf{Vec}_{\mathbb{K}}$-valued algebraic quantum field theories on $\mathbf{Sp}^{\perp}$ is equivalent to the category $\mathbf{Alg}_{\mathcal{P}_{\mathbf{Sp}^{\perp}}}(\mathbf{Alg}_{\mathbb{K}})$ of $\mathbf{Alg}_{\mathbb{K}}$-valued prefactorization algebras over $\mathbf{Sp}^{\perp}$.*

Remark 3.2.2. Theorem 3.2.1 provides an interesting description of $\perp$-commutativity (see Definition 1.2.11). An algebraic quantum field theory can be interpreted as a prefactorization algebra $\mathfrak{F} : \mathcal{P}_{\mathbf{Sp}^{\perp}} \to \mathbf{Alg}_{\mathbb{K}}$ with two kinds of compatible multiplications, i.e. a multiplication $\mu_{\mathfrak{F}(s)} : \mathfrak{F}(s) \otimes \mathfrak{F}(s) \to \mathfrak{F}(s)$ and factorization products $\mathfrak{F}(\underline{f}) : \mathfrak{F}(s_1) \otimes \cdots \otimes \mathfrak{F}(s_n) \to \mathfrak{F}(s)$ (see Definition 2.1.2). Therefore, $\perp$-commutativity is an instance of an Eckmann-Hilton argument. More precisely, consider two morphisms $(f_1 : s_1 \to s) \perp (f_2 : s_2 \to s)$ then, since $\mathfrak{F}(\underline{f})$ is an algebra morphism, we obtain that $\mathfrak{F}(\underline{f})(a_1 a_1' \otimes a_2 a_2') = \mathfrak{F}(\underline{f})(a_1 \otimes a_2) \cdot \mathfrak{F}(\underline{f})(a_1' \otimes a_2')$, where μ_{s_1}, μ_{s_2} and μ_s are the multiplications of the associative and unital algebras $\mathfrak{F}(s_1)$, $\mathfrak{F}(s_2)$ and $\mathfrak{F}(s)$ respectively. Furthermore, since Theorem 3.2.1 implies that factorization products $\mathfrak{F}(\underline{f})$ for $\underline{f} = (f_1, \ldots, f_n) \in \mathcal{P}_{\mathbf{Sp}^{\perp}}\binom{t}{\underline{c}}$ factorize as

$$\mathfrak{F}(\underline{f}) \, : \, \bigotimes_{i=1}^{n} \mathfrak{F}(c_i) \xrightarrow{\otimes_i \mathfrak{F}(f_i)} \mathfrak{F}(t)^{\otimes n} \xrightarrow{\mu^n_{\mathfrak{F}(t)}} \mathfrak{F}(t) \quad , \tag{3.2.1}$$

substituting $a_1' = 1_{\mathfrak{F}(s_1)}$ and $a_2 = 1_{\mathfrak{F}(s_2)}$, we obtain that $\mathfrak{F}(f_1)a_1 \cdot \mathfrak{F}(f_2)a_2 = \mathfrak{F}(f_2)a_2 \cdot \mathfrak{F}(f_1)a_1$. $\triangle$

As mentioned in the introduction to this chapter a natural choice of target for 2-algebraic quantum field theories is the 2-category $\mathbf{Pr}_{\mathbb{K}}$ of locally presentable $\mathbb{K}$-linear categories. In particular, in light of Theorem 3.2.1, we will define a *2-algebraic quantum field theory* $\mathfrak{A}$ to be a pseudo-multifunctor $\mathfrak{A} : \mathcal{P}_{\mathbf{Sp}^\perp} \to \mathbf{Pr}_{\mathbb{K}}$. Before explicitly stating this definition let us recall a few facts regarding locally presentable $\mathbb{K}$-linear categories.

A $\mathbb{K}$-linear category is a category $\mathbf{D}$ enriched over the closed symmetric monoidal category of vector spaces $\mathbf{Vec}_{\mathbb{K}}$. In particular, this means that in a $\mathbb{K}$-linear category $\mathbf{D}$ the sets of $\mathbf{D}$-morphisms $\mathbf{D}(d,d')$ are vector spaces, i.e. $\mathbf{D}(d,d') \in \mathbf{Vec}_{\mathbb{K}}$, and that the composition maps $\gamma : \mathbf{D}(d',d'') \times \mathbf{D}(d,d') \to \mathbf{D}(d,d'')$ are $\mathbb{K}$-bilinear. Given $\mathbb{K}$-linear categories $\mathbf{D}$ and $\mathbf{E}$, a $\mathbb{K}$-linear functor is a functor $F : \mathbf{D} \to \mathbf{E}$ such that $F_d^{d'} : \mathbf{D}(d,d') \to \mathbf{E}(Fd, Fd')$ is $\mathbb{K}$-linear for all d, $d' \in \mathbf{Ob}(\mathbf{D})$.

Definition 3.2.3 ([BCJF15])**.** A locally presentable $\mathbb{K}$-linear category is a $\mathbb{K}$-linear category $\mathbf{D}$ that:

(a) Is cocomplete, i.e. it has all small colimits.

(b) Contains a set $\Gamma \subseteq \mathbf{Ob}(\mathbf{D}_0)$ of small objects such that every object in $\mathbf{D}$ is a κ-directed colimit of objects of Γ for some regular cardinal κ (a cardinal κ is called *regular* if the cardinality of any disjoint union indexed by a set of cardinality smaller than κ of sets of cardinality smaller than κ, is smaller than κ).

Notice that here the word *locally* in "locally presentable category" refers to the objects of the category and not to the category itself. In particular, local presentability can be thought as the categorical analogue of being finitely generated for modules.

Definition 3.2.4. We denote by $\mathbf{Pr}_{\mathbb{K}}$ the 2-category of locally presentable $\mathbb{K}$-linear categories, co-continuous $\mathbb{K}$-linear functors and natural transformations.

Definition 3.2.5. The $\mathbf{Cat}$-enriched multicategory (see Definition 3.1.2) $\mathbf{Pr}_{\mathbb{K}}$ of *locally presentable $\mathbb{K}$-linear categories* is defined by the following data:

(a) The collection of objects $\mathbf{Pr}_{\mathbb{K}_0}$ consists of all the locally presentable $\mathbb{K}$-linear categories.

(b) For every $\mathbf{T} \in \mathbf{Pr}_{\mathbb{K}_0}$, $n \geq 0$, $\underline{\mathbf{D}} = (\mathbf{D}_1, \dots, \mathbf{D}_n) \in \mathbf{Pr}^n_{\mathbb{K}_0}$, the categories

$$\mathbf{Pr}_{\mathbb{K}}\big(\tbinom{\mathbf{T}}{\underline{\mathbf{D}}}\big) \subseteq \mathbf{Fun}\Big(\prod_{i=1}^{n} \mathbf{D}_i, \mathbf{T}\Big), \tag{3.2.2}$$

where $\mathbf{Pr}_{\mathbb{K}}\big(\tbinom{\mathbf{T}}{\underline{\mathbf{D}}}\big)$ is the full subcategory of $\mathbf{Fun}\Big(\prod_{i=1}^{n} \mathbf{D}_i, \mathbf{T}\Big)$ consisting of all the functors $F : \prod_{i=1}^{n} \mathbf{D}_i \to \mathbf{T}$ that are $\mathbb{K}$-linear and co-continuous in each variable. For the empty tuple $\underline{\mathbf{D}} = \emptyset$, we set $\mathbf{Pr}_{\mathbb{K}}\big(\tbinom{\mathbf{T}}{\emptyset}\big) := \mathbf{Fun}(\mathbf{1}, \mathbf{T})$, where $\mathbf{1}$ is the category with only one object and the identity morphism.

(c) For every $\mathbf{T} \in \mathbf{Pr}_{\mathbb{K}_0}$, $n \geq 1$, $\underline{\mathbf{D}} \in \mathbf{Pr}^n_{\mathbb{K}_0}$, $m_i \geq 0$, $\underline{E}_i \in \mathbf{Pr}^{m_i}_{\mathbb{K}_0}$ for $i = 1, \ldots, n$, the composition functors $\gamma : \mathbf{Pr}_{\mathbb{K}}\binom{\mathbf{T}}{\underline{\mathbf{D}}} \times \prod_{i=1}^n \mathbf{Pr}_{\mathbb{K}}\binom{\mathbf{D}_i}{\underline{\mathbf{E}}_i} \to \mathbf{Pr}_{\mathbb{K}}\binom{\mathbf{T}}{\underline{\mathbf{E}}}$ given by composition of functors and (horizontal) composition of natural transformations, i.e.

$$\gamma\big(F, (G_1, \ldots, G_n)\big) := F\,\underline{G} := F \prod_{i=1}^n G_i \quad , \tag{3.2.3a}$$

$$\gamma\big(\alpha, (\beta_1, \ldots, \beta_n)\big) := \alpha * \underline{\beta} := \alpha * \prod_{i=1}^n \beta_i \quad . \tag{3.2.3b}$$

(d) For every $\mathbf{T} \in \mathbf{Pr}_{\mathbb{K}_0}$, functors $\mathbb{1}_{\mathbf{T}} : \mathbf{1} \to \mathbf{Pr}_{\mathbb{K}}\binom{\mathbf{T}}{\mathbf{T}}$ picking out the identity functors $\mathbb{1}_{\mathbf{T}} := \mathrm{id}_{\mathbf{T}} \in \mathbf{Pr}_{\mathbb{K}}\binom{\mathbf{T}}{\mathbf{T}} \subseteq \mathbf{Fun}(\mathbf{T}, \mathbf{T})$.

(e) The right action of Σ_n on the collection of categories of n-operations are given by functors $\mathbf{Pr}_{\mathbb{K}}(\sigma) : \mathbf{Pr}_{\mathbb{K}}\binom{\mathbf{T}}{\underline{\mathbf{D}}} \to \mathbf{Pr}_{\mathbb{K}}\binom{\mathbf{T}}{\underline{\mathbf{D}}\sigma}$ defined by

$$\mathbf{Pr}_{\mathbb{K}}(\sigma)(F) := F\,\mathrm{flip}_\sigma \quad , \qquad \mathbf{Pr}_{\mathbb{K}}(\sigma)(\alpha) := \alpha * \mathrm{Id}_{\mathrm{flip}_\sigma} \quad , \tag{3.2.4}$$

where $\mathrm{flip}_\sigma : \prod_{i=1}^n \mathbf{D}_{\sigma(i)} \to \prod_{i=1}^n \mathbf{D}_i$ is the permutation functor and $\mathrm{Id}_{\mathrm{flip}_\sigma} : \mathrm{flip}_\sigma \Rightarrow \mathrm{flip}_\sigma$ the identity natural transformation.

Remark 3.2.6. Notice that we are abusing notation by indicating the 2-category $\mathbf{Pr}_{\mathbb{K}}$ of locally presentable $\mathbb{K}$-linear categories, co-continuous $\mathbb{K}$-linear functors and natural transformations (see Definition 3.2.3), and the $\mathbf{Cat}$-enriched multicategory $\mathbf{Pr}_{\mathbb{K}}$ with the same symbol. The reason for this choice relies on the fact that the 2-category $\mathbf{Pr}_{\mathbb{K}}$ is closed symmetric monoidal with respect to the Kelly-Deligne tensor product $\mathbf{D} \boxtimes \mathbf{E}$ of locally presentable $\mathbb{K}$-linear categories and the monoidal unit given by $\mathbf{Vec}_{\mathbb{K}}$. In fact, the $\mathbf{Cat}$-enriched multicategory $\mathbf{Pr}_{\mathbb{K}}$ is then the 2-multicategory assigned via the 2-categorical analogue of Example 1.1.3 to this symmetric monoidal 2-category. In particular, we have that:

$$\mathbf{Pr}_{\mathbb{K}}\binom{\mathbf{T}}{\underline{\mathbf{D}}} \simeq \mathbf{Pr}_{\mathbb{K}}\Big(\boxtimes_{i=1}^n \mathbf{D}_i, \mathbf{T}\Big) \tag{3.2.5}$$

It should be (hopefully) always clear from the context whether, by the symbol $\mathbf{Pr}_{\mathbb{K}}$, we mean the 2-category from Definition 3.2.3 or the $\mathbf{Cat}$-enriched multicategory from Definition 3.2.5. △

Similarly to what we have done in Chapter 2 for prefactorization algebras and morphisms between those, we will give both multicategorical and explicit descriptions of 2-algebraic quantum field theories, 1-morphisms between those and 2-morphisms between 1-morphisms of 2-algebraic quantum field theories, both for practical convenience as for the sake of clarity. In particular, we will bounce back and forth between interpretations, hopefully leading to more transparency.

Definition 3.2.7. Let $\mathbf{Sp}^\perp = (\mathbf{Sp}, \perp)$ be an orthogonal category. We denote by $\mathbf{2AQFT}(\mathbf{Sp}^\perp)$ the 2-category $[\mathcal{P}_{\mathbf{Sp}^\perp}, \mathbf{Pr}_{\mathbb{K}}]$ (see Remark 3.1.7), where $\mathbf{Pr}_{\mathbb{K}}$ denotes the $\mathbf{Cat}$-enriched multicategory from Definition 3.2.5. In particular, we call an object $\mathfrak{A} \in \mathbf{2AQFT}$ a 2-*algebraic quantum field theory on* $\mathbf{Sp}^\perp$.

More explicitly, a 2-algebraic quantum field theory consists of the following data:

(a) For each $s \in \mathbf{Sp}$, a locally presentable $\mathbb{K}$-linear category $\mathfrak{A}(s) \in \mathbf{Pr}_{\mathbb{K}}$.

(b) For each tuple $\underline{f} = (f_1, \ldots, f_n) \in \mathcal{P}_{\mathbf{Sp}^\perp}\binom{t}{\underline{s}}$ of pairwise orthogonal $\mathbf{Sp}$-morphisms, a functor (called *factorization product*)

$$\mathfrak{A}(\underline{f}) \,:\, \prod_{i=1}^{n} \mathfrak{A}(s_i) \;\longrightarrow\; \mathfrak{A}(t) \tag{3.2.6}$$

that is $\mathbb{K}$-linear and co-continuous in each variable. For the empty tuple $\underline{s} = \varnothing$, this defines an object $\mathbf{a}_t := \mathfrak{A}(*_t) \in \mathfrak{A}(t)$ (called *pointing*, see e.g. [BZBJ18a]) that is associated to the only element $*_t \in \mathcal{P}_{\mathbf{Sp}^\perp}\binom{t}{\varnothing}$.

(c) For each $\underline{f} \in \mathcal{P}_{\mathbf{Sp}}\binom{t}{\underline{a}}$ and $\underline{\underline{g}} = (\underline{g}_1, \ldots, \underline{g}_n) \in \prod_{i=1}^n \mathcal{P}_{\mathbf{Sp}^\perp}\binom{a_i}{\underline{b}_i}$, a natural isomorphism

$$\begin{tikzcd}
\prod\limits_{i=1}^{n}\prod\limits_{j=1}^{m_i} \mathfrak{A}(b_{ij}) \arrow[r, "\mathfrak{A}(\underline{\underline{g}}) := \prod_i \mathfrak{A}(\underline{g}_i)"] \arrow[dr, "\mathfrak{A}(\underline{f}\,\underline{\underline{g}})"'] & \prod\limits_{i=1}^{n} \mathfrak{A}(a_i) \arrow[d, "\mathfrak{A}(\underline{f})"] \arrow[dl, phantom, "\swArrow\,\mathfrak{A}^2_{(\underline{f},\underline{\underline{g}})}"] \\
 & \mathfrak{A}(t)
\end{tikzcd} \tag{3.2.7}$$

(d) For each $t \in \mathbf{Sp}$, a natural isomorphism

$$\begin{tikzcd}
\mathfrak{A}(t) \arrow[r, bend left, "\mathrm{id}_{\mathfrak{A}(t)}"] \arrow[r, bend right, "\mathfrak{A}(\mathrm{id}_t)"'] \arrow[r, phantom, "\Downarrow\mathfrak{A}^0_t"] & \mathfrak{A}(t)
\end{tikzcd} \tag{3.2.8}$$

(e) For each $\underline{f} \in \mathcal{P}_{\mathbf{Sp}^\perp}\binom{t}{\underline{s}}$ and permutation $\sigma \in \Sigma_n$, a natural isomorphism

$$\begin{tikzcd}
\prod\limits_{i=1}^{n} \mathfrak{A}(s_{\sigma(i)}) \arrow[r, "\mathrm{flip}_\sigma"] \arrow[dr, "\mathfrak{A}(\underline{f}\sigma)"'] & \prod\limits_{i=1}^{n} \mathfrak{A}(s_i) \arrow[d, "\mathfrak{A}(\underline{f})"] \arrow[dl, phantom, "\swArrow\,\mathfrak{A}^\sigma_{\underline{f}}"] \\
 & \mathfrak{A}(t)
\end{tikzcd} \tag{3.2.9}$$

Satisfying the axioms from Definition 3.1.3.

A 1-morphism $\zeta : \mathfrak{A} \to \mathfrak{B}$ between $\mathfrak{A}, \mathfrak{B} \in \mathbf{2AQFT}(\mathbf{Sp}^\perp)$ is given by the following data (see Definition 3.1.5):

(a) For each $s \in \mathbf{Sp}$, a co-continuous $\mathbb{K}$-linear functor $\zeta_s : \mathfrak{A}(s) \to \mathfrak{B}(s)$.

(b) For each $\underline{f} \in \mathcal{P}_{\mathbf{Sp}^\perp}\binom{t}{\underline{s}}$, a natural isomorphism

$$\begin{tikzcd} \prod_{i=1}^{n} \mathfrak{A}(s_i) \arrow[r, "\prod_i \zeta_{s_i}"] \arrow[d, "\mathfrak{A}(\underline{f})"'] & \prod_{i=1}^{n} \mathfrak{B}(s_i) \arrow[d, "\mathfrak{B}(\underline{f})"] \arrow[dl, Rightarrow, "\zeta_{\underline{f}}"'] \\ \mathfrak{A}(t) \arrow[r, "\zeta_t"'] & \mathfrak{B}(t) \end{tikzcd} \tag{3.2.10}$$

Notice that, for $\underline{f} = *_t \in \mathcal{P}_{\overline{\mathbf{C}}}\binom{t}{\varnothing}$, this amounts to a $\mathfrak{B}(t)$-isomorphism ζ_{*_t} : $\mathbf{b}_t \xrightarrow{\cong} \zeta_t(\mathbf{a}_t)$ from the pointing $\mathbf{b}_t = \mathfrak{B}(*_t) \in \mathfrak{B}(t)$ to the image of the pointing $\mathbf{a}_t = \mathfrak{A}(*_t) \in \mathfrak{A}(t)$ under the functor $\zeta_t : \mathfrak{A}(t) \to \mathfrak{B}(t)$.

Satisfying the axioms from Definition 3.1.5.

A 2-morphism $\Gamma : \zeta \Rightarrow \kappa$ between 1-morphisms $\zeta, \kappa : \mathfrak{A} \to \mathfrak{B}$ in $\mathbf{2AQFT}(\mathbf{Sp}^\perp)$ is given by the following data:

(a) For each $s \in \mathbf{Sp}$, a natural transformation

$$\begin{tikzcd} \mathfrak{A}(s) \arrow[r, bend left, "\zeta_s"] \arrow[r, bend right, "\kappa_s"'] \arrow[r, phantom, "\Gamma_s \Downarrow"] & \mathfrak{B}(s) \end{tikzcd} \tag{3.2.11}$$

These data are required to satisfy the axioms from Definition 3.1.6.

Remark 3.2.8. **Cat**-valued prefactorization algebras have appeared in [BZBJ18a, BZBJ18b]) in the context of factorization homology. Notice that we can interpret the examples in those papers in terms of 2-algebraic quantum field theories by considering the orthogonal category $\mathbf{Man}_2^{\perp_d} = (\mathbf{Man}_2, \perp_d)$, where $\mathbf{Man}_2$ is the category of 2-dimensional (oriented or framed) manifolds and $\perp_d$ is the orthogonality relation given by disjointness, and by considering locally constant prefactorization algebras, i.e. prefactorization algebras $\mathfrak{F}$ such that $\mathfrak{F}(f)$ is an equivalence in the 2-category $\mathbf{Pr}_{\mathbb{K}}$ for every isotopy equivalence $(f : M \to N) \in \mathrm{Mor}(\mathbf{Man}_2)$, i.e. for every $(f : M \to N) \in \mathrm{Mor}(\mathbf{Man}_2)$ such that there exists $(g : N \to M) \in \mathrm{Mor}(\mathbf{Man}_2)$ with the property that $f \circ g$ and $g \circ f$ are isotopically equivalent respectively to the identities $\mathrm{id}_N : N \to N$, $\mathrm{id}_M : M \to M$. △

3.3 BIADJUNCTION

The aim of this section is to show the existence of a biadjunction

$$\iota : \mathbf{AQFT}(\mathbf{Sp}^\perp) \leftrightarrows \mathbf{2AQFT}(\mathbf{Sp}^\perp) : \pi$$

by explicitly building pseudo-functors $\pi : \mathbf{2AQFT}(\mathbf{Sp}^\perp) \to \mathbf{AQFT}(\mathbf{Sp}^\perp)$ and $\iota : \mathbf{AQFT}(\mathbf{Sp}^\perp) \to \mathbf{2AQFT}(\mathbf{Sp}^\perp)$, which we call respectively the *inclusion* and *truncation* (not to be confused with the functor $\pi : \mathbf{MULT} \to \mathbf{CAT}$ from Remark 1.1.13) pseudo-functors.

Intuitively, the truncation 2-functor π associates to any $\mathfrak{A} \in \mathbf{2AQFT}(\mathbf{Sp}^{\perp})$ a 1-algebraic quantum field theory $\pi(\mathfrak{A}) \in \mathbf{AQFT}(\mathbf{Sp}^{\perp})$ defined by sending each object $s \in \mathbf{Sp}^{\perp}$ to the algebra $\pi(\mathfrak{A})(s) := \mathrm{End}(\mathbf{a}_s)$, where $\mathbf{a}_s$ denotes the pointing of $\mathfrak{A}(s)$ (see Definition 3.2.7) and the $\mathbb{K}$-algebra structure on $\pi(\mathfrak{A})(s)$ is given by compositions and $\mathbb{K}$-linear sums of endomorphisms (remember that $\mathfrak{A}(s)$ is a $\mathbb{K}$-linear category).

The inclusion pseudo-functor $\iota : \mathbf{AQFT}(\mathbf{Sp}^{\perp}) \to \mathbf{2AQFT}(\mathbf{Sp}^{\perp})$ assigns to any $\mathfrak{A} \in \mathbf{AQFT}(\mathbf{Sp}^{\perp})$ the 2-algebraic quantum field theory $\iota(\mathfrak{A}) \in \mathbf{2AQFT}$ defined on objects $s \in \mathbf{Sp}^{\perp}$ by $\iota(\mathfrak{A})(s) := \mathbf{Mod}_{\mathfrak{A}(s)}$, where $\mathbf{Mod}_{\mathfrak{A}(s)}$ denotes the $\mathbb{K}$-linear category of right $\mathfrak{A}(s)$-modules. Therefore, given $\mathfrak{A} \in \mathbf{AQFT}(\mathbf{Sp}^{\perp})$, we obtain that $\mathfrak{A}(s) \cong \pi\iota(\mathfrak{A}(s))$, recovering the intuition from the introduction that when X is an affine scheme the algebra of functions $\mathcal{O}(X)$ can be recovered from the category of quasi-coherent sheaves $\mathbf{QCoh}(X) \cong \mathbf{Mod}_{\mathcal{O}(X)}$ as the endomorphism $\mathbb{K}$-algebra $\mathrm{End}(\mathcal{O}(X))$.

We will see that the biadjunction $\iota \dashv \pi$ exhibits the category $\mathbf{AQFT}(\mathbf{Sp}^{\perp})$ as a coreflective full 2-subcategory of the 2-category $\mathbf{2AQFT}(\mathbf{Sp}^{\perp})$, i.e. the unit $\eta : \mathrm{id} \to \pi\iota$ of the biadjunction is an equivalence. In particular, this implies that the category $\mathbf{AQFT}(\mathbf{Sp}^{\perp})$ can be equivalently studied inside the 2-category $\mathbf{2AQFT}(\mathbf{Sp}^{\perp})$ and that any $\mathfrak{A} \in \mathbf{AQFT}(\mathbf{Sp}^{\perp})$ can be recovered from its associated 2-algebraic quantum field theory $\iota(\mathfrak{A}) \in \mathbf{2AQFT}(\mathbf{Sp}^{\perp})$.

The biadjunction $\iota \dashv \pi$ is not, in general, an adjoint equivalence as we will see in Section 3.4. Therefore, there exist examples of 2-algebraic quantum field theories that are *non-truncated*, i.e. are not equivalent to a 2-algebraic quantum field theory of the form $\iota(\mathfrak{A})$.

The outline of this section is as follows:

(a) In Subsection 3.3.1 we introduce the *truncation* 2-functor $\pi : \mathbf{2AQFT}(\mathbf{Sp}^{\perp}) \to \mathbf{AQFT}(\mathbf{Sp}^{\perp})$.

(b) In Subsection 3.3.2 we introduce the *inclusion* pseudo-functor $\iota : \mathbf{AQFT}(\mathbf{Sp}^{\perp}) \to \mathbf{2AQFT}(\mathbf{Sp}^{\perp})$.

(c) In Subsection 3.3.3 we prove that ι and π form the data of a biadjunction $\iota \dashv \pi$.

3.3.1 *Truncation*

The goal of this subsection is to introduce the truncation 2-functor $\pi : \mathbf{2AQFT}(\mathbf{Sp}^{\perp}) \to \mathbf{AQFT}(\mathbf{Sp}^{\perp})$ mentioned in the introduction to Section 3.3. More precisely, we build a 2-functor $\pi : \mathbf{2AQFT}(\mathbf{Sp}^{\perp}) \to \mathbf{Alg}_{\mathcal{P}_{\mathbf{Sp}^{\perp}}}(\mathbf{Alg}_{\mathbb{K}}) \cong \mathbf{AQFT}(\mathbf{Sp}^{\perp})$ (see Theorem 3.2.1).

Proposition 3.3.1. *Let* $\mathbf{Sp}^{\perp} = (\mathbf{Sp}, \perp)$ *be an orthogonal category. The following data defines a 2-functor* $\pi : \mathbf{2AQFT}(\mathbf{Sp}^{\perp}) \to \mathbf{AQFT}(\mathbf{Sp}^{\perp})$*, which we call* truncation 2-functor*:*

(a) On objects (2-algebraic quantum field theories)*: Given* $\mathfrak{A} \in \mathbf{2AQFT}$*,* $\pi(\mathfrak{A}) \in \mathbf{Alg}_{\mathcal{P}_{\mathbf{Sp}^{\perp}}}(\mathbf{Alg}_{\mathbb{K}})$ *is given by the following data:*

(a.1) It assigns to each $s \in \mathbf{Sp}$, the associative and unital $\mathbb{K}$-algebra

$$\pi(\mathfrak{A})(s) := \mathrm{End}(\mathbf{a}_s) := \mathfrak{A}(s)(\mathbf{a}_s, \mathbf{a}_s)\,, \tag{3.3.1}$$

where $\mathbf{a}_s \in \mathfrak{A}(s)$ is the pointing of $\mathfrak{A}(s)$.

(a.2) It assigns to every non empty tuple of morphisms $\underline{f} \in \mathcal{P}_{\mathbf{Sp}^\perp}\binom{t}{\underline{s}}$ the $\mathbf{Alg}_\mathbb{K}$-morphism

$$\begin{aligned}\pi(\mathfrak{A})(\underline{f}) \,:\, \bigotimes_{i=1}^{n} \pi(\mathfrak{A})(s_i) &\longrightarrow \pi(\mathfrak{A})(t) \quad ,\\ h_1 \otimes \cdots \otimes h_n &\longmapsto \mathfrak{A}^2_{(\underline{f},*_{\underline{s}})} \circ \mathfrak{A}(\underline{f})(h_1, \dots, h_n) \circ (\mathfrak{A}^2_{(\underline{f},*_{\underline{s}})})^{-1} \quad ,\end{aligned} \tag{3.3.2}$$

where $\mathfrak{A}(\underline{f}) : \bigotimes_{i=1}^n \mathrm{End}(\mathbf{a}_{s_i}) \to \mathrm{End}\big(\mathfrak{A}(\underline{f})(\mathbf{a}_{s_1}, \dots, \mathbf{a}_{s_n})\big)$ denotes the $\mathbb{K}$-algebra map induced from the universal property of tensor products of algebras by the restriction of the functor $\mathfrak{A}(\underline{f}) : \prod_{i=1}^n \mathfrak{A}(s_i) \to \mathfrak{A}(t)$ to the endomorphism algebras $\mathfrak{A}(\underline{f}) : \prod_{i=1}^n \mathrm{End}(\mathbf{a}_{s_i}) \to \mathrm{End}\big(\mathfrak{A}(\underline{f})(\mathbf{a}_{s_1}, \dots, \mathbf{a}_{s_n})\big)$, and where $\mathfrak{A}^2_{(\underline{f},*_s)}$ is the coherence isomorphism $\mathfrak{A}^2_{(\underline{f},*_{\underline{s}})} : \mathfrak{A}(\underline{f})(\mathbf{a}_{s_1}, \dots, \mathbf{a}_{s_n}) \to \mathbf{a}_t$ associated to the composition $(\underline{f}, *_{\underline{s}}) := (\underline{f}, (*_{s_1}, \dots, *_{s_n}))$.

To a 0-operation $*_t \in \mathcal{P}_{\mathbf{Sp}^\perp}\binom{t}{\varnothing}$, $\pi(\mathfrak{A})$ assigns the $\mathbf{Alg}_\mathbb{K}$-morphism $\pi(\mathfrak{A})(*_t) : \mathbb{K} \to \pi(\mathfrak{A})(t)$ that picks out the identity $\mathrm{id}_{\mathbf{a}_t}$ of $\pi(\mathfrak{A})(t)$.

To check that $\pi(\mathfrak{A}) \in \mathbf{Ob}(\mathbf{AQFT}(\mathbf{Sp}^\perp))$ is an easy (but tedious) exercise which requires leveraging the axioms of Definition 3.2.7 and performing some diagram chasing.

(b) On 1-morphisms*: To a 1-morphism $\zeta : \mathfrak{A} \to \mathfrak{B}$ of 2-algebraic quantum field theories π assigns the $\mathbf{Alg}_\mathbb{K}$-morphism $\pi(\zeta) : \pi(\mathfrak{A}) \to \pi(\mathfrak{B})$ defined by*

$$\begin{aligned}\pi(\zeta)_s \,:\, \pi(\mathfrak{A})(s) &\longrightarrow \pi(\mathfrak{B})(s) \quad ,\\ h &\longmapsto (\zeta_{*_s})^{-1} \circ \zeta_s(h) \circ \zeta_{*_s} \quad .\end{aligned} \tag{3.3.3}$$

where $\zeta_{*_s} : \mathbf{b}_s \to \zeta_s(\mathbf{a}_s)$ is the coherence map from Equation (3.2.10) *and $\zeta_s : \mathrm{End}(\mathbf{a}_s) \to \mathrm{End}(\zeta_s(\mathbf{a}_s))$ is the restriction of the functor $\zeta_s : \mathfrak{A}(s) \to \mathfrak{B}(s)$ to endomorphism algebras.*

Checking that $\pi(\zeta)$ is a well defined morphism of $\mathbf{Alg}_\mathbb{K}$-valued prefactorization algebras (or 1-algebraic quantum field theories) is an exercise of diagram chasing which involves using the axioms from Definition 3.2.7.

(c) On 2-morphisms*: For all 1-morphisms $\zeta, \kappa : \mathfrak{A} \to \mathfrak{B}$ of 2-algebraic quantum field theories $\mathfrak{A}, \mathfrak{B} \in \mathbf{2AQFT}(\mathbf{Sp}^\perp)$, π assigns to each 2-morphism $\Gamma : \zeta \to \kappa$ the identity map $\pi(\Gamma) := \mathrm{Id} : \pi(\zeta) \Rightarrow \pi(\kappa)$. To check that $\pi(\Gamma)$ is well defined consider the commutative diagram :*

$$\begin{tikzcd} \mathbf{b}_s \arrow[r, "="] \arrow[d, "\zeta_{*_s}"', "\cong"] & \mathbf{b}_s \arrow[d, "\kappa_{*_s}", "\cong"'] \\ \zeta_s(\mathbf{a}_s) \arrow[r, "\Gamma_s"'] & \kappa_s(\mathbf{a}_s) \end{tikzcd} \tag{3.3.4}$$

obtained from the axioms in Definition 3.2.7. From (3.3.4) *we deduce that* Γ_s *is an isomorphism and leveraging Equations* (3.3.3) *and* (3.3.4) *we obtain the following chain of equalities:*

$$\begin{aligned} \pi(\kappa)_s(h) &= (\kappa_{*_s})^{-1} \circ \kappa_s(h) \circ \kappa_{*_s} = (\zeta_{*_s})^{-1} \circ (\Gamma_s)^{-1} \circ \kappa_s(h) \circ \Gamma_s \circ \zeta_{*_s} \\ &= (\zeta_{*_s})^{-1} \circ \zeta_s(h) \circ \zeta_{*_s} = \pi(\zeta)_s(h) \quad , \end{aligned} \tag{3.3.5}$$

where in the third step we used that (3.2.11) *is a natural transformation. Therefore, we conclude that* $\pi(\zeta)$ *and* $\pi(\kappa)$ *define the same* 1*-morphism of* 1*-algebraic quantum field theories, i.e* $\pi(\Gamma) := \mathrm{Id} : \pi(\zeta) \Rightarrow \pi(\kappa)$ *is well defined.*

3.3.2 *Inclusion*

In this subsection we define an *inclusion* pseudo-functor $\iota : \mathbf{AQFT}(\mathbf{Sp}^\perp) \to \mathbf{2AQFT}(\mathbf{Sp}^\perp)$. More precisely, similarly to Subsection 3.3.1, we introduce a pseudo-functor $\iota : \mathbf{Alg}_{\mathcal{P}_{\mathbf{Sp}^\perp}}(\mathbf{Alg}_\mathbb{K}) \to \mathbf{2AQFT}(\mathbf{Sp}^\perp)$ and leverage Theorem 3.2.1 to obtain a pseudo-functor $\iota : \mathbf{AQFT}(\mathbf{Sp}^\perp) \to \mathbf{2AQFT}(\mathbf{Sp}^\perp)$ (notice the slight abuse of notations).

Proposition 3.3.2. *Let* $\mathbf{Sp}^\perp = (\mathbf{Sp}, \perp)$ *be an orthogonal category. The following data defines a pseudo-functor* $\iota : \mathbf{Alg}_{\mathcal{P}_{\mathbf{Sp}^\perp}}(\mathbf{Alg}_\mathbb{K}) \to \mathbf{2AQFT}(\mathbf{Sp}^\perp)$ *(or equivalently a pseudo-functor* $\iota : \mathbf{AQFT}(\mathbf{Sp}^\perp) \to \mathbf{2AQFT}(\mathbf{Sp}^\perp)$*):*

(a) On objects*: Given* $\mathfrak{A} \in \mathbf{Alg}_{\mathcal{P}_{\mathbf{Sp}^\perp}}(\mathbf{Alg}_\mathbb{K}) \cong \mathbf{AQFT}(\mathbf{Sp}^\perp)$*,* $\iota(\mathfrak{A}) \in \mathbf{2AQFT}(\mathbf{Sp}^\perp)$ *is defined by the following data:*

*(a.*1*) It assigns to every* $s \in \mathcal{P}_{\mathbf{Sp}_0^\perp}$ *the locally presentable* $\mathbb{K}$*-linear category*

$$\iota(\mathfrak{A})(s) \; := \; \mathbf{Mod}_{\mathfrak{A}(s)} \tag{3.3.6}$$

where $\mathbf{Mod}_{\mathfrak{A}(s)}$ *denotes the locally presentable* $\mathbb{K}$*-linear category of right* $\mathfrak{A}(s)$*-modules (see [BCJF15]).*

*(a.*2*) It assigns to every non empty tuple of morphisms* $\underline{f} \in \mathcal{P}_{\mathbf{Sp}^\perp}\binom{t}{\underline{s}}$ *the co-continuous* $\mathbb{K}$*-linear functor*

$$\iota(\mathfrak{A})(\underline{f}) \; : \; \prod_{i=1}^{n} \mathbf{Mod}_{\mathfrak{A}(s_i)} \xrightarrow{\;\otimes^n\;} \mathbf{Mod}_{\otimes_{i=1}^n \mathfrak{A}(s_i)} \xrightarrow{\;\mathfrak{A}(\underline{f})_!\;} \mathbf{Mod}_{\mathfrak{A}(t)} \quad , \tag{3.3.7}$$

where $\otimes^n : \prod_{i=1}^n \mathbf{Vec}_\mathbb{K} \to \mathbf{Vec}_\mathbb{K}$ *is the functor that sends n* $\mathbb{K}$*-vector spaces* $V_1, \ldots, V_n \in \mathbf{Vec}_\mathbb{K}$ *to their tensor product* $V_1 \otimes \cdots \otimes V_n$ *as* $\mathbb{K}$*-vector spaces and where*

$$\mathfrak{A}(\underline{f})_! = (-) \otimes_{\otimes_{i=1}^n \mathfrak{A}(s_i)} \mathfrak{A}(t) \; : \; \mathbf{Mod}_{\otimes_{i=1}^n \mathfrak{A}(s_i)} \longrightarrow \mathbf{Mod}_{\mathfrak{A}(t)} \quad . \tag{3.3.8}$$

is the induced module functor*, i.e. the left adjoint of the restriction functor* $\mathfrak{A}(\underline{f})^* : \mathbf{Mod}_{\mathfrak{A}(t)} \to \mathbf{Mod}_{\otimes_{i=1}^n \mathfrak{A}(s_i)}$ *induced by the* $\mathbf{Alg}_\mathbb{K}$*-morphism* $\mathfrak{A}(\underline{f}) : \bigotimes_{i=1}^n \mathfrak{A}(s_i) \to \mathfrak{A}(t)$*.*

Moreover, $\iota(\mathfrak{A})$ assigns to the unique morphism $*_t \in \mathcal{P}_{\mathbf{Sp}^\perp}\binom{t}{\varnothing}$ the pointing $\iota(\mathfrak{A})(*_t) := \mathfrak{A}(t) \in \mathbf{Mod}_{\mathfrak{A}(t)}$ determined by the rank 1 free $\mathfrak{A}(t)$-module $\mathfrak{A}(t)$.

(a.3) *The coherence natural isomorphisms for compositions from (3.2.7) are given by pasting of*

$$
\begin{tikzcd}
\prod_{i=1}^{n}\prod_{j=1}^{m_i} \mathbf{Mod}_{\mathfrak{A}(b_{ij})} \arrow[r, "\prod_i \otimes^{m_i}"] \arrow[dr, "\otimes^{\Sigma m_i}"'] & \prod_{i=1}^{n} \mathbf{Mod}_{\otimes_{j=1}^{m_i}\mathfrak{A}(b_{ij})} \arrow[r, "\prod_i \mathfrak{A}(\underline{g}_i)_!"] \arrow[d, "\otimes^n"'] \arrow[dl, phantom, "\Swarrow (\star)"] & \prod_{i=1}^{n} \mathbf{Mod}_{\mathfrak{A}(a_i)} \arrow[d, "\otimes^n"] \arrow[dl, phantom, "\Swarrow (\star\star)"] \\
 & \mathbf{Mod}_{\otimes_{i=1}^n \otimes_{j=1}^{m_i}\mathfrak{A}(b_{ij})} \arrow[r, "(\otimes_i \mathfrak{A}(\underline{g}_i))_!"] \arrow[dr, "\mathfrak{A}(\underline{f}\,\underline{\underline{g}})_!"'] & \mathbf{Mod}_{\otimes_{i=1}^n \mathfrak{A}(a_i)} \arrow[d, "\mathfrak{A}(\underline{f})_!"] \arrow[dl, phantom, "\Swarrow (\star\star\star)"] \\
 & & \mathbf{Mod}_{\mathfrak{A}(t)}
\end{tikzcd}
\tag{3.3.9}
$$

Where $(\star)$ and $(\star\star)$ are the natural isomorphisms determined by the coherence isomorphisms of the tensor product, while $(\star\star\star)$ is the natural isomorphism determined by uniqueness (up to unique isomorphism) of left adjoints and the (strict) composition property $(\bigotimes_i \mathfrak{A}(\underline{g}_i))^* \, \mathfrak{A}(\underline{f})^* = \big(\mathfrak{A}(\underline{f}) \, \bigotimes_i \mathfrak{A}(\underline{g}_i)\big)^* = \mathfrak{A}(\underline{f}\,\underline{\underline{g}})^*$ of the right adjoints.

(a.4) *The coherence natural isomorphisms for identities in (3.2.8) are canonically determined by uniqueness of left adjoint functors and the strict identity property $\mathfrak{A}(\mathrm{id}_t)^* = \mathrm{id}^*_{\mathfrak{A}(t)} = \mathrm{id}_{\mathbf{Mod}_{\mathfrak{A}(t)}}$ of the right adjoints.*

(a.5) *The coherence natural isomorphisms for the permutations actions in (3.2.9) are given by pasting of*

$$
\begin{tikzcd}
\prod_{i=1}^{n} \mathbf{Mod}_{\mathfrak{A}(s_{\sigma(i)})} \arrow[r, "\mathrm{flip}_\sigma"] \arrow[d, "\otimes^n"'] & \prod_{i=1}^{n} \mathbf{Mod}_{\mathfrak{A}(s_i)} \arrow[d, "\otimes^n"] \arrow[dl, phantom, "\Swarrow (\star)"] \\
\mathbf{Mod}_{\otimes_{i=1}^n \mathfrak{A}(s_{\sigma(i)})} \arrow[r, "(\tau_\sigma)_!"] \arrow[dr, "\mathfrak{A}(\underline{f}\sigma)_!"'] & \mathbf{Mod}_{\otimes_{i=1}^n \mathfrak{A}(s_i)} \arrow[d, "\mathfrak{A}(\underline{f})_!"] \arrow[dl, phantom, "\Swarrow (\star\star)"] \\
 & \mathbf{Mod}_{\mathfrak{A}(t)}
\end{tikzcd}
\tag{3.3.10}
$$

To check that this data defines a 2-algebraic quantum field theory $\iota(\mathfrak{A}) \in \mathbf{2AQFT}(\mathbf{Sp}^\perp)$ it suffices to notice that all the coherences in ((a.3)-(a.5)) are canonically given by coherence isomorphisms.

(b) On 1-morphisms*: Given a 1-morphism $\zeta : \mathfrak{A} \to \mathfrak{B}$ of $\mathbf{Alg}_{\mathbb{K}}$-valued prefactorization algebras, $\iota(\zeta) : \iota(\mathfrak{A}) \to \iota(\mathfrak{B})$ is defined by the following data:*

(b.1) *For every $s \in \mathcal{P}_{\mathbf{Sp}_0^\perp}$*

$$\iota(\zeta)_s \;:=\; (\zeta_s)_! \;:\; \mathbf{Mod}_{\mathfrak{A}(s)} \;\longrightarrow\; \mathbf{Mod}_{\mathfrak{B}(s)} \tag{3.3.11}$$

is the $\mathbb{K}$-linear and co-continuous induced module functor along the $\mathbf{Alg}_{\mathbb{K}}$-morphism $\zeta_s : \mathfrak{A}(s) \to \mathfrak{B}(s)$.

(b.2) *The coherence natural isomorphisms in* (3.2.10) *are given by pasting of*

$$\begin{tikzcd}
\prod_{i=1}^{n} \mathbf{Mod}_{\mathfrak{A}(s_i)} \arrow[r, "\prod_i (\zeta_{s_i})_!"] \arrow[d, "\otimes^n"'] & \prod_{i=1}^{n} \mathbf{Mod}_{\mathfrak{B}(s_i)} \arrow[d, "\otimes^n"] \arrow[dl, Rightarrow, "(\star)"'] \\
\mathbf{Mod}_{\otimes_{i=1}^n \mathfrak{A}(s_i)} \arrow[r, "(\otimes_i \zeta_{s_i})_!"] \arrow[d, "\mathfrak{A}(\underline{f})_!"'] & \mathbf{Mod}_{\otimes_{i=1}^n \mathfrak{B}(s_i)} \arrow[d, "\mathfrak{B}(\underline{f})_!"] \arrow[dl, Rightarrow, "(\star\star)"'] \\
\mathbf{Mod}_{\mathfrak{A}(t)} \arrow[r, "(\zeta_t)_!"'] & \mathbf{Mod}_{\mathfrak{B}(t)}
\end{tikzcd} \tag{3.3.12}$$

where $(\star)$ is canonically determined by the coherence isomorphisms for tensor products and $(\star\star)$ is canonically determined by uniqueness of left adjoint functors and the strict naturality property $(\bigotimes_i \zeta_{s_i})^* \; \mathfrak{B}(\underline{f})^* \;=\; \big(\mathfrak{B}(\underline{f}) \;\; \bigotimes_i \zeta_{s_i}\big)^* \;=\; \big(\zeta_t \; \mathfrak{A}(\underline{f})\big)^* = \mathfrak{A}(\underline{f})^* \; (\zeta_t)^*$ of the right adjoints.

3.3.3 *Biadjunction*

The aim of this subsection is to prove that $\iota : \mathbf{AQFT}(\mathbf{Sp}^\perp) \leftrightarrows \mathbf{2AQFT}(\mathbf{Sp}^\perp) : \pi$ forms a biadjoint pair of pseudo-functors. To achieve this goal, we will define a functor between Hom-categories

$$\widetilde{(-)} \;:\; \mathbf{2AQFT}(\mathbf{Sp}^\perp)\big(\iota(\mathfrak{A}), \boldsymbol{\mathfrak{B}}\big) \;\longrightarrow\; \mathbf{AQFT}(\mathbf{Sp}^\perp)\big(\mathfrak{A}, \pi(\boldsymbol{\mathfrak{B}})\big) \quad , \tag{3.3.13}$$

for every $\mathfrak{A} \in \mathbf{AQFT}(\mathbf{Sp}^\perp)$ and $\boldsymbol{\mathfrak{B}} \in \mathbf{2AQFT}(\mathbf{Sp}^\perp)$ and prove that this is an equivalence of categories natural both in $\mathfrak{A}$ and $\boldsymbol{\mathfrak{B}}$.

In order to build such functor we will define a natural transformation $\eta : \mathrm{id} \to \pi\iota$, which plays the role of the unit of the biadjunction, and use it to define (3.3.13) as the functor that assigns to a 1-morphism $\zeta : \iota(\mathfrak{A}) \to \boldsymbol{\mathfrak{B}}$ in $\mathbf{2AQFT}(\mathbf{Sp}^\perp)$, the $\mathbf{AQFT}(\mathbf{Sp}^\perp)$-morphism

$$\widetilde{\zeta} \,:\, \mathfrak{A} \xrightarrow{\eta_{\mathfrak{A}}} \pi\,\iota(\mathfrak{A}) \xrightarrow{\pi(\zeta)} \pi(\boldsymbol{\mathfrak{B}}) \quad . \tag{3.3.14}$$

and to a 2-morphism $\Gamma : \zeta \Rightarrow \kappa$ between 1-morphisms $\zeta, \kappa : \iota(\mathfrak{A}) \to \boldsymbol{\mathfrak{B}}$ in $\mathbf{2AQFT}(\mathbf{Sp}^\perp)$ the identity $\widetilde{\Gamma} = \mathrm{Id} : \widetilde{\zeta} \Rightarrow \widetilde{\zeta} = \widetilde{\kappa}$ (notice that the latter assignment is legit since, as we have seen in Subsection 3.3.1 , $\pi(\zeta) = \pi(\kappa)$).

In order to construct the natural transformation $\eta : \mathrm{id} \to \pi\iota$, which will correspond to the unit of the biadjunction, we notice that the action of the composed pseudo-functor $\pi\iota : \mathbf{AQFT}(\mathbf{Sp}^\perp) \to \mathbf{AQFT}(\mathbf{Sp}^\perp)$ (see Subections 3.3.1 and 3.3.2 for an explicit construction of π and ι respectively) on a given $\mathbf{Alg}_{\mathbb{K}}$-valued prefactorization algebra $\mathfrak{A} \in \mathbf{AQFT}(\mathbf{Sp}^\perp)$ at an object $s \in \mathcal{P}_{\mathbf{Sp}_0^\perp}$ is given by

$$(\pi\,\iota(\mathfrak{A}))(s) \;=\; \mathrm{End}(\mathfrak{A}(s)) \;=\; \mathbf{Mod}_{\mathfrak{A}(s)}\big(\mathfrak{A}(s), \mathfrak{A}(s)\big) \tag{3.3.15}$$

where $\mathrm{End}(\mathfrak{A}(s))$ is the endomorphism algebra of $\mathfrak{A}(s) \in \mathbf{Mod}_{\mathfrak{A}(s)}$. Therefore, we define $\eta : \mathrm{id} \to \pi\iota$ to be the natural transformation determined on each component $\eta_{\mathfrak{A}} : \mathfrak{A} \to \pi\,\iota(\mathfrak{A})$ by the $\mathbf{Alg}_{\mathbb{K}}$-morphism

$$(\eta_{\mathfrak{A}})_s : \mathfrak{A}(s) \longrightarrow \mathrm{End}(\mathfrak{A}(s)) \quad , \quad a \longmapsto \mu_{\mathfrak{A}(s)}(a \otimes -) \quad , \tag{3.3.16}$$

where $\mu_{\mathfrak{A}(s)}(a \otimes -) : \mathfrak{A}(s) \to \mathfrak{A}(s)\,,\ a' \mapsto a\,a'$ is the right $\mathfrak{A}(s)$-module endomorphism given by left multiplication by $a \in \mathfrak{A}(s)$ for every $s \in \mathcal{P}_{\mathbf{Sp}_0^\perp}$.

Theorem 3.3.3. *Let $\mathbf{Sp}^\perp = (\mathbf{Sp}, \perp)$ be an orthogonal category. Then the functor* (3.3.13) *is an equivalence of categories natural in $\mathfrak{A} \in \mathbf{AQFT}(\mathbf{Sp}^\perp)$ and $\mathfrak{B} \in \mathbf{2AQFT}(\mathbf{Sp}^\perp)$. Therefore, we obtain a biadjunction*

$$\iota : \mathbf{AQFT}(\mathbf{Sp}^\perp) \rightleftarrows \mathbf{2AQFT}(\mathbf{Sp}^\perp) : \pi \quad , \tag{3.3.17}$$

where the left biadjoint is the inclusion pseudo-functor from Proposition 3.3.2 and the right biadjoint is the truncation 2-functor from Proposition 3.3.1. Moreover, the unit $\eta : \mathrm{id} \Rightarrow \pi\,\iota$ of this biadjunction is a natural isomorphism, exhibiting the category $\mathbf{AQFT}(\mathbf{Sp}^\perp)$ of ordinary algebraic quantum field theories as a coreflective full 2-subcategory of the 2-category $\mathbf{2AQFT}(\mathbf{Sp}^\perp)$.

Proof. Let us recall from [BCJF15] the following fact: For any associative and unital $\mathbb{K}$-algebra $A \in \mathbf{Alg}_{\mathbb{K}}$, denote by $\mathbf{B}\mathrm{End}(A)$ the full $\mathbb{K}$-linear subcategory of $\mathbf{Mod}_A \in \mathbf{Pr}_{\mathbb{K}}$ on the object $A \in \mathbf{Mod}_A$. (Note that $\mathbf{B}\mathrm{End}(A)$ is just the endomorphism algebra $\mathrm{End}(A)$ regarded as a $\mathbb{K}$-linear category with only one object.) Then, for any locally presentable $\mathbb{K}$-linear category $\mathbf{D} \in \mathbf{Pr}_{\mathbb{K}}$, the restriction along the inclusion $\mathbf{B}\mathrm{End}(A) \subseteq \mathbf{Mod}_A$ of $\mathbb{K}$-linear categories induces an equivalence (i.e. a fully faithful and essentially surjective functor)

$$\mathbf{Lin}_{\mathbb{K},c}\big(\mathbf{Mod}_A, \mathbf{D}\big) \xrightarrow{\ \simeq\ } \mathbf{Lin}_{\mathbb{K}}\big(\mathbf{B}\mathrm{End}(A), \mathbf{D}\big) \tag{3.3.18}$$

from the full subcategory of $\mathbf{Fun}(\mathbf{Mod}_A, \mathbf{D})$ that consists of $\mathbb{K}$-linear and cocontinuous functors to the full subcategory of $\mathbf{Fun}(\mathbf{B}\mathrm{End}(A), \mathbf{D})$ that consists of $\mathbb{K}$-linear functors. Using this result, we can check that the functor (3.3.13) is fully faithful and essentially surjective as claimed.

Faithful: Let $\Gamma, \Delta : \zeta \Rrightarrow \kappa$ be 2-morphisms between the 1-morphisms $\zeta, \kappa : \iota(\mathfrak{A}) \to \mathfrak{B}$ in $\mathbf{2AQFT}(\mathbf{Sp}^\perp)$. (Note that $\widetilde{\Gamma} = \widetilde{\Delta}$ is automatic.) From (3.3.4) we deduce that, for every $s \in \mathcal{P}_{\mathbf{Sp}_0^\perp}$, the morphisms $\Gamma_s = \Delta_s : \zeta_s(\mathfrak{A}(s)) \to \kappa_s(\mathfrak{A}(s))$ in $\mathfrak{B}(s)$ coincide. This means that the two natural transformations $\Gamma_s, \Delta_s : \zeta_s \Rightarrow \kappa_s$ between the cocontinuous $\mathbb{K}$-linear functors $\zeta_s, \kappa_s : \mathbf{Mod}_{\mathfrak{A}(s)} \to \mathfrak{B}(s)$ have the same restriction along the inclusion $\mathbf{B}\mathrm{End}(\mathfrak{A}(s)) \subseteq \mathbf{Mod}_{\mathfrak{A}(s)}$. Recalling that the restriction functor (3.3.18) is faithful, we conclude that $\Gamma_s = \Delta_s : \zeta_s \Rightarrow \kappa_s$ coincide as natural transformations, for all $s \in \mathcal{P}_{\mathbf{Sp}_0^\perp}$, and hence that $\Gamma = \Delta : \zeta \Rrightarrow \kappa$ coincide as 2-morphisms in $\mathbf{2AQFT}(\mathbf{Sp}^\perp)$. This shows that the functor (3.3.13) is faithful.

Full: Let $\zeta, \kappa : \iota(\mathfrak{A}) \to \mathfrak{B}$ be 1-morphisms in $\mathbf{2AQFT}(\mathbf{Sp}^\perp)$ such that $\widetilde{\zeta} = \widetilde{\kappa} : \mathfrak{A} \to \pi(\mathfrak{B})$ in $\mathbf{AQFT}(\mathbf{Sp}^\perp)$. (Recall that $\mathbf{AQFT}(\mathbf{Sp}^\perp)$ only has identity 2-morphisms.) For each $s \in \mathcal{P}_{\mathbf{Sp}_0^\perp}$, consider the morphism $\kappa_{*_s} \circ (\zeta_{*_s})^{-1} : \zeta_s(\mathfrak{A}(s)) \to \kappa_s(\mathfrak{A}(s))$ in $\mathfrak{B}(s)$.

Using $\widetilde{\zeta} = \widetilde{\kappa}$, one shows that this defines a natural transformation between the restrictions along the inclusion functor $\mathbf{B}\mathrm{End}(\mathfrak{A}(s)) \subseteq \mathbf{Mod}_{\mathfrak{A}(s)}$ of the co-continuous $\mathbb{K}$-linear functors $\zeta_s, \kappa_s : \mathbf{Mod}_{\mathfrak{A}(s)} \to \mathfrak{B}(s)$. Recalling that the restriction functor (3.3.18) is full, there exists a natural transformation $\Gamma_s : \zeta_s \Rightarrow \kappa_s$ whose $\mathfrak{A}(s)$-component is $\kappa_{*_s} \circ (\zeta_{*_s})^{-1}$. We still have to prove that the collection Γ_s, for all $s \in \mathcal{P}_{\mathbf{Sp}_0^\perp}$, defines a 2-morphism $\Gamma : \zeta \Rightarrow \kappa$ between the 1-morphisms $\zeta, \kappa : \iota(\mathfrak{A}) \to \mathfrak{B}$ in $\mathbf{2AQFT}(\mathbf{Sp}^\perp)$. This amounts to showing that the diagram

$$\begin{tikzcd} \mathfrak{B}(\underline{f}) \prod_i \zeta_{s_i} \arrow[r, "\mathrm{Id} * \prod_i \Gamma_{s_i}"] \arrow[d, "\zeta_{\underline{f}}"'] & \mathfrak{B}(\underline{f}) \prod_i \kappa_{s_i} \arrow[d, "\kappa_{\underline{f}}"] \\ \zeta_s\, \iota(\mathfrak{A})(\underline{f}) \arrow[r, "\Gamma_t * \mathrm{Id}"'] & \kappa_t\, \iota(\mathfrak{A})(\underline{f}) \end{tikzcd} \tag{3.3.19}$$

of natural transformations commutes, for all $\underline{f} \in \mathcal{P}_{\mathbf{Sp}^\perp}\binom{t}{\underline{s}}$. Since this diagram lives in the category $\mathbf{Pr}_{\mathbb{K}}\binom{\mathfrak{B}(t)}{\iota(\mathfrak{A})(\underline{s})}$, i.e. all functors are $\mathbb{K}$-linear and co-continuous in each variable, we deduce from the equivalences in (3.2.5) and (3.3.18) that the diagram (3.3.19) of natural transformations commutes if and only if the corresponding component on the object $\prod_{i=1}^n \mathfrak{A}(s_i) \in \prod_{i=1}^n \mathbf{Mod}_{\mathfrak{A}(s_i)}$ commutes. This can be checked directly by using that $\zeta, \kappa : \iota(\mathfrak{A}) \to \mathfrak{B}$ are 1-morphisms in $\mathbf{2AQFT}(\mathbf{Sp}^\perp)$. (Here the axioms of Definition 3.1.5 enter explicitly.) This shows that the functor (3.3.13) is full.

Essentially surjective: Let $\zeta : \mathfrak{A} \to \pi(\mathfrak{B})$ be any $\mathbf{AQFT}(\mathbf{Sp}^\perp)$-morphism. We denote its component $\mathbf{Alg}_{\mathbb{K}}$-morphisms by $\zeta_s : \mathfrak{A}(s) \to \mathrm{End}(\mathbf{b}_s)$, for all $s \in \mathcal{P}_{\mathbf{Sp}_0^\perp}$. Recalling that $\mathfrak{A}(s) \in \mathbf{Alg}_{\mathbb{K}}$ is naturally isomorphic via η (cf. (3.3.16)) to the endomorphism algebra $\mathrm{End}(\mathfrak{A}(s))$ of the object $\mathfrak{A}(s) \in \mathbf{Mod}_{\mathfrak{A}(s)}$, we define a functor $\widehat{\zeta}_s : \mathbf{B}\mathrm{End}(\mathfrak{A}(s)) \to \mathfrak{B}(s)$ that sends the only object $\mathfrak{A}(s) \in \mathbf{B}\mathrm{End}(\mathfrak{A}(s))$ to $\mathbf{b}_s \in \mathfrak{B}(s)$ and each $\mathbf{B}\mathrm{End}(\mathfrak{A}(s))$-morphism $h \in \mathrm{End}(\mathfrak{A}(s))$ to the $\mathfrak{B}(s)$-morphism $\widehat{\zeta}_s(h) := \zeta_s((\eta_{\mathfrak{A}})_s^{-1}(h))$. This functor is by construction $\mathbb{K}$-linear, i.e. $\widehat{\zeta}_s \in \mathbf{Lin}_{\mathbb{K}}(\mathbf{B}\mathrm{End}(\mathfrak{A}(s)), \mathfrak{B}(s))$. Since the functor (3.3.18) is essentially surjective, there exists a $\mathbb{K}$-linear and co-continuous functor $\kappa_s \in \mathbf{Lin}_{\mathbb{K},c}(\mathbf{Mod}_{\mathfrak{A}(s)}, \mathfrak{B}(s))$ and a natural isomorphism κ_{*_s} from the functor $\widehat{\zeta}_s$ to the restriction along the inclusion $\mathbf{B}\mathrm{End}(\mathfrak{A}(s)) \subseteq \mathbf{Mod}_{\mathfrak{A}(s)}$ of the functor κ_s. Because $\mathfrak{A}(s) \in \mathbf{B}\mathrm{End}(\mathfrak{A}(s))$ is the only object, the natural isomorphism κ_{*_s} consists of a single $\mathfrak{B}(s)$-isomorphism $\kappa_{*_s} : \mathbf{b}_s \to \kappa_s(\mathfrak{A}(s))$, with naturality being encoded in the condition $\kappa_s(h) \circ \kappa_{*_s} = \kappa_{*_s} \circ \widehat{\zeta}_s(h)$, for all $h \in \mathrm{End}(\mathfrak{A}(s))$. Note that we have just constructed part of the data defining a 1-morphism $\kappa : \iota(\mathfrak{A}) \to \mathfrak{B}$ in $\mathbf{2AQFT}(\mathbf{Sp}^\perp)$ (cf. Definition 3.2.7). To complete the data, we have to construct, for each $\underline{f} \in \mathcal{P}_{\mathbf{Sp}^\perp}\binom{t}{\underline{s}}$, a natural isomorphism $\kappa_{\underline{f}} : \mathfrak{B}(\underline{f}) \prod_i \kappa_{s_i} \Rightarrow \kappa_t\, \iota(\mathfrak{A})(\underline{f})$ between functors from $\prod_{i=1}^n \mathbf{Mod}_{\mathfrak{A}(s_i)}$ to $\mathfrak{B}(t)$ that are $\mathbb{K}$-linear and co-continuous in each variable. Using again the equivalences in (3.2.5) and (3.3.18), this problem is equivalent to constructing a $\mathfrak{B}(t)$-isomorphism, denoted with a slight abuse of notation also by $\kappa_{\underline{f}} : \mathfrak{B}(\underline{f})\, \big(\prod_i \kappa_{s_i}\big(\prod_i \mathfrak{A}(s_i)\big)\big) \to \kappa_t\, \Big(\iota(\mathfrak{A})(\underline{f})\big(\prod_i \mathfrak{A}(s_i)\big)\Big)$, fulfilling the naturality condition $\kappa_t\Big(\iota(\mathfrak{A})(\underline{f})(\underline{h})\Big) \circ \kappa_{\underline{f}} = \kappa_{\underline{f}} \circ \mathfrak{B}(\underline{f})\left(\prod_i \kappa_{s_i}(\underline{h})\right)$, for all $\underline{h} \in \prod_{i=1}^n \mathrm{End}(\mathfrak{A}(s_i))$. We define the $\mathfrak{B}(t)$-isomorphism $\kappa_{\underline{f}}$ according to

$$\begin{tikzcd}
\mathfrak{B}(\underline{f})\left(\prod_i \kappa_{s_i}\left(\prod_i \mathfrak{A}(s_i)\right)\right) \arrow[rr, dashed, "\kappa_{\underline{f}}"] & & \kappa_t\Big(\iota(\mathfrak{A})(\underline{f})\left(\prod_i \mathfrak{A}(s_i)\right)\Big) \arrow[d, "\kappa_t\left(\iota(\mathfrak{A})^2_{(\underline{f},*_{\underline{s}})}\right)", "\cong"'] \\
\mathfrak{B}(\underline{f})\left(\prod_i \mathbf{b}_{s_i}\right) \arrow[u, "\mathfrak{B}(\underline{f})\left(\prod_i \kappa_{*_{s_i}}\right)", "\cong"'] \arrow[r, "\mathfrak{B}^2_{(\underline{f},*_{\underline{s}})}"'] & \mathbf{b}_t \arrow[r, "\kappa_{*_t}"'] & \kappa_t(\mathfrak{A}(t))
\end{tikzcd} \tag{3.3.20}$$

and observe that the required naturality condition for $\kappa_{\underline{f}}$ follows from naturality of κ_{*_s} and of ζ. This provides us with the desired natural isomorphism $\kappa_{\underline{f}} : \mathfrak{B}(\underline{f}) \prod_i \kappa_{s_i} \Rightarrow \kappa_t\, \iota(\mathfrak{A})(\underline{f})$ and hence completes the data needed to define a 1-morphism $\kappa : \iota(\mathfrak{A}) \to \mathfrak{B}$ in $\mathbf{2AQFT}(\mathbf{Sp}^\perp)$. It remains to check that the relevant axioms hold (cf. Definition 3.2.7 and Definition 3.1.5). Using once again the equivalences in (3.2.5) and (3.3.18), confirming these axioms can be reduced to checking that certain diagrams in $\mathfrak{B}(t)$ commute. This can be done directly by using that $\iota(\mathfrak{A})$ and $\mathfrak{B}$ are objects in $\mathbf{2AQFT}(\mathbf{Sp}^\perp)$. (Here the axioms for pseudo-multifunctors from Definition 3.1.3 enter explicitly.) Since by construction the $\mathbf{AQFT}(\mathbf{Sp}^\perp)$-morphisms $\widetilde{\kappa} = \zeta : \mathfrak{A} \to \pi(\mathfrak{B})$ coincide, this shows that the functor (3.3.13) is essentially surjective. $\square$

The counit $\epsilon : \iota\pi \to \mathrm{id}$ of the biadjunction in Theorem 3.3.3 is retrieved implicitly as the pseudo-natural transformation whose $\mathfrak{B}$-components are the 1-morphisms $\epsilon_{\mathfrak{B}} : \iota\pi(\mathfrak{B}) \to \mathfrak{B}$ of 2-algebraic quantum field theories corresponding under the equivalence of categories in (3.3.13) to the identities $\widetilde{\epsilon_{\mathfrak{B}}} = \mathrm{id}_{\pi(\mathfrak{B})} : \pi(\mathfrak{B}) \to \pi(\mathfrak{B})$ in $\mathbf{AQFT}(\mathbf{Sp}^\perp)$ (see [LN16, Definition 2.5 and Remark 2.6] for further details on biadjunctions). Notice further that to determine whether a 2-algebraic quantum field theory $\mathfrak{B}$ is equivalent to some $\iota(\mathfrak{A})$, where $\mathfrak{A}$ denotes any ordinary AQFT, is sufficient to see if the $\mathfrak{B}$-component of the counit $\epsilon_{\mathfrak{B}} : \iota\pi(\mathfrak{B}) \to \mathfrak{B}$ is an equivalence in the 2-category $\mathbf{2AQFT}(\mathbf{Sp}^\perp)$. If this is the case the 2-algebraic quantum field theory $\mathfrak{B}$ is fully determined by the 1-algebraic quantum field theory $\pi(\mathfrak{B})$. Therefore, we give the following definition:

Definition 3.3.4. Let $\mathbf{Sp}^\perp = (\mathbf{Sp}, \perp)$ be an orthogonal category. We say that a 2-algebraic quantum field theory $\mathfrak{B}$ is *truncated* if the $\mathfrak{B}$-component $\epsilon_{\mathfrak{B}} : \iota\pi(\mathfrak{B}) \to \mathfrak{B}$ of the counit $\epsilon : \iota\pi \to \mathrm{id}$ of adjunction in Theorem 3.3.3 is an equivalence in the 2-category $\mathbf{2AQFT}(\mathbf{Sp}^\perp)$.

The question now is whether genuine *non*-truncated 2-algebraic quantum field theories exist. We will show in Section 3.4 that the answer to this interrogative is affirmative, therefore proving that the biadjunction in Theorem 3.3.3 is not, in general, a biadjoint equivalence.

3.4 GAUGING CONSTRUCTION AND ORBIFOLD 2AQFTS

The aim of this section is to show the existence of genuine (i.e. non-truncated) 2-algebraic quantum field theories through introducing a functor

$$(-)^G : G\text{-}\mathbf{AQFT}(\mathbf{Sp}^\perp) \to \mathbf{2AQFT}(\mathbf{Sp}^\perp) \quad ,$$

called *gauging construction*, that associates to every G-equivariant 1-algebraic quantum field theory, i.e. to every algebraic quantum field theory $\mathfrak{A}$ endowed with an action $\rho : G \to \mathrm{Aut}(\mathfrak{A})$ of a finite group G, a 2-algebraic quantum field theory $\mathfrak{A}^G$, called the *categorified orbifold theory* of $(\mathfrak{A}, \rho)$, which should be understood from a physical perspective as a local gauging of $\mathfrak{A}$ with respect to G (see Remark 3.4.6). Furthermore, we give concrete criteria to establish when a categorified orbifold theory $\mathfrak{A}^G$ is non-truncated.

Notice that some of the definitions that follow can be generalized to deal with infinite groups but since the main result of this section, i.e. Theorem 3.4.12, holds just for finite groups we prefer not to discuss such generalizations.

Let us begin by introducing some nomenclature.

Definition 3.4.1. Let $\mathbf{Sp}^\perp = (\mathbf{Sp}, \perp)$ be an orthogonal category and G a finite group. A *G-equivariant 1-algebraic quantum field theory* is a pair $(\mathfrak{A}, \rho)$ where $\mathfrak{A} \in \mathbf{AQFT}(\mathbf{Sp}^\perp)$ is a 1-algebraic quantum field theory and $\rho : G \to \mathrm{Aut}(\mathfrak{A})$ is a representation of G as natural automorphisms of $\mathfrak{A}$. A *morphism $\zeta : (\mathfrak{A}, \rho) \to (\mathfrak{B}, \sigma)$ of G-equivariant algebraic quantum field theories* is a morphism $\zeta : \mathfrak{A} \to \mathfrak{B}$ of algebraic quantum field theories compatible with the G-actions, i.e. $\zeta\, \rho(g) = \sigma(g)\, \zeta$, for all $g \in G$. We denote by G-$\mathbf{AQFT}(\mathbf{Sp}^\perp)$ the category consisting of G-equivariant algebraic quantum field theories and their morphisms.

Remark 3.4.2. Consider the category $\mathbf{Rep}_{\mathbb{K}}(G)$ consisting of $\mathbb{K}$-linear representations of G endowed with the symmetric monoidal structure $(\mathbf{Rep}_{\mathbb{K}}(G), \otimes, \mathbb{K}, \tau)$, where $\otimes$ denotes the tensor product $V \otimes W$ of representations, $\mathbb{K}$ denotes the trivial representation $G \to \mathrm{Aut}(\mathbb{K})$ sending each $g \in G$ to the identity $id_{\mathbb{K}} : \mathbb{K} \to \mathbb{K}$ and τ is the symmetric braiding given by the flip map. Moreover, denote by G-$\mathbf{Alg}_{\mathbb{K}}$ the category of *G-equivariant associative and unital $\mathbb{K}$-algebras*, i.e. the category $\mathbf{Alg}_{\mathsf{As}}(\mathbf{Rep}_{\mathbb{K}}(G))$ of monoids in the symmetric monoidal category $(\mathbf{Rep}_{\mathbb{K}}(G), \otimes, \mathbb{K}, \tau)$ (see Remark 1.1.21). This category is symmetric monoidal with respect to the tensor product algebra $A \otimes B$ (see Section 3.2) endowed with the tensor product G-action, monoidal unit $\mathbb{K}$ endowed with the trivial representation and symmetric braiding given by the flip map.

It is not difficult to show that a G-equivariant algebraic quantum field theory $(\mathfrak{A}, \rho) \in G$-$\mathbf{AQFT}(\mathbf{Sp}^\perp)$ can be equivalently defined as a $\perp$-commutative functor (see Definition 1.2.11) $\mathfrak{A} : \mathbf{Sp} \to G$-$\mathbf{Alg}_{\mathbb{K}}$ and that morphisms of G-equivariant algebraic quantum field theories can be equivalently defined as natural transformations between $\perp$-commutative functors $\mathbf{Sp} \to G$-$\mathbf{Alg}_{\mathbb{K}}$. Therefore, our choice of terminology is coherent. △

Given any G-equivariant algebraic quantum field theory $(\mathfrak{A}, \rho) \in G$-$\mathbf{AQFT}(\mathbf{Sp}^\perp)$ we can construct its associated *orbifold theory* $\mathfrak{A}_0^G \in \mathbf{AQFT}(\mathbf{Sp}^\perp)$, i.e. the algebraic quantum field theory that associates to every $s \in \mathbf{Sp}$ the associative and unital $\mathbb{K}$-algebra $\mathfrak{A}_0^G(s)$ given by the G-invariants of $\mathfrak{A}(s)$ ([Xu00, Mug05]). As anticipated in the beginning of this section, we will show that $\mathfrak{A}_0^G$ is the truncation $\mathfrak{A}_0^G \cong \pi(\mathfrak{A}^G)$ (see Subsection 3.3.1) of a 2-algebraic quantum field theory $\mathfrak{A}^G$, called *categorified orbifold construction*, obtained from $\mathfrak{A}$ by applying a functor $(-)^G : G$-$\mathbf{AQFT}(\mathbf{Sp}^\perp) \to \mathbf{2AQFT}(\mathbf{Sp}^\perp)$ called *gauging construction*. Conceptually, the functor $(-)^G : G$-$\mathbf{AQFT}(\mathbf{Sp}^\perp) \to \mathbf{2AQFT}(\mathbf{Sp}^\perp)$ is a G-equivariant generalization of the functor ι from Subsection 3.3.2 obtained by sending each $\mathfrak{A} \in G\text{-}\mathbf{AQFT}(\mathbf{Sp}^{\perp})$ to the the 2-algebraic quantum field theory $\mathfrak{A}^G$ which associates to each $s \in \mathcal{P}_{\mathbf{Sp}_0^{\perp}}$ the $\mathbb{K}$-linear category $G\text{-}\mathbf{Mod}_{\mathfrak{A}(s)}$ of G-equivariant right $\mathfrak{A}(s)$-modules.

Remark 3.4.3. Notice that the construction of the category $\mathbf{Mod}_A(\mathbf{Vec}_{\mathbb{K}})$ of right A-modules for a given algebra $A \in \mathbf{Alg}_{\mathbb{K}} \cong \mathbf{Alg}_{\mathsf{As}}(\mathbf{Vec}_{\mathbb{K}})$ can be generalized to any symmetric monoidal category $\mathbf{C}$. More precisely, given a symmetric monoidal category $(\mathbf{C}, \otimes, I)$ and its associated category of monoids $\mathbf{Alg}_{\mathsf{As}}(\mathbf{C})$ it is possible to consider for every monoid (c, μ_c, η_c) the category $\mathbf{Mod}_c(\mathbf{C})$ of right c-modules in $\mathbf{C}$.

If $\mathbf{C} = \mathbf{Rep}_{\mathbb{K}}(G)$ is the symmetric monoidal category (see Remark 3.4.2) of $\mathbb{K}$-linear G-representations of a finite group G, the symmetric monoidal category $\mathbf{Alg}_{\mathsf{As}}(\mathbf{Rep}_{\mathbb{K}}(G)) := G\text{-}\mathbf{Alg}_{\mathbb{K}}$ is the category of G-equivariant associative algebras (see Remark 3.4.2). For any $A \in G\text{-}\mathbf{Alg}_{\mathbb{K}}$ we call the $\mathbb{K}$-linear locally presentable category $G\text{-}\mathbf{Mod}_A := \mathbf{Mod}_A(\mathbf{Rep}_{\mathbb{K}}(G))$, the *category of G-equivariant right A-modules*. Explicitly, $G\text{-}\mathbf{Mod}_A$ is the category whose objects are couples $(V, V \otimes A \to V)$ where $V \in \mathbf{Rep}_{\mathbb{K}}(G)$ and $V \otimes A \to V$ is a $\mathbf{Rep}_{\mathbb{K}}(G)$-morphism satisfying the axioms of a G-action.

Analogously to the non-equivariant case, given a $G\text{-}\mathbf{Alg}_{\mathbb{K}}$-morphism $\kappa : A \to B$, there exists a $\mathbb{K}$-linear co-continuous functor $\kappa_! = (-) \otimes_A B : G\text{-}\mathbf{Mod}_A \to G\text{-}\mathbf{Mod}_B$, left adjoint to the restriction functor $\kappa^* : G\text{-}\mathbf{Mod}_B \to G\text{-}\mathbf{Mod}_A$ induced by $\kappa : A \to B$, called the *induced G-module functor*. △

Proposition 3.4.4. *Let $\mathbf{Sp}^{\perp} = (\mathbf{Sp}, \perp)$ be an orthogonal category. Then, the following data defines a pseudo-functor $(-)^G : G\text{-}\mathbf{AQFT}(\mathbf{Sp}^{\perp}) \to \mathbf{2AQFT}(\mathbf{Sp}^{\perp})$ called the* gauging construction*:*

(a) On objects*: Given any G-equivariant algebraic quantum field theory $\mathfrak{A} \in G\text{-}\mathbf{AQFT}(\mathbf{Sp}^{\perp})$ the functor $(-)^G$ assigns:*

(a.1) To each $s \in \mathcal{P}_{\mathbf{Sp}_0^{\perp}}$ the locally presentable $\mathbb{K}$-linear category

$$\mathfrak{A}^G(s) := G\text{-}\mathbf{Mod}_{\mathfrak{A}(s)} \tag{3.4.1}$$

of G-equivariant right $\mathfrak{A}(s)$-modules (see Remark 3.4.3).

(a.2) To each non-empty tuple $\underline{f} = (f_1, \dots, f_n) \in \mathcal{P}_{\mathbf{Sp}^{\perp}}\binom{t}{\underline{s}}$ of mutually orthogonal $\mathbf{Sp}$-morphisms $\mathfrak{A}^G$, the functor

$$\mathfrak{A}^G(\underline{f}) \,:\, \prod_{i=1}^{n} G\text{-}\mathbf{Mod}_{\mathfrak{A}(s_i)} \xrightarrow{\ \otimes^n\ } G\text{-}\mathbf{Mod}_{\bigotimes_{i=1}^n \mathfrak{A}(s_i)} \xrightarrow{\ \mathfrak{A}(\underline{f})_!\ } G\text{-}\mathbf{Mod}_{\mathfrak{A}(t)} \quad , \tag{3.4.2}$$

where $\otimes^n : (V_1, \dots, V_n) \mapsto V_1 \otimes \cdots \otimes V_n$ is the functor that assigns to all $V_1 \in G\text{-}\mathbf{Mod}_{\mathfrak{A}(s_1)}, \dots, V_n \in G\text{-}\mathbf{Mod}_{\mathfrak{A}(s_n)}$ their tensor product as representations with the induced G-equivariant structure over the tensor product of algebras and where $\mathfrak{A}(\underline{f})_!$ is the induced G-module functor along the $G\text{-}\mathbf{Alg}_{\mathbb{K}}$-morphism $\mathfrak{A}(\underline{f}) : \bigotimes_{i=1}^n \mathfrak{A}(s_i) \to \mathfrak{A}(t)$ (see Remark 3.4.3). In particular, $\mathfrak{A}^G(\underline{f})$ is $\mathbb{K}$-linear and co-continuous in each variable, i.e. defines a 1-morphism in $\mathbf{Pr}_{\mathbb{K}}$. To a 0-operation $*_s \in \mathcal{P}_{Sp^{\perp}}\binom{s}{\varnothing}$, $\mathfrak{A}^G$ associates the pointing $\mathfrak{A}^G(*_s) := \mathfrak{A}(s) \in G\text{-}\mathbf{Mod}_{\mathfrak{A}(s)}$.

(a.3-a.5) The coherences are analogous to those of the inclusion functor $\iota : \mathbf{AQFT}(\mathbf{Sp}^{\perp}) \to \mathbf{2AQFT}(\mathbf{Sp}^{\perp})$ *from Subsection 3.3.2.*

(b) On 1-morphisms*: To any* 1*-morphism* $\zeta : \mathfrak{A} \to \mathfrak{B}$ *of G-equivariant algebraic quantum field theories* $\mathfrak{A}^G$ *assigns the* 1*-morphism* $\zeta^G : \mathfrak{A}^G \to \mathfrak{B}^G$ *of* 2*-algebraic quantum field theories given for every* $s \in \mathcal{P}_{\mathbf{Sp}_0^{\perp}}$ *by the induced G-module functor (*$\mathbb{K}$*-linear and co-continuous, see Remark 3.4.3) along the G-*$\mathbf{Alg}_{\mathbb{K}}$ *morphism* $\zeta_s : \mathfrak{A}(s) \to \mathfrak{B}(s)$

$$(\zeta^G)_s \,:=\, (\zeta_s)_! \,:\, G\text{-}\mathbf{Mod}_{\mathfrak{A}(s)} \,\longrightarrow\, G\text{-}\mathbf{Mod}_{\mathfrak{B}(s)}. \tag{3.4.3}$$

The coherences for $\zeta^G : \mathfrak{A}^G \to \mathfrak{B}^G$ *are analogous to those of* $\iota(\nu)$*, where* ν *denotes any* 1*-morphism of* 1*-algebraic quantum field theories and* ι *is the inclusion pseudo-functor from Subsection 3.3.2.*

Proposition 3.4.5. *Let* $(\mathfrak{A}, \rho) \in G\text{-}\mathbf{AQFT}(\mathbf{Sp}^{\perp})$ *be a G-equivariant algebraic quantum field theory. Then, there exists an isomorphism* $\mathfrak{A}_0^G \cong \pi(\mathfrak{A}^G)$ *between the orbifold theory* $\mathfrak{A}_0^G$ *and the truncation (see Subsection 3.3.1)* $\pi(\mathfrak{A}^G)$ *of the categorified orbifold construction* $\mathfrak{A}^G$*. This isomorphism is natural in* $(\mathfrak{A}, \rho) \in G\text{-}\mathbf{AQFT}(\mathbf{Sp}^{\perp})$*.*

Proof. From the description of the truncation 2-functor in Subsection 3.3.1, we obtain that $\pi(\mathfrak{A}^G)(s) = \mathrm{End}(\mathfrak{A}(s))$ is the endomorphism algebra of the G-equivariant module $\mathfrak{A}(s) \in G\text{-}\mathbf{Mod}_{\mathfrak{A}(s)}$, for each $s \in \mathbf{Sp}$. Since morphisms in $G\text{-}\mathbf{Mod}_{\mathfrak{A}(c)}$ preserve by definition the G-action, it follows that $\mathrm{End}(\mathfrak{A}(s))$ is isomorphic to the subalgebra of G-action invariants in $\mathfrak{A}(s)$, hence $\pi(\mathfrak{A}^G)(s) \cong \mathfrak{A}_0^G(s)$ is isomorphic to the algebra that is assigned by the traditional orbifold theory $\mathfrak{A}_0^G$. Using further the explicit description of the factorization products of $\pi(\mathfrak{A}^G)$ from Section 3.3.1, one checks that this family of $\mathbf{Alg}_{\mathbb{K}}$-isomorphisms defines an $\mathbf{AQFT}(\mathbf{Sp}^{\perp})$-isomorphism $\pi(\mathfrak{A}^G) \cong \mathfrak{A}_0^G$. Naturality of this isomorphism in $(\mathfrak{A}, \rho) \in G\text{-}\mathbf{AQFT}(\mathbf{Sp}^{\perp})$ is obvious. □

Remark 3.4.6. Proposition 3.4.5 suggests that we can really think of the gauging construction $\mathfrak{A}^{\mathfrak{G}}$ as a categorified orbifold theory for $(\mathfrak{A}, \rho) \in G\text{-}\mathbf{AQFT}(\mathbf{Sp}^{\perp})$. However, this is not the only justification for our choice of nomenclature as the following (informal) argument should explain: The field configurations of a classical σ-model are maps $\phi : \Sigma \to X$ where Σ is a world-sheet and X is a target space. Furthermore, if X is endowed with an action of a finite group G, we can form the quotient stack (orbifold) $X//G$ and consider the associated *orbifold σ-model* whose field configurations are maps $\phi : \Sigma \to X//G$. The space of field configurations is the mapping stack

$$\mathrm{Fields}(\Sigma) \,:=\, \mathrm{Map}\big(\Sigma, X//G\big) \quad . \tag{3.4.4}$$

To study local aspects of this field theory we restrict our attention to the orthogonal category $\mathbf{Disk}(\Sigma)^{\perp_d} = (\mathbf{Disk}(\Sigma), \perp_d)$ (see Example 1.1.8). For spaces $U \in \mathbf{Ob}(\mathbf{Disk}(\Sigma))$ the mapping stack commutes with the stacky quotient, i.e.

$$\mathrm{Fields}(U) \,\simeq\, \mathrm{Map}(U, X//G) \,\simeq\, \mathrm{Map}(U, X)//G. \tag{3.4.5}$$

and $\mathrm{Map}(U, X)$ is an *ordinary* mapping space. Therefore, if we decide to ignore for a moment the stacky quotient by G, we are in the usual scenario where the space of field configurations $\mathsf{Map}(U,X)$ is an *ordinary* space and not a stack, hence, formal deformation quantization of such a field theory leads to an 1-algebraic quantum field theory $\mathfrak{A} \in \mathbf{AQFT}(\mathbf{Disk}(\Sigma)^{\perp_d})$, which, in the case of no anomalies, can be endowed with a G-action $\rho : G \to \mathrm{Aut}(\mathfrak{A})$, i.e. $(\mathfrak{A},\rho) \in G\text{-}\mathbf{AQFT}(\mathbf{Disk}(\Sigma)^{\perp_d})$. By construction, $\mathfrak{A}(U)$ is a deformation quantization of a suitable G-equivariant function algebra $\mathcal{O}(\mathrm{Map}(U,X))$.

Taking the perspective of [Lur04, Bra14], if we do consider the stacky quotient by G we should assign to the stack $\mathrm{Fields}(U)$ its category of quasi-coherent sheaves

$$\mathbf{QCoh}\big(\mathrm{Fields}(U)\big) \simeq \mathbf{QCoh}\big(\mathrm{Map}(U,X)//G\big) \simeq G\text{-}\mathbf{Mod}_{\mathcal{O}(\mathrm{Map}(U,X))} \quad . \tag{3.4.6}$$

Considering the aforementioned algebra $\mathfrak{A}(U)$, obtained via formal quantization of $\mathcal{O}(\mathsf{Map}(U,X))$, we can legitimately think of the pointed category $\mathfrak{A}^G(U) = G\text{-}\mathbf{Mod}_{\mathfrak{A}(U)}$ as a quantization of the category $G\text{-}\mathbf{Mod}_{\mathcal{O}(\mathrm{Map}(U,X))}$ (see Proposition 3.4.4). Therefore, our gauging construction encodes local aspects of orbifold σ-models. △

Notice that the categorified orbifold construction from Proposition 3.4.4 enables us to introduce of simple toy-models of non-truncated 2-algebraic quantum field theories (see Definition 3.3.4) for non trivial groups G.

Example 3.4.7. Let $G \neq e$ be a non trivial group and consider the G-equivariant algebraic quantum field theory $(\mathbb{K},\rho) \in G\text{-}\mathbf{AQFT}(\mathbf{Sp}^{\perp})$ where $\mathbb{K} \in \mathbf{AQFT}(\mathbf{Sp}^{\perp})$ is the trivial 1-algebraic quantum field theory associating to every $s \in \mathbf{Sp}$ the field $\mathbb{K}$ and where $\rho : G \to \mathrm{Aut}(\mathbb{K})$ is the trivial G-action sending any element $g \in G$ to the identity $\mathrm{id}_{\mathbb{K}}$. The categorified orbifold theory $\mathbb{K}^G$ (see Proposition 3.4.4) assigns, by definition, to each $s \in \mathbf{Sp}$ the locally presentable $\mathbb{K}$-linear category $\mathbb{K}^G(s) = G\text{-}\mathbf{Mod}_{\mathbb{K}} = \mathbf{Rep}_{\mathbb{K}}(G)$ of $\mathbb{K}$-linear G-representations with pointing given by $\mathbb{K}$ with the trivial representation. Our claim is that $\mathbb{K}^G$ is non-truncated, therefore, we proceed to show that the counit $\epsilon : \iota\pi \to \mathrm{id}$ at $\mathbb{K}^G$ is *not* an equivalence (see Definition 3.3.4). We begin by noticing that for every $s \in \mathbf{Sp}$, $\iota\pi(\mathbb{K}^G)(s) \cong \mathbf{Vec}_{\mathbb{K}}$, and that the $\mathbb{K}$-linear functor $\epsilon_s : \iota\pi(\mathbb{K}^G)(s) \cong \mathbf{Vec}_{\mathbb{K}} \to \mathbf{Rep}_{\mathbb{K}}(G)$ preserves the pointing up to coherence isomorphisms, i.e. $\epsilon_s(\mathbb{K}) \cong \mathbb{K}$. Therefore, since ϵ_s is co-continuous and since any vector space V over the field $\mathbb{K}$ is isomorphic to a direct sum of copies of $\mathbb{K}$, i.e. $V \cong \oplus_{i\in I}\mathbb{K}$ for some index set I, we obtain that the image $\epsilon_s(\mathbf{Vec}_{\mathbb{K}}) \subseteq \mathbf{Rep}_{\mathbb{K}}(G)$ is contained in the subcategory of $\mathbf{Rep}_{\mathbb{K}}(G)$ consisting of trivial representations, proving our claim. ▽

Example 3.4.7 provides instances of simple toy-models of non-truncated 2-algebraic quantum field theories, proving that, generally, the 2-category $\mathbf{2AQFT}(\mathbf{Sp}^{\perp})$ is bigger than the category $\mathbf{AQFT}(\mathbf{Sp}^{\perp})$ of ordinary algebraic quantum field theories. What is more surprising is that we can characterize exactly those G-equivariant algebraic quantum field theories $(\mathfrak{A},\rho) \in G\text{-}\mathbf{AQFT}(\mathbf{Sp}^{\perp})$ that are truncated. We will find out that this characterization relies on whether or not the inclusion of the G-action invariants of the algebra $\mathfrak{A}_0^G(s) \subseteq \mathfrak{A}(s)$ is a $\mathcal{O}(G)$-Hopf-Galois extension for every $s \in \mathbf{Sp}$.

Remark 3.4.8 (see [DT89, Mon09]). The purpose of this Remark is to recall some basic terminology from Hopf-Galois theory that will be needed in what follows.

A *Hopf algebra* $H = (\mu, \eta, \Delta, \epsilon, S)$ over $\mathbb{K}$ is an associative and co-associative $\mathbb{K}$-bialgebra together with a $\mathbb{K}$-linear antipode map $S : H \to H$ satisfying the following equation: $\mu \circ (\mathrm{id} \otimes S) \circ \Delta = \mu \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \epsilon$.

Given a Hopf algebra H, consider the monoidal category $\mathbf{Mod}^H$ of right H-comodules and its associated category of algebras $\mathbf{Alg}_{\mathbb{K}}^H = \mathbf{Alg}_{\mathsf{As}}(\mathbf{Mod}^H)$. We call the objects of the latter category, i.e. the algebras $A \in \mathbf{Alg}_{\mathbb{K}}$ endowed with an $\mathbf{Alg}_{\mathbb{K}}$-morphism $\delta : A \to A \otimes H$ satisfying the axioms of a right H-coaction, *right H-comodule algebras*.

Given an H-comodule algebra A, we can consider the category $\mathbf{Mod}_A^H$ of right (H, A)-Hopf comodules, i.e. the category of right A-modules in $\mathbf{Mod}^H$ (see Remark 3.4.3). The objects of $\mathbf{Mod}_A^H$ are the right A-modules $V \in \mathbf{Mod}_A$ with a compatible H-coaction $\delta^V : V \to V \otimes H$ (i.e. $\delta^V(va) = \delta^V(v)\delta(a)$). The morphisms of $\mathbf{Mod}_A^H$ are the A-action and H-coaction preserving $\mathbb{K}$-linear maps. △

Definition 3.4.9. Let H be a Hopf algebra over $\mathbb{K}$, let $A \in \mathbf{Alg}_{\mathbb{K}}^H$ be a right H-comodule algebra with H-coaction $\delta : A \to A \otimes H$ and denote by $B := A^{\mathrm{co}H} := \{a \in A : \delta(a) = a \otimes 1_H\} \subseteq A$ the subalgebra of *H-coaction invariants*. We say that the extension $B := A^{\mathrm{co}H} \subseteq A$ is *Hopf-Galois*, if the map

$$\beta \,:\, A \otimes_B A \longrightarrow A \otimes H\,, \quad a \otimes_B a' \longmapsto (a \otimes 1_H)\,\delta(a') \tag{3.4.7}$$

is bijective.

Given a Hopf algebra H over $\mathbb{K}$ any right H-comodule algebra A induces the following adjunction:

$$\Phi \,:\, \mathbf{Mod}_B \rightleftarrows \mathbf{Mod}_A^H \,:\, \Psi \quad , \tag{3.4.8}$$

where $\Phi : \mathbf{Mod}_B \to \mathbf{Mod}_A^H$ is the *induced Hopf-module functor* that sends a right $B(:= A^{\mathrm{co}H})$-module V to the (H, A)-Hopf module $V \otimes_B A$ endowed with the natural right H-coaction $\mathrm{id} \otimes_B \delta$ ($\delta : A \to A \otimes H$ is the right H-coaction on A), and where $\Psi : \mathbf{Mod}_A^H \to \mathbf{Mod}_B$ is the functor assigning to each right (H, A)-Hopf module $W \in \mathbf{Mod}_A^H$, the right B-module $V^{\mathrm{co}H} := \{v \in V : \delta^V(v) = v \otimes 1_H\}$ of H-coaction invariants.

As we mentioned earlier, we will see that whether the categorified orbifold construction of a G-equivariant algebraic quantum field theory is truncated or not relies on checking if certain algebra extensions are $\mathcal{O}(G)$-Hopf Galois. Before restricting our attention to the Hopf algebra of functions $\mathcal{O}(G)$, we need the following general result (see [Mon09, Theorem 5.6] or [DT89]):

Theorem 3.4.10. *Let H be a finite dimensional Hopf algebra over $\mathbb{K}$ and let A be an H-comodule algebra. Then, the extension $B = A^{\mathrm{co}H} \subseteq A$ is H-Hopf-Galois if and only if the counit $\epsilon : \Phi\Psi \Rightarrow \mathrm{id}$ of the adjunction* (3.4.8) *is a natural isomorphism.*

In the case where the Hopf algebra over $\mathbb{K}$ under consideration is the Hopf algebra of functions $H = \mathsf{Map}(G, \mathbb{K}) := \mathcal{O}(G)$, which is finite dimensional for finite groups G, the category $\mathbf{Mod}_A^H$ of right (H, A)-Hopf modules is equivalent to the category

G-$\mathbf{Mod}_A$ of G-equivariant right A-modules. To see this, notice that any right $\mathcal{O}(G)$-coaction $\delta^V : V \to V \otimes \mathcal{O}(G)$ determines a group action $\rho : G \to \mathrm{Aut}(V)$ defined by $\rho(g)(v) = v_{(0)} \langle v_{(1)}, g\rangle$, where we used Sweedler notation $\delta^V(v) = v_{(0)} \otimes v_{(1)}$ and the duality pairing $\langle\cdot,\cdot\rangle : \mathcal{O}(G) \otimes \mathbb{K}[G] \to \mathbb{K}$. Moreover, notice that the subalgebra $B := A^{\mathrm{co}H} \subseteq A$ of H-coaction invariants coincides with the subalgebra of G-action invariants $A_0^G \subseteq A$, i.e. $B = A_0^G$. These observations lead us to reinterpret the adjunction in equation (3.4.8) as an adjunction $\Phi : \mathbf{Mod}_B \leftrightarrows G\text{-}\mathbf{Mod}_A : \Psi$.

Corollary 3.4.11. *Let G be a finite group, let $H = \mathcal{O}(G) = \mathsf{Map}(G, \mathbb{K})$ denote the function Hopf algebra of G and let $B = A^{\mathrm{co}H} \subseteq A$ be the subalgebra of A of H-coaction invariants. Then, the adjunction in equation* (3.4.8) *reads as*

$$\Phi \ : \ \mathbf{Mod}_B \ \xrightleftharpoons{} \ G\text{-}\mathbf{Mod}_A \ : \ \Psi \quad , \tag{3.4.9}$$

and it is an adjoint equivalence in $\mathbf{Pr}_{\mathbb{K}}$ if and only if the extension $B = A_0^G \subseteq A$ is $\mathcal{O}(G)$-Hopf-Galois.

Proof. The left adjoint functor $\Phi = (-) \otimes_B A$ is clearly $\mathbb{K}$-linear and co-continuous, i.e. a 1-morphism in $\mathbf{Pr}_{\mathbb{K}}$. The right adjoint functor $\Psi = (-)^{\mathrm{co}H} = (-)_0^G$ assigns the G-invariants (given by a categorical limit), which for actions of finite groups G and $\mathrm{char}(\mathbb{K}) = 0$ coincides with the G-*co*invariants (i.e. a categorical colimit). Hence, the right adjoint Ψ is a $\mathbb{K}$-linear and co-continuous functor too and the adjunction (3.4.9) is in the 2-category $\mathbf{Pr}_{\mathbb{K}}$.

The unit $\eta : \mathrm{id} \Rightarrow \Psi\,\Phi$ of the adjunction (3.4.9) is given by the components $\eta_W : W \to (W \otimes_B A)_0^G$, $w \mapsto w \otimes_B 1_A$, for all $W \in \mathbf{Mod}_B$. Using again that forming G-invariants coincides with forming G-*co*invariants, we find that $\eta : \mathrm{id} \Rightarrow \Psi\,\Phi$ is a natural isomorphism. Our claim then follows from Theorem 3.4.10. □

Theorem 3.4.12. *Let G be a finite group, $\mathbf{Sp}^\perp = (\mathbf{Sp}, \perp)$ be an orthogonal category and let $(\mathfrak{A}, \rho) \in G\text{-}\mathbf{AQFT}(\mathbf{Sp}^\perp)$ be a G-equivariant algebraic quantum field theory. Then, the categorified orbifold construction $\mathfrak{A}^G \in \mathbf{2AQFT}(\mathbf{Sp}^\perp)$ is truncated if and only if the extension $B := \mathfrak{A}(s)_0^G \subseteq \mathfrak{A}(s)$ is $\mathcal{O}(G)$-Hopf-Galois for every $s \in \mathcal{P}_{\mathbf{Sp}_0^\perp}$.*

Proof. Recalling Definition 3.3.4, the 2AQFT $\mathfrak{A}^G \in \mathbf{2AQFT}(\mathbf{Sp}^\perp)$ is by definition truncated if the corresponding component $\epsilon_{\mathfrak{A}^G} : \iota\,\pi(\mathfrak{A}^G) \to \mathfrak{A}^G$ of the counit of the inclusion-truncation biadjunction from Theorem 3.3.3 is an equivalence in $\mathbf{2AQFT}(\mathbf{Sp}^\perp)$. The component $\epsilon_{\mathfrak{A}^G}$ of the counit is determined uniquely (up to invertible 2-morphisms in $\mathbf{2AQFT}(\mathbf{Sp}^\perp)$) by the condition $\widetilde{\epsilon_{\mathfrak{A}^G}} = \mathrm{id}_{\pi(\mathfrak{A}^G)} : \pi(\mathfrak{A}^G) \to \pi(\mathfrak{A}^G)$ on its adjunct under (3.3.13). Using the explicit description of the inclusion and truncation pseudo-functors from Section 3.1 and the one of the gauging construction from the present section, one observes that the induced module functors $\Phi_s = (-) \otimes_{\mathfrak{A}_0^G(s)} \mathfrak{A}(s) : \iota\,\pi(\mathfrak{A}^G)(s) \cong \mathbf{Mod}_{\mathfrak{A}_0^G(s)} \to \mathfrak{A}^G(s) = G\text{-}\mathbf{Mod}_{\mathfrak{A}(s)}$ (together with the obvious coherence isomorphisms) define a 1-morphism $\Phi : \iota\,\pi(\mathfrak{A}^G) \to \mathfrak{A}^G$ in $\mathbf{2AQFT}(\mathbf{Sp}^\perp)$ that satisfies $\widetilde{\Phi} = \mathrm{id}_{\pi(\mathfrak{A}^G)} : \pi(\mathfrak{A}^G) \to \pi(\mathfrak{A}^G)$. Hence, $\Phi \cong \epsilon_{\mathfrak{A}^G}$ and we can equivalently investigate if Φ is an equivalence in $\mathbf{2AQFT}(\mathbf{Sp}^\perp)$.

By a straightforward but slightly lengthy calculation, one proves that a 1-morphism in $\mathbf{2AQFT}(\mathbf{Sp}^\perp)$ is an equivalence if and only if all its components are equivalences in the 2-category $\mathbf{Pr}_{\mathbb{K}}$. (In this proof one uses that every equivalence in any 2-category (here $\mathbf{Pr}_{\mathbb{K}}$) can be upgraded to an adjoint equivalence in order to define quasi-inverse 1-morphisms in $\mathbf{2AQFT}(\mathbf{Sp}^{\perp})$.) Thus, to prove that $\mathfrak{A}^G \in \mathbf{2AQFT}(\mathbf{Sp}^{\perp})$ is truncated we can equivalently study the components $\Phi_s = (-) \otimes_{\mathfrak{A}_0^G(s)} \mathfrak{A}(s) : \mathbf{Mod}_{\mathfrak{A}_0^G(s)} \to G\text{-}\mathbf{Mod}_{\mathfrak{A}(s)}$, for all $s \in \mathbf{Sp}$. By Corollary 3.4.11, these components are equivalences in $\mathbf{Pr}_{\mathbb{K}}$ if and only if the algebra extension $\mathfrak{A}_0^G(s) \subseteq \mathfrak{A}(s)$ is $\mathcal{O}(G)$-Hopf-Galois, for all $s \in \mathbf{Sp}$. This completes the proof. □

Remark 3.4.13. Notice that Theorem 3.4.12 matches the physical intuition from Remark 3.4.6. In particular, it shows that we can think of the Hopf-Galois extension condition as an analogue for algebras of a free G-action on spaces. More precisely, when the action of G on a space X is free, the stacky quotient $X//G$ is nothing else but the ordinary quotient space X/G. Therefore, the resulting σ-model is an ordinary σ-model and no higher categorical features appear. Analogously, when a G-equivariant algebraic quantum field theory satisfies the point-wise Hopf-Galois condition, its categorified orbifold construction is an ordinary algebraic quantum field theory and no higher categorical features appear. △

To conclude this section we show an example of a non-truncated classical field theory that quantizes to a truncated quantum field theory.

Example 3.4.14. Consider the orthogonal category $\mathbf{Disk}(\mathbb{S}^1)^{\perp_d}$ from Example 1.1.8 and the algebraic quantum field theory $\mathfrak{A} \in \mathbf{AQFT}(\mathbf{Disk}(\mathbb{S}^1)^{\perp_d})$, called *chiral free boson*, defined by the following data:

(a) *On objects*: To each open interval $I \subset \mathbb{S}^1$, $\mathfrak{A}$ assigns the canonical commutation relations algebra (CCR):

$$\mathfrak{A}(I) := T_{\mathbb{C}}^{\otimes} C_c^{\infty}(I) \Big/ \Big\langle \varphi_1 \otimes \varphi_2 - \varphi_2 \otimes \varphi_1 - i\hbar \int_I \varphi_1 \, \mathrm{d}\varphi_2 \, 1 \Big\rangle \in \mathbf{Alg}_{\mathbb{C}} \quad , \tag{3.4.10}$$

where $\hbar \in \mathbb{R}$ is the Planck's constant treated as a number and not as a formal deformation parameter, $C_c^{\infty}(I)$ is the vector space of compactly supported real-valued functions on $I \subset \mathbb{S}^1$ and $T_{\mathbb{C}}^{\otimes} C_c^{\infty}(I) := \bigoplus_{n=0}^{\infty} (C_c^{\infty}(I) \otimes_{\mathbb{R}} \mathbb{C})^{\otimes n} \in \mathbf{Alg}_{\mathbb{C}}$ is the complexified free algebra.

(b) *On morphisms*: To each inclusion $\iota_I^J : I \to J$, $\mathfrak{A}$ assigns the $\mathbf{Alg}_{\mathbb{K}}$-morphism $\mathfrak{A}(\iota_I^J) : \mathfrak{A}(I) \to \mathfrak{A}(J)$ defined on the generators by pushforward (i.e. extension by zero) of compactly supported functions.

Furthermore, endow $\mathfrak{A}$ with the $\mathbb{Z}_2$-action $\rho : \mathbb{Z}_2 \to \mathrm{Aut}(\mathfrak{A})$ defined for every open $I \subset \mathbb{S}^1$ by $\rho(\pm)(\phi) = \pm\phi$ for every generator ϕ of $\mathfrak{A}(I)$, obtaining a $\mathbb{Z}_2$-equivariant algebraic quantum field theory $(\mathfrak{A}, \rho) \in \mathbb{Z}_2\text{-}\mathbf{AQFT}(\mathbf{Disk}(\mathbb{S}^1)^{\perp_d})$.

Our goal is to find whether the categorified orbifold construction $\mathfrak{A}^G$ is truncated using Theorem 3.4.12. Let's begin by studying the subalgebra $B := A^{\mathrm{co}H} = A_0^{\mathbb{Z}_2} \subseteq A$ of H-coaction invariants and how the Hopf-Galois map $\beta : A \otimes_B A \to A \otimes H$ (see Definition 3.4.9) acts. Having a look at equation (3.4.10) makes clear that B consists of the even part of the algebra (3.4.10). Moreover, since we can decompose A into its even part B and odd part V, i.e. $A = B \oplus V$ and since $A \otimes \mathcal{O}(G) \cong \prod_{g \in G} A$, β reads as:

$$\begin{aligned} \beta \,:\, B \oplus (V \otimes_B V) \oplus V \oplus V \;&\longrightarrow\; \prod_{g \in \mathbb{Z}_2} A \quad , \\ b + v \otimes_B v' + v_1 + v_2 \;&\longmapsto\; \begin{pmatrix} b + v\,v' + v_1 + v_2 \\ b - v\,v' + v_1 - v_2 \end{pmatrix} \quad . \end{aligned} \tag{3.4.11}$$

Notice that from equation (3.4.11) we get that $\beta : A \otimes_B A \to A \otimes H$ is a bijection if and only if $\mu : V \otimes_B V \to B\,,\; v \otimes_B v' \mapsto v\,v'$ is. Hence, to study whether the chiral free boson theory $\mathfrak{A}$ is truncated, is sufficient to study μ.

Let's begin by considering the case $\hbar = 0$, i.e. the case in which the chiral free boson theory $\mathfrak{A}$ is a classical field theory. In this case (3.4.10) is a complexified symmetric algebra over $C^\infty_c(I)$ and the map $\mu : V \otimes_B V \to B$ is *not* surjective because its image is at least quadratic in the generators. Therefore, using Theorem 3.4.12, we obtain that the classical chiral free boson theory is *non*-truncated.

When $\hbar \neq 0$, μ is bijective. To prove that μ is surjective notice that one can always find two generators $\varphi_1, \varphi_2 \in C^\infty_c(I) \subseteq V \subseteq A$ that satisfy $[\varphi_1, \varphi_2] = i\hbar\, 1$. Then, given any $b \in B$ the following equation holds:

$$\mu\Big(b\,\frac{1}{i\hbar}(\varphi_1 \otimes_B \varphi_2 - \varphi_2 \otimes_B \varphi_1)\Big) = b\,\frac{1}{i\hbar}[\varphi_1, \varphi_2] = b \quad , \tag{3.4.12}$$

proving surjectivity. For injectivity, consider $\sum_j v_j \otimes_B v'_j \in V \otimes_B V$ such that $\sum_j v_j\, v'_j = 0$ and observe that

$$\begin{aligned} \sum_j v_j \otimes_B v'_j &= \frac{1}{i\hbar} \sum_j [\varphi_1, \varphi_2]\, v_j \otimes_B v'_j \\ &= \frac{1}{i\hbar}\, \varphi_1 \otimes_B \varphi_2 \sum_j v_j\, v'_j - \frac{1}{i\hbar}\, \varphi_2 \otimes_B \varphi_1 \sum_j v_j\, v'_j = 0 \quad , \end{aligned} \tag{3.4.13}$$

where we used that $\varphi_1\, v_j \in B$ and $\varphi_2\, v_j \in B$.

Therefore, when $\hbar \neq 0$, the categorified orbifold construction is truncated (see Theorem 3.4.12). This example shows that quantization can "destroy" classical higher categorical features.

$\triangledown$

3.5 FREDENHAGEN'S UNIVERSAL CATEGORY

The aim of this section is twofold: first of all we want to introduce a categorification of Fredenhagen's universal algebra (see Example 1.2.20) that should be considered as an analogue of Factorization Homology for topological quantum field theories ([AF15, BZBJ18a, BZBJ18b]). Secondly, we want to show some examples of Fredenhagen's universal categories on the circle $\mathbb{S}^1$. Before introducing the aforementioned categorification we would like to recall what Fredenhagen's universal algebra is.

We have seen in Example 1.2.20 that multicategorical left Kan extension can be used to extend any ordinary algebraic quantum field theory $\mathfrak{A} \in \mathbf{AQFT}(\mathbf{Loc}_\diamond^{\perp_c})$ to an ordinary algebraic quantum field theory $J_!(\mathfrak{A}) \in \mathbf{AQFT}(\mathbf{Loc}^{\perp_c})$, where $\mathbf{Loc}_\diamond^{\perp_c}$ is the orthogonal category of diamond globally hyperbolic Lorentzian manifolds. This construction can be generalized to more general orthogonal categories by recalling that a $\mathbf{Vec}_{\mathbb{K}}$-valued algebraic quantum field theory on a generic orthogonal category $\mathbf{Sp}^{\perp} = (\mathbf{Sp}, \perp)$ can be equivalently interpreted as an $\mathbf{Alg}_{\mathbb{K}}$-valued prefactorization algebra (see Theorem 3.2.1). In particular, remembering from Example 1.1.14, that given orthogonal categories $\mathbf{D}^{\perp} = (\mathbf{D}, \perp_{\mathbf{D}})$ and $\mathbf{E}^{\perp} = (\mathbf{E}, \perp_{\mathbf{E}})$ and an orthogonal functor $J : \mathbf{D}^{\perp} \to \mathbf{E}^{\perp}$, there exists an induced multifunctor $J : \mathcal{P}_{\mathbf{D}^{\perp}} \to \mathcal{P}_{\mathbf{E}^{\perp}}$ (notice the slight abuse of notations) and recalling Theorem 1.1.26 we obtain the following adjunction:

$$J_! \,:\, \mathbf{AQFT}(\mathbf{D}^{\perp}) \cong \mathbf{Alg}_{\mathcal{P}_{\mathbf{D}^{\perp}}}(\mathbf{Alg}_{\mathbb{K}}) \rightleftarrows \mathbf{Alg}_{\mathcal{P}_{\mathbf{E}^{\perp}}}(\mathbf{Alg}_{\mathbb{K}}) \cong \mathbf{AQFT}(\mathbf{E}^{\perp}) \,:\, J^* \quad , \tag{3.5.1}$$

where the right adjoint functor J^* is given by restriction of AQFTs along J and $J_!$ is obtained via multicategorical left Kan extension.

Given $e \in \mathcal{P}_{\mathbf{E}^{\perp}}$ and an $\mathfrak{A} \in \mathbf{AQFT}(\mathbf{D}^{\perp})$, we say that $J_!(\mathfrak{A})(e)$ is *Fredenhagen's universal algebra* along J for $\mathfrak{A}$ on e.

To categorify Fredenhagen's universal algebra we will use a 2-categorical generalization of the multicategorical left Kan extensions (see Theorem 1.1.26) along multifunctors $J : \mathcal{P}_{\mathbf{D}^{\perp}} \to \mathcal{P}_{\mathbf{E}^{\perp}}$ induced by maps of orthogonal categories $J : \mathbf{D}^{\perp} \to \mathbf{E}^{\perp}$. More precisely, we will leverage the biadjunction

$$J_! \,:\, \mathbf{2AQFT}(\mathbf{D}^{\perp}) \rightleftarrows \mathbf{2AQFT}(\mathbf{E}^{\perp}) \,:\, J^* \quad , \tag{3.5.2}$$

where the right adjoint 2-functor J^* is given by restriction of 2AQFTs along J and the left adjoint pseudo-functor $J_!$ is a 2-categorical generalization of multicategorical left Kan extension along J. Analogously to the 1-categorical case, given $\boldsymbol{\mathfrak{A}} \in \mathbf{2AQFT}(\mathbf{D}^{\perp})$ and an object $e \in \mathcal{P}_{\mathbf{E}_0^{\perp}}$, we call the category $J_!(\boldsymbol{\mathfrak{A}})(e)$, *Fredenhagen's universal category* along J for $\boldsymbol{\mathfrak{A}}$ on e.

Let us outline briefly the content of this section.

In Subsection 3.5.1 we introduce a bicolimit formula, analogous to that of Remark 1.1.27, to compute Fredenhagen's universal category in explicit terms. Finally, in Subsection 3.5.2 we give some examples of Fredenhagen universal categories along the inclusion orthogonal functor $\iota : \mathbf{Disk}(\mathbb{S}^1)^{\perp_d} \hookrightarrow \mathbf{Open}(\mathbb{S}^1)^{\perp_d}$. More precisely, we will focus on Fredenhagen's universal categories along ι for categorified orbifold algebraic quantum field theories (see Section 3.4).

3.5.1 *Extension*

The aim of this subsection is to obtain an explicit description of Fredenhagen's universal category. In order to achieve this goal we begin by describing in explicit terms the monoidal pseudo-functor $\underline{\boldsymbol{\mathfrak{A}}} : \mathcal{P}^{\otimes}_{\mathbf{D}^{\perp}} \to \mathbf{Pr}_{\mathbb{K}}$ associated to any $\boldsymbol{\mathfrak{A}} \in \mathbf{2AQFT}(\mathbf{D}^{\perp})$ by the universal property of monoidal envelopes (see Remarks 1.1.16 and 1.1.27) and proceed by introducing a 2-categorical analogue of the colimit formula in Remark 1.1.27 to compute $J_!(\boldsymbol{\mathfrak{A}}) \in \mathbf{2AQFT}(\mathbf{E}^{\perp})$, where $J_!$ is the left adjoint in (3.5.2), i.e. the *2-multicategorical left Kan-extension pseudo-functor along J*.

Remark 3.5.1. We can associate to every $\mathfrak{A} \in \mathbf{2AQFT}(\mathbf{D}^{\perp})$ a symmetric monoidal pseudo-functor

$$\underline{\mathfrak{A}} \,:\, \mathcal{P}^{\otimes}_{\mathbf{D}^{\perp}} \longrightarrow \mathbf{Pr}_{\mathbb{K}} \tag{3.5.3a}$$

given by the universal property of monoidal envelopes (see Remarks 1.1.16 and 1.1.27). This pseudo-functor is defined by the following data:

(a) *On objects*: It assigns to each non empty tuple $\underline{d} = (d_1, \ldots, d_n) \in \mathcal{P}^{\otimes}_{\mathbf{D}^{\perp}}$ the n-ary Kelly-Deligne tensor product

$$\underline{\mathfrak{A}}(\underline{d}) \,:=\, \boxtimes_{i=1}^{n} \mathfrak{A}(d_i) \tag{3.5.3b}$$

of the locally presentable $\mathbb{K}$-linear categories $\mathfrak{A}(d_i) \in \mathbf{Pr}_{\mathbb{K}}$ (see Remark 3.2.6). The empty tuple, i.e. the monoidal unit of $\mathcal{P}^{\otimes}_{\mathbf{D}^{\perp}}$, is assigned, by convention the monoidal unit of $\mathbf{Pr}_{\mathbb{K}}$, i.e. $\underline{\mathfrak{A}}(\emptyset) := \mathbf{Vec}_{\mathbb{K}}$.

(b) *On morphisms*: It assigns to each $(\alpha, \underline{\underline{f}}) : \underline{d} \to \underline{t}$ in $\mathcal{P}^{\otimes}_{\mathbf{D}^{\perp}}$, the functor

$$\underline{\mathfrak{A}}(\alpha, \underline{\underline{f}}) \,:\, \underline{\mathfrak{A}}(\underline{d}) = \boxtimes_{i=1}^{n} \mathfrak{A}(d_i) \xrightarrow{\;\simeq_\alpha\;} \boxtimes_{j=1}^{m} \underline{\mathfrak{A}}(\underline{d}_{\alpha,j}) \xrightarrow{\;\boxtimes_j \mathfrak{A}(\underline{f}_j)\;} \boxtimes_{j=1}^{m} \mathfrak{A}(t_j) = \underline{\mathfrak{A}}(\underline{t}) \quad , \tag{3.5.3c}$$

where $\simeq_\alpha$ is the equivalence in the symmetric monoidal 2-category $\mathbf{Pr}_{\mathbb{K}}$ associated to the permutation induced by α.

(c) The coherences are canonically given by the coherences for $\mathfrak{A} \in \mathbf{2AQFT}(\mathbf{D}^{\perp})$ and the symmetric monoidal structure on $\mathbf{Pr}_{\mathbb{K}}$.

$\triangle$

Remark 3.5.2. In Remark 1.1.27 we have seen that, given a multifunctor $\phi : \mathcal{O} \to \mathcal{P}$, the multicategorical left Kan-extension $\phi_!(\mathfrak{A}) \in \mathbf{Alg}_{\mathcal{P}}(\mathbf{C})$ of a multifunctor $\mathfrak{A} \in \mathbf{Alg}_{\mathcal{O}}(\mathbf{C})$ can be computed point-wise as a particular colimit. Analogously, given an orthogonal functor $J : \mathbf{D}^{\perp} \to \mathbf{E}^{\perp}$ and denoting by $J_! : \mathbf{2AQFT}(\mathbf{D}^{\perp}) \to \mathbf{2AQFT}(\mathbf{E}^{\perp})$ the 2-multicategorical left Kan-extension along J, i.e. the left adjoint in Equation (3.5.2), we can compute $J_!(\mathfrak{A})(e)$, for every $e \in \mathcal{P}_{\mathbf{E}^{\perp}_0}$, via the following bicolimit in $\mathbf{Pr}_{\mathbb{K}}$ (which always exists since $\mathbf{Pr}_{\mathbb{K}}$ is bicomplete, see e.g. [BCJF15, Lemma 2.4]):

$$J_!(\mathfrak{A})(e) \,:=\, \mathrm{bicolim}\left(J^{\otimes}/(e) \xrightarrow{\text{forget}} \mathcal{P}^{\otimes}_{\mathbf{D}^{\perp}} \xrightarrow{\;\underline{\mathfrak{A}}\;} \mathbf{Pr}_{\mathbb{K}} \right) \tag{3.5.4a}$$

where $J^{\otimes}/(e)$ is the slice category for the functor $J^{\otimes} : \mathcal{P}^{\otimes}_{\mathbf{D}^{\perp}} \to \mathcal{P}^{\otimes}_{\mathbf{E}^{\perp}}$ over the 1-tuple $(e) \in \mathcal{P}^{\otimes}_{\mathbf{E}^{\perp}}$ and $\underline{\mathfrak{A}} : \mathcal{P}^{\otimes}_{\mathbf{D}^{\perp}} \to \mathbf{Pr}_{\mathbb{K}}$ is the functor from Remark 3.5.1.

Describing the functor

$$J_!(\mathfrak{A})(\underline{g}) \,:\, \prod_{i=1}^{n} J_!(\mathfrak{A})(e_i) \longrightarrow J_!(\mathfrak{A})(t) \tag{3.5.4b}$$

associated to an operation $\underline{g} = (g_1, \ldots, g_n) \in \mathcal{P}_{\mathbf{E}^\perp}\binom{t}{\underline{e}}$ is a bit more complicated. Consider the diagram

$$\begin{tikzcd}
\prod\limits_{i=1}^{n} J^{\otimes}/(e_i) \arrow[r, "\prod_i \mathrm{forget}"] \arrow[d, "\underline{g}_*"'] & \prod\limits_{i=1}^{n} \mathcal{P}^{\otimes}_{\mathbf{D}^{\perp}\mathbf{D}} \arrow[r, "\prod_i \underline{\mathfrak{A}}"] \arrow[d, "\otimes^n"'] & \prod\limits_{i=1}^{n} \mathbf{Pr}_{\mathbb{K}} \arrow[d, "\boxtimes^n"] \arrow[dl, Rightarrow, "(\star)"'] \\
J^{\otimes}/(t) \arrow[r, "\mathrm{forget}"'] & \mathcal{P}^{\otimes}_{\mathbf{D}^{\perp}\mathbf{D}} \arrow[r, "\underline{\mathfrak{A}}"'] & \mathbf{Pr}_{\mathbb{K}}
\end{tikzcd} \tag{3.5.4c}$$

where $\underline{g}_* : \prod_{i=1}^{n} J^{\otimes}/(e_i) \to J^{\otimes}/(t)$ is the functor induced by post-composition with $\underline{g}$ in the multicategory $\mathcal{P}_{\mathbf{E}^\perp}$. The left square of this diagram commutes by direct inspection. In the right square, the natural isomorphism $(\star)$ between the functors given by the clockwise and counter-clockwise paths is obtained using the symmetric monoidal structure on the pseudo-functor $\underline{\mathfrak{A}}$. Passing to bicolimits and recalling that the Kelly-Deligne tensor product $\boxtimes$ commutes with bicolim (in each variable) provides a co-continuous $\mathbb{K}$-linear functor $\boxtimes_{i=1}^{n} J_!(\mathfrak{A})(e_i) \to J_!(\mathfrak{A})(t)$. Pre-composition with the canonical functor $\prod_{i=1}^{n} J_!(\mathfrak{A})(e_i) \to \boxtimes_{i=1}^{n} J_!(\mathfrak{A})(e_i)$, which is $\mathbb{K}$-linear and co-continuous in each variable, completes the construction of the functor in Equation (3.5.4b). The $\mathbb{K}$-linear co-continuous functor associated to the only operation $*_e \in \mathcal{P}_{\mathbf{E}^\perp\mathbf{E}}\binom{e}{\varnothing}$, i.e. the pointing $J_!(\mathfrak{A})(*_e) \in J_!(\mathfrak{A})(e)$ of Fredenhagen's universal category $J_!(\mathfrak{A})(e)$, is obtained in the same fashion from (3.5.4c).

The coherences (see Remark 3.2.7) for the extended 2-algebraic quantum field theory $J_!(\mathfrak{A}) \in \mathbf{2AQFT}(\mathbf{E}^\perp)$ are obtained canonically from the construction above and the symmetric monoidal pseudo-functor $\underline{\mathfrak{A}} : \mathcal{P}^{\otimes}_{\mathbf{D}^\perp} \to \mathbf{Pr}_{\mathbb{K}}$. △

In order to obtain an explicit description of $J_!(\mathfrak{A})(e)$, i.e. Fredenhagen's universal category along J for $\mathfrak{A}$ on e, we compute the bicolimit in (3.5.4a) using Lemma 2.4 of [BCJF15].

Lemma 3.5.3 (Lemma 2.4 [BCJF15])**.** *The 2-category of locally presentable $\mathbb{K}$-linear categories is bicomplete and bicocomplete. In particular, bicolimits are computed by replacing every 1-morphism by its right adjoint (which exists because every co-continuous $\mathbb{K}$-linear functor between locally presentable categories admits a right adjoint by the special adjoint functor theorem [AR94, BCJF15]), therefore obtaining a contravariant bifunctor $\mathbf{Pr}_{\mathbb{K}} \to \mathbf{Cat}$ and by computing the corresponding bilimit in $\mathbf{Cat}$, ignoring that the arrows happen to be right adjoint.*

In particular, Lemma 2.4 of [BCJF15] states that $J_!(\mathfrak{A})(e)$ can be computed via the following two step procedure:

(a) First, we consider the pseudo-functor $\underline{\mathfrak{A}}^{\mathrm{R}} : (\mathcal{P}^{\otimes}_{\mathbf{D}^\perp})^{\mathrm{op}} \to \mathbf{Cat}$ obtained from the pseudo-functor $\underline{\mathfrak{A}} : \mathcal{P}_{\mathbf{D}^\perp} \to \mathbf{Pr}_{\mathbb{K}}$ in the following way:

  (a.1) *On objects*: it assigns to every object $\underline{d} \in \mathbf{Ob}(\mathcal{P}^{\otimes}_{\mathbf{D}^\perp})$ the category $\underline{\mathfrak{A}}^{\mathrm{R}}(\underline{d}) = \underline{\mathfrak{A}}(\underline{d})$.

  (a.2) *On morphisms*: it assigns to every morphism $(\alpha, \underline{\underline{f}}) \in \mathcal{P}^{\otimes}_{\mathbf{D}^\perp}\binom{\underline{t}}{\underline{d}}$ the right adjoint $\underline{\mathfrak{A}}^{\mathrm{R}} : \underline{\mathfrak{A}}^{\mathrm{R}}(\underline{t}) \to \underline{\mathfrak{A}}^{\mathrm{R}}(\underline{d})$ of the $\mathbb{K}$-linear co-continuous functor $\underline{\mathfrak{A}} : \underline{\mathfrak{A}}^{\mathrm{R}}(\underline{d}) \to \underline{\mathfrak{A}}^{\mathrm{R}}(\underline{t})$, which exists because every co-continuous $\mathbb{K}$-linear functor between locally presentable categories admits a right adjoint (which is not generally co-continuous) by the special adjoint functor theorem ([AR94, BCJF15]).

(b) $J_!(\mathfrak{A})(e)$ is the bilimit in **Cat** of the pseudo-functor $\underline{\mathfrak{A}}^{\mathrm{R}} \circ \mathrm{forget} : (J^{\otimes}/(e))^{\mathrm{op}} \to$ **Cat**, which is a locally presentable $\mathbb{K}$-linear category in a canonical way ([BCJF15]).

An explicit description of the bilimit of $\underline{\mathfrak{A}}^{\mathrm{R}} \circ \mathrm{forget} : (J^{\otimes}/(e))^{\mathrm{op}} \to$ **Cat** can be obtained by applying the models for bilimits of pseudo-functors in [Str80, LN16]. In particular, $J_!(\mathfrak{A})(\underline{e})$ is the following category:

Objects: An object

$$(V, \xi^V) := \big(\{V_{\underline{h}}\}, \{\xi^V_{(\alpha,\underline{\underline{f}})}\}\big) \in J_!(\mathfrak{A})(e) \tag{3.5.5}$$

consists of the following data:

(1) For each object $\big(\underline{h} := (*, \underline{h}) : \underline{d} \to (e)\big) \in J^{\otimes}/(e)$, where $* : \{1, \dots, n\} \to \{1\}$ denotes the unique map of sets to the singleton $\{1\}$, an object

$$V_{\underline{h}} \in \underline{\mathfrak{A}}(\underline{d}) = \boxtimes_{i=1}^{n} \mathfrak{A}(d_i) \quad . \tag{3.5.6a}$$

(2) For each morphism $(\alpha, \underline{\underline{f}}) : \underline{h} \to \underline{h}'$ in $J^{\otimes}/(e)$, an isomorphism

$$\xi^V_{(\alpha,\underline{\underline{f}})} : \underline{\mathfrak{A}}^{\mathrm{R}}(\alpha, \underline{\underline{f}})(V_{\underline{h}'}) \longrightarrow V_{\underline{h}} \tag{3.5.6b}$$

in the category $\underline{\mathfrak{A}}(\underline{d})$.

These data have to satisfy the following cocycle conditions:

(i) For all objects $\big(\underline{h} : \underline{d} \to (e)\big) \in J^{\otimes}/(e)$, the diagram

$$\begin{tikzcd} \underline{\mathfrak{A}}^{\mathrm{R}}(\mathrm{id}_{\underline{h}})(V_{\underline{h}}) \arrow[r, "\xi^V_{\mathrm{id}_{\underline{h}}}"] \arrow[d, "\underline{\mathfrak{A}}^{\mathrm{R}\,0}_{\underline{e}}"', "\cong"] & V_{\underline{h}} \\ V_{\underline{h}} \arrow[ur, "\mathrm{id}_{V_{\underline{h}}}"'] & \end{tikzcd} \tag{3.5.7a}$$

in $\underline{\mathfrak{A}}(\underline{d})$ commutes, where $\underline{\mathfrak{A}}^{\mathrm{R}\,0}_{\underline{e}}$ denotes the coherence isomorphisms for identities that are associated with the pseudo-functor $\underline{\mathfrak{A}}^{\mathrm{R}}$.

(ii) For all composable pairs of morphisms $(\alpha, \underline{\underline{f}}) : \underline{h} \to \underline{h}'$ and $(\beta, \underline{\underline{g}}) : \underline{h}' \to \underline{h}''$ in $J^{\otimes}/(e)$, the diagram

$$\begin{tikzcd} \underline{\mathfrak{A}}^{\mathrm{R}}(\alpha, \underline{\underline{f}})\, \underline{\mathfrak{A}}^{\mathrm{R}}(\beta, \underline{\underline{g}})(V_{\underline{h}''}) \arrow[r, "\underline{\mathfrak{A}}^{\mathrm{R}}(\alpha,\underline{\underline{f}})(\xi^V_{(\beta,\underline{\underline{g}})})"] \arrow[d, "\underline{\mathfrak{A}}^{\mathrm{R}\,2}_{((\beta,\underline{\underline{g}}),(\alpha,\underline{\underline{f}}))}"', "\cong"] & \underline{\mathfrak{A}}^{\mathrm{R}}(\alpha, \underline{\underline{f}})(V_{\underline{h}'}) \arrow[d, "\xi^V_{(\alpha,\underline{\underline{f}})}"] \\ \underline{\mathfrak{A}}^{\mathrm{R}}\big((\beta, \underline{\underline{g}}) \circ (\alpha, \underline{\underline{f}})\big)(V_{\underline{h}''}) \arrow[r, "\xi^V_{(\beta,\underline{\underline{g}})\circ(\alpha,\underline{\underline{f}})}"'] & V_{\underline{h}} \end{tikzcd} \tag{3.5.7b}$$

in $\underline{\mathfrak{A}}(\underline{d})$ commutes, where $\underline{\mathfrak{A}}^{\mathrm{R}2}_{((\beta,\underline{\underline{g}}),(\alpha,\underline{\underline{f}}))}$ denotes the coherence isomorphisms for compositions that are associated with the pseudo-functor $\underline{\mathfrak{A}}^{\mathrm{R}}$.

Morphisms: A morphism

$$\Gamma := \{\Gamma_{\underline{h}}\} : (V,\xi^V) \longrightarrow (W,\xi^W) \tag{3.5.8}$$

in $J_!(\mathfrak{A})(e)$ consists of a family of $\underline{\mathfrak{A}}(\underline{d})$-morphisms

$$\Gamma_{\underline{h}} : V_{\underline{h}} \longrightarrow W_{\underline{h}} \quad , \tag{3.5.9a}$$

for all $\big(\underline{h} : \underline{d} \to (e)\big) \in J^{\otimes}/(e)$, such that the diagrams

$$\begin{tikzcd}
\underline{\mathfrak{A}}^{\mathrm{R}}(\alpha,\underline{\underline{f}})(V_{\underline{h}'}) \arrow[r, "\underline{\mathfrak{A}}^{\mathrm{R}}(\alpha,\underline{\underline{f}})(\Gamma_{\underline{h}'})"] \arrow[d, "\xi^V_{(\alpha,\underline{\underline{f}})}"'] & \underline{\mathfrak{A}}^{\mathrm{R}}(\alpha,\underline{\underline{f}})(W_{\underline{h}'}) \arrow[d, "\xi^W_{(\alpha,\underline{\underline{f}})}"] \\
V_{\underline{h}} \arrow[r, "\Gamma_{\underline{h}}"'] & W_{\underline{h}}
\end{tikzcd} \tag{3.5.9b}$$

in $\underline{\mathfrak{A}}(\underline{d})$ commute, for all morphisms $(\alpha,\underline{\underline{f}}) : \underline{h} \to \underline{h}'$ in $J^{\otimes}/(e)$.

Identities and compositions: Identities and composition are defined component-wise.

3.5.2 *Examples on $M = \mathbb{S}^1$*

The goal of this subsection is to provide examples of Fredenhagen's universal categories along the orthogonal inclusion functor $\iota : \mathbf{Disk}(\mathbb{S}^1)^{\perp_d} \to \mathbf{Open}(\mathbb{S}^1)^{\perp_d}$ (see Example 1.1.8). More precisely, we study extensions of categorified orbifold theories defined on $\mathbf{Disk}(\mathbb{S}^1)$ to the whole circle $\mathbb{S}^1$, i.e. locally presentable $\mathbb{K}$-linear categories of the form $J_!(\mathfrak{A})(\mathbb{S}^1)$ with $\mathfrak{A} \in \mathbf{2AQFT}(\mathbf{Disk}(\mathbb{S}^1))$. Therefore, we will be particularly interested in the following bicolimit:

$$J_!(\mathfrak{A})(\mathbb{S}^1) = \mathrm{bicolim}\left(J^{\otimes}/(\mathbb{S}^1) \xrightarrow{\text{forget}} \mathcal{P}^{\otimes}_{\mathbf{Disk}(\mathbb{S}^1)^{\perp_d}} \xrightarrow{\mathfrak{A}} \mathbf{Pr}_{\mathbb{K}} \right) \tag{3.5.10}$$

To compute it we will leverage the explicit description of Fredenhagen's universal category from Subsection 3.5.1 and the fact that the slice category $J^{\otimes}/(\mathbb{S}^1)$ for the functor $\iota : \mathcal{P}_{\mathbf{Disk}(\mathbb{S}^1)^{\perp_d}} \hookrightarrow \mathcal{P}_{\mathbf{Open}(\mathbb{S}^1)^{\perp_d}}$ over $(\mathbb{S}^1)$ admits the following, easy, description:

(a) *Objects*: The objects are tuples $\underline{I} = (I_1,\dots,I_n)$ of pairwise disjoint open intervals $I_i \subset \mathbb{S}^1$ (notice that the inclusion $I_i \subset \mathbb{S}^1$ is strict, i.e $I_i \neq \mathbb{S}^1$), i.e. $I_i \cap I_j = \varnothing$ for all $i \neq j$.

(b) *Morphisms*: A morphism $\alpha : \underline{I} = (I_1,\dots,I_n) \to \underline{J} = (J_1,\dots,J_m)$ is a map of sets $\alpha : \{1,\dots,n\} \to \{1,\dots,m\}$ such that $I_i \subseteq J_{\alpha(i)}$, for all $i = 1,\dots,n$.

Let us outline the content of the remainder of this subsection:

(a) Firstly, we discuss extensions $J_!(\mathfrak{A}) \in \mathbf{2AQFT}(\mathbf{Open}(\mathbb{S}^1)^{\perp_d})$ of truncated algebraic quantum field theories $\mathfrak{A} \in \mathbf{2AQFT}(\mathbf{Disk}(\mathbb{S}^1)^{\perp_d})$. More precisely, we realize that $J_!(\iota(\mathfrak{B})) \in \mathbf{2AQFT}(\mathbf{Open}(\mathbb{S}^1)^{\perp_d})$ computed on $\mathbb{S}^1$, where ι is the inclusion functor from Subsection 3.3.2 and $\mathfrak{B} \in \mathbf{AQFT}(\mathbf{Disk}(\mathbb{S}^1)^{\perp_d})$, is equivalent to the category $\mathbf{Mod}_{J_!(\mathfrak{B})(\mathbb{S}^1)}$ of right modules over Fredenhagen's universal *algebra* $J_!(\mathfrak{B})(\mathbb{S}^1)$ (notice the abuse of notations in denoting with the same name the extension functors $\mathbf{AQFT}(\mathbf{Disk}(\mathbb{S}^1)^{\perp_d}) \to \mathbf{AQFT}(\mathbf{Open}(\mathbb{S}^1)^{\perp_d})$ and $\mathbf{2AQFT}(\mathbf{Disk}(\mathbb{S}^1)^{\perp_d}) \to \mathbf{2AQFT}(\mathbf{Open}(\mathbb{S}^1)))$.

(b) Secondly, we discuss the extension $J_!(\mathbb{K}^G) \in \mathbf{2AQFT}(\mathbf{Open}(\mathbb{S}^1)^{\perp_d})$ of the categorified orbifold category $\mathbb{K}^G \in \mathbf{2AQFT}(\mathbf{Disk}(\mathbb{S}^1)^{\perp_d})$ from Example 3.4.7. In particular, we realize that $J_!(\mathbb{K}^G)(\mathbb{S}^1)$ is the category $G\text{-}\mathbf{Mod}_{\mathcal{O}(G)}$ of G-equivariant right modules over the function Hopf algebra $\mathcal{O}(G)$. We will notice that this identification has something to do with the fact that the category $\mathbf{QCoh}(Bund_G(\mathbb{S}^1))$ of quasi-coherent sheaves over the stack of principal G-bundles on $\mathbb{S}^1$ is equivalent to $G\text{-}\mathbf{Mod}_{\mathcal{O}(G)}$.

(c) Finally, we study extensions $J_!(\mathfrak{A}^G) \in \mathbf{2AQFT}(\mathbf{Open}(\mathbb{S}^1)^{\perp_d})$ of gaugings $\mathfrak{A}^G \in \mathbf{2AQFT}(\mathbf{Disk}(\mathbb{S}^1)^{\perp_d})$ of generic G-equivariant algebraic quantum field theories $(\mathfrak{A}, \rho) \in G\text{-}\mathbf{AQFT}(\mathbf{Disk}(\mathbb{S}^1)^{\perp_d})$. Unluckily, since our 2-algebraic quantum field theories $\mathfrak{A}^G \in \mathbf{2AQFT}(\mathbf{Disk}(\mathbb{S}^1)^{\perp_d})$ are not *locally constant* (i.e. do not in general assign equivalences to interval inclusions), the description of Fredenhagen's universal category $J_!(\mathfrak{A}^G) \in \mathbf{2AQFT}(\mathbf{Open}(\mathbb{S}^1)^{\perp_d})$ will be more complicated.

Example 3.5.4. Let $\mathfrak{A} \in \mathbf{2AQFT}(\mathbf{Disk}(\mathbb{S}^1)^{\perp_d})$ be a truncated algebraic quantum field theory, i.e. $\mathfrak{A} \cong \iota(\mathfrak{A})$, with $\mathfrak{A} \in \mathbf{AQFT}(\mathbf{Disk}(\mathbb{S}^1)^{\perp_d})$. To compute Fredenhagen's universal category $J_!(\mathfrak{A})(\mathbb{S}^1)$ (see the beginning of this section) consider the following square of biadjunctions:

$$\begin{tikzcd}
\mathbf{AQFT}(\mathbf{Disk}(\mathbb{S}^1)^{\perp_d}) \arrow[r, shift left, "\iota"] \arrow[d, shift right, "J_!"'] & \mathbf{2AQFT}(\mathbf{Disk}(\mathbb{S}^1)^{\perp_d}) \arrow[l, shift left, "\pi"] \arrow[d, shift right, "J_!"'] \\
\mathbf{AQFT}(\mathbf{Open}(\mathbb{S}^1)^{\perp_d}) \arrow[r, shift left, "\iota"] \arrow[u, shift right, "J^*"'] & \mathbf{2AQFT}(\mathbf{Open}(\mathbb{S}^1)^{\perp_d}) \arrow[l, shift left, "\pi"] \arrow[u, shift right, "J^*"']
\end{tikzcd} \tag{3.5.11}$$

where the horizontal biadjunctions are the inclusion/truncation biadjunctions from Section 3.3 (notice that we are abusing notations by denoting with $J_! \dashv J_*$ the restriction/extension adjunctions both for ordinary algebraic quantum field theories and 2-algebraic quantum field theories). It is not difficult to see that the square in Equation (3.5.11) formed by the right adjoints commutes, i.e. $\pi\, J^* = J^*\, \pi$, therefore, the square formed by the left adjoints commutes up to equivalence, i.e $\iota\, J_! \simeq J_!\, \iota$. This means, not surprisingly, that the extension $J_!(\mathfrak{A})$ of a $\mathfrak{A} = \iota(\mathfrak{A})$ is truncated. Therefore, Fredenhagen's universal *category* along J on $\mathbb{S}^1$ is equivalent to the category $\mathbf{Mod}_{J_!(\mathfrak{A})(\mathbb{S}^1)}$ of right modules over Fredenhagen's universal *algebra*

$$J_!(\mathfrak{A})(\mathbb{S}^1) = \operatorname{colim}\left(J^{\otimes}/(\mathbb{S}^1) \xrightarrow{\text{forget}} \mathcal{P}^{\otimes}_{\mathbf{Disk}(\mathbb{S}^1)^{\perp_d}} \xrightarrow{\mathfrak{A}} \mathbf{Alg}_{\mathbb{K}} \right) \quad . \tag{3.5.12}$$

To build more intuition and obtain a better understanding of Fredenhagen's universal category we provide a detailed description of the equivalence $F : \mathbf{Mod}_{J_!(\mathfrak{A})(\mathbb{S}^1)} \cong J_!(\iota(\mathfrak{A}))(\mathbb{S}^1) : F^{-1}$ by explicitly building the functor $F : \mathbf{Mod}_{J_!(\mathfrak{A})(\mathbb{S}^1)} \to J_!(\iota(\mathfrak{A}))(\mathbb{S}^1)$. Notice that we will rely on the explicit description of Fredenhagen's universal category in Subsection 3.5.1 to describe the category $J_!(\iota(\mathfrak{A}))(\mathbb{S}^1)$.

The functor $F : \mathbf{Mod}_{J_!(\mathfrak{A})(\mathbb{S}^1)} \to J_!(\iota(\mathfrak{A}))(\mathbb{S}^1)$ is given by the following data:

(a) *On objects*: It assigns to every right $J_!(\mathfrak{A})(\mathbb{S}^1)$-module $V \in \mathbf{Mod}_{J_!(\mathfrak{A})(\mathbb{S}^1)}$ the object $(V, \xi^V) ;= (\{V_{\underline{I}}\}, \{\xi^V_\alpha\}) \in J_!(\iota(\mathfrak{A}))(\mathbb{S}^1)$ defined in the following way:

(a.1) For every $\underline{I} \in J^{\otimes}/(\mathbb{S}^1)$, the right $\underline{\mathfrak{A}}(\underline{I}) := \bigotimes_{i=1}^n \mathfrak{A}(I_i)$-module $V_{\underline{I}}$ ($\bigotimes_{i=1}^n \mathfrak{A}(I_i)$ denotes the tensor product algebra) is defined by

$$V_{\underline{I}} := \chi^*_{\underline{I}}(V) \in \mathbf{Mod}_{\underline{\mathfrak{A}}(\underline{I})} \tag{3.5.13}$$

where $\chi^*_{\underline{I}}$ is the restriction of modules associated to the canonical $\mathbf{Alg}_{\mathbb{K}}$-morphism $\chi_{\underline{I}} : \underline{\mathfrak{A}}(\underline{I}) = \bigotimes_{i=1}^n \mathfrak{A}(I_i) \to J_!(\mathfrak{A})(\mathbb{S}^1)$ to the colimit (3.5.12), for every $\underline{I} \in J^{\otimes}/(\mathbb{S}^1)$.

(a.2)

$$\xi^V_\alpha := \mathrm{id}_{V_{\underline{I}}} : \iota(\mathfrak{A})^{\mathrm{R}}(\alpha)\big(V_{\underline{J}}\big) \longrightarrow V_{\underline{I}} \tag{3.5.14}$$

is an identity for every $\alpha : \underline{I} \to \underline{J}$. The reason why this assignment is legit is the following: For any morphism $\alpha : \underline{I} \to \underline{J}$ in $J^{\otimes}/(\mathbb{S}^1)$, the functor $\iota(\mathfrak{A})^{\mathrm{R}}(\alpha) = \underline{\mathfrak{A}}(\alpha)^* : \mathbf{Mod}_{\underline{\mathfrak{A}}(\underline{J})} \to \mathbf{Mod}_{\underline{\mathfrak{A}}(\underline{I})}$ is given by restriction of modules along the $\mathbf{Alg}_{\mathbb{K}}$-morphism $\underline{\mathfrak{A}}(\alpha) : \underline{\mathfrak{A}}(\underline{I}) \to \underline{\mathfrak{A}}(\underline{J})$. Because $\{\chi_{\underline{I}}\}_{\underline{I}}$ is a co-cone, we obtain

$$\iota(\mathfrak{A})^{\mathrm{R}}(\alpha)\big(V_{\underline{J}}\big) = \underline{\mathfrak{A}}(\alpha)^* \chi^*_{\underline{J}}(V) = \big(\chi_{\underline{J}}\, \underline{\mathfrak{A}}(\alpha)\big)^*(V) = \chi^*_{\underline{I}}(V) = V_{\underline{I}} \tag{3.5.15a}$$

Therefore, an object in Fredenhagen's universal category can be described as a family of modules over the algebras $\underline{\mathfrak{A}}(\underline{I}) = \bigotimes_{i=1}^n \mathfrak{A}(I_i)$ whose restrictions along inclusions $\alpha : \underline{I} \to \underline{J}$ coincide.

(b) *On morphisms*: F associates to any $\mathbf{Mod}_{J_!(\mathfrak{A})(\mathbb{S}^1)}$-morphism $L : V \to W$ the morphism $L : (V, \xi^V) \to (W, \xi^W)$ in $J_!(\iota(\mathfrak{A}))(\mathbb{S}^1)$ defined by

$$L_{\underline{I}} := \chi^*_{\underline{I}}(L) : V_{\underline{I}} = \chi^*_{\underline{I}}(V) \longrightarrow \chi^*_{\underline{I}}(W) = W_{\underline{I}} \quad , \tag{3.5.16}$$

for all $\underline{I} \in J^{\otimes}/(\mathbb{S}^1)$ (it is easily seen that L satisfies the coherence conditions in (3.5.9)).

Therefore, the morphisms in Fredenhagen's universal category can be described as families of module morphisms (see (3.5.16)), whose restrictions along inclusions $\alpha : \underline{I} \to \underline{J}$ coincide.

$\triangledown$

Example 3.5.5. In Example 3.4.7 we have shown that the categorified orbifold theory $\mathbb{K}^G \in \mathbf{2AQFT}(\mathbf{Disk}(\mathbb{S}^1)^{\perp_d})$, assigning to every open non empty interval $I \subset \mathbb{S}^1$ the locally presentable $\mathbb{K}$-linear category $\mathbf{Rep}_{\mathbb{K}}(G)$, is a genuine non-truncated and *locally constant* (i.e. $\mathbb{K}^G(\iota_I^J) : \mathbb{K}^G(I) \to \mathbb{K}^G(J)$ is an equivalence in $\mathbf{Pr}_{\mathbb{K}}$ for every interval inclusion $\iota_I^J : I \to J$) 2-algebraic quantum field theory whenever G is non trivial, i.e. $G \neq \{e\}$.

Before computing Fredenhagen's universal category $J_!(\mathbb{K}^G)(\mathbb{S}^1)$ along J for $\mathbb{K}^G$ on $\mathbb{S}^1$, we would like to discuss how it should look intuitively: By Remark 3.4.6 we can interpret $\mathbb{K}^G \in \mathbf{2AQFT}(\mathbf{Disk}(\mathbb{S}^1)^{\perp_d})$ as an orbifold σ-model defined on the non-empty intervals $I \subset \mathbb{S}^1$ of the circle with target the classifying stack $\mathbf{B}G = \{*\}//G$ of G. In fact, the stack of fields on an interval $I \subset \mathbb{S}^1$ is $\mathrm{Fields}(I) = \mathrm{Map}(I, \mathbf{B}G) \simeq \{*\}//G$ and its category of quasi-coherent sheaves is $\mathbf{QCoh}(\mathrm{Fields}(I)) \simeq \mathbf{Rep}_{\mathbb{K}}(G)$. On the whole circle $\mathbb{S}^1$, the stack of fields $\mathrm{Fields}(\mathbb{S}^1)$ coincides with the stack $\mathsf{Map}(\mathbb{S}^1, \mathbf{B}G)$, which, by the universal property of the classifying stack, is equivalent to the stack $\mathrm{Bun}_G(\mathbb{S}^1)$ of principal G-bundles on $\mathbb{S}^1$. The non-trivial bundles of $\mathrm{Bun}_G(\mathbb{S}^1)$ can be thought as "twisted sectors" of the orbifold σ-model. The category of quasi-coherent sheaves $\mathbf{QCoh}(\mathrm{Fields}(\mathbb{S}^1))$ on $\mathrm{Bun}_G(\mathbb{S}^1)$ is equivalent to the category $G\text{-}\mathbf{Mod}_{\mathcal{O}(G)}$ of G-equivariant right $\mathcal{O}(G)$-modules. Hence, our expectation is that $J_!(\mathbb{K}^G)(\mathbb{S}^1) \cong G\text{-}\mathbf{Mod}_{\mathcal{O}(G)}$. Let us verify it.

Since $\mathbb{K}^G \in \mathbf{2AQFT}(\mathbf{Disk}(\mathbb{S}^1)^{\perp_d})$ is locally constant we can use techniques from Factorization Homology, in particular [AF15, Theorem 3.19], to compute Fredenhagen's universal category $J_!(\mathbb{K}^G)(\mathbb{S}^1)$ along J over $\mathbb{S}^1$. More precisely, we obtain that $J_!(\mathbb{K}^G)(\mathbb{S}^1)$ is equivalent to $\mathrm{HH}_\bullet(\mathbf{Rep}_{\mathbb{K}}(G))$, where $\mathrm{HH}_\bullet(\mathbf{Rep}_{\mathbb{K}}(G))$ denotes Hochschild homology of the associative and unital algebra $(\mathbf{Rep}_{\mathbb{K}}(G), \otimes, \mathbb{K}) \in \mathbf{Alg}_{\mathsf{As}}(\mathbf{Pr}_{\mathbb{K}})$ (see Remark 3.4.2 for an explicit description of the monoidal structure of $\mathbf{Rep}_{\mathbb{K}}(G)$ and notice that $\otimes : \mathbf{Rep}_{\mathbb{K}}(G) \boxtimes \mathbf{Rep}_{\mathbb{K}}(G) \to \mathbf{Rep}_{\mathbb{K}}(G)$ is co-continuous $\mathbb{K}$-linear). Hochschild homology of $\mathbf{Rep}_{\mathbb{K}}(G)$ can be obtained as the bicolimit of the simplicial diagram associated to $(\mathbf{Rep}_{\mathbb{K}}(G), \otimes, \mathbb{K}) \in \mathbf{Alg}_{\mathsf{As}}(\mathbf{Pr}_{\mathbb{K}})$ (see [BZFN10, Section 5.1]), i.e.

$$\mathrm{HH}_\bullet(\mathbf{Rep}_{\mathbb{K}}(G)) = \mathrm{bicolim}\Big(\mathbf{Rep}_{\mathbb{K}}(G) \leftleftarrows \mathbf{Rep}_{\mathbb{K}}(G)^{\boxtimes 2} \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; \mathbf{Rep}_{\mathbb{K}}(G)^{\boxtimes 3} \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow}\; \cdots \Big) \tag{3.5.17}$$

where we are suppressing the degeneracy maps as usual. Notice, moreover, that, in order to compute the bicolimit, the simplicial diagram in Equation (3.5.17) can be truncated to $\mathbf{Rep}_{\mathbb{K}}(G)^{\boxtimes 3}$. The bicolimit in (3.5.17) can then be computed, by taking right adjoints of face and degeneracy maps in (3.5.17), as the bilimit in the 2-category $\mathbf{Cat}$ of small categories of the truncated co-simplicial diagram (Lemma 3.5.3):

$$\mathrm{HH}_\bullet(\mathbf{Rep}_{\mathbb{K}}(G)) = \mathrm{bilim}\Big(\mathbf{Rep}_{\mathbb{K}}(G) \rightrightarrows \mathbf{Rep}_{\mathbb{K}}(G^2) \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathbf{Rep}_{\mathbb{K}}(G^3). \Big) \tag{3.5.18}$$

where G^2 and G^3 appear because, for a generic $n \in \mathbb{N}$, the n-ary Kelly-Deligne tensor product $\mathbf{Rep}_{\mathbb{K}}(G)^{\boxtimes n}$ is equivalent to $\mathbf{Rep}_{\mathbb{K}}(G^n)$, the category of $\mathbb{K}$-linear representations of G^n.

To obtain an explicit description of the bilimit in (3.5.18), i.e. of the category $\mathrm{HH}_\bullet(\mathbf{Rep}_{\mathbb{K}}(G))$, it is helpful to describe the co-face and co-degeneracy maps in

(3.5.18). In particular, co-faces and co-degeneracies are obtained as co-induced representation functors $\phi_* : \mathbf{Rep}_{\mathbb{K}}(G') \to \mathbf{Rep}_{\mathbb{K}}(G'')$ for suitable group maps $\phi : G' \to G''$. More explicitly:

(a) The co-face map

$$\delta^0 = \delta^1 = \Delta_* \; : \; \mathbf{Rep}_{\mathbb{K}}(G) \; \longrightarrow \; \mathbf{Rep}_{\mathbb{K}}(G^2) \tag{3.5.19}$$

is the co-induced representation functor associated to the map $\Delta : G \to G^2$, $g \mapsto (g, g)$.

(b) The co-face maps

$$\delta^i = \phi^i_* \; : \; \mathbf{Rep}_{\mathbb{K}}(G^2) \; \longrightarrow \; \mathbf{Rep}_{\mathbb{K}}(G^3) \tag{3.5.20a}$$

For $1 = 1, 2, 3$ are co-induced representation functors determined by

$$\phi^i \; : \; G^2 \; \longrightarrow \; G^3 \quad , \quad (g_1, g_2) \; \longmapsto \; \begin{cases} (g_1, g_1, g_2) & , \text{ for } i = 0 \; , \\ (g_1, g_2, g_2) & , \text{ for } i = 1 \; , \\ (g_1, g_2, g_1) & , \text{ for } i = 2 \; . \end{cases} \tag{3.5.20b}$$

(c) The co-degeneracy map $\epsilon^0 : \mathbf{Rep}_{\mathbb{K}}(G^2) \to \mathbf{Rep}_{\mathbb{K}}(G)$ is given by the co-induced representation functor for $G^2 \to G$, $(g_1, g_2) \mapsto g_1$.

We can then describe the category $\mathrm{HH}_\bullet(\mathbf{Rep}_{\mathbb{K}}(G))$, i.e. the bilimit in (3.5.18), as the category given by the following data:

(a) *Objects*: An object is a couple (V, θ^V), where $V \in \mathbf{Rep}_{\mathbb{K}}(G)$ is a $\mathbb{K}$-linear representation of G and $\theta^V : \delta^1(V) \to \delta^0(V)$ is a $\mathbf{Rep}_{\mathbb{K}}(G^2)$-isomorphism such that $\epsilon^0(\theta^V) = \mathrm{id}_V$ and $\delta^0(\theta^V) \circ \delta^2(\theta^V) = \delta^1(\theta^V)$ in $\mathbf{Rep}_{\mathbb{K}}(G^3)$ (ϵ^0 denotes the co-degeneracy map $\epsilon^0 : \mathbf{Rep}_{\mathbb{K}}(G^2) \to \mathbf{Rep}_{\mathbb{K}}(G)$).

(b) *Morphisms*: A morphism $L : (V, \theta^V) \to (W, \theta^W)$ is a morphism $L : V \to W$ in $\mathbf{Rep}_{\mathbb{K}}(G)$, such that the diagram

$$\begin{tikzcd} \delta^1(V) \arrow[r, "\delta^1(L)"] \arrow[d, "\theta^V"'] & \delta^1(W) \arrow[d, "\theta^W"] \\ \delta^0(V) \arrow[r, "\delta^0(L)"'] & \delta^0(W) \end{tikzcd} \tag{3.5.21}$$

in $\mathbf{Rep}_{\mathbb{K}}(G^2)$ commutes.

Our claim is that this category is equivalent to the category $G\text{-}\mathbf{Mod}_{\mathcal{O}(G)}$ of G-equivariant right $\mathcal{O}(G)$-modules. In fact, we can simplify its description by noticing that, since $\mathbb{K}$ is a field of characteristic 0 and since G is a finite group, the co-induced representation functors $\phi_* : \mathbf{Rep}_{\mathbb{K}}(G') \to \mathbf{Rep}_{\mathbb{K}}(G'')$ are naturally isomorphic to the induced representation functors $\phi_! : \mathbf{Rep}_{\mathbb{K}}(G') \to \mathbf{Rep}_{\mathbb{K}}(G'')$ which are easy to describe: Considering $\mathbf{Rep}_{\mathbb{K}}(G') = {}_{\mathbb{K}[G']}\mathbf{Mod}$ and $\mathbf{Rep}_{\mathbb{K}}(G'') = {}_{\mathbb{K}[G'']}\mathbf{Mod}$ respectively as the categories of left $\mathbb{K}[G']$-modules and left $\mathbb{K}[G'']$-modules, where $\mathbb{K}[G']$ and $\mathbb{K}[G'']$ denote the Hopf-group algebras of G' and G'', the functor $\phi_! : \mathbf{Rep}_{\mathbb{K}}(G') \to \mathbf{Rep}_{\mathbb{K}}(G'')$ sends an object $V \in \mathbf{Rep}_{\mathbb{K}}(G'')$ to the object $\phi_!(V) := \mathbb{K}[G''] \otimes_{\mathbb{K}[G']} V \in \mathbf{Rep}_{\mathbb{K}}(G'')$. Therefore, we can use the explicit description of the co-induced representation functors $\phi_* : \mathbf{Rep}_{\mathbb{K}}(G') \to \mathbf{Rep}_{\mathbb{K}}(G'')$ in terms of the induced representation functors $\phi_! : \mathbf{Rep}_{\mathbb{K}}(G') \to \mathbf{Rep}_{\mathbb{K}}(G'')$ to deduce that for any object $(V, \theta^V) \in \mathrm{HH}_\bullet\big(\mathbf{Rep}_{\mathbb{K}}(G)\big)$, the map $\theta^V : \mathbb{K}[G^2] \otimes_{\mathbb{K}[G]} V \to \mathbb{K}[G^2] \otimes_{\mathbb{K}[G]} V$ is completely determined by a $\mathbb{K}$-linear map $\vartheta^V : V \to \mathbb{K}[G] \otimes V$ via $\theta^V(1 \otimes 1 \otimes v) = 1 \otimes \vartheta^V(v)$, which is G-equivariant with respect to the adjoint action on $\mathbb{K}[G]$ and satisfies the axioms of a left $\mathbb{K}[G]$-coaction. Moreover, we deduce that a morphism in $\mathrm{HH}_\bullet\big(\mathbf{Rep}_{\mathbb{K}}(G)\big)$ is a G-equivariant map that preserves these $\mathbb{K}[G]$-coactions. Hence, we get the following chain of equivalences

$$J_!(\mathbb{K}^G)(\mathbb{S}^1) \simeq \mathrm{HH}_\bullet\big(\mathbf{Rep}_{\mathbb{K}}(G)\big) \simeq G\text{-}{}^{\mathbb{K}[G]}\mathbf{Mod} \simeq G\text{-}\mathbf{Mod}_{\mathcal{O}(G)} \quad , \tag{3.5.22}$$

where the last equivalence holds since left $\mathbb{K}[G]$-comodules are equivalent to right modules over the dual Hopf algebra $\mathcal{O}(G)$ of functions on G (the G-action on $\mathcal{O}(G)$ is again the adjoint action). ▽

Example 3.5.6. As a last example we consider extensions to the circle $J_!(\mathfrak{A}^G)(\mathbb{S}^1)$ of gaugings $\mathfrak{A}^G$ of generic $(\mathfrak{A}, \rho) \in G\text{-}\mathbf{AQFT}(\mathbf{Disk}(\mathbb{S}^1)^{\perp_d})$. Unluckily, since our 2-algebraic quantum field theories *are not* generally locally constant, we cannot borrow techniques from Factorization Homology to simplify the description of Fredenhagen's universal category $J_!(\mathfrak{A}^G)(\mathbb{S}^1)$ as we did in Example 3.5.5. Anyhow, we would like to specify the explicit construction in Subsection 3.5.1 to the case at hand to build some intuition.

An object $(V, \xi^V) \in J_!(\mathfrak{A}^G)(\mathbb{S}^1)$ consists of the following data:

(a) For each tuple $\underline{I} = (I_1, \dots, I_n) \in J^{\otimes}/(\mathbb{S}^1)$ of mutually disjoint intervals, a G^n-equivariant right $\underline{\mathfrak{A}}(\underline{I}) = \bigotimes_{i=1}^n \mathfrak{A}(I_i)$-module

$$V_{\underline{I}} \in G^n\text{-}\mathbf{Mod}_{\underline{\mathfrak{A}}(\underline{I})} \tag{3.5.23}$$

where $\underline{\mathfrak{A}}(\underline{I}) = \bigotimes_{i=1}^n \mathfrak{A}(I_i)$ denotes the tensor product of algebras and the G^n-action on the tensor product of algebras is given by the component-wise G-actions.

Notice that $V_{\underline{I}}$ is a right $\underline{\mathfrak{A}}(\underline{I}) = \bigotimes_{i=1}^n \mathfrak{A}(I_i)$-module, with the I_i's pairwise disjoint, endowed with a *separate G-action* for each connected component. In particular, the group G is allowed to act differently on different intervals, which is a feature of *local* gauge symmetries.

(b) For each morphism $\alpha : \underline{I} = (I_1, \dots, I_n) \to \underline{J} = (J_1, \dots, J_m)$ in $J^{\otimes}/(\mathbb{S}^1)$, a $G^n\text{-}\mathbf{Mod}_{\underline{\mathfrak{A}}(\underline{I})}$-isomorphism

$$\xi^V_\alpha : (\mathfrak{A}^G)^{\mathrm{R}}(\alpha)\big(V_{\underline{J}}\big) \longrightarrow V_{\underline{I}} \quad . \tag{3.5.24}$$

Here $(\mathfrak{A}^G)^{\mathrm{R}}(\alpha) : G^m\text{-}\mathbf{Mod}_{\underline{\mathfrak{A}}(\underline{J})} \to G^n\text{-}\mathbf{Mod}_{\underline{\mathfrak{A}}(\underline{I})}$ is the right adjoint of the functor

$$\begin{tikzcd} G^n\text{-}\mathbf{Mod}_{\underline{\mathfrak{A}}(\underline{I})} \arrow[rr, "\mathfrak{A}^G(\alpha)"] \arrow[dr, "\Delta_\alpha^*"'] & & G^m\text{-}\mathbf{Mod}_{\underline{\mathfrak{A}}(\underline{J})} \\ & G^m\text{-}\mathbf{Mod}_{\Delta_\alpha^*(\underline{\mathfrak{A}}(\underline{I}))} \arrow[ur, "\underline{\mathfrak{A}}(\alpha)_!"'] & \end{tikzcd} \tag{3.5.25}$$

where $\Delta_\alpha : G^m \to G^n\,,\ (g_1, \dots, g_m) \mapsto (g_{\alpha(1)}, \dots, g_{\alpha(n)})$ is the group map determined by $\alpha : \{1, \dots, n\} \to \{1, \dots, m\}$, $\Delta_\alpha^* : \mathbf{Rep}_{\mathbb{K}}(G^n) \to \mathbf{Rep}_{\mathbb{K}}(G^m)$ denotes the corresponding restricted representation functor and $\underline{\mathfrak{A}}(\alpha)_!$ is the induced module functor for the G^m-equivariant algebra morphism $\underline{\mathfrak{A}}(\alpha) : \Delta_\alpha^*(\underline{\mathfrak{A}}(\underline{I})) \to \underline{\mathfrak{A}}(\underline{J})$. Explicitly, one finds that $(\mathfrak{A}^G)^{\mathrm{R}}(\alpha)$ is given by the composition

$$\begin{tikzcd} G^m\text{-}\mathbf{Mod}_{\underline{\mathfrak{A}}(\underline{J})} \arrow[r, "(\mathfrak{A}^G)^{\mathrm{R}}(\alpha)"] \arrow[d, "\underline{\mathfrak{A}}(\alpha)^*"'] & G^n\text{-}\mathbf{Mod}_{\underline{\mathfrak{A}}(\underline{I})} \\ G^m\text{-}\mathbf{Mod}_{\Delta_\alpha^*(\underline{\mathfrak{A}}(\underline{I}))} \arrow[r, "\Delta_{\alpha *}"'] & G^n\text{-}\mathbf{Mod}_{\Delta_{\alpha *}\Delta_\alpha^*(\underline{\mathfrak{A}}(\underline{I}))} \arrow[u, "\eta_{\underline{\mathfrak{A}}(\underline{I})}^*"'] \end{tikzcd} \tag{3.5.26}$$

where η denotes the unit of the adjunction $\Delta_\alpha^* : \mathbf{Rep}_{\mathbb{K}}(G^n) \rightleftarrows \mathbf{Rep}_{\mathbb{K}}(G^m) : \Delta_{\alpha *}$. These data have to satisfy the coherence conditions (3.5.7).

To get a better feeling for the coherence maps ξ^V(3.5.24), we study the case where we include two intervals $I_1, I_2 \subset \mathbb{S}^1$ into a single bigger interval $J \subset \mathbb{S}^1$, i.e. $\alpha : \underline{I} = (I_1, I_2) \to J$. In this case $\Delta_\alpha = \Delta : G \to G^2$ is the diagonal map and (3.5.24) is given by a G^2-$\mathbf{Mod}_{\underline{\mathfrak{A}}(\underline{I})}$-isomorphism

$$\xi_\alpha^V \ : \ \eta_{\underline{\mathfrak{A}}(\underline{I})}^*\, \Delta_*\, \underline{\mathfrak{A}}(\alpha)^*\big(V_J\big) \ \longrightarrow \ V_{\underline{I}} \quad . \tag{3.5.27}$$

Using as in Example 3.5.5 that the co-induced representation functor $\Delta_* : \mathbf{Rep}_{\mathbb{K}}(G) \to \mathbf{Rep}_{\mathbb{K}}(G^2)$ is naturally isomorphic to the induced representation functor $\Delta_! : \mathbf{Rep}_{\mathbb{K}}(G) \to \mathbf{Rep}_{\mathbb{K}}(G^2)$, we obtain that ξ_α^V is completely determined by a $\mathbb{K}$-linear map $\kappa_\alpha^V : V_J \to V_{\underline{I}}$ via $\xi_\alpha^V(1 \otimes 1 \otimes v) = \kappa_\alpha^V(v)$, for all $v \in V_J$. This $\mathbb{K}$-linear map has to satisfy the following conditions: 1.) G-equivariance: $\kappa_\alpha^V(g\, v) = (g, g)\, \kappa_\alpha^V(v)$, for all $v \in V_J$ and $g \in G$. 2.) Preservation of the $\underline{\mathfrak{A}}(\underline{I})$-actions:

$$\kappa_\alpha^V(v) \cdot (a_1 \otimes a_2) = \sum_{(g_1, g_2) \in G^2} (g_1^{-1}, g_2^{-1})\, \kappa_\alpha^V\Big(v \cdot \Big(\mathfrak{A}(\iota_{I_1}^J)(g_1\, a_1)\, \mathfrak{A}(\iota_{I_2}^J)(g_2\, a_2)\Big)\Big) \quad , \tag{3.5.28}$$

for all $a_1 \otimes a_2 \in \mathfrak{A}(I_1) \otimes \mathfrak{A}(I_2)$ and $v \in V_J$, where $\iota_{I_i}^J : I_i \to J$ denote the interval inclusions. (The sum over G^2 comes from the unit η of the adjunction $\Delta^* \dashv \Delta_*$ when we use $\Delta_!$ as a model for Δ_*.) Comparing (3.5.28) with the truncated case from Example 3.5.4, we observe that there is a component-wise G^2-action on the algebra element $a_1 \otimes a_2 \in \mathfrak{A}(I_1) \otimes \mathfrak{A}(I_2)$ on a pair of intervals before it acts on the module element $v \in V_J$ on the single bigger interval. From a superficial point of view, this behaviour resembles the twisted representations of G-equivariant AQFTs by Müger [Mug05]. Unfortunately, we do not understand at the moment if there exists a precise relationship between Fredenhagen's universal category $J_!(\mathfrak{A}^G)(\mathbb{S}^1)$ for categorified orbifold theories and the results in [Mug05]. $\triangledown$

4

SMOOTH 1-DIMENSIONAL AQFTS

An m-dimensional algebraic quantum field theory is a functor $\mathfrak{A} : \mathbf{Loc}_m \to {}^*\mathbf{Alg}_{\mathbb{C}}$ (see Definition 1.2.7) satisfying some physically motivated axioms, e.g. *Einstein causality*. This perspective on quantum field theories, although leading to very interesting model-independent results and to a prolific exchange of techniques between AQFT and Category Theory as we have seen in the previous chapters, has the following insufficiency: the axioms of an algebraic quantum field theory do not entail a suitable concept of *smoothness*, in the sense that there is no prescription on how observable algebras should respond to smooth changes of spacetimes. Let us be slightly more precise: Let's say we have a globally hyperbolic Lorentzian manifold M_s depending smoothly on a parameter s (in some appropriate sense, as we will see later). Such a family could be generated for example by changing smoothly the coefficients of the metric tensor. We would expect then the family of $*$-algebras $\mathfrak{A}(M_s)$, obtained by evaluating $\mathfrak{A}$ on each M_s, to vary smoothly accordingly, since, from a physical perspective, we can imagine small changes of the spacetimes to not affect too much observable algebras, but, to our knowledge, there is no suitable concept of smoothness for $*$-algebras. To solve this inadequacy, our long term plan is to carry out a program similar to the one conducted in the context of functorial quantum field theory [ST11, BEP15, BW21, LS21], i.e. by using *stacks* to introduce smooth refinements of the categories $\mathbf{Loc}_m$ and ${}^*\mathbf{Alg}_{\mathbb{C}}$. In order to make a first step toward such goal we decide to avoid technical complications of both categorical and analytical nature by restricting our attention to 1-dimensional globally hyperbolic Lorentzian manifolds (recall that a 1-dimensional globally hyperbolic Lorentzian manifold is just an open interval together with a differential 1-form, see Remark 1.2.8) in which context ordinary algebraic quantum field theories boil down to functors $\mathbf{Loc}_1 \to {}^*\mathbf{Alg}_{\mathbb{C}}$ since Einstein causality does not enter the picture.

More precisely, in this chapter, drawing framework and results from our paper [BPS20], we introduce stacks $\mathbf{Loc}_1^\infty : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ and ${}^*\mathbf{Alg}_{\mathbb{C}}^\infty : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ representing smooth refinements of $\mathbf{Loc}_1$ and ${}^*\mathbf{Alg}_{\mathbb{C}}$, i.e. pseudo-functors (see Remark 3.1.8) from the opposite category of manifolds and smooth maps to the category of small categories satisfying a suitable categorification of the descent condition for sheaves, and define smooth 1-dimensional algebraic quantum field theories to be stack morphisms (i.e. pseudo-natural transformations) $\mathfrak{A} : \mathbf{Loc}_1^\infty \to {}^*\mathbf{Alg}_{\mathbb{C}}^\infty$. Leveraging the right biadjoint to the cartesian product of stacks, i.e. the *mapping stack*, we introduce a stack $\mathbf{AQFT}_1^\infty$ of smooth 1-dimensional algebraic quantum field theories adding further layers of smoothness to our setting. In particular, this stack will enable us to talk about smooth families of smooth 1-dimensional algebraic quantum field theories (via the *functor of points* approach) and will permit us to assign a *smooth automorphism group* to each smooth 1-dimensional algebraic quantum field theory.

In what follows, we shall carefully describe such constructions and illustrate them via simple examples.

Let us outline in more detail the content of the remainder of this chapter.

In Section 4.1 we discuss the relevant concepts from the theory of stacks. In particular, we introduce the *descent category*, i.e. the category of descent data for a pseudo-functor on an open cover of a manifold, the 2-*Yoneda Lemma*, a 2-categorical analogue of the Yoneda Lemma, and the *mapping stack*.

In Section 4.2 we define the aforementioned smooth refinements of the categories $\mathbf{Loc}_1$ and ${}^*\mathbf{Alg}_{\mathbb{C}}$, i.e. the stacks $\mathbf{Loc}_1^\infty : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ and ${}^*\mathbf{Alg}_{\mathbb{C}}^\infty : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$, and we introduce the mapping stack $\mathbf{AQFT}_1^\infty := \mathsf{Map}(\mathbf{Loc}_1^\infty, {}^*\mathbf{Alg}_{\mathbb{C}}^\infty)$. Moreover, we introduce *smooth 1-dimensional algebraic quantum field theories* as "global points" $\underline{\{*\}} \to \mathbf{AQFT}_1^\infty$, i.e. as stack morphisms $\mathbf{Loc}_1^\infty \to {}^*\mathbf{Alg}_{\mathbb{C}}^\infty$, and assign to each of them a smooth automorphism group. Furthermore, we will introduce smooth G-equivariant 1-dimensional algebraic quantum field theories, smooth analogues of ordinary G-equivariant AQFTs.

In Section 4.3 we discuss smooth refinements of the canonical commutation relation (CCR) functor and of the canonical anti-commutation relation (CAR) functor.

In Section 4.4 we introduce smooth generalizations of retarded and advanced Green operators which we harness to build examples of smooth 1-dimensional algebraic quantum field theories. In particular, we discuss a smooth analogue of the 1-dimensional massive scalar field (quantum harmonic oscillator) in the presence of a smooth variation of the mass (frequency) parameter and an example of a $U(1)$-equivariant smooth 1-dimensional AQFT, which can be interpreted physically as a smooth counterpart of the 1-dimensional massless Dirac field together with its global $U(1)$-symmetry.

4.1 PRELIMINARIES ON STACKS OF CATEGORIES

The aim of this section is to introduce the relevant concepts from the theory of stacks that will be needed in the remainder of this chapter. In particular, we recall the notion of *stack*, 1-*morphism* (or simply *morphism*) between stacks and 2-*morphism* between 1-morphisms of stacks. The interested reader can find a thorough treatment of the theory of stacks in [Vis05].

A stack is, roughly speaking, a 2-categorical analogue of a sheaf where the *functoriality*, *locality* and *gluing* axioms hold up to coherent isomorphisms. While both the concepts of stack and sheaf can be introduced over generic *sites* (i.e. categories with a suitable notion of covering for their objects) we will avoid such complication and focus our attention on stacks and sheaves defined on the site **Man**, i.e. the category of manifolds and smooth maps endowed with the *open cover Grothendieck topology*, where such concepts become pretty straightforward. For instance, a **Set**-valued sheaf on **Man** is just a contravariant functor $F : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Set}$ such that $F|_M$, where $F|_M$ denotes the restriction of F to the subcategory $\mathbf{Open}(M)^{\mathrm{op}} \subseteq \mathbf{Man}^{\mathrm{op}}$ (see Example 1.1.7), is a sheaf on M for every $M \in \mathbf{Man}$.

Before defining stacks, we have to introduce the 2-categorical analogue of presheaves on **Man**, i.e. *prestacks* on **Man**. As per usual, after introducing the relevant 2-categorical objects, we will explicitly spell out their data (see Remark 4.1.2).

Definition 4.1.1. The 2-category of *prestacks* (of categories) $[\mathbf{Man}^{\mathrm{op}}, \mathbf{Cat}]$ consists of pseudo-functors $\mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$, pseudo-natural transformations and modifications (see Remark 3.1.8). The objects of this category, i.e. the pseudo-functors $\mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$, are called *prestacks*, while we will refer to pseudo-natural transformations and modifications as (1-)*morphisms* and 2-*morphisms* of prestacks respectively.

Remark 4.1.2. A prestack (of categories) $X : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ consists of the following data:

(a) For each object $U \in \mathbf{Man}$, a category $X(U)$.

(b) For each morphism $h : U \to U'$ in **Man**, a functor $X(h) : X(U') \to X(U)$.

(c) For each pair of composable morphisms $h : U \to U'$ and $h' : U' \to U''$ in **Man**, a natural isomorphism $X_{h',h} : X(h)\,X(h') \Rightarrow X(h'\,h)$ of functors from $X(U'')$ to $X(U)$.

(d) For each object $U \in \mathbf{Man}$, a natural isomorphism $X_U : \mathrm{id}_{X(U)} \Rightarrow X(\mathrm{id}_U)$ of functors from $X(U)$ to $X(U)$.

Satisfying the following axioms:

(a) For all triples of composable morphisms $h : U \to U'$, $h' : U' \to U''$ and $h'' : U'' \to U'''$ in **Man**, the diagram

$$\begin{tikzcd} X(h)\,X(h')\,X(h'') \arrow[r, Rightarrow, "X_{h',h}*\mathrm{Id}"] \arrow[d, Rightarrow, "\mathrm{Id}*X_{h'',h'}"'] & X(h'\,h)\,X(h'') \arrow[d, Rightarrow, "X_{h'',h'h}"] \\ X(h)\,X(h''\,h') \arrow[r, Rightarrow, "X_{h''h',h}"'] & X(h''\,h'\,h) \end{tikzcd} \tag{4.1.1}$$

of natural transformations commutes. (The capital Id denotes identity natural transformations and $*$ denotes horizontal composition of natural transformations.)

(b) For all morphisms $h : U \to U'$ in **Man**, the two diagrams

$$\begin{tikzcd} \mathrm{id}_{X(U)}\,X(h) \arrow[d, Rightarrow, "X_U*\mathrm{Id}"'] \arrow[dr, equal] & \\ X(\mathrm{id}_U)\,X(h) \arrow[r, Rightarrow, "X_{h,\mathrm{id}_U}"'] & X(h) \end{tikzcd} \qquad \begin{tikzcd} X(h)\,\mathrm{id}_{X(U')} \arrow[d, Rightarrow, "\mathrm{Id}*X_{U'}"'] \arrow[dr, equal] & \\ X(h)\,X(\mathrm{id}_{U'}) \arrow[r, Rightarrow, "X_{\mathrm{id}_{U'},h}"'] & X(h) \end{tikzcd} \tag{4.1.2}$$

of natural transformations commute.

From now on we will often suppress the coherence isomorphisms $X_{h',h}$, X_U of a prestack $X \in [\mathbf{Man}^{\mathrm{op}}, \mathbf{Cat}]$.

A *morphism* $F : X \to Y$ of prestacks consists of the following data:

(a) For each $U \in \mathbf{Man}$, a functor $F_U : X(U) \to Y(U)$.

(b) For each morphism $h : U \to U'$ in **Man**, a natural isomorphism

$$\begin{tikzcd} X(U') \arrow[r, "F_{U'}"] \arrow[d, "X(h)"'] & Y(U') \arrow[d, "Y(h)"] \arrow[dl, Rightarrow, "F_h"'] \\ X(U) \arrow[r, "F_U"'] & Y(U) \end{tikzcd} \tag{4.1.3}$$

These data have to satisfy the following axioms:

(a) For all pairs of composable morphisms $h : U \to U'$ and $h' : U' \to U''$ in **Man**, the diagram

$$\begin{tikzcd} Y(h)\,Y(h')\,F_{U''} \arrow[r, Rightarrow, "\mathrm{Id}*F_{h'}"] \arrow[d, Rightarrow, "Y_{h',h}*\mathrm{Id}"'] & Y(h)\,F_{U'}\,X(h') \arrow[r, Rightarrow, "F_h*\mathrm{Id}"] & F_U\,X(h)\,X(h') \arrow[d, Rightarrow, "\mathrm{Id}*X_{h',h}"] \\ Y(h'\,h)F_{U''} \arrow[rr, Rightarrow, "F_{h'h}"'] & & F_U\,X(h'\,h) \end{tikzcd} \tag{4.1.4}$$

of natural transformations commutes.

(b) For all $U \in \mathbf{Man}$, the diagram

$$\begin{tikzcd} \mathrm{id}_{Y(U)}\,F_U \arrow[r, equal] \arrow[d, Rightarrow, "Y_U*\mathrm{Id}"'] & F_U\,\mathrm{id}_{X(U)} \arrow[d, Rightarrow, "\mathrm{Id}*X_U"] \\ Y(\mathrm{id}_U)\,F_U \arrow[r, Rightarrow, "F_{\mathrm{id}_U}"'] & F_U\,X(\mathrm{id}_U) \end{tikzcd} \tag{4.1.5}$$

of natural transformations commutes.

A 2-*morphism* $\zeta : F \Rightarrow G$ between two prestack morphisms $F, G : X \to Y$ consists of the following data:

(a) For each $U \in \mathbf{Man}$, a natural transformation $\zeta_U : F_U \Rightarrow G_U$ of functors from $X(U)$ to $Y(U)$.

These data have to satisfy the following axioms:

(a) For all morphisms $h : U \to U'$ in **Man**, the diagram

$$\begin{tikzcd} Y(h)\,F_{U'} \arrow[r, Rightarrow, "\mathrm{Id}*\zeta_{U'}"] \arrow[d, Rightarrow, "F_h"'] & Y(h)\,G_{U'} \arrow[d, Rightarrow, "G_h"] \\ F_U\,X(h) \arrow[r, Rightarrow, "\zeta_U*\mathrm{Id}"'] & G_U\,X(h) \end{tikzcd} \tag{4.1.6}$$

of natural transformations commutes.

△

As we mentioned earlier, a sheaf (of sets) on **Man** is a presheaf $X : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Set}$ satisfying a descent condition (the combination of locality and gluing axioms) for any open cover $\{U_\alpha \subseteq U\}$ of any manifold $U \in \mathbf{Man}$. This descent condition can be interpreted in terms of limits or in a more explicit way by defining a *descent set* $X(\{U_\alpha \subseteq U\})$ and checking whether a certain canonically induced map $X(U) \to X\{U_\alpha \subseteq U\}$ is a bijection. In order to define *stacks* on **Man** we decide to follow a categorification of the latter approach but we would like to point out that also a categorification of the former in terms of bilimits is feasible (and equivalent to the first, see [Vis05]).

Definition 4.1.3. Let $X : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ be a prestack and $U \in \mathbf{Man}$ be a manifold. For any open cover $\{U_\alpha \subseteq U\}$ there exists a category $X(\{U_\alpha \subseteq U\})$, called *descent category* of X on the open covering $\{U_\alpha \subseteq U\}$, defined by the following data:

(a) *Objects*: An object is a tuple

$$\Big(\{x_\alpha\}, \{\varphi_{\alpha\beta} : x_\beta|_{U_{\alpha\beta}} \to x_\alpha|_{U_{\alpha\beta}}\}\Big) \in X(\{U_\alpha : \alpha \in A\}) \tag{4.1.7a}$$

of families of objects $x_\alpha \in X(U_\alpha)$ and isomorphisms $\varphi_{\alpha\beta}$ in $X(U_{\alpha\beta})$ satisfying

$$\begin{tikzcd}[column sep=small]
 & x_\beta|_{U_{\beta\gamma}}|_{U_{\alpha\beta\gamma}} \arrow[dr, "\cong"] & \\
x_\gamma|_{U_{\beta\gamma}}|_{U_{\alpha\beta\gamma}} \arrow[ur, "\varphi_{\beta\gamma}|_{U_{\alpha\beta\gamma}}"] \arrow[d, "\cong"'] & & x_\beta|_{U_{\alpha\beta}}|_{U_{\alpha\beta\gamma}} \arrow[d, "\varphi_{\alpha\beta}|_{U_{\alpha\beta\gamma}}"] \\
x_\gamma|_{U_{\alpha\gamma}}|_{U_{\alpha\beta\gamma}} \arrow[dr, "\varphi_{\alpha\gamma}|_{U_{\alpha\beta\gamma}}"'] & & x_\alpha|_{U_{\alpha\beta}}|_{U_{\alpha\beta\gamma}} \arrow[dl, "\cong"] \\
 & x_\alpha|_{U_{\alpha\gamma}}|_{U_{\alpha\beta\gamma}} &
\end{tikzcd}
\qquad
\begin{tikzcd}
x_\alpha|_{U_{\alpha\alpha}} \arrow[r, "\varphi_{\alpha\alpha}"] \arrow[d, "\cong"'] & x_\alpha|_{U_{\alpha\alpha}} \arrow[d, "\cong"] \\
x_\alpha \arrow[r, "\mathrm{id}_{x_\alpha}"'] & x_\alpha
\end{tikzcd} \tag{4.1.7b}$$

for all α, β, γ, where we denoted by $U_{\alpha_1\alpha_2\cdots\alpha_n} := U_{\alpha_1} \cap U_{\alpha_2} \cap \cdots \cap U_{\alpha_n}$ the intersection of open subsets, by $|_{\tilde{U}} := X(\iota^U_{\tilde{U}}) : X(U) \to X(\tilde{U})$ the functor obtained by evaluating X on a subset inclusion **Man**-morphism $\iota^U_{\tilde{U}} : \tilde{U} \to U$ and where the unlabelled isomorphisms are given by the coherence isomorphisms of the pseudo-functor $X : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$.

(b) *Morphisms*: A $X(\{U_\alpha : \alpha \in A\})$-morphism

$$\{\psi_\alpha\} \,:\, \big(\{x_\alpha\}, \{\varphi_{\alpha\beta}\}\big) \longrightarrow \big(\{x'_\alpha\}, \{\varphi'_{\alpha\beta}\}\big) \tag{4.1.8a}$$

is a family of morphisms $\psi_\alpha : x_\alpha \to x'_\alpha$ in $X(U_\alpha)$ satisfying

$$\begin{tikzcd}
x_\beta|_{U_{\alpha\beta}} \arrow[r, "\psi_\beta|_{U_{\alpha\beta}}"] \arrow[d, "\varphi_{\alpha\beta}"'] & x'_\beta|_{U_{\alpha\beta}} \arrow[d, "\varphi'_{\alpha\beta}"] \\
x_\alpha|_{U_{\alpha\beta}} \arrow[r, "\psi_\alpha|_{U_{\alpha\beta}}"'] & x'_\alpha|_{U_{\alpha\beta}}
\end{tikzcd} \tag{4.1.8b}$$

for all α, β.

Given a prestack $X : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ and an open cover $\{U_\alpha \subseteq U\}$ of a manifold $U \in \mathbf{Man}$, there exists a canonical functor

$$\begin{aligned} X(U) &\longrightarrow X(\{U_\alpha \subseteq U\}) \quad , \\ x &\longmapsto \big(\{x|_{U_\alpha}\}, \{x|_{U_\beta}|_{U_{\alpha\beta}} \xrightarrow{\cong} x|_{U_\alpha}|_{U_{\alpha\beta}}\}\big) \quad , \\ \psi &\longmapsto \{\psi|_{U_\alpha}\} \quad . \end{aligned} \tag{4.1.9}$$

Definition 4.1.4. A *stack* (of categories) on **Man** is a prestack $X : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ satisfying the *descent condition* for stacks, i.e. such that for each manifold $U \in \mathbf{Man}$ and each open cover $\{U_\alpha \subseteq U\}$ the functor $X(U) \to X(\{U_\alpha \subseteq U\})$ in (4.1.9) is an equivalence of categories.

Example 4.1.5. A classical example of stack is the stack $\mathbf{Sh} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ of sheaves, i.e. the prestack assigning to each manifold U the category $\mathbf{Sh}(U)$ of **Set**-valued sheaves over it and to each **Man**-morphism $h : U \to U'$ the functor $h^* : \mathbf{Sh}(U') \to \mathbf{Sh}(U)$ sending each sheaf on U' to its inverse image sheaf on U. The coherence isomorphisms of **Sh** are determined by the universal property of the inverse image. The descent condition can then be interpreted by saying that to construct a sheaf X on U is sufficient to produce a compatible collection of sheaves X_{U_α} for each open U_α of an open cover $\{U_\alpha \subseteq U\}$. ▽

Example 4.1.6. Another standard example is the stack of vector bundles $\mathbf{VecBun}_{\mathbb{K}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ assigning to each manifold $U \in \mathbf{Man}$ the category $\mathbf{VecBun}_{\mathbb{K}}(U)$ of locally trivializable and finite rank $\mathbb{K}$-vector bundles over U with morphisms given by base space-preserving vector bundle maps, and to each **Man**-morphism $h : U \to U'$ the functor $h^* : X(U') \to X(U)$ sending each vector bundle $(\pi : M \to U')$ over U' to its pullback vector bundle $h^*(\pi : M \to U')$ over U. The coherence isomorphisms of $\mathbf{VecBun}_{\mathbb{K}}$ are determined by the universal property of pullbacks.

What the descent condition for $\mathbf{VecBun}_{\mathbb{K}}$ describes is the well-known fact that a vector bundle $(\pi : M \to U)$ on U can be built from a family of vector bundles $(\pi_\alpha : M_\alpha \to U_\alpha)$ on an open cover $\{U_\alpha \subseteq U\}$, together with transition functions on the overlaps satisfying the cocycle condition (see [Lee13, Problem 10-6]). ▽

Definition 4.1.7. We denote by **St** the full 2-subcategory of the category of prestacks $[\mathbf{Man}^{\mathrm{op}}, \mathbf{Cat}]$ consisting of stacks.

We conclude this section by recalling a few constructions that will be exploited in the remainder of this chapter, namely:

(a) The 2-Yoneda Lemma, which, as the name suggests, is a 2-categorical analogue of the 1-categorical Yoneda Lemma.

(b) The product of stacks $X \times Y$.

(c) The Mapping stack, i.e the internal-Hom for the 2-cartesian closed category **St**.

THE 2-YONEDA EMBEDDING AND THE 2-YONEDA LEMMA: There exists a fully faithful 2-functor

$$\underline{(-)} \,:\, \mathbf{Man} \;\longrightarrow\; \mathbf{St} \tag{4.1.10}$$

called 2-*Yoneda embedding* that assigns to any manifold $N \in \mathbf{Ob}(\mathbf{Man})$ the stack $\underline{N} := \mathbf{Man}(-,N) : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$, $U \mapsto \mathbf{Man}(U,N)$ where the set $\mathbf{Man}(U,N) = C^\infty(U,N)$ of smooth maps is considered as a category with only identity morphisms (see Example 3.1.4).

The 2-Yoneda Lemma states that given any stack $X : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ and a representable stack $\underline{U}$, there exists a canonical equivalence of categories

$$\mathbf{St}(\underline{U},X) \;\simeq\; X(U) \tag{4.1.11}$$

natural in X and U. Therefore, we can interpret $X(U)$ as the category of "smooth maps" $\underline{U} \to X$. For instance, $X(\{*\})$ can be understood as the category of "global points" of X, while $X(\mathbb{R})$ can be interpreted as the category of "smooth maps" $\underline{\mathbb{R}} \to X$.

PRODUCTS OF STACKS Given two stacks $X, Y \in \mathbf{St}$, the *product stack* $X \times Y$ is defined by:

$$\begin{aligned} X \times Y \,:\, \mathbf{Man}^{\mathrm{op}} &\longrightarrow \mathbf{Cat} \quad ,\\ U &\longmapsto X(U) \times Y(U) \quad ,\\ \big(h : U \to U'\big) &\longmapsto \big(X(h) \times Y(h) : X(U') \times Y(U') \to X(U) \times Y(U)\big) \end{aligned} \tag{4.1.12}$$

with coherence isomorphisms induced by those of X and Y.

The product stack is, as the name suggests, the bicategorical product of the stacks X and Y. Notice that the product of two representable stacks $\underline{M} \times \underline{N}$ is equivalent to the stack $\underline{M \times N}$.

MAPPING STACK Given stacks $X, Y \in \mathbf{St}$ the *mapping stack* $\mathsf{Map}(X,Y)$ is defined by :

$$\begin{aligned} \mathsf{Map}(X,Y) \,:\, \mathbf{Man}^{\mathrm{op}} &\longrightarrow \mathbf{Cat} \quad ,\\ U &\longmapsto \mathbf{St}(X \times \underline{U}, Y) \quad ,\\ \big(h : U \to U'\big) &\longmapsto \big((\mathrm{id} \times \underline{h})^* : \mathbf{St}(X \times \underline{U'}, Y) \to \mathbf{St}(X \times \underline{U}, Y)\big) \quad , \end{aligned} \tag{4.1.13}$$

where $(\mathrm{id} \times \underline{h})^* := (-) \circ (\mathrm{id} \times \underline{h})$ denotes pre-composition.

The mapping stack $\mathsf{Map}(X,Y)$ can be considered as the space of "smooth maps" $X \to Y$. More precisely, we will be able to interpret stack morphisms $\underline{\tilde{U}} \to \mathsf{Map}(X,Y)$ as "smooth $\tilde{U}$-families of smooth maps $X \to Y$" (this approach is called *functor of points* perspective, see [BS19a, Section 3.2] and [BS17] for a review in the context of field theory). To justify this statement let us consider the case in which $X = \underline{M}$ and $Y = \underline{N}$ are representable stacks (manifolds).

In this setting the category of "global points" $\mathbf{St}(\underline{\{*\}}, \mathsf{Map}(\underline{M}, \underline{N})) \cong C^\infty(M, N)$ (notice that we are using the 2-Yoneda Lemma and the definition of the mapping stack) reduces to the set of smooth maps $M \to N$, while, for a generic $\tilde{U} \in \mathbf{Man}$, an $\tilde{U}$-family of smooth maps $\underline{\tilde{U}} \to \mathsf{Map}(\underline{M}, \underline{N})$ boils down to an element of the set $\mathbf{St}(\underline{\tilde{U}}, \mathsf{Map}(\underline{M}, \underline{N}) \cong C^\infty(\tilde{U} \times M, N)$, i.e. a smooth map $\tilde{U} \times M \to N$.

Notice that the mapping stack is right biadjoint to the cartesian product of stacks and endows $\mathbf{St}$ with the structure of a closed 2-category (the same statement is true replacing "stack" with "prestack"). This implies that for all stacks X, Y and Z, there exists an equivalence $\mathbf{St}(X \times Y, Z) \cong \mathbf{St}(X, \mathsf{Map}(Y, Z))$ (analogously for prestacks).

4.2 SMOOTH 1-DIMENSIONAL AQFTS

An m-dimensional algebraic quantum field theory is a $\perp_c$-commutative functor $\mathbf{Loc}_m \to {}^*\mathbf{Alg}_{\mathbb{C}}$, i.e. a law that assigns $*$-algebras of observables to m-dimensional spacetimes in a functorial way, satisfying the Einstein causality axiom (see Definition 1.2.12). We have seen in the previous chapters that m-dimensional algebraic quantum field theories can equivalently be interpreted as $\mathbf{Vec}_{\mathbb{K}}$-valued $\mathcal{O}_{\mathbf{Loc}_m^{\perp_c}}$-algebras (see Subsection 1.2.3) or $\mathbf{Alg}_{\mathbb{K}}$-valued $\mathcal{P}_{\mathbf{Loc}_m^{\perp_c}}$-algebras (see Section 3.2.7), where $\mathcal{O}_{\mathbf{Loc}_m^{\perp_c}}$ and $\mathcal{P}_{\mathbf{Loc}_m^{\perp_c}}$ denote the restrictions $\mathcal{O}_{\mathbf{Loc}^{\perp_c}}$ and $\mathcal{P}_{\mathbf{Loc}^{\perp_c}}$ to m-dimensional manifolds.

As mentioned in the introduction to this chapter our long-term goal is to define a suitable axiomatization of m-dimensional algebraic quantum field theories that takes smoothness into account. The idea is to introduce smooth algebraic quantum field theories as morphisms between smooth (*lax*)stacks of multicategories, taking advantage of the aforementioned multicategorical descriptions of AQFTs. However, this goal comes with challenges of both categorical and analytical nature (see Chapter 5). Therefore, as a first step, we restrict our attention to 1-dimensional algebraic quantum field theories. In the 1-dimensional case, in fact, spacetimes are just time intervals together with a differential 1-form and the axiom of Einstein causality disappears from the picture (see Remark 1.2.8). In this context, the idea is to introduce smooth refinements $\mathbf{Loc}_1^\infty$ and ${}^*\mathbf{Alg}_{\mathbb{C}}^\infty$ of the categories of spacetimes $\mathbf{Loc}_1$ and algebras ${}^*\mathbf{Alg}_{\mathbb{C}}$ respectively, i.e. stacks $\mathbf{Loc}_1^\infty : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ and ${}^*\mathbf{Alg}_{\mathbb{C}}^\infty : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$, and to call *smooth algebraic quantum field theories* the stack morphisms $\mathfrak{A} : \mathbf{Loc}_1^\infty \to {}^*\mathbf{Alg}_{\mathbb{C}}^\infty$ between them.

Notice that the category of smooth algebraic quantum field theories and 2-morphisms between those is equivalent to the category of global points $\mathsf{Map}(\mathbf{Loc}_1^\infty, {}^*\mathbf{Alg}_{\mathbb{C}}^\infty)(\{*\}) \cong \mathbf{St}(\mathbf{Loc}_1^\infty, {}^*\mathbf{Alg}_{\mathbb{C}}^\infty)$ of the mapping stack $\mathsf{Map}(\mathbf{Loc}_1^\infty, {}^*\mathbf{Alg}_{\mathbb{C}}^\infty)$, therefore suggesting a further level of smoothness: the mapping stack $\mathsf{Map}(\mathbf{Loc}_1^\infty, {}^*\mathbf{Alg}_{\mathbb{C}}^\infty)$ plays the role of a stack of smooth algebraic quantum field theories $\mathbf{AQFT}_1^\infty$, hence permitting us, through the functor of points approach (see Section 4.1), to make sense of questions like "what is a smooth curve of smooth 1-dimensional algebraic quantum field theories?", or more generally "what is a smooth $\tilde{U}$-family of smooth 1-dimensional algebraic quantum field theories?".

Let us outline briefly the content of this section:

(a) In Subsection 4.2.1 we introduce the stack $\mathbf{Loc}_1^\infty$ of smooth 1-dimensional globally hyperbolic Lorentzian manifolds, a smooth refinement of the category $\mathbf{Loc}_1$ of 1-dimensional globally hyperbolic Lorentzian manifolds (see Remark 1.2.8).

(b) In Subsection 4.2.2 we define the stack ${}^*\mathbf{Alg}_{\mathbb{C}}^\infty$ of smooth associative and unital algebras, a smooth refinement of the category ${}^*\mathbf{Alg}_{\mathbb{C}}$ of associative and unital $*$-algebras (see Remark 1.1.33).

(c) In Subsection 4.2.3 we define the stack $\mathbf{AQFT}_1^\infty$ of smooth algebraic quantum field theories, a smooth refinement of the category $\mathbf{AQFT}_1 := \mathbf{AQFT}(\mathbf{Loc}_1^{\perp_d})$, and the smooth automorphism group of a smooth 1-dimensional algebraic quantum field theory. Moreover, we discuss G-equivariant smooth 1-dimensional AQFTs.

4.2.1 *The stack* $\mathbf{Loc}_1^\infty$

The aim of this subsection is to introduce the stack $\mathbf{Loc}_1^\infty : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$, a smooth refinement of the category $\mathbf{Loc}_1$ from Remark 1.2.8.

The idea is that, given a manifold U, the objects of the category $\mathbf{Loc}_1^\infty(U)$ should represent smooth U-families of 1-dimensional spacetimes, i.e. bundles with vertical fibers given by 1-dimensional globally hyperbolic Lorentzian manifolds.

Definition 4.2.1. A *smooth U-family of 1-dimensional spacetimes* is a pair $(\pi : M \to U, E)$ consisting of a (locally trivializable) fiber bundle $\pi : M \to U$ with typical fiber an open interval $I \subseteq \mathbb{R}$ and a non-degenerate vertical 1-form $E \in \Omega^1_{\mathrm{v}}(M)$ on the total space.

Notice that for $U = \{*\}$, the category $\mathbf{Loc}_1^\infty(\{x\})$ is equivalent to the category $\mathbf{Loc}_1$.

Remark 4.2.2. A smooth U-family of 1-dimensional spacetimes $(\pi : M \to U, E)$ (see Definition 4.2.1) can be interpreted as a U-parametrized family of smoothly varying 1-dimensional spacetimes. In fact, each fiber $(M|_x, E|_x) = (\pi^{-1}(\{x\}), E|_{\pi^{-1}(\{x\})})$ is a 1-dimensional globally hyperbolic Lorentzian manifold and the bundle structure encodes the smooth variation of such spacetimes over the base space U. △

Recall that a $\mathbf{Loc}_1$-morphism $f : (I, e) \to (I', e')$ is an open embedding $f : I \to I'$ preserving 1-forms, i.e. $f^*(e') = e$ (Remark 1.2.8). Therefore, in order to obtain a suitable notion of morphism between smooth U-families of 1-dimensional spacetimes we have to take into consideration the following elements:

(a) We want a morphism $f : (\pi : M \to U, E) \to (\pi' : M' \to U, E')$ of smooth U-families of 1-dimensional spacetimes to be smooth, i.e. to be a fiber bundle map

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "\pi"'] & & M' \arrow[dl, "\pi'"] \\ & U & \end{tikzcd} \tag{4.2.1}$$

(b) We want $f : (\pi : M \to U, E) \to (\pi' : M' \to U, E')$ to preserve 1-forms, i.e. $f^*(E') = E$.

(c) Furthermore, we want $f : (\pi : M \to U, E) \to (\pi' : M' \to U, E')$ to be, in some suitable sense, an "open embedding of fiber bundles".

Notice that there are a priori different ways to implement this last requirement. We could ask for example the fiber bundle map $f : (\pi : M \to U) \to (\pi' : M' \to U)$ underlying $f : (\pi : M \to U, E) \to (\pi' : M' \to U, E')$ to be such that $f|_x : M|_x \to M'_{f(x)}$ is an open embedding for every $x \in U$. However, this notion is incompatible with pushing forward vertically compactly supported functions on the total spaces (cf. Section 4.4), which is pivotal to construct examples of smooth AQFTs. Hence, what we are looking for is a more "U-uniform" notion of open embedding between smooth U-families of 1-dimensional spacetimes. Such considerations lead us naturally to three possible candidates that are, luckily, equivalent:

Lemma 4.2.3. *Let $(\pi : M \to U)$ and $(\pi' : M' \to U)$ be locally trivializable fiber bundles and let $f : (\pi : M \to U) \to (\pi' : M' \to U)$ be a fiber bundle map. Then, the following three statements are equivalent:*

(a) For each $x \in U$, there exists an open neighbourhood $U_x \subseteq U$, such that the restricted map $f|_{U_x} : M|_{U_x} \to M'|_{U_x}$ is an open embedding of manifolds.

(b) The map $f : M \to M'$ is an open embedding of manifolds.

(c) The restricted maps $f|_{\tilde{U}} : M|_{\tilde{U}} \to M'|_{\tilde{U}}$ are open embeddings of manifolds for each open subset $\tilde{U} \subseteq U$.

Proof. (a) $\Rightarrow$ (b): From the hypothesis it is clear that $f : M \to f(M)$ is a bijection of sets. Furthermore, for each $x \in U$, there exists an open neighbourhood $U_x \subseteq U$ such that $f|_{U_x} : M|_{U_x} \to M'|_{U_x}$ is an open embedding, which implies that $f : M \to f(M)$ is a diffeomorphism. To show that the image $f(M) \subseteq M'$ is open, observe that $f|_{U_x}(M|_{U_x}) \subseteq M'|_{U_x}$ is by hypothesis open and that $M'|_{U_x} \subseteq M'$ is open too. Hence, $f(M) = \bigcup_{x \in U} f|_{U_x}(M|_{U_x}) \subseteq M'$ is open.

(b) $\Rightarrow$ (c): The open embedding $f : M \to M'$ factors as a diffeomorphism $f : M \to f(M)$ followed by an open inclusion $f(M) \subseteq M'$. Take any open subset $\tilde{U} \subseteq U$ and consider the restriction $f|_{\tilde{U}} : M|_{\tilde{U}} \to M'|_{\tilde{U}}$, which factors as a map $f|_{\tilde{U}} : M|_{\tilde{U}} \to f(M|_{\tilde{U}})$ followed by an inclusion $f(M|_{\tilde{U}}) \subseteq M'|_{\tilde{U}}$. Because f is a fiber bundle map, we have that $f(M|_{\tilde{U}}) = f(M) \cap M'|_{\tilde{U}}$, which implies that $f|_{\tilde{U}} : M|_{\tilde{U}} \to f(M|_{\tilde{U}})$ is a diffeomorphism and that $f(M|_{\tilde{U}}) \subseteq M'|_{\tilde{U}}$ is an open inclusion. Hence, $f|_{\tilde{U}} : M|_{\tilde{U}} \to M'|_{\tilde{U}}$ is an open embedding.

(c). $\Rightarrow$ (a).: Trivial. □

Definition 4.2.4. Let $U \in \mathbf{Man}$ be a manifold. We denote by $\mathbf{Loc}_1^\infty(U)$ the category whose objects are smooth U-families of 1-dimensional spacetimes $(\pi : M \to U, E)$ and whose morphisms $f : (\pi : M \to U, E) \to (\pi' : M' \to U, E')$ are fiber bundle maps $f : (\pi : M \to U) \to (\pi' : M' \to U)$ preserving 1-forms, i.e. $f^*(E') = E$, and satisfying one of the equivalent conditions from Lemma 4.2.3.

Definition 4.2.4 describes the stack $\mathbf{Loc}_1^\infty : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ on objects (i.e. on manifolds $U \in \mathbf{Man}$). It remains then to describe it on the morphisms $h : U \to U'$ (see Remark 4.1.2), i.e. we have to establish functors

$$h^* := \mathbf{Loc}_1^\infty(h) : \mathbf{Loc}_1^\infty(U') \longrightarrow \mathbf{Loc}_1^\infty(U) \quad . \qquad (4.2.2)$$

Recall that given a fiber bundle $(\pi : M \to U')$ we can form the *pullback bundle*

$$\begin{tikzcd} h^*M \arrow[r, dashed, "\overline{h}^M"] \arrow[d, dashed, "\pi_h"'] & M \arrow[d, "\pi"] \\ U \arrow[r, "h"'] & U' \end{tikzcd} \tag{4.2.3}$$

which is a locally trivializable fiber bundle with the same typical fiber as $\pi : M \to U'$. Therefore, we define, for each **Man**-morphism $h : U \to U'$, $h^* = X(h) : \mathbf{Loc}_1^\infty(U') \to \mathbf{Loc}_1^\infty(U)$ to be the functor given by the following data:

(a) *On objects*: it assigns to any smooth U'-family of 1-dimensional spacetimes $(\pi : M \to U', E)$ the U-family

$$h^*\big(\pi : M \to U', E\big) := \big(\pi_h : h^*M \to U, \overline{h}^{M*}(E)\big) \tag{4.2.4a}$$

(b) *On morphisms*: it assigns to any $\mathbf{Loc}_1^\infty(U')$-morphism $f : (\pi : M \to U', E) \to (\pi' : M' \to U', E')$ the $\mathbf{Loc}_1^\infty(U)$-morphism

$$h^*f \,:\, \big(\pi_h : h^*M \to U, \overline{h}^{M*}(E)\big) \longrightarrow \big(\pi'_h : h^*M' \to U, \overline{h}^{M'*}(E')\big) \quad , \tag{4.2.4b}$$

determined from the universal property of pullback bundles by the commutative diagram

$$\begin{tikzcd} & h^*M' \arrow[rr, "\overline{h}^{M'}"] \arrow[dd, "\pi'_h"] & & M' \arrow[dd, "\pi'"] \\ h^*M \arrow[ur, dashed, "h^*f"] \arrow[rr, "\overline{h}^M"] \arrow[dr, "\pi_h"'] & & M \arrow[ur, "f"] \arrow[dr, "\pi"] & \\ & U \arrow[rr, "h"'] & & U' \end{tikzcd} \tag{4.2.4c}$$

To check that h^*f is actually a $\mathbf{Loc}_1^\infty$-morphism we have to verify, first of all, that h^*f preserves 1-forms, i.e. that $(h^*f)^*\overline{h}^{M'*}(E') = \overline{h}^{M*}(E)$, which is a consequence of diagram (4.2.4c) and the fact that $f^*(E') = E$. Secondly, we need to control that h^*f satisfies one of the conditions from Lemma 4.2.3. This is easily checked by noticing that since f satisfies condition (a) of Lemma 4.2.3, h^*f does.

Proposition 4.2.5. *The prestack* $\mathbf{Loc}_1^\infty : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ *defined by sending each manifold* U *to the category* $\mathbf{Loc}_1^\infty(U)$ *and any* **Man***-morphism* $h : U \to U'$ *to the functor* h^* *in* (4.2.2)*, with coherence isomorphisms obtained from the universal property of pullback bundles, is a stack.*

Proof. This is a direct consequence of descent for fiber bundles and differential forms, and the fact that the first condition on the fiber bundle morphisms stated in Lemma 4.2.3 is a local condition on $U \in \mathbf{Man}$. Therefore, we do not have to spell out the details. □

4.2.2 *The stack* ${}^*\mathbf{Alg}^\infty_{\mathbb{C}}$

The aim of this section is to introduce a smooth refinement ${}^*\mathbf{Alg}^\infty_{\mathbb{C}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ of the category ${}^*\mathbf{Alg}_{\mathbb{C}}$ of associative and unital $*$-algebras (see Subsection 1.1.2 and in particular Remark 1.1.39). As we have seen in Proposition 1.1.39, ${}^*\mathbf{Alg}_{\mathbb{C}}$ can be interpreted as the category $*\text{-}\mathbf{Mon}_{\mathrm{rev}}(\mathbf{Vec}_{\mathbb{C}})$ of order-reversing $*$-monoids in the category of complex vector spaces $\mathbf{Vec}_{\mathbb{C}}$. The idea is therefore to define a suitable smooth analogue of the category $\mathbf{Vec}_{\mathbb{C}}$ of vector spaces and leverage the 2-functor $*\text{-}\mathbf{Mon}_{\mathrm{rev}}$ from Proposition 1.1.2 to define the stack ${}^*\mathbf{Alg}^\infty_{\mathbb{C}}$.

The first, naive, attempt we make to define a smooth refinement of $\mathbf{Vec}_{\mathbb{K}}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ (for the sake of generality), is to consider the stack $\mathbf{VecBun}_{\mathbb{K}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ from Example 4.1.6 assigning the category $\mathbf{VecBun}_{\mathbb{K}}(U)$ to each manifold U. However, we immediately find an obstacle. In fact, the fibers of the vector bundles in $\mathbf{VecBun}_{\mathbb{K}}(U)$ consist of *finite-dimensional* vector spaces, while, for the scopes of AQFT, we need infinite-dimensional ones (think for example about canonical commutation relation algebras). But not everything is lost, since there exists a natural way to solve this insufficiency, consisting in enlarging the category $\mathbf{VecBun}_{\mathbb{K}}(U)$, through the sheaf of sections functor, to the category $\mathbf{Sh}_{C^\infty_{\mathbb{K}}}(U)$ of sheaves of $C^\infty_{\mathbb{K},U}$-modules, where $C^\infty_{\mathbb{K},U} : \mathbf{Open}(U)^{\mathrm{op}} \to \mathbf{Alg}_{\mathbb{K}}\,,\ (\tilde{U} \subseteq U) \mapsto C^\infty_{\mathbb{K}}(\tilde{U})$ is the sheaf of $\mathbb{K}$-valued smooth functions on U.

(Notice that $\mathbf{VecBun}_{\mathbb{K}}(U)$ is equivalent to the subcategory of $\mathbf{Sh}_{C^\infty_{\mathbb{K}}}(U)$ consisting of locally free $C^\infty_{\mathbb{K},U}$-modules of finite rank, see e.g. [Ram05, Chapter 2].)

Proposition 4.2.6. *Let $\mathbb{K}$ be either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. Then, the following data defines a stack $\mathbf{Sh}_{C^\infty_{\mathbb{K}}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$:*

(a) To any manifold U, $\mathbf{Sh}_{C^\infty_{\mathbb{K}}}$ assigns the category $\mathbf{Sh}_{C^\infty_{\mathbb{K}}}(U)$ of sheaves of $C^\infty_{\mathbb{K},U}$-modules over U.

(b) To every $\mathbf{Man}$-morphism $h : U \to U'$, $\mathbf{Sh}_{C^\infty_{\mathbb{K}}}$ assigns the functor

$$h^* := \mathbf{Sh}_{C^\infty_{\mathbb{K}}}(h) : \mathbf{Sh}_{C^\infty_{\mathbb{K}}}(U') \longrightarrow \mathbf{Sh}_{C^\infty_{\mathbb{K}}}(U) \tag{4.2.5a}$$

that acts on $V \in \mathbf{Sh}_{C^\infty_{\mathbb{K}}}(U')$ as

$$h^*V := h^{-1}(V) \otimes_{h^{-1}(C^\infty_{\mathbb{K},U'})} C^\infty_{\mathbb{K},U} \quad , \tag{4.2.5b}$$

where $\otimes_{h^{-1}(C^\infty_{\mathbb{K},U'})}$ denotes the relative tensor product and h^{-1} the inverse image functor.

(c) The coherences are those associated to the relative tensor product $\otimes_{h^{-1}(C^\infty_{\mathbb{K},U'})}$ and the inverse image functor h^{-1}.

(Notice that the category $\mathbf{Sh}_{C^\infty_{\mathbb{K}}}(\{*\})$ of global points $\{*\} \to \mathbf{Sh}_{C^\infty_{\mathbb{K}}}$ is equivalent to the category of vector spaces $\mathbf{Vec}_{\mathbb{K}}$.)

Proof. This is a standard result in stack theory. See for example [KS06, Proposition 19.4.7]. □

Remark 4.2.7. The reader might wonder why we did not choose other approaches for enlarging $\mathbf{VecBun}_{\mathbb{K}}(U)$ such as considering fiber bundles on U with bornological or diffeological fibers or full subcategories of $\mathbf{Sh}_{C^\infty_{\mathbb{K}}}(U)$ that describe sheaves of modules with additional properties such as projectivity or finitely generatedness. The reason for our choice $\mathbf{Sh}_{C^\infty_{\mathbb{K}}}(U)$ is that it is known to be a stack, while such other more restrictive approaches would require further checks of the descent property. △

As mentioned earlier, our plan is to leverage the stack $\mathbf{Sh}_{C^\infty_{\mathbb{C}}}$ from Proposition 4.2.6 and the 2-functor $\ast\text{-}\mathbf{Mon}_{\mathrm{rev}} : \mathbf{ISMCat} \to \mathbf{Cat}$ from Proposition 1.1.39 to define a smooth analogue of the category ${}^{\ast}\mathbf{Alg}_{\mathbb{C}} \cong \ast\text{-}\mathbf{Mon}_{\mathrm{rev}}(\mathbf{Vec}_{\mathbb{C}})$ of associative and unital $\ast$-algebras over $\mathbb{C}$. More precisely, we want to do the following:

(a) Firstly, we want to prove that the stack $\mathbf{Sh}_{C^\infty_{\mathbb{K}}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ can be promoted to a stack $\mathbf{Sh}_{C^\infty_{\mathbb{K}}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{SMCat}$.

(b) Secondly, we want to prove that in the case $\mathbb{K} = \mathbb{C}$ the stack $\mathbf{Sh}_{C^\infty_{\mathbb{C}}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{SMCat}$ can be promoted to a stack $\mathbf{Sh}_{C^\infty_{\mathbb{C}}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{ISMCat}$.

(c) Finally, we want to pre-compose the 2-functor $\ast\text{-}\mathbf{Mon}_{\mathrm{rev}} : \mathbf{ISMCat} \to \mathbf{Cat}$ with the stack $\mathbf{Sh}_{C^\infty_{\mathbb{C}}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{ISMCat}$ from the previous point to define ${}^{\ast}\mathbf{Alg}^{\infty}_{\mathbb{C}} := \ast\text{-}\mathbf{Mon}_{\mathrm{rev}} \circ \mathbf{Sh}_{C^\infty_{\mathbb{C}}}$.

Our first task is then to prove that the stack $\mathbf{Sh}_{C^\infty_{\mathbb{K}}} : \mathbf{Man} \to \mathbf{Cat}$ naturally lifts to a stack $\mathbf{Sh}_{C^\infty_{\mathbb{K}}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{SMCat}$. This is easily seen by noticing the following facts:

(a) For each manifold U, the category $\mathbf{Sh}_{C^\infty_{\mathbb{K}}}(U)$ is symmetric monoidal with respect to the monoidal product defined by the relative tensor product

$$V \otimes_{C^\infty_{\mathbb{K},U}} V' \in \mathbf{Sh}_{C^\infty_{\mathbb{K}}}(U) \quad , \tag{4.2.6}$$

for all $V, V' \in \mathbf{Sh}_{C^\infty_{\mathbb{K}}}(U)$, and the monoidal unit given by $C^\infty_{\mathbb{K}}(U) \in \mathbf{Sh}_{C^\infty_{\mathbb{K}}}(U)$ considered as a sheaf of $C^\infty_{\mathbb{K},U}$-modules.

(b) For any $\mathbf{Man}$-morphism $h : U \to U'$, the functor $h^\ast$ from (4.2.5) is strong symmetric monoidal with coherence isomorphisms given by

$$\begin{aligned} h^\ast\big(V \otimes_{C^\infty_{\mathbb{K},U'}} V'\big) &= h^{-1}\big(V \otimes_{C^\infty_{\mathbb{K},U'}} V'\big) \otimes_{h^{-1}(C^\infty_{\mathbb{K},U'})} C^\infty_{\mathbb{K},U} \\ &\cong h^{-1}(V) \otimes_{h^{-1}(C^\infty_{\mathbb{K},U'})} h^{-1}(V') \otimes_{h^{-1}(C^\infty_{\mathbb{K},U'})} C^\infty_{\mathbb{K},U} \\ &\cong \big(h^{-1}(V) \otimes_{h^{-1}(C^\infty_{\mathbb{K},U'})} C^\infty_{\mathbb{K},U}\big) \otimes_{C^\infty_{\mathbb{K},U}} \big(h^{-1}(V') \otimes_{h^{-1}(C^\infty_{\mathbb{K},U'})} C^\infty_{\mathbb{K},U}\big) \\ &= (h^\ast V) \otimes_{C^\infty_{\mathbb{K},U}} (h^\ast V') \end{aligned} \tag{4.2.7a}$$

and

$$h^\ast C^\infty_{\mathbb{K},U'} = h^{-1}(C^\infty_{\mathbb{K},U'}) \otimes_{h^{-1}(C^\infty_{\mathbb{K},U'})} C^\infty_{\mathbb{K},U} \cong C^\infty_{\mathbb{K},U} \quad . \tag{4.2.7b}$$

Moreover, the canonical coherence isomorphisms of the stack $\mathbf{Sh}_{C^\infty_{\mathbb{K}}}$ are monoidal natural transformations.

Corollary 4.2.8. *The stack* $\mathbf{Sh}_{C^\infty_{\mathbb{K}}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ *can be promoted canonically to a stack* $\mathbf{Sh}_{C^\infty_{\mathbb{K}}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{SMCat}$.

Our second task is to prove that when $\mathbb{K} = \mathbb{C}$, the stack $\mathbf{Sh}_{C^\infty_{\mathbb{C}}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{SMCat}$ lifts to a stack $\mathbf{Sh}_{C^\infty_{\mathbb{C}}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{ISMCat}$.

(a) We begin by noticing that there exists an involution endofunctor (squaring to the identity) $\overline{(-)} : \mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U) \to \mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U)$ defined by sending each $V \in \mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U)$ to the complex conjugate sheaf $\overline{V}$ of right $C^\infty_{\mathbb{C},U}$-modules whose underlying sheaf is V and whose $C^\infty_{\mathbb{C},U}$-module structure is given by conjugation of $\mathbb{C}$-valued functions as $\overline{v} \cdot a := \overline{v \cdot a^*}$, for all $\overline{v} \in \overline{V}$ and $a \in C^\infty_{\mathbb{C},U}$ (here $\overline{v \cdot a^*}$ is the element $v \cdot a^*$ of V considered as an element of $\overline{V}$, see Section 1.1.2 for similar notations in the context of the complex conjugate vector space). Moreover, we notice that $\overline{(-)} : \mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U) \to \mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U)$ is a strong symmetric monoidal functor with respect to the monoidal structure on $\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U)$ discussed earlier. Therefore, $\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U)$ is an involutive symmetric monoidal category for every manifold U.

(b) We proceed by noticing that for each $\mathbf{Man}$-morphism $h : U \to U'$ in $\mathbf{Man}$, the symmetric monoidal functor (4.2.5) is involutive via the coherence isomorphisms

$$\begin{aligned} h^*\overline{V} &= h^{-1}(\overline{V}) \otimes_{h^{-1}(C^\infty_{\mathbb{C},U'})} C^\infty_{\mathbb{C},U} \cong \overline{h^{-1}(V)} \otimes_{h^{-1}(C^\infty_{\mathbb{C},U'})} \overline{C^\infty_{\mathbb{C},U}} \\ &\cong \overline{h^{-1}(V) \otimes_{h^{-1}(C^\infty_{\mathbb{C},U'})} C^\infty_{\mathbb{C},U}} = \overline{h^*V} \quad , \end{aligned} \tag{4.2.8}$$

for all $V \in \mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U')$, where in the second step we used complex conjugation $* : C^\infty_{\mathbb{C},U} \to \overline{C^\infty_{\mathbb{C},U}}$.

Corollary 4.2.9. *The stack* $\mathbf{Sh}_{C^\infty_{\mathbb{C}}}$ *in Corollary 4.2.8 lifts to a stack* $\mathbf{Sh}_{C^\infty_{\mathbb{C}}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{ISMCat}$ *valued in the 2-category* $\mathbf{ISMCat}$ *of involutive symmetric monoidal categories, involutive strong symmetric monoidal functors and involutive monoidal natural transformations.*

As promised, we define a prestack

$$^*\mathbf{Alg}^\infty_{\mathbb{C}} := *\text{-}\mathbf{Mon}_{\mathrm{rev}} \circ \mathbf{Sh}_{C^\infty_{\mathbb{C}}} : \mathbf{Man}^{\mathrm{op}} \longrightarrow \mathbf{Cat} \quad . \tag{4.2.9}$$

Explicitly $^*\mathbf{Alg}^\infty_{\mathbb{C}}$ is given by the following data:

(a) It assigns to each manifold $U \in \mathbf{Ob}(\mathbf{Man})$ the category $^*\mathbf{Alg}^\infty_{\mathbb{C}}(U) := *\text{-}\mathbf{Mon}_{\mathrm{rev}}(\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U))$. Recall (see Remark 1.1.33) that an object of $^*\mathbf{Alg}^\infty_{\mathbb{C}}(U)$ is a quadruple $(A, \mu, \eta, *)$, where $A \in \mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U)$ is a sheaf of $C^\infty_{\mathbb{C},U}$-modules on U and

$$\mu : A \otimes_{C^\infty_{\mathbb{C},U}} A \longrightarrow A \quad , \qquad \eta : C^\infty_{\mathbb{C},U} \longrightarrow A \quad , \qquad * : A \longrightarrow \overline{A} \tag{4.2.10}$$

are $\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U)$-morphisms satisfying the axioms of an associative and unital $*$-algebra.

(b) It assigns to each **Man**-morphism $h : U \to U'$ the functor

$$h^* := {}^*\mathbf{Alg}^\infty_{\mathbb{C}}(h) : {}^*\mathbf{Alg}^\infty_{\mathbb{C}}(U') \longrightarrow {}^*\mathbf{Alg}^\infty_{\mathbb{C}}(U) \tag{4.2.11}$$

sending each $(A, \mu, \eta, *) \in {}^*\mathbf{Alg}^\infty_{\mathbb{C}}(U')$ to the object $h^*A \in \mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U)$ given in (4.2.5b), endowed with the following order-reversing $*$-monoid structure:

$$(h^*A) \otimes_{C^\infty_{\mathbb{C},U}} (h^*A) \cong h^*\big(A \otimes_{C^\infty_{\mathbb{C},U'}} A\big) \xrightarrow{h^*\mu} h^*A \quad , \tag{4.2.12a}$$

$$C^\infty_{\mathbb{C},U} \cong h^* C^\infty_{\mathbb{C},U'} \xrightarrow{h^*\eta} h^*A \quad , \tag{4.2.12b}$$

$$h^*A \xrightarrow{h^**} h^*\overline{A} \cong \overline{h^*A} \quad , \tag{4.2.12c}$$

obtained by harnessing the coherence isomorphisms of the involutive symmetric monoidal stack $\mathbf{Sh}_{C^\infty_{\mathbb{C}}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{ISMCat}$ (see Corollary 4.2.9).

Proposition 4.2.10. *The prestack* ${}^*\mathbf{Alg}^\infty_{\mathbb{C}} := *\text{-}\mathbf{Mon}_{\mathrm{rev}} \circ \mathbf{Sh}_{C^\infty_{\mathbb{C}}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ *is a stack. (Notice that the category* ${}^*\mathbf{Alg}^\infty_{\mathbb{C}}(\{*\})$ *of global points* $\underline{\{*\}} \to {}^*\mathbf{Alg}^\infty_{\mathbb{C}}$ *is equivalent to the category* ${}^*\mathbf{Alg}_{\mathbb{C}}$ *of associative and unital $*$-algebras.)*

Proof. Let $\{U_\alpha : \alpha \in A\}$ be any open cover of any $U \in \mathbf{Man}$. The key step is to realize that the descent category ${}^*\mathbf{Alg}^\infty_{\mathbb{C}}(\{U_\alpha : \alpha \in A\})$ coincides with the category $*\text{-}\mathbf{Mon}_{\mathrm{rev}}\big(\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(\{U_\alpha : \alpha \in A\})\big)$ of order-reversing $*$-monoids in the descent category $\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(\{U_\alpha : \alpha \in A\})$, which we endow with the involutive symmetric monoidal structure given by

$$\big(\{V_\alpha\}, \{\varphi_{\alpha\beta}\}\big) \otimes \big(\{V'_\alpha\}, \{\varphi'_{\alpha\beta}\}\big) := \big(\{V_\alpha \otimes_{C^\infty_{\mathbb{C},U_\alpha}} V'_\alpha\}, \{\varphi_{\alpha\beta} \otimes_{C^\infty_{\mathbb{C},U_{\alpha\beta}}} \varphi'_{\alpha\beta}\}\big) \tag{4.2.13a}$$

$$\overline{\big(\{V_\alpha\}, \{\varphi_{\alpha\beta}\}\big)} := \big(\{\overline{V_\alpha}\}, \{\overline{\varphi_{\alpha\beta}}\}\big) \quad , \tag{4.2.13b}$$

where we have suppressed the coherence isomorphisms (4.2.7) and (4.2.8). Fully explicitly, the conjugated cocycle $\overline{\varphi_{\alpha\beta}}$ is given by

$$\overline{V_\beta}|_{U_{\alpha\beta}} \cong \overline{V_\beta|_{U_{\alpha\beta}}} \xrightarrow{\overline{\varphi_{\alpha\beta}}} \overline{V_\alpha|_{U_{\alpha\beta}}} \cong \overline{V_\alpha}|_{U_{\alpha\beta}} \quad , \tag{4.2.14}$$

and similarly for the tensor product cocycle $\varphi_{\alpha\beta} \otimes_{C^\infty_{\mathbb{C},U_{\alpha\beta}}} \varphi'_{\alpha\beta}$. The functor to the descent category $\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U) \to \mathbf{Sh}_{C^\infty_{\mathbb{C}}}(\{U_\alpha : \alpha \in A\})$ given in (4.1.9) carries a canonical involutive symmetric monoidal structure and it is an equivalence in **ISMCat** because $\mathbf{Sh}_{C^\infty_{\mathbb{C}}}$ is a stack. Applying the 2-functor $*\text{-}\mathbf{Mon}_{\mathrm{rev}} : \mathbf{ISMCat} \to \mathbf{Cat}$ that takes order-reversing $*$-monoids then yields the equivalence of categories ${}^*\mathbf{Alg}^\infty_{\mathbb{C}}(U) \to {}^*\mathbf{Alg}^\infty_{\mathbb{C}}(\{U_\alpha : \alpha \in A\})$ that proves descent for ${}^*\mathbf{Alg}^\infty_{\mathbb{C}}$. □

4.2.3 *The stack* $\mathbf{AQFT}^\infty$

The aim of this subsection is to define a smooth refinement (i.e. a stack) $\mathbf{AQFT}^\infty_1 : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ of the category $\mathbf{AQFT}_1$ of 1-dimensional algebraic quantum field theories and study some of the consequences such definition entails. In particular, we introduce the "global points" of $\mathbf{AQFT}_1^\infty$, i.e. smooth 1-dimensional algebraic quantum field theories, and verify they capture "smooth changes of observable algebras corresponding to smooth variations of spacetimes". Moreover, we describe from the functor of points perspective "smooth $\tilde{U}$-families ($\tilde{U}$ is a manifold) of smooth 1-dimensional algebraic quantum field theories", i.e. stack morphisms $\underline{\tilde{U}} \to \mathbf{AQFT}_1^\infty$, and we introduce the smooth automorphism group $\mathrm{Aut}(\mathfrak{A})$ of a smooth 1-dimensional algebraic quantum field theory $\mathfrak{A}$. Finally, we define what a G-equivariant smooth 1-dimensional algebraic quantum field theory, for G a smooth Lie group, is. In particular, we prove that this definition represents, in a suitable sense, a generalization of the notion of G-equivariant ordinary algebraic quantum field theory (see Section 3.4), i.e. an ordinary algebraic quantum field theory $\mathfrak{B}$ together with a group action $\rho : G \to \mathrm{Aut}(\mathfrak{B})$.

We begin by recalling that $\mathbf{AQFT}_1$ is the functor category $[\mathbf{Loc}_1, {}^*\mathbf{Alg}_{\mathbb{C}}]$. This fact suggests the following definition:

Definition 4.2.11. Let $\mathbf{Loc}_1^\infty : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ be the stack from Proposition 4.2.5 and ${}^*\mathbf{Alg}_{\mathbb{C}}^\infty : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ be the stack from Proposition 4.2.10. We call the mapping stack (see Section 4.1)

$$\mathbf{AQFT}_1^\infty := \mathsf{Map}\big(\mathbf{Loc}_1^\infty, {}^*\mathbf{Alg}_{\mathbb{C}}^\infty\big) \in \mathbf{St} \quad , \tag{4.2.15}$$

the *stack of smooth 1-dimensional algebraic quantum field theories*.

Moreover, we call a global point $\{*\} \to \mathbf{AQFT}_1^\infty$ a *smooth 1-dimensional algebraic quantum field theory*.

In order to show that that smooth algebraic quantum field theories capture "smooth changes of observables algebras associated to smooth variations of spacetimes", recall that a smooth 1-dimensional algebraic quantum field theory, which by definition (see Definition 4.2.11) is a global point $\mathfrak{A} : \underline{\{*\}} \to \mathbf{AQFT}_1^\infty$, can be equivalently described (using 2-Yoneda Lemma) as an object of the category $\mathbf{AQFT}^\infty(\{*\}) \cong \mathbf{St}(\mathbf{Loc}_1^\infty \times \underline{\{*\}}, {}^*\mathbf{Alg}_{\mathbb{C}}^\infty) \cong \mathbf{St}(\mathbf{Loc}_1^\infty, {}^*\mathbf{Alg}_{\mathbb{C}}^\infty)$, i.e. as a stack morphism $\mathfrak{A} : \mathbf{Loc}_1^\infty \to {}^*\mathbf{Alg}_{\mathbb{C}}^\infty$.

Explicitly, a smooth 1-dimensional algebraic quantum field theory $\mathfrak{A} : \mathbf{Loc}_1^\infty \to {}^*\mathbf{Alg}_{\mathbb{C}}^\infty$ consists of the following data:

(a) For every $U \in \mathbf{Man}$, functors

$$\mathfrak{A}_U \,:\, \mathbf{Loc}_1^\infty(U) \longrightarrow {}^*\mathbf{Alg}_{\mathbb{C}}^\infty(U) \quad . \tag{4.2.16a}$$

(b) For every $\mathbf{Man}$-morphism $h : U \to U'$, natural isomorphisms

$$\begin{tikzcd} \mathbf{Loc}_1^\infty(U') \arrow[r, "\mathfrak{A}_{U'}"] \arrow[d, "h^*"'] & {}^*\mathbf{Alg}_{\mathbb{C}}^\infty(U') \arrow[d, "h^*"] \arrow[dl, Rightarrow, "\mathfrak{A}_h"'] \\ \mathbf{Loc}_1^\infty(U) \arrow[r, "\mathfrak{A}_U"'] & {}^*\mathbf{Alg}_{\mathbb{C}}^\infty(U) \end{tikzcd} \tag{4.2.16b}$$

satisfying the coherence axioms for stack morphisms from Remark 4.1.2.

Therefore, interpreting the objects of $\mathbf{Loc}_1^\infty(U)$ and ${}^*\mathbf{Alg}_\mathbb{C}^\infty(U)$ as smooth U-families of spacetimes and algebras respectively, we conclude that it is the functors $\mathfrak{A}_U$ which capture the response of "observables algebras to smooth variations of spacetimes", hence convincing us of the suitability of our definition.

The mapping stack $\mathbf{AQFT}_1^\infty$ from Definition 4.2.11 adds a further level of "smoothness" to the picture by enabling us talking about "smooth curves of smooth AQFTs", i.e. stack morphisms $\underline{\mathbb{R}} \to \mathbf{AQFT}_1^\infty$, or more generally about "smooth $\tilde{U}$-families of smooth 1-dimensional algebraic quantum field theories" for a generic manifold $\tilde{U} \in \mathbf{Man}$, i.e. stack morphisms $\mathfrak{B} : \underline{\tilde{U}} \to \mathbf{AQFT}_1^\infty$. In particular, using 2-Yoneda Lemma, we obtain that a smooth $\tilde{U}$-family of 1-dimensional AQFTs $\mathfrak{B} : \underline{\tilde{U}} \to \mathbf{AQFT}_1^\infty$ can be equivalently described as an object of the category $\mathfrak{B} \in \mathbf{AQFT}_1^\infty(\tilde{U})$, which, by definition of mapping stack is just a stack morphism $\mathfrak{B} : \mathbf{Loc}_1^\infty \times \underline{\tilde{U}} \to {}^*\mathbf{Alg}_\mathbb{C}^\infty$. Furthermore, using the fact that the mapping stack is right adjoint to the cartesian product of stacks we can interpret $\mathfrak{B} : \mathbf{Loc}_1^\infty \times \underline{\tilde{U}} \to {}^*\mathbf{Alg}_\mathbb{C}^\infty$ as a stack morphism

$$\mathfrak{B} \,:\, \mathbf{Loc}_1^\infty \longrightarrow \mathsf{Map}\big(\underline{\tilde{U}}, {}^*\mathbf{Alg}_\mathbb{C}^\infty\big) \quad . \tag{4.2.17}$$

Finally, using again 2-Yoneda Lemma on $\mathfrak{B} : \mathbf{Loc}_1^\infty \to \mathsf{Map}(\underline{\tilde{U}}, {}^*\mathbf{Alg}_\mathbb{C})$, we obtain that a smooth $\tilde{U}$-family of smooth algebraic quantum field theories is equivalent to the following data:

(a) For each manifold $U \in \mathbf{Man}$, functors

$$\mathfrak{B}_U \,:\, \mathbf{Loc}_1^\infty(U) \longrightarrow {}^*\mathbf{Alg}_\mathbb{C}^\infty(U \times \tilde{U}) \quad , \tag{4.2.18a}$$

(b) For each $\mathbf{Man}$-morphism $h : U \to U'$, natural isomorphisms

$$\begin{tikzcd} \mathbf{Loc}_1^\infty(U') \arrow[r, "\mathfrak{B}_{U'}"] \arrow[d, "h^*"'] & {}^*\mathbf{Alg}_\mathbb{C}^\infty(U' \times \tilde{U}) \arrow[d, "(h\times\mathrm{id})^*"] \arrow[dl, Rightarrow, "\mathfrak{B}_h"'] \\ \mathbf{Loc}_1^\infty(U) \arrow[r, "\mathfrak{B}_U"'] & {}^*\mathbf{Alg}_\mathbb{C}^\infty(U \times \tilde{U}) \end{tikzcd} \tag{4.2.18b}$$

Satisfying the conditions from Remark 4.1.2.

The functors $\mathfrak{B}(U) : \mathbf{Loc}_1^\infty(U) \to {}^*\mathbf{Alg}_\mathbb{C}^\infty(U \times \tilde{U})$ capture both the "smooth response of observable algebras to smooth U-variations of spacetimes" as well as "smooth $\tilde{U}$-variations of the smooth algebraic quantum field theory itself", which is in line with the intuition of how a smooth $\tilde{U}$-family of smooth 1-dimensional AQFTs should look like.

To conclude this section we discuss the smooth automorphism group $\mathrm{Aut}(\mathfrak{A}) : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ of a smooth 1-dimensional algebraic quantum field theory $\mathfrak{A} : \mathbf{Loc}_1^\infty \to {}^*\mathbf{Alg}_\mathbb{C}^\infty$.

Definition 4.2.12. Let $\mathfrak{A}$ be a smooth 1-dimensional algebraic quantum field theory. We call the following bicategorical pullback (*loop stack*):

$$\begin{tikzcd} \mathrm{Aut}(\mathfrak{A}) \arrow[r, dashed] \arrow[d, dashed] & \underline{\{*\}} \arrow[d, "\mathfrak{A}"] \\ \underline{\{*\}} \arrow[r, "\mathfrak{A}"'] & \mathbf{AQFT}_1^\infty \end{tikzcd} \tag{4.2.19}$$

the *smooth automorphism group of* $\mathfrak{A}$.

Notice that this pullback exists for any smooth 1-dimensional quantum field theory $\mathfrak{A}$ since **St** admits all bicategorical limits, see [Fio04, Theorem 5.1].

Remark 4.2.13. Computing the bicategorical limit $\mathrm{Aut}(\mathfrak{A}) : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ in diagram (4.2.19) we obtain a **Set**-valued sheaf on **Man**, i.e. the categories $\mathrm{Aut}(\mathfrak{A})(U)$ are discrete (i.e. sets) for each manifold U, endowed with a group action. Recall that any **Set**-valued sheaf $X : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Set}$ endowed with a group action can be equivalently described as a **Grp**-valued sheaf $X : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Grp}$, where **Grp** is the category of groups and group morphisms, since the free-forgetful adjunction $\mathbf{Set} \leftrightarrows \mathbf{Grp}$ is monadic. Therefore, we can alternatively describe the bicategorical limit in diagram (4.2.19) as a **Grp**-valued sheaf $\mathrm{Aut}(\mathfrak{A}) : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Grp}$. △

Remark 4.2.13 suggests a way to generalize Definition 3.4.1 to smooth 1-dimensional algebraic quantum field theories. In fact, given a Lie group G, we could define a G-equivariant smooth 1-dimensional algebraic quantum field theory to be a couple $(\mathfrak{A}, \rho)$, where $\mathfrak{A}$ is a smooth 1-dimensional algebraic quantum field theory and ρ is a stack morphism $\rho : \underline{G} \to \mathrm{Aut}(\mathfrak{A})$ (where the Lie group is considered as a stack via the 2-Yoneda embedding) to the smooth automorphism group that is also a morphism of group objects (the group structure on $\underline{G}$ is given by point-wise multiplication of functions). However, we decide to follow a different (but equivalent, as we will see later in Remark 4.2.15) path (more suitable for eventual generalizations) and give a definition of G-equivariant smooth 1-dimensional AQFT based on the following fact: a group action on an object d in a category **D** can be equivalently described in terms of a functor $\mathbf{B}G \to \mathbf{D}$, where $\mathbf{B}G$ denotes the groupoid with a single object and G as morphisms.

Definition 4.2.14. Let G be a Lie group and let $[\{*\}/G]$ denote the quotient stack

$$[\{*\}/G] := \mathrm{bicolim}_{\mathbf{St}} \Big(\underline{\{*\}} \leftrightarrows \underline{G} \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \underline{G}^2 \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \cdots \Big) \in \mathbf{St} \tag{4.2.20}$$

associated with the trivial action of G, where $\mathrm{bicolim}_{\mathbf{St}}$ denotes the bicategorical colimit in **St**.

Then, we call any stack morphism

$$\mathfrak{A}^{\mathrm{eq}} : [\{*\}/G] \longrightarrow \mathbf{AQFT}_1^\infty \quad . \tag{4.2.21}$$

a *G-equivariant smooth 1-dimensional algebraic quantum field theory*.

Remark 4.2.15. By the universal property of bicategorical colimits, we obtain that a G-equivariant smooth 1-dimensional AQFT $\mathfrak{A}^{\mathrm{eq}} : [\{*\}/G] \to \mathbf{AQFT}_1^\infty$ can be equivalently described as a smooth 1-dimensional AQFT $\mathfrak{A} : \underline{\{*\}} \to \mathbf{AQFT}_1^\infty$ together with a 2-automorphism $\mathfrak{A}_2$ of the stack morphism $\underline{G} \to \underline{\{*\}} \xrightarrow{\mathfrak{A}} \mathbf{AQFT}_1^\infty$ satisfying compatibility conditions arising from the face and degeneracy maps in (4.2.20) that at the level of components $\mathfrak{A}_{2U}$, where $U \in \mathbf{Man}$, reads as follows: Since $\underline{G}(U)$ is a discrete category (a set with identity morphisms for each object), $\mathfrak{A}_{2U}$ boils down to a family of $\mathbf{AQFT}_1^\infty(U)$-isomorphisms $\mathfrak{A}_{2U_g} : \mathfrak{A}_U(*) \to \mathfrak{A}_U(*)$ labelled by elements $g \in \underline{G}(U) = C^\infty(U, G)$. The compatibility conditions then imply that this labelling is compatible with the group structure on $\underline{G}(U) = C^\infty(U, G)$ given by point-wise product of functions, i.e. $\mathfrak{A}_{2U_{g \cdot g'}} = \mathfrak{A}_{2U_g} \circ \mathfrak{A}_{2U_{g'}}$, for all $g, g' \in \underline{G}(U)$, and $\mathfrak{A}_{2U_e} = \mathrm{id}$, for the identity element $e \in \underline{G}(U)$. From this we obtain a bicategorical cone

$$\begin{tikzcd} \underline{G} \arrow[r] \arrow[d] & \underline{\{*\}} \arrow[d, "\mathfrak{A}"] \arrow[dl, Rightarrow, "\mathfrak{A}_2"'] \\ \underline{\{*\}} \arrow[r, "\mathfrak{A}"'] & \mathbf{AQFT}_1^\infty \end{tikzcd} \tag{4.2.22}$$

and, by the universal property of the bicategorical pullback in (4.2.19), a stack morphism $\underline{G} \to \mathrm{Aut}(\mathfrak{A})$ to the smooth automorphism group of $\mathfrak{A}$, which, due to the compatibility conditions of $\mathfrak{A}_2$ is also a morphism of group objects. △

We conclude by giving an alternative description of G-equivariant smooth 1-dimensional algebraic quantum field theories. More precisely, we will see that any G-equivariant smooth 1-dimensional AQFT $\mathfrak{A}^{\mathrm{eq}} : [\{*\}/G] \to \mathbf{AQFT}_1^\infty$ can be equivalently interpreted as a prestack morphism

$$\tilde{\mathfrak{A}} \,:\, \mathbf{Loc}_1^\infty \times [\{*\}/G]_{\mathrm{pre}} \longrightarrow {}^*\mathbf{Alg}_{\mathbb{C}}^\infty \tag{4.2.23}$$

where $[\{*\}/G]_{\mathrm{pre}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ denotes the prestack assigning to each $U \in \mathbf{Man}$ the groupoid

$$[\{*\}/G]_{\mathrm{pre}}(U) = \begin{cases} \mathrm{Obj}: & * \\ \mathrm{Mor}: & C^\infty(U, G) \end{cases} \tag{4.2.24}$$

and to each $\mathbf{Man}$-morphism $h : U \to U'$ the functor given by pullback of functions $[\{*\}/G]_{\mathrm{pre}}(h) := h^*$, a fact that will turn out to be very helpful in concrete computations.

In order to obtain the aforementioned description we need to recall that similarly to the case of $\mathbf{Set}$-valued sheaves on $\mathbf{Man}$, where the universal property of the sheafification functor $(-)^+ : \mathbf{PSh}(\mathbf{Man}) \to \mathbf{Sh}(\mathbf{Man})$ (recall that the sheafification functor exhibits $\mathbf{Sh}(\mathbf{Man})$ as a reflective full subcategory of $\mathbf{PSh}(\mathbf{Man})$) tells us that given a presheaf P and a sheaf X, the set of sheaf morphisms $\mathbf{Sh}(P^+, X)$ is in bijection with the set of presheaf morphisms $\mathbf{PSh}(P, X)$. There exists a pseudo-functor $(-)^+ : [\mathbf{Man}^{\mathrm{op}}, \mathbf{Cat}] \to \mathbf{St}$, called *stackyfication*, such that for every prestack P and every stack X, the categories of stack morphisms $\mathbf{St}(P^+, X)$ and of prestack morphisms $[\mathbf{Man}^{\mathrm{op}}, \mathbf{Cat}](P, X)$ are equivalent.

The stack $[\{*\}/G]_{\mathrm{pre}}^+$ is equivalent to the stack $[\{*\}/G]$ from (4.2.20). Hence, from the universal property of stackyfication we obtain that a G-equivariant smooth 1-dimensional smooth algebraic quantum field theory $\mathfrak{A}^{\mathrm{eq}} : [\{*\}/G] \to \mathbf{AQFT}_1^\infty$ is equivalent to a prestack morphism $\tilde{\mathfrak{A}} : [\{*\}/G]_{\mathrm{pre}} \to \mathbf{AQFT}_1^\infty$. Moreover, from the universal property of the mapping prestack we get that $\tilde{\mathfrak{A}} : [\{*\}/G]_{\mathrm{pre}} \to \mathbf{AQFT}_1^\infty$ is equivalent to a prestack morphism

$$\tilde{\mathfrak{A}} \,:\, \mathbf{Loc}_1^\infty \times [\{*\}/G]_{\mathrm{pre}} \;\longrightarrow\; {}^*\mathbf{Alg}_{\mathbb{C}}^\infty \quad .$$

4.3 SMOOTH CANONICAL QUANTIZATION

The aim of this section is to introduce smooth refinements of the canonical commutation relation (CCR) functor and anti-commutation relation (CAR) functor (see [BGP07, BG12]), which are central for the construction of ordinary free field theories. In particular, in Subsection 4.3.1 we introduce a smooth refinement $\mathbf{PoVec}_{\mathbb{R}}^\infty$ of the category $\mathbf{PoVec}_{\mathbb{R}}$ of Poisson vector spaces and a commutation relation stack morphism $\mathfrak{CCR} : \mathbf{PoVec}_{\mathbb{R}}^\infty \to {}^*\mathbf{Alg}_{\mathbb{C}}^\infty$, while, in Subsection 4.3.2 we introduce a smooth refinement $\mathbf{IPVec}_{\mathbb{C}}^\infty$ of the category $\mathbf{IPVec}_{\mathbb{C}}$ of inner product vector spaces and a canonical anti-commutation relation stack morphism $\mathfrak{CAR} : \mathbf{IPVec}_{\mathbb{C}}^\infty \to {}^*\mathbf{Alg}_{\mathbb{C}}^\infty$.

For the sake of completeness, let us recall how the categories $\mathbf{PoVec}_{\mathbb{R}}$, $\mathbf{IPVec}_{\mathbb{C}}$, and the (ordinary) functors $\mathfrak{CCR} : \mathbf{PoVec}_{\mathbb{R}} \to {}^*\mathbf{Alg}_{\mathbb{C}}$, $\mathfrak{CAR} : \mathbf{IPVec}_{\mathbb{C}} \to {}^*\mathbf{Alg}_{\mathbb{C}}$ are defined.

Definition 4.3.1. $\mathbf{PoVec}_{\mathbb{R}}$ is the category given by the following data:

(a) *Objects*: The collection of objects $\mathbf{Ob}(\mathbf{PoVec}_{\mathbb{R}})$ consists of couples (W, τ) where $W \in \mathbf{Vec}_{\mathbb{R}}$ is a vector space and $\tau : W \otimes_{\mathbb{R}} W \to \mathbb{R}$ is an antisymmetric $\mathbf{Vec}_{\mathbb{R}}$-morphism called *Poisson structure*.

(b) *Morphisms*: For any (W, τ), (W', τ'), a morphism $\psi : (W, \tau) \to (W', \tau')$ consists of a $\mathbf{Vec}_{\mathbb{R}}$-morphism $\psi : W \to W'$ satisfying $\tau' \circ (\psi \otimes_{\mathbb{R}} \psi) = \tau$.

The CCR functor $\mathfrak{CCR} : \mathbf{PoVec}_{\mathbb{R}} \to {}^*\mathbf{Alg}_{\mathbb{C}}$ assigns to a Poisson vector space $(W, \tau) \in \mathbf{PoVec}_{\mathbb{R}}$ the associative and unital $*$-algebra

$$\mathfrak{CCR}(W,\tau) \;:=\; \bigoplus_{n\geq 0} \big(W \otimes_{\mathbb{R}} \mathbb{C}\big)^{\otimes_{\mathbb{C}} n} \Big/ \mathcal{I}^{\mathrm{CCR}}_{(W,\tau)} \,\in\, {}^*\mathbf{Alg}_{\mathbb{C}} \quad , \tag{4.3.1}$$

where $\mathcal{I}^{\mathrm{CCR}}_{(W,\tau)}$ denotes the 2-sided $*$-ideal generated by the canonical commutation relations $w \otimes w' - w' \otimes w = \mathrm{i}\, \tau(w, w')$, for all $w, w' \in W$, and where $\mathrm{i} \in \mathbb{C}$ denotes the imaginary unit, with $*$-involution given by $w^* = w$, for all $w \in W$.

Remark 4.3.2. In order to construct a smooth refinement of the CCR functor it will be helpful to describe more abstractly how the associative and unital $*$-algebra $\mathfrak{CCR}(W, \tau)$ is obtained:

(a) First of all, given a Poisson vector space (W, τ), one needs to consider the complexification $W \otimes \mathbb{C}$ of the real vector space W with the $*$-object structure $\mathrm{id} \otimes_{\mathbb{R}} * : W \otimes_{\mathbb{R}} \mathbb{C} \to \overline{W \otimes_{\mathbb{R}} \mathbb{C}} = W \otimes_{\mathbb{R}} \overline{\mathbb{C}}$ determined by complex conjugation on $\mathbb{C}$. Therefore, $(W \otimes_{\mathbb{R}} \mathbb{C}, \mathrm{id} \otimes_{\mathbb{R}} *) \in {}^*\mathbf{Obj}(\mathbf{Vec}_{\mathbb{C}})$ defines a $*$-object in the involutive symmetric monoidal category of complex vector spaces (see Definition 1.1.32).

(b) Secondly, one considers the free order-reversing $*$-monoid on

$$(W \otimes_{\mathbb{R}} \mathbb{C}, \mathrm{id} \otimes_{\mathbb{R}} *) \in {}^*\mathbf{Obj}(\mathbf{Vec}_{\mathbb{C}}),$$

which is the associative and unital $*$-algebra $\bigoplus_{n \geq 0} \big(W \otimes_{\mathbb{R}} \mathbb{C}\big)^{\otimes_{\mathbb{C}} n} \in {}^*\mathbf{Alg}_{\mathbb{C}}$.

(c) Thirdly, one implements the canonical commutation relations associated to the antisymmetric $\mathbf{Vec}_{\mathbb{R}}$-morphism $\tau : W \otimes_{\mathbb{R}} W \to \mathbb{R}$ via a coequalizer in ${}^*\mathbf{Alg}_{\mathbb{C}}$.

$\triangle$

Definition 4.3.3. The category $\mathbf{IPVec}_{\mathbb{C}}$ is defined by the following data:

1. *Objects*: The collection of objects $\mathbf{Ob}(\mathbf{IPVec}_{\mathbb{C}})$ consists of triples $(V, *, \langle \cdot, \cdot \rangle)$, where $(V, *) \in \mathbf{Ob}({}^*\mathbf{Obj}(\mathbf{Vec}_{\mathbb{C}}))$ is a $*$-object in the involutive symmetric monoidal category $\mathbf{Vec}_{\mathbb{C}}$ and where $\langle \cdot, \cdot \rangle : (V, *) \otimes (V, *) \to (\mathbb{C}, *)$ is a symmetric $*$-morphism. Recall from Definition 1.1.32 that a $*$-morphism $\langle \cdot, \cdot \rangle : (V, *) \otimes (V, *) \to (\mathbb{C}, *)$ is a $\mathbb{C}$-linear map $\langle \cdot, \cdot \rangle : V \otimes V \to \mathbb{C}$ satisfying the following equation:

$$\begin{tikzcd} V \otimes V \arrow[r, "\langle \cdot, \cdot \rangle"] \arrow[d, "* \otimes *"'] & \mathbb{C} \arrow[d, "*"] \\ \overline{V} \otimes \overline{V} \cong \overline{V \otimes V} \arrow[r, "\overline{\langle \cdot, \cdot \rangle}"'] & \overline{\mathbb{C}} \end{tikzcd} \tag{4.3.2}$$

which concretely reads: $\langle v, v' \rangle^* = \langle v^*, v'^* \rangle$, for all $v, v' \in V$.

2. *Morphisms*: For all objects $(V, *, \langle \cdot, \cdot \rangle)$, $(V', *', \langle \cdot, \cdot \rangle') \in \mathbf{Ob}(\mathbf{IPVec}_{\mathbb{C}})$, a morphism $\psi : (V, *, \langle \cdot, \cdot \rangle) \to (V', *', \langle \cdot, \cdot \rangle')$ consists of a $*$-morphism $\psi : (V, *) \to (V', *')$ satisfying $\langle \cdot, \cdot \rangle' \circ (\psi \otimes \psi) = \langle \cdot, \cdot \rangle$.

The CAR functor $\mathfrak{CAR} : \mathbf{IPVec}_{\mathbb{C}} \to {}^*\mathbf{Alg}_{\mathbb{C}}$ assigns to an object $(V, *, \langle \cdot, \cdot \rangle) \in \mathbf{Ob}(\mathbf{IPVec}_{\mathbb{C}})$ the associative and unital $*$-algebra

$$\mathfrak{CAR}(V, *, \langle \cdot, \cdot \rangle) := \bigoplus_{n \geq 0} V^{\otimes n} \Big/ \mathcal{I}^{\mathrm{CAR}}_{(V, *, \langle \cdot, \cdot \rangle)} \in {}^*\mathbf{Alg}_{\mathbb{C}} \quad , \tag{4.3.3}$$

where $\mathcal{I}^{\mathrm{CAR}}_{(V, *, \langle \cdot, \cdot \rangle)}$ is the 2-sided $*$-ideal generated by the canonical anti-commutation relations $v \otimes v' + v' \otimes v = \langle v, v' \rangle$, for all $v, v' \in V$.

Remark 4.3.4. In order to obtain a smooth refinement of the CAR functor it will be helpful to describe more abstractly how the associative and unital $*$-algebra $\mathfrak{CAR}(V, *, \langle \cdot, \cdot \rangle)$ is obtained:

(a) Firstly, given an object $(V, *, \langle \cdot, \cdot \rangle) \in \mathbf{Ob}(\mathbf{IPVec}_{\mathbb{C}})$, one considers the free order-reversing $*$-monoid $\bigoplus_{n \geq 0} V^{\otimes n} \in {}^*\mathbf{Alg}_{\mathbb{C}}$ of $(V, *) \in \mathbf{Ob}({}^*\mathbf{Obj}(\mathbf{Vec}_{\mathbb{C}}))$ (see Remark 4.3.2).

(b) Secondly, one implements the canonical anti-commutation relations associated with $\langle \cdot, \cdot \rangle$ by a coequalizer in ${}^*\mathbf{Alg}_{\mathbb{C}}$.

$\triangle$

4.3.1 *Canonical commutation relations stack morphism*

The aim of this subsection is to discuss a smooth refinement $\mathfrak{CCR} : \mathbf{PoVec}^\infty_{\mathbb{R}} \to {}^*\mathbf{Alg}^\infty_{\mathbb{C}}$ of the CCR functor $\mathfrak{CCR} : \mathbf{PoVec}_{\mathbb{R}} \to {}^*\mathbf{Alg}_{\mathbb{C}}$. In order to introduce it, we begin by describing how to produce a smooth refinement of the category $\mathbf{PoVec}_{\mathbb{R}}$ of Poisson vector spaces.

In Subsection 4.2.3 we have seen that a sensible notion of smooth refinement of the category of vector spaces $\mathbf{Vec}_{\mathbb{R}}$ is the stack $\mathbf{Sh}_{C^\infty_{\mathbb{R}}}$. Therefore, a suitable refinement of $\mathbf{PoVec}_{\mathbb{R}}$ should be obtained from this stack. This is pretty straightforward:

Proposition 4.3.5. *The prestack $\mathbf{PoVec}^\infty_{\mathbb{R}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ is given by the following data:*

(a) On objects*: It assigns to each manifold $U \in \mathbf{Man}$ the category $\mathbf{PoVec}^\infty_{\mathbb{R}}(U)$ whose objects are couples (W,τ), where $W \in \mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U)$ and $\tau : W \otimes_{C^\infty_{\mathbb{R},U}} W \to C^\infty_{\mathbb{R},U}$ is an antisymmetric $\mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U)$-morphism called* Poisson structure*, and whose morphisms $\psi : (W,\tau) \to (W',\tau')$ consist of a $\mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U)$-morphisms $\psi : W \to W'$ satisfying $\tau' \circ (\psi \otimes_{C^\infty_{\mathbb{R},U}} \psi) = \tau$.*

(b) On 1-morphisms*: It assigns to each $\mathbf{Man}$-morphism $h : U \to U'$ the functor*

$$h^* := \mathbf{PoVec}^\infty_{\mathbb{R}}(h) \ : \ \mathbf{PoVec}^\infty_{\mathbb{R}}(U') \longrightarrow \mathbf{PoVec}^\infty_{\mathbb{R}}(U) \tag{4.3.4}$$

sending an object $(W,\tau) \in \mathbf{PoVec}^\infty_{\mathbb{R}}(U')$ to the $C^\infty_{\mathbb{R},U}$-module $h^*W \in \mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U)$ (see (4.2.5b)*) endowed with the Poisson structure*

$$(h^*W) \otimes_{C^\infty_{\mathbb{R},U}} (h^*W) \cong h^*\big(W \otimes_{C^\infty_{\mathbb{R},U'}} W\big) \xrightarrow{h^*\tau} h^* C^\infty_{\mathbb{R},U'} \cong C^\infty_{\mathbb{R},U} \quad , \tag{4.3.5}$$

where $\cong$ denotes the coherence isomorphisms in (4.2.7)*.*

This prestack is a stack.

Proof. This follows from the fact that $\mathbf{Sh}_{C^\infty_{\mathbb{R}}}$ is a stack, see Proposition 4.2.6. Indeed, spelling out descent for objects $(W,\tau) \in \mathbf{PoVec}^\infty_{\mathbb{R}}(U)$, one observes that it involves descent for the underlying objects $W \in \mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U)$ and also for the underlying $\mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U)$-morphisms $\tau : W \otimes_{C^\infty_{\mathbb{R},U}} W \to C^\infty_{\mathbb{R},U}$, which are both simple consequences of descent for the stack $\mathbf{Sh}_{C^\infty_{\mathbb{R}}}$. Similarly, descent for $\mathbf{PoVec}^\infty_{\mathbb{R}}(U)$-morphisms $\psi : (W,\tau) \to (W',\tau')$ involves descent for the underlying $\mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U)$-morphisms $\psi : W \to W'$ and the verification that $\tau' \circ (\psi \otimes_{C^\infty_{\mathbb{R},U}} \psi) = \tau$ coincide as $\mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U)$-morphisms, which are again both consequences of descent for the stack $\mathbf{Sh}_{C^\infty_{\mathbb{R}}}$ and of the fact that the descent data have this property. □

Now that we have defined a smooth refinement $\mathbf{PoVec}^\infty_{\mathbb{R}}$ of the category $\mathbf{PoVec}_{\mathbb{R}}$ we can formalize a CCR stack morphism $\mathfrak{CCR} : \mathbf{PoVec}^\infty_{\mathbb{R}} \to {}^*\mathbf{Alg}^\infty_{\mathbb{C}}$. To do so, we begin by abstracting the 3-steps construction from Remark 4.3.2 to obtain functors

$$\mathfrak{CCR}_U \ : \ \mathbf{PoVec}^\infty_{\mathbb{R}}(U) \longrightarrow {}^*\mathbf{Alg}^\infty_{\mathbb{C}}(U) \quad , \tag{4.3.6}$$

for every manifold $U \in \mathbf{Man}$ and proceed by defining the coherence isomorphisms for $\mathfrak{CCR}$.

(a) First of all, given $(W,\tau) \in \mathbf{PoVec}^\infty_{\mathbb{R}}(U)$, one needs to consider its complexification $W \otimes_{C^\infty_{\mathbb{R},U}} C^\infty_{\mathbb{C},U} \in \mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U)$ with the $*$-object structure $\mathrm{id} \otimes_{C^\infty_{\mathbb{R},U}} *$ determined by complex conjugation $* : C^\infty_{\mathbb{C},U} \to \overline{C^\infty_{\mathbb{C},U}}$. Notice that this data defines a functor $L_U : \mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U) \to {}^*\mathbf{Obj}\big(\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U)\big)$ that is part of an adjunction

$$L_U \,:\, \mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U) \rightleftarrows {}^*\mathbf{Obj}\big(\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U)\big) \,:\, R_U \quad . \tag{4.3.7}$$

where the right adjoint R_U assigns to a $*$-object $(V,*)$ in $\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U)$ the sheaf of $*$-invariants $R_U(V,*) = \ker\big(V_{\mathbb{R}} \xrightarrow{*-\mathrm{id}} V_{\mathbb{R}}\big) \in \mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U)$ and where $V_{\mathbb{R}} \in \mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U)$ denotes the restriction of $V \in \mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U)$ to a sheaf of $C^\infty_{\mathbb{R},U}$-modules via the morphism $C^\infty_{\mathbb{R},U} \to C^\infty_{\mathbb{C},U}$.

(b) Secondly, one considers the free order reversing $*$-monoid $\bigoplus_{n\geq 0}(W \otimes_{C^\infty_{\mathbb{R},U}} C^\infty_{\mathbb{C},U})^{\otimes n}$ on $(W \otimes_{C^\infty_{\mathbb{R},U}} C^\infty_{\mathbb{C},U}, \mathrm{id} \otimes_{C^\infty_{\mathbb{R},U}} *)$, where tensor products and coproducts are formed in the symmetric monoidal category $\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U)$, with order-reversing $*$-structure determined by canonical extension of the $*$-structure on the generators of $(W \otimes_{C^\infty_{\mathbb{R},U}} C^\infty_{\mathbb{C},U}, \mathrm{id} \otimes_{C^\infty_{\mathbb{R},U}} *)$.

Notice that there exists a functor F_U assigning to each $*$-object $(V,*)$ the free order-reversing $*$-monoid $F_U(V,*) := \bigoplus_{n\geq 0} V^{\otimes n}$ with order-reversing $*$-structure defined by the canonical extension of the $*$-structure on the generators $(V,*)$, whose right adjoint is the forgetful functor G_U assigning to an associative and unital $*$-algebra $(A,\mu,\eta,*)$ in $\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U)$ its underlying $*$-object $(A,*)$, (i.e. it forgets multiplication μ and unit η):

$$F_U \,:\, {}^*\mathbf{Obj}\big(\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U)\big) \rightleftarrows {}^*\mathbf{Alg}^\infty_{\mathbb{C}}(U) \,:\, G_U \quad . \tag{4.3.8}$$

(c) Thirdly, we need to implement the commutations relations determined by the antisymmetric $\mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U)$-morphism $\tau : W \otimes_{C^\infty_{\mathbb{R},U}} W \to C^\infty_{\mathbb{R},U}$. We do so via the following coequalizer in ${}^*\mathbf{Alg}^\infty_{\mathbb{C}}(U)$:

$$\mathfrak{CCR}_U(W,\tau) := \mathrm{colim}\Big(\, F_U L_U(W \otimes_{C^\infty_{\mathbb{R},U}} W) \overset{r_1}{\underset{r_2}{\rightrightarrows}} F_U L_U(W) \,\Big) \quad , \tag{4.3.9}$$

where r_1, r_2 are associated to (we suppress the subscripts $_U$ to simplify notations)

$$\begin{tikzcd}
W \otimes_{C^\infty_{\mathbb{R},U}} W \arrow[rrr, "\tilde{r}_1"] \arrow[d, "\mathrm{unit}\ L\dashv R"'] & & & RGFL(W) \\
RL(W \otimes_{C^\infty_{\mathbb{R},U}} W) \arrow[r, "\cong"'] & R\big(L(W) \otimes_{C^\infty_{\mathbb{C},U}} L(W)\big) \arrow[rr, "\mathrm{unit}\ F\dashv G"'] & & R\big(GFL(W) \otimes_{C^\infty_{\mathbb{C},U}} GFL(W)\big) \arrow[u, "R(\mu-\mu^{\mathrm{op}})"']
\end{tikzcd} \tag{4.3.10a}$$

and

$$\begin{tikzcd}
W \otimes_{C^\infty_{\mathbb{R},U}} W \arrow[rr, "\tilde{r}_2"] \arrow[d, "\tau"'] & & RGFL(W) \\
C^\infty_{\mathbb{R},U} \arrow[r, "\mathrm{unit}\ L\dashv R"'] & RL(C^\infty_{\mathbb{R},U}) \arrow[r, "\cong"'] & R(C^\infty_{\mathbb{C},U}, *) \arrow[u, "R(\mathrm{i}\,\eta)"']
\end{tikzcd} \tag{4.3.10b}$$

under the adjunctions in (4.3.7) and (4.3.8) and where $\mu^{(\mathrm{op})}$ and η denote the (opposite multiplication) and unit element of $FL(W)$ respectively. Since the coequalizer in (4.3.9) is functorial with respect to morphisms $\psi : (W, \tau) \to (W', \tau')$ in $\mathbf{PoVec}^\infty_{\mathbb{R}}(U)$, we can conclude that $\mathfrak{CCR}_U$ defines a functor $\mathfrak{CCR}_U : \mathbf{PoVec}^\infty_{\mathbb{R}}(U) \to {}^*\mathbf{Alg}^\infty_{\mathbb{C}}(U)$.

Remark 4.3.6. Notice that if $U = \{*\}$ is a point, the algebra $\mathfrak{CCR}_U(W, \tau)$ from (4.3.9) coincides with the CCR algebra from (4.3.1). △

To conclude the definition of the stack morphism $\mathfrak{CCR} : \mathbf{PoVec}^\infty_{\mathbb{R}} \to {}^*\mathbf{Alg}^\infty_{\mathbb{C}}$, we need to introduce, for every $\mathbf{Man}$-morphisms $h : U \to U'$, coherence isomorphisms (see Remark 4.1.2)

$$\begin{tikzcd}
\mathbf{PoVec}^\infty_{\mathbb{R}}(U') \arrow[r, "\mathfrak{CCR}_{U'}"] \arrow[d, "h^*"'] & {}^*\mathbf{Alg}^\infty_{\mathbb{C}}(U') \arrow[d, "h^*"] \arrow[dl, Rightarrow, "\mathfrak{CCR}_h"'] \\
\mathbf{PoVec}^\infty_{\mathbb{R}}(U) \arrow[r, "\mathfrak{CCR}_U"'] & {}^*\mathbf{Alg}^\infty_{\mathbb{C}}(U)
\end{tikzcd} \tag{4.3.11}$$

These can be built from the coherence isomorphisms for the left adjoint functors in (4.3.7) and (4.3.8)

$$\begin{tikzcd}
\mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U') \arrow[r, "L_{U'}"] \arrow[d, "h^*"'] & {}^*\mathbf{Obj}\big(\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U')\big) \arrow[r, "F_{U'}"] \arrow[d, "h^*"'] \arrow[dl, Rightarrow, "L_h"'] & {}^*\mathbf{Alg}^\infty_{\mathbb{C}}(U') \arrow[d, "h^*"] \arrow[dl, Rightarrow, "F_h"'] \\
\mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U) \arrow[r, "L_U"'] & {}^*\mathbf{Obj}\big(\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U)\big) \arrow[r, "F_U"'] & {}^*\mathbf{Alg}^\infty_{\mathbb{C}}(U)
\end{tikzcd} \tag{4.3.12}$$

by pasting them. Let us describe why this works more precisely. For $W \in \mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U')$, the isomorphism L_h is given by

$$\begin{aligned}
h^* L_{U'}(W) &= h^*\big(W \otimes_{C^\infty_{\mathbb{R},U'}} C^\infty_{\mathbb{C},U'}, \mathrm{id} \otimes *\big) \\
&\cong \Big(h^{-1}(W) \otimes_{h^{-1}(C^\infty_{\mathbb{R},U'})} h^{-1}(C^\infty_{\mathbb{C},U'}) \otimes_{h^{-1}(C^\infty_{\mathbb{C},U'})} C^\infty_{\mathbb{C},U}, \mathrm{id} \otimes * \otimes *\Big) \\
&\cong \Big(h^{-1}(W) \otimes_{h^{-1}(C^\infty_{\mathbb{R},U'})} C^\infty_{\mathbb{C},U}, \mathrm{id} \otimes *\Big) \\
&\cong \Big(h^{-1}(W) \otimes_{h^{-1}(C^\infty_{\mathbb{R},U'})} C^\infty_{\mathbb{R},U} \otimes_{C^\infty_{\mathbb{R},U}} C^\infty_{\mathbb{C},U}, \mathrm{id} \otimes \mathrm{id} \otimes *\Big) = L_U h^*(W) \quad ,
\end{aligned} \tag{4.3.13a}$$

and for $(V, *) \in {}^*\mathbf{Obj}\big(\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U')\big)$, the isomorphism F_h is given by

$$\begin{aligned}
h^* F_{U'}(V, *) = h^*\Big(\bigoplus_{n \geq 0} V^{\otimes_{C^\infty_{\mathbb{C},U'}} n}\Big) &\cong \bigoplus_{n \geq 0} h^*\Big(V^{\otimes_{C^\infty_{\mathbb{C},U'}} n}\Big) \\
&\cong \bigoplus_{n \geq 0} \big(h^*(V)\big)^{\otimes_{C^\infty_{\mathbb{C},U}} n} = F_U h^*(V, *) \quad ,
\end{aligned} \tag{4.3.13b}$$

where in the second step of (4.3.13b) we have used that h^* preserves coproducts being a left adjoint and in the third step of (4.3.13b) we have used the coherence isomorphisms of the involutive symmetric monoidal stack $\mathbf{Sh}_{C^\infty_{\mathbb{C}}}$ from Corollary 4.2.9. Pasting the natural isomorphisms in (4.3.12) defines a natural isomorphism $(FL)_h : h^* F_{U'} L_{U'} \Rightarrow F_U L_U h^*$. For every object $(W, \tau) \in \mathbf{PoVec}^\infty_{\mathbb{R}}(U')$, the associated isomorphism $h^* F_{U'} L_{U'}(W) \cong F_U L_U h^*(W)$ descends to the CCR algebras (4.3.9) and thereby defines the natural isomorphism $\mathfrak{CCR}_h$ in (4.3.11).

Proposition 4.3.7. *The construction above defines a stack morphism $\mathfrak{CCR} : \mathbf{PoVec}^\infty_{\mathbb{R}} \to {}^*\mathbf{Alg}^\infty_{\mathbb{C}}$.*

4.3.2 *Canonical anti-commutation relations stack morphism*

The aim of this subsection is to define a smooth refinement $\mathfrak{CAR} : \mathbf{IPVec}^\infty_{\mathbb{C}} \to {}^*\mathbf{Alg}^\infty_{\mathbb{C}}$ of the CAR functor $\mathfrak{CAR} : \mathbf{IPVec}_{\mathbb{C}} \to {}^*\mathbf{Alg}_{\mathbb{C}}$ (see its explicit description at the beginning of this section). In order to achieve this goal we proceed similarly to Subsection 4.3.1. In particular, we begin by providing a smooth refinement $\mathbf{IPVec}^\infty_{\mathbb{C}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ of the category $\mathbf{IPVec}_{\mathbb{C}}$ (see Definition 4.3.3).

Proposition 4.3.8. *The prestack $\mathbf{IPVec}^\infty_{\mathbb{C}} : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Cat}$ is given by the following data:*

(a) On objects*: It assigns to each manifold $U \in \mathbf{Man}$ the category $\mathbf{IPVec}^\infty_{\mathbb{C}}(U)$ whose objects are triples $(V, *, \langle \cdot, \cdot \rangle)$ consisting of a $*$-object $(V, *) \in {}^*\mathbf{Obj}(\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U))$ and a symmetric $*$-morphism $\langle \cdot, \cdot \rangle : (V, *) \otimes_{C^\infty_{\mathbb{C},U}} (V, *) \to (C^\infty_{\mathbb{C},U}, *)$, and whose morphisms $\psi : (V, *, \langle \cdot, \cdot \rangle) \to (V', *', \langle \cdot, \cdot \rangle')$ are ${}^*\mathbf{Obj}(\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U))$-morphisms $\psi : (V, *) \to (V', *')$ satisfying $\langle \cdot, \cdot \rangle' \circ (\psi \otimes_{C^\infty_{\mathbb{C},U}} \psi) = \langle \cdot, \cdot \rangle$.*

(b) On 1-morphisms*: It assigns to any $\mathbf{Man}$-morphism $h : U \to U'$ the functor*

$$h^* := \mathbf{IPVec}^\infty_{\mathbb{C}}(h) \ : \ \mathbf{IPVec}^\infty_{\mathbb{C}}(U') \longrightarrow \mathbf{IPVec}^\infty_{\mathbb{C}}(U) \tag{4.3.14}$$

assigning to each $(V, *, \langle \cdot, \cdot \rangle) \in \mathbf{IPVec}^\infty_{\mathbb{C}}(U')$ the object in $\mathbf{IPVec}^\infty_{\mathbb{C}}(U)$ determined by $h^*(V, *) \in {}^*\mathbf{Obj}(\mathbf{Sh}_{C^\infty_{\mathbb{C}}}(U))$ and

$$\begin{aligned} (h^*(V,*)) \otimes_{C^\infty_{\mathbb{C},U}} (h^*(V,*)) \ &\cong \ h^*\big((V,*) \otimes_{C^\infty_{\mathbb{C},U'}} (V,*)\big) \\ &\xrightarrow{h^*\langle \cdot,\cdot \rangle} h^*(C^\infty_{\mathbb{C},U'}, *) \ \cong \ (C^\infty_{\mathbb{C},U}, *) \quad , \end{aligned} \tag{4.3.15}$$

where we have used the coherence isomorphisms of the involutive symmetric monoidal stack $\mathbf{Sh}_{C^\infty_{\mathbb{C}}}$ from Corollary 4.2.9.

This prestack is a stack.

Proof. Analogous to the proof of Proposition 4.3.5. □

To define the stack morphism $\mathfrak{CAR} : \mathbf{IPVec}^\infty_{\mathbb{C}} \to {}^*\mathbf{Alg}^\infty_{\mathbb{C}}$ we proceed similarly to Subsection 4.3.1, defining its component functors $\mathfrak{CAR}_U : \mathbf{IPVec}^\infty_{\mathbb{C}}(U) \to {}^*\mathbf{Alg}^\infty_{\mathbb{C}}(U)$ and coherence isomorphisms.

Proposition 4.3.9. *The following data defines a stack morphism $\mathfrak{CAR} : \mathbf{IPVec}^\infty_{\mathbb{C}} \to {}^*\mathbf{Alg}^\infty_{\mathbb{C}}$:*

(a) Functors*: For all manifolds $U \in \mathbf{Man}$, functors $\mathfrak{CAR}_U : \mathbf{IPVec}^\infty_{\mathbb{C}}(U) \to {}^*\mathbf{Alg}^\infty_{\mathbb{C}}(U)$, defined on $(V, *, \langle \cdot, \cdot \rangle) \in \mathbf{IPVec}^\infty_{\mathbb{C}}(U)$ by the coequalizer in ${}^*\mathbf{Alg}^\infty_{\mathbb{C}}(U)$*

$$\mathfrak{CAR}_U\big(V, *, \langle \cdot, \cdot \rangle\big) := \mathrm{colim}\Big(F_U\big((V,*) \otimes_{C^\infty_{\mathbb{C},U}} (V,*)\big) \underset{s_2}{\overset{s_1}{\rightrightarrows}} F_U\big(V,*\big) \Big) \quad , \tag{4.3.16}$$

where the relations s_1, s_2 are associated to (we suppress the subscripts $_U$ to simplify notations)

$$\begin{tikzcd} (V,*) \otimes_{C^\infty_{\mathbb{C},U}} (V,*) \arrow[r, "\tilde{s}_1"] \arrow[d, "\text{unit } F \dashv G"'] & GF(V,*) \\ GF(V,*) \otimes_{C^\infty_{\mathbb{C},U}} GF(V,*) \arrow[ur, "\mu + \mu^{\mathrm{op}}"'] & \end{tikzcd} \tag{4.3.17a}$$

and

$$\begin{tikzcd} (V,*) \otimes_{C^\infty_{\mathbb{C},U}} (V,*) \arrow[r, "\tilde{s}_2"] \arrow[d, "\langle\cdot,\cdot\rangle"'] & GF(V,*) \\ (C^\infty_{\mathbb{C},U}, *) \arrow[ur, "\eta"'] & \end{tikzcd} \tag{4.3.17b}$$

under the adjunction in (4.3.8).

(b) Coherences*: for any $\mathbf{Man}$-morphisms $h : U \to U'$, coherence isomorphisms*

$$\begin{tikzcd} \mathbf{IPVec}^\infty_{\mathbb{C}}(U') \arrow[r, "\mathfrak{CAR}_{U'}"] \arrow[d, "h^*"'] & {}^*\mathbf{Alg}^\infty_{\mathbb{C}}(U') \arrow[d, "h^*"] \arrow[dl, Rightarrow, "\mathfrak{CAR}_h"'] \\ \mathbf{IPVec}^\infty_{\mathbb{C}}(U) \arrow[r, "\mathfrak{CAR}_U"'] & {}^*\mathbf{Alg}^\infty_{\mathbb{C}}(U) \end{tikzcd} \tag{4.3.18}$$

defined similarly to those in Subsection 4.3.1.

4.4 ILLUSTRATION THROUGH FREE THEORIES

The goal of this section is to introduce smooth refinements of the Fermionic and Bosonic free field theories discussed in [BGP07, BG12]. Roughly speaking, we introduce:

BOSONIC MODELS: a stack morphism

$$\begin{tikzcd} \mathbf{Loc}^\infty_1 \arrow[rr, "\mathfrak{A}^{\mathrm{b}}"] \arrow[dr, "\mathfrak{L}^{\mathrm{b}}"'] & & {}^*\mathbf{Alg}^\infty_{\mathbb{C}} \\ & \mathbf{PoVec}^\infty_{\mathbb{R}} \arrow[ur, "\mathfrak{CCR}"'] & \end{tikzcd} \tag{4.4.1}$$

defined by pre-composing the stack morphism $\mathfrak{L}^{\mathsf{b}} : \mathbf{Loc}_1^\infty \to \mathbf{PoVec}_{\mathbb{R}}^\infty$ (see Subsection 4.4.2), assigning linear observables with their Poisson structure determined by a suitable smooth refinement of the concept of retarded/advanced Green operators $G^\pm$ (see Subsection 4.4.1), with the CCR stack morphism $\mathfrak{CCR} : \mathbf{PoVec}_{\mathbb{R}}^\infty \to {}^*\mathbf{Alg}_{\mathbb{C}}^\infty$ from Proposition 4.3.7.

FERMIONIC MODELS: a stack morphism

$$\begin{tikzcd}
\mathbf{Loc}_1^\infty \arrow[rr, "\mathfrak{A}^{\mathsf{f}}"] \arrow[dr, "\mathfrak{L}^{\mathsf{f}}"'] & & {}^*\mathbf{Alg}_{\mathbb{C}}^\infty \\
& \mathbf{IPVec}_{\mathbb{C}}^\infty \arrow[ur, "\mathfrak{CAR}"'] &
\end{tikzcd} \tag{4.4.2}$$

defined by pre-composing the stack morphism $\mathfrak{L}^{\mathsf{f}} : \mathbf{Loc}_1^\infty \to \mathbf{IPVec}_{\mathbb{C}}^\infty$ (see Subsection 4.4.3), assigning linear observables with their bilinear product determined by a suitable smooth refinement of the concept of retarded/advanced Green operators $G^\pm$ (see Subsection 4.4.1), with the CAR stack morphism $\mathfrak{CAR} : \mathbf{IPVec}_{\mathbb{C}}^\infty \to {}^*\mathbf{Alg}_{\mathbb{C}}^\infty$ from Proposition 4.3.9.

More precisely:

(a) In Subsection 4.4.1 we introduce smooth refinements of retarded and advanced Green operators $G^\pm$ and give concrete examples of these constructions. More precisely, we study the retarded and advanced Green operators for a differential operator $\tilde{P}_{(\pi,E)} : C^\infty_{\pi\times\mathrm{id}} \to C^\infty_{\pi\times\mathrm{id}}$ describing a smooth U-family of 1-dimensional scalar fields and for a differential operator $D_{(\pi,E)} : C^\infty_\pi \otimes \mathbb{C}^2 \to C^\infty_\pi \otimes \mathbb{C}^2$ representing a smooth generalization of the massless Dirac field. Moreover, we discuss smoothly parametrized initial value problems.

(b) In Subsection 4.4.2 we introduce a smooth $\tilde{U}$-family of smooth AQFTs (in the sense of (4.2.17)) describing a family of 1-dimensional massive scalar fields with smoothly varying mass parameter $m \in C^\infty(\tilde{U}, \mathbb{R}^{>0})$, i.e. a stack morphism $\mathfrak{B} = \mathsf{Map}(\underline{\tilde{U}}, \mathfrak{CCR}) \circ \mathfrak{W} : \mathbf{Loc}_1^\infty \to \mathsf{Map}(\underline{\tilde{U}}, {}^*\mathbf{Alg}_{\mathbb{C}}^\infty)$

$$\begin{tikzcd}
\mathbf{Loc}_1^\infty \arrow[rr, "\mathfrak{B}"] \arrow[dr, "\mathfrak{W}"'] & & \mathsf{Map}\big(\underline{\tilde{U}}, {}^*\mathbf{Alg}_{\mathbb{C}}^\infty\big) \\
& \mathsf{Map}\big(\underline{\tilde{U}}, \mathbf{PoVec}_{\mathbb{R}}^\infty\big) \arrow[ur, "\mathsf{Map}(\underline{\tilde{U}}, \mathfrak{CCR})"'] &
\end{tikzcd} \quad , \tag{4.4.3}$$

where $\mathfrak{W}$, representing a "smooth $\tilde{U}$-parametrized refinement of the functor associating Poisson vector space of linear observables", reduces to the stack morphism $\mathfrak{L}^{\mathsf{b}}$ from (4.4.1) when $U = \{*\}$ (therefore also $\mathfrak{B}$ reduces to $\mathfrak{A}^{\mathsf{b}}$ in this case).

(c) In Subsection 4.4.3 we introduce the smooth 1-dimensional massless Dirac field as a smooth AQFT, i.e. the stack morphism $\mathfrak{A}^{\mathsf{f}} = \mathfrak{CAR} \circ \mathfrak{L}^{\mathsf{f}}$ from (4.4.2), and verify that its global $U(1)$-symmetry is realized in terms of smooth automorphisms in the sense of (4.2.19).

4.4.1 *Green operators, solutions and initial data*

In this subsection we introduce smooth analogues of retarded and advanced Green operators $G^{\pm}$ (see [BDH13] for an excellent introduction to these topics). More precisely, we will begin by introducing an object

$$C^{\infty}_{\pi} \otimes \mathbb{K}^n \,\in\, \mathbf{Sh}_{C^{\infty}_{\mathbb{K}}}(U) \tag{4.4.4}$$

representing the field configuration space of a vector-valued field on the smooth family of spacetimes $(\pi : M \to U, E) \in \mathbf{Loc}^{\infty}_1(U)$ and will consider as equations of motion $\mathbf{Sh}_{C^{\infty}_{\mathbb{K}}}(U)$-morphisms $P : C^{\infty}_{\pi} \otimes \mathbb{K}^n \to C^{\infty}_{\pi} \otimes \mathbb{K}^n$ determined by *vertical differential operators* on $\pi : M \to U$. We will then introduce a concept of *retarded/advanced Green operators* $G^{\pm} : C^{\infty}_{\pi\,\mathrm{vpc/vfc}} \otimes \mathbb{K}^n \to C^{\infty}_{\pi\,\mathrm{vpc/vfc}} \otimes \mathbb{K}^n$ for any $\mathbf{Sh}_{C^{\infty}_{\mathbb{K}}}(U)$-morphism $P : C^{\infty}_{\pi} \otimes \mathbb{K}^n \to C^{\infty}_{\pi} \otimes \mathbb{K}^n$, and by giving some examples of such constructions.

The object $C^{\infty}_{\pi} \in \mathbf{Sh}_{C^{\infty}_{\mathbb{R}}}(U)$, that we interpret as the field configuration space of a real scalar field on the smooth family of spacetimes $(\pi : M \to U, E) \in \mathbf{Loc}^{\infty}_1(U)$, is the sheaf of sets

$$C^{\infty}_{\pi} \,:\, \mathbf{Open}(U)^{\mathrm{op}} \;\longrightarrow\; \mathbf{Set} \tag{4.4.5}$$

defined by the following data:

(a) *On objects*: It associates to every $U' \subseteq U$ the set $C^{\infty}(M|_{U'})$.

(b) *On* 1*-morphisms*: It associates to every $\mathbf{Open}(U)$-morphism $\iota^{U''}_{U'} : U' \subseteq U''$, i.e. to every inclusion $U' \subseteq U''(\subseteq U)$, the restriction map $C^{\infty}_{\pi}(\iota^{U''}_{U'}) : C^{\infty}_{\pi}(U'') \to C^{\infty}_{\pi}(U')$.

endowed with the $C^{\infty}_{\mathbb{R},U}$-module structure induced by pullback of functions.

As mentioned earlier, we interpret

$$C^{\infty}_{\pi} \otimes \mathbb{K}^n \,\in\, \mathbf{Sh}_{C^{\infty}_{\mathbb{K}}}(U) \tag{4.4.6}$$

as the field configuration space of a vector-valued field on the smooth family of spacetimes $(\pi : M \to U, E) \in \mathbf{Loc}^{\infty}_1(U)$, where $n \in \mathbb{Z}_{\geq 1}$ denotes the number of field components.

Definition 4.4.1. A *vertical differential operator* on $\pi : M \to U$ is a differential operator on M that differentiates only along fibers.

Remark 4.4.2. Notice that every vertical differential operator on $\pi : M \to U$ determines an equation of motion given by the $\mathbf{Sh}_{C^{\infty}_{\mathbb{K}}}(U)$-morphism $P : C^{\infty}_{\pi} \otimes \mathbb{K}^n \to C^{\infty}_{\pi} \otimes \mathbb{K}^n$. △

In order to define Green operators for a $\mathbf{Sh}_{C^{\infty}_{\mathbb{K}}}(U)$-morphism $P : C^{\infty}_{\pi} \otimes \mathbb{K}^n \to C^{\infty}_{\pi} \otimes \mathbb{K}^n$ determined by a vertical differential operator, we need to introduce subsheaves $C^{\infty}_{\pi\,\mathrm{vc}}, C^{\infty}_{\pi\,\mathrm{vpc}}, C^{\infty}_{\pi\,\mathrm{vfc}} \in \mathbf{Sh}_{C^{\infty}_{\mathbb{R}}}(U)$ of the sheaf of modules C^{∞}_{π}, representing functions with restrictions on their support.

Definition 4.4.3. Let $(\pi : M \to U, E) \in \mathbf{Loc}_1^\infty(U)$ be a smooth U-family of 1-dimensional spacetimes and let $S \subseteq M$ be a subset of the total space. We denote by $J_{\mathrm{v}}^{\pm}(S)$ the *vertical future / past* of S i.e. the set of points that can be reached from S via future/past directed causal curves with respect to the orientation given by E.

Definition 4.4.4. Let $(\pi : M \to U, E) \in \mathbf{Loc}_1^\infty(U)$ be a smooth U-family of 1-dimensional spacetimes and let $U' \subseteq U$ be an open subset of U. We say that a function $\phi \in C_\pi^\infty(U') := C^\infty(M|_{U'})$ is *vertically future* (*past*) *compactly supported* if there is a section $\sigma : U' \to M|_{U'}$ such that the support of ϕ is contained in $J_{\mathrm{v}}^-(\sigma(U'))$ $(J_{\mathrm{v}}^+(\sigma(U')))$. Moreover, we say that a function ϕ is *vertically compactly supported* if it is both future and past compactly supported, i.e. if there exist two sections $\sigma_1 : U' \to M|_{U'}$, $\sigma_2 : U' \to M|_{U'}$, such that the support of ϕ is contained in $J_{\mathrm{v}}^-(\sigma_1(U')) \cap J_{\mathrm{v}}^+(\sigma_2(U'))$.

We denote the sets of vertically future/past compactly supported functions $\phi \in C_\pi^\infty(U')$ with the symbol $\widetilde{C}^\infty_{\pi\,\mathrm{vf/pc}}(U')$ and the set of vertically compactly supported functions by $\widetilde{C}^\infty_{\pi\,\mathrm{vc}}(U')$.

(Notice that our fiber bundles admit sections because the fibers are open intervals, see e.g. [Ste51, Sections 12.2 and 6.7].)

Definition 4.4.4 enables us to define the following *pre*sheaves associated to a U-family of 1-dimensional globally hyperbolic Lorentzian manifolds $(\pi : M \to U, E) \in \mathbf{Loc}_1^\infty(U)$:

(a) The presheaf $\widetilde{C}^\infty_{\pi\,\mathrm{vc}}$ of vertically compactly supported functions, i.e. the subpresheaf of C_π^∞ assigning to each $U' \subseteq U$ the set $\widetilde{C}^\infty_{\pi\,\mathrm{vc}}(U')$.

(b) The presheaf $\widetilde{C}^\infty_{\pi\,\mathrm{vpc}}$ of vertically past compactly supported functions, i.e. the subpresheaf of C_π^∞ assigning to each $U' \subseteq U$ the set $\widetilde{C}^\infty_{\pi\,\mathrm{vpc}}(U')$.

(c) The presheaf $\widetilde{C}^\infty_{\pi\,\mathrm{vfc}}$ of vertically future compactly supported functions, i.e. the subpresheaf of C_π^∞ assigning to each $U' \subseteq U$ the set $\widetilde{C}^\infty_{\pi\,\mathrm{vfc}}(U')$.

Definition 4.4.5. We denote by $C^\infty_{\pi\,\mathrm{vfc}}$, $C^\infty_{\pi\,\mathrm{vpc}}$ and $C^\infty_{\pi\,\mathrm{vc}}$ the sheafification (see the discussion at the end of Subsection 4.2.3) $\widetilde{C}^{\infty,+}_{\pi\,\mathrm{vfc}}$ of the presheaf $\widetilde{C}^\infty_{\pi\,\mathrm{vfc}}$, the sheafification $\widetilde{C}^{\infty,+}_{\pi\,\mathrm{vpc}}$ of the presheaf $\widetilde{C}^\infty_{\pi\,\mathrm{vpc}}$ and the sheafification $\widetilde{C}^{\infty,+}_{\pi\,\mathrm{vc}}$ of the presheaf $\widetilde{C}^\infty_{\pi\,\mathrm{vc}}$ respectively.

Remark 4.4.6. Since they are obtained under sheafification, we can interpret the sheaves from Definition 4.4.5 as the "local datum of their presheaves". For instance, the sheaf $C^\infty_{\pi\,\mathrm{vfc}}$ of vertically future compactly supported functions evaluated at $U' \subseteq U$ is the subset $C^\infty_{\pi\,\mathrm{vfc}}(U') \subseteq C^\infty_\pi(U')$ of functions ϕ that admit for every $x \in U'$ an open neighbourhood $U_x \subseteq U'$ of x such that when restricted to U_x are vertically future compact. Analogous statements hold for the sheaves $C^\infty_{\pi\,\mathrm{vpc}}$ and $C^\infty_{\pi\,\mathrm{vc}}$. △

We can now introduce the advanced and retarded Green operators.

Definition 4.4.7. Let $P : C_\pi^\infty \otimes \mathbb{K}^n \to C_\pi^\infty \otimes \mathbb{K}^n$ be a $\mathbf{Sh}_{C^\infty_{\mathbb{K}}}(U)$-morphism determined by a vertical differential operator on $\pi : M \to U$ (see Definition 4.4.1) and suppose that the restrictions $P : C^\infty_{\pi\,\mathrm{vpc/vfc}} \otimes \mathbb{K}^n \to C^\infty_{\pi\,\mathrm{vpc/vfc}} \otimes \mathbb{K}^n$ of P to the subsheaves of vertically past/future compactly supported functions are invertible with inverses $G^{\pm} : C^{\infty}_{\pi\,\mathrm{vpc/vfc}} \otimes \mathbb{K}^n \to C^{\infty}_{\pi\,\mathrm{vpc/vfc}} \otimes \mathbb{K}^n$. Then, we say that $G^{\pm} : C^{\infty}_{\pi\,\mathrm{vpc/vfc}} \otimes \mathbb{K}^n \to C^{\infty}_{\pi\,\mathrm{vpc/vfc}} \otimes \mathbb{K}^n$ are *retarded/advanced Green operators* for P if for each open subset $U' \subseteq U$ and $\varphi \in C^{\infty}_{\pi\,\mathrm{vpc/vfc}}(U') \otimes \mathbb{K}^n$, we have $\mathrm{supp}(G^{\pm}\varphi) \subseteq J^{\pm}_{\mathrm{v}}(\mathrm{supp}(\varphi))$.

If P admits both retarded and advanced Green operators, we call the $\mathbf{Sh}_{C^{\infty}_{\mathbb{K}}}(U)$-morphism $G := G^+ - G^- : C^{\infty}_{\pi\,\mathrm{vc}} \otimes \mathbb{K}^n \to C^{\infty}_{\pi} \otimes \mathbb{K}^n$ the *causal propagator*.

The usual exact sequence (see [BGP07, BD15]) for P and G generalizes to our context.

Remark 4.4.8. Before introducing the aforementioned exact sequence we would like to point out that an exact sequence of sheaves in $\mathbf{Sh}_{C^{\infty}_{\mathbb{K}}}(U)$ might not be an exact sequence of presheaves. △

Proposition 4.4.9. *Let $P : C^{\infty}_{\pi} \otimes \mathbb{K}^n \to C^{\infty}_{\pi} \otimes \mathbb{K}^n$ be a $\mathbf{Sh}_{C^{\infty}_{\mathbb{K}}}(U)$-morphism determined by a vertical differential operator on $\pi : M \to U$ that admits retarded and advanced Green operators $G^{\pm} : C^{\infty}_{\pi\,\mathrm{vpc/vfc}} \otimes \mathbb{K}^n \to C^{\infty}_{\pi\,\mathrm{vpc/vfc}} \otimes \mathbb{K}^n$. Then, the associated sequence*

$$0 \longrightarrow C^{\infty}_{\pi\,\mathrm{vc}} \otimes \mathbb{K}^n \xrightarrow{P} C^{\infty}_{\pi\,\mathrm{vc}} \otimes \mathbb{K}^n \xrightarrow{G} C^{\infty}_{\pi} \otimes \mathbb{K}^n \xrightarrow{P} C^{\infty}_{\pi} \otimes \mathbb{K}^n \longrightarrow 0 \tag{4.4.7}$$

in $\mathbf{Sh}_{C^{\infty}_{\mathbb{K}}}(U)$ is exact. Even stronger, the corresponding sequence of presheaves is exact, i.e. for each open subset $U' \subseteq U$, the sequence

$$0 \longrightarrow C^{\infty}_{\pi\,\mathrm{vc}}(U') \otimes \mathbb{K}^n \xrightarrow{P} C^{\infty}_{\pi\,\mathrm{vc}}(U') \otimes \mathbb{K}^n \xrightarrow{G} C^{\infty}_{\pi}(U') \otimes \mathbb{K}^n \xrightarrow{P} C^{\infty}_{\pi}(U') \otimes \mathbb{K}^n \longrightarrow 0 \tag{4.4.8}$$

of $C^{\infty}_{\mathbb{K}}(U')$-modules is exact.

Proof. We prove the second (stronger) statement, which implies the first. Let $U' \subseteq U$ be any open subset. To prove exactness at the first term, consider any $\varphi \in C^{\infty}_{\pi\,\mathrm{vc}}(U') \otimes \mathbb{K}^n$ such that $P\varphi = 0$ and note that $0 = G^{\pm}P\varphi = \varphi$ by Definition 4.4.7. For the second term, let $\varphi \in C^{\infty}_{\pi\,\mathrm{vc}}(U') \otimes \mathbb{K}^n$ be such that $G\varphi = 0$. Then $G^+\varphi = G^-\varphi =: \rho \in C^{\infty}_{\pi\,\mathrm{vc}}(U') \otimes \mathbb{K}^n$ because of the support properties of Green operators and the definition of vertically compact support. Hence, $P\rho = PG^{\pm}\varphi = \varphi$ by Definition 4.4.7.

For the third term, let $\Phi \in C^{\infty}_{\pi}(U') \otimes \mathbb{K}^n$ be such that $P\Phi = 0$. Choosing two non-intersecting sections $\sigma_{\pm} : U' \to M|_{U'}$ such that σ_+ lies in the vertical future of σ_-, we obtain an open cover $\{M|_{U'} \setminus J^-_{\mathrm{v}}(\sigma_-(U')),\, M|_{U'} \setminus J^+_{\mathrm{v}}(\sigma_+(U'))\}$ of $M|_{U'}$. Choosing a partition of unity subordinate to this cover, we can decompose $\Phi = \Phi_+ + \Phi_-$ with $\Phi_{\pm} \in C^{\infty}_{\pi\,\mathrm{vpc/vfc}}(U') \otimes \mathbb{K}^n$. Then $\rho := P\Phi_+ = -P\Phi_- \in C^{\infty}_{\pi\,\mathrm{vc}}(U') \otimes \mathbb{K}^n$ is vertically compactly supported and $G\rho = G^+\rho - G^-\rho = G^+P\Phi_+ + G^-P\Phi_- = \Phi_+ + \Phi_- = \Phi$ by Definition 4.4.7.

For the last term, take any $\Phi \in C^{\infty}_{\pi}(U') \otimes \mathbb{K}^n$ and decompose as before $\Phi = \Phi_+ + \Phi_-$ with $\Phi_{\pm} \in C^{\infty}_{\pi\,\mathrm{vpc/vfc}}(U') \otimes \mathbb{K}^n$. Defining $\rho := G^+\Phi_+ + G^-\Phi_-$, we obtain $P\rho = PG^+\Phi_+ + PG^-\Phi_- = \Phi_+ + \Phi_- = \Phi$ by Definition 4.4.7. □

Remark 4.4.10. Proposition 4.4.9 tells us that the cokernel sheaf

$$\tfrac{C^\infty_{\pi\,\mathrm{vc}}\otimes\mathbb{K}^n}{P(C^\infty_{\pi\,\mathrm{vc}}\otimes\mathbb{K}^n)} \;:=\; \mathrm{coker}\big(P : C^\infty_{\pi\,\mathrm{vc}}\otimes\mathbb{K}^n \to C^\infty_{\pi\,\mathrm{vc}}\otimes\mathbb{K}^n\big) \;\in\; \mathbf{Sh}_{C^\infty_{\mathbb{K}}}(U) \tag{4.4.9a}$$

is isomorphic to the presheaf quotient $\mathrm{coker}\big(P : C^\infty_{\pi\,\mathrm{vc}}\otimes\mathbb{K}^n \to C^\infty_{\pi\,\mathrm{vc}}\otimes\mathbb{K}^n\big) \in \mathbf{PSh}_{C^\infty_{\mathbb{K}}}(U)$, i.e.

$$\tfrac{C^\infty_{\pi\,\mathrm{vc}}\otimes\mathbb{K}^n}{P(C^\infty_{\pi\,\mathrm{vc}}\otimes\mathbb{K}^n)}(U') \;=\; C^\infty_{\pi\,\mathrm{vc}}(U')\otimes\mathbb{K}^n/P(C^\infty_{\pi\,\mathrm{vc}}(U')\otimes\mathbb{K}^n) \quad , \tag{4.4.9b}$$

for every open subset $U'\subseteq U$. Furthermore, this sheaf is isomorphic to the solution sheaf $\mathsf{Sol}_\pi := \ker(P : C^\infty_\pi\otimes\mathbb{K}^n \to C^\infty_\pi\otimes\mathbb{K}^n) \in \mathbf{Sh}_{C^\infty_{\mathbb{K}}}(U)$ via the causal propagator

$$G \;:\; \tfrac{C^\infty_{\pi\,\mathrm{vc}}\otimes\mathbb{K}^n}{P(C^\infty_{\pi\,\mathrm{vc}}\otimes\mathbb{K}^n)} \xrightarrow{\;\cong\;} \mathsf{Sol}_\pi \tag{4.4.10}$$

△

Let us now study some examples of Green operators.

Example 4.4.11. In this example we study the retarded and advanced Green operators for a vertical differential operator $P : C^\infty_\pi \to C^\infty_\pi$ describing a smooth U-family of 1-dimensional scalar fields (harmonic oscillators) with a fixed mass/frequency parameter m, on time intervals whose geometry depends on the point $x\in U$.

Let $(\pi : M\to U, E)\in \mathbf{Loc}^\infty_1(U)$ be a smooth U-family of 1-dimensional spacetimes, let $m\in(0,\infty)$ be a fixed parameter, let $\mathrm{d_v}$ denote the vertical de Rham differential on $\pi : M\to U$ and let $*_\mathrm{v}$ denote the vertical Hodge operator induced by $E\in\Omega^1_\mathrm{v}(M)$. As equation of motion we take the vertical differential operator

$$P_{(\pi,E)} \;:=\; *_\mathrm{v}\,\mathrm{d_v}\,*_\mathrm{v}\,\mathrm{d_v} + m^2 \;:\; C^\infty_\pi \;\longrightarrow\; C^\infty_\pi \quad . \tag{4.4.11}$$

In order to prove that (4.4.11) admits retarded and advanced Green operators, we consider an open cover $\{U_\alpha : \alpha\in A\}$ that locally trivializes the bundle $\pi : M\to U$, i.e. such that the restricted fiber bundles $M|_{U_\alpha}\to U_\alpha$ admit trivializations $M|_{U_\alpha}\cong \mathbb{R}\times U_\alpha$, and show that each $\mathcal{P}_{\alpha,(\pi,E)} : C^\infty_\pi|_{U_\alpha}\to C^\infty_\pi|_{U_\alpha}$ admits retarded and advanced Green operators $G^\pm_\alpha : C^\infty_{\pi\,\mathrm{vpc/vfc}}|_{U_\alpha}\to C^\infty_{\pi\,\mathrm{vpc/vfc}}|_{U_\alpha}$. This is sufficient, because, since the uniqueness of the advanced and retarded Green operators implies that the G_αs form naturally a morphism in the descent category $\mathbf{Sh}_{C^\infty_{\mathbb{K}}}(\{U_\alpha\subseteq U\})$ (see Definition 4.1.3), and since $\mathbf{Sh}_{C^\infty_{\mathbb{K}}}$ is a stack, we can "glue" the G_αs to obtain retarded and advanced Green operators $G^\pm : C^\infty_{\pi\,\mathrm{vpc/vfc}}\to C^\infty_{\pi\,\mathrm{vpc/vfc}}$ on all U.

With respect to this trivializing cover $\{U_\alpha : \alpha\in A\}$, we have that $E|_{U_\alpha}\cong\rho\,\mathrm{d}t$, where $\rho\in C^\infty(\mathbb{R}\times U_\alpha,\mathbb{R}^{>0})$ is a positive function and where $t\in\mathbb{R}$ is a time coordinate on $\mathbb{R}$. In particular, P_α reads as $P_\alpha = \rho^{-1}\,\partial_t\,\rho^{-1}\partial_t + m^2$. In order to simplify the expression for P_α we introduce a new time coordinate $T(t,x)$ (notice the dependence on $x\in U_\alpha$) such that $\mathrm{d_v}T = \rho\,\mathrm{d}t$. More precisely, the equation of motion operator reads as $P_\alpha = \partial_T^2 + m^2$, which admits the following retarded/advanced Green operator

$$(G^\pm_\alpha\,\varphi)(T,x) \;=\; \int_{T(\mp\infty,x)}^{T} m^{-1}\sin\big(m\,(T-S)\big)\,\varphi(S,x)\,\mathrm{d}S \quad . \tag{4.4.12}$$

Notice that the integral in (4.4.12) exists, because $\varphi \in C^\infty_{\pi\,\mathrm{vpc/vfc}}|_{U_\alpha}$ is vertically past/future compactly supported, and depends smoothly on both $T \in (T(-\infty, x), T(\infty, x))$ and $x \in U_\alpha$.

$\triangledown$

Example 4.4.12. This example constitutes a generalization to "smooth mass parameters" $m \in C^\infty(\tilde{U}, \mathbb{R}^{>0})$ of Example 4.4.11. We will harness it to introduce the smooth $\tilde{U}$-family of smooth 1-dimensional algebraic quantum field theories presented in Subsection 4.4.2.

Given a manifold U and a smooth U-family of 1-dimensional spacetimes $(\pi : M \to U, E) \in \mathbf{Loc}_1^\infty(U)$, we introduce the object $(\pi \times \mathrm{id} : M \times \tilde{U} \to U \times \tilde{U}, \mathrm{pr}_M^*(E)) \in \mathbf{Loc}_1^\infty(U \times \tilde{U})$ and the vertical differential operator

$$\tilde{P}_{(\pi,E)} := *_\mathrm{v}\, \mathrm{d}_\mathrm{v} *_\mathrm{v} \mathrm{d}_\mathrm{v} + \mathrm{pr}_{\tilde{U}}^*(m^2) \,:\, C^\infty_{\pi\times\mathrm{id}} \longrightarrow C^\infty_{\pi\times\mathrm{id}} \quad , \tag{4.4.13}$$

where $\mathrm{pr}_M : M \times \tilde{U} \to M$ and $\mathrm{pr}_{\tilde{U}} : M \times \tilde{U} \to \tilde{U}$ denote the projection maps. Analogously to Example 4.4.11, we proceed by choosing an open cover $\{U_\alpha : \alpha \in A\}$ that trivializes $(\pi \times \mathrm{id} : M \times \tilde{U})$, i.e $(M \times \tilde{U})|_{U_\alpha \times \tilde{U}} \cong \mathbb{R} \times U_\alpha \times \tilde{U}$ for every U_α, and prove the existence of Green operators for $\tilde{P}_\alpha : C^\infty_{\pi\times\mathrm{id}}|_{U_\alpha} \to C^\infty_{\pi\times\mathrm{id}}|_{U_\alpha}$. Again, in complete analogy with Example 4.4.11, we choose the time coordinate T obtained by solving $\mathrm{d}_\mathrm{v} T = \rho\, \mathrm{d}t = \mathrm{pr}_M^*(E)|_{U_\alpha \times \tilde{U}}$ and derive $\tilde{P}_\alpha = \partial_T^2 + m^2(\tilde{x})$ (notice the dependence on $\tilde{x} \in \tilde{U}$). This operator admits a retarded/advanced Green operator given by

$$(\tilde{G}^\pm_\alpha \varphi)(T, x, \tilde{x}) = \int_{T(\mp\infty, x)}^{T} m(\tilde{x})^{-1} \sin\big(m(\tilde{x})\,(T-S)\big)\, \varphi(S, x, \tilde{x})\, \mathrm{d}S \quad , \tag{4.4.14}$$

for all $\varphi \in C^\infty_{\pi\times\mathrm{id}\,\mathrm{vpc/vfc}}|_{U_\alpha \times \tilde{U}}$. $\triangledown$

Example 4.4.13. In this example we describe the advanced and retarded Green operators associated to the smooth 1-dimensional massless Dirac field.

Let $U \in \mathbf{Man}$ be a manifold and let $(\pi : M \to U, E) \in \mathbf{Loc}_1^\infty(U)$ be a smooth U-family of 1-dimensional spacetimes. We introduce the massless Dirac vertical differential operator

$$D_{(\pi,E)} := \begin{pmatrix} \mathrm{i} *_\mathrm{v} \mathrm{d}_\mathrm{v} & 0 \\ 0 & -\mathrm{i} *_\mathrm{v} \mathrm{d}_\mathrm{v} \end{pmatrix} : C^\infty_\pi \otimes \mathbb{C}^2 \longrightarrow C^\infty_\pi \otimes \mathbb{C}^2 \quad , \tag{4.4.15}$$

where $\mathrm{i} \in \mathbb{C}$ is the imaginary unit and the elements $\binom{\Psi}{\overline{\Psi}} \in C^\infty_\pi \otimes \mathbb{C}^2$ should be interpreted as the Dirac field Ψ and its Dirac conjugate $\overline{\Psi}$.

We then proceed similarly to Examples 4.4.11 and 4.4.12: we consider a trivializing cover $\{U_\alpha \subseteq U\}$ and the local time coordinate T, obtaining the following formula for the local Dirac operators

$$D_\alpha = \begin{pmatrix} \mathrm{i}\,\partial_T & 0 \\ 0 & -\mathrm{i}\,\partial_T \end{pmatrix} \tag{4.4.16}$$

Therefore, the retarded/advanced Green operators associated to the D_αs are given by

$$\left(S^\pm_\alpha \binom{\psi}{\overline{\psi}}\right)(T, x) = \int_{T(\mp\infty, x)}^{T} \begin{pmatrix} -\mathrm{i}\,\psi(S, x) \\ \mathrm{i}\,\overline{\psi}(S, x) \end{pmatrix} \mathrm{d}S \quad , \tag{4.4.17}$$

where $T(\mp\infty, x)$ was defined in Example 4.4.11.

$\triangledown$

We conclude this subsection with a few remarks on smoothly parametrized initial value problems. In particular, we want to show that the operators introduced in Examples 4.4.11, 4.4.12 and 4.4.13 have a *well-posed initial value problem*. The question now is: what does it mean that a vertical differential operator has a well-posed initial value problem? We are particularly interested in answering this question for the vertical differential operators introduced in the examples of this subsection, i.e. on particular instances of vertical differential operators of first and second order.

Remark 4.4.14. Let $U \in \mathbf{Man}$, let $(\pi : M \to U, E) \in \mathbf{Loc}_1^\infty(U)$ and let $P : C^\infty_\pi \to C^\infty_\pi$ be the $\mathbf{Sh}_{C^\infty_\mathbb{K}}(U)$-morphism determined by a second order vertical differential operator. Then, for every section $\sigma : U \to M$, we can define a $\mathbf{Sh}_{C^\infty_\mathbb{K}}(U)$-morphism

$$\mathrm{data}^{\mathrm{2nd}}_\sigma \,:\, \mathsf{Sol}_\pi \;\longrightarrow\; \left(C^\infty_{\mathbb{R},U} \otimes \mathbb{K}^n\right)^{\oplus 2} \tag{4.4.18a}$$

defined, for each open subset $U' \subseteq U$, by assigning to any $\Phi \in \mathsf{Sol}_\pi(U') \subseteq C^\infty_\pi(U') \otimes \mathbb{K}^n = C^\infty_\mathbb{R}(M|_{U'}) \otimes \mathbb{K}^n$ (see Remark 4.4.10) its initial data

$$\mathrm{data}^{\mathrm{2nd}}_\sigma(\Phi) \;:=\; \big(\sigma^*(\Phi), \sigma^*(*_\mathrm{v}\mathrm{d}_\mathrm{v}\Phi)\big) \;\in\; \left(C^\infty_\mathbb{R}(U') \otimes \mathbb{K}^n\right)^{\oplus 2} \tag{4.4.18b}$$

on $\sigma(U') \subseteq M|_{U'}$.

We say that P has a *well-posed initial value problem* if (4.4.18) is an isomorphism in $\mathbf{Sh}_{C^\infty_\mathbb{K}}(U)$. $\triangle$

Remark 4.4.15. Let $U \in \mathbf{Ob}(\mathbf{Man})$, let $(\pi : M \to U, E) \in \mathbf{Loc}_1^\infty(U)$ and let $P : C^\infty_\pi \to C^\infty_\pi$ be the $\mathbf{Sh}_{C^\infty_\mathbb{K}}(U)$-morphism determined by a first order vertical differential operator. Then, for any section $\sigma : U \to M$, we can define a $\mathbf{Sh}_{C^\infty_\mathbb{K}}(U)$-morphism

$$\mathrm{data}^{\mathrm{1st}}_\sigma \,:\, \mathsf{Sol}_\pi \;\longrightarrow\; C^\infty_{\mathbb{R},U} \otimes \mathbb{K}^n \tag{4.4.19a}$$

defined, for each open subset $U' \subseteq U$, by assigning to a solution $\Phi \in \mathsf{Sol}_\pi(U')$ its initial data

$$\mathrm{data}^{\mathrm{1st}}_\sigma(\Phi) \;:=\; \sigma^*(\Phi) \;\in\; C^\infty_\mathbb{R}(U') \otimes \mathbb{K}^n \tag{4.4.19b}$$

on $\sigma(U') \subseteq M|_{U'}$.

Analogously to Remark 4.4.14, we say that P has a *well-posed initial value problem* if (4.4.19) is an isomorphism in $\mathbf{Sh}_{C^\infty_\mathbb{K}}(U)$. $\triangle$

Example 4.4.16. The aim of this example is to show that the operators in Examples 4.4.11 and 4.4.12 have a well-posed initial value problem. In particular, we will focus on the vertical differential operator from the latter example since the vertical differential operator from the former can be obtained from it by imposing $\tilde{U} = \{*\}$. In order to check that the sheaf morphism in (4.4.18) is an isomorphism, it is sufficient, since $\mathbf{Sh}_{C^\infty_\mathbb{K}}$ is a stack (see Example 4.4.11), to check it locally, i.e. on an open cover $\{U_\alpha : \alpha \in A\}$. Once more, we choose said open cover to be trivializing and introduce a suitable time coordinate T, obtaining local vertical differential operators: $\tilde{P}_\alpha = \partial_T^2 + m^2(\tilde{x})$.

Therefore, for every $(\Phi_0, \Phi_1) \in (C^\infty_{\mathbb{R}, U_\alpha \times \tilde{U}})^{\oplus 2}$, the inverse of the restriction of the initial data map $\mathrm{data}^{2\mathrm{nd}}_\sigma$ to $U_\alpha \subseteq U$ is given by

$$\begin{aligned}\mathrm{solve}_\sigma\big(\Phi_0, \Phi_1\big)(T, x, \tilde{x}) \;:=\; & \Phi_0(x, \tilde{x}) \cos\big(m(\tilde{x})\,(T - T_\sigma(x))\big) \\ & + \Phi_1(x, \tilde{x})\, m(\tilde{x})^{-1} \sin\big(m(\tilde{x})\,(T - T_\sigma(x))\big) \quad , \end{aligned} \tag{4.4.20}$$

where the initial time $T_\sigma(x) \in (T(-\infty, x), T(\infty, x))$ is determined by the (local) coordinate expression $\sigma(x) = (T_\sigma(x), x)$ of the section σ, for all $x \in U_\alpha$. $\triangledown$

Showing that the Dirac vertical differential operator from Example 4.4.13 has a well-posed initial value problem, in the sense of Remark 4.4.15, is an easy exercise.

4.4.2 1-*dimensional scalar field*

The aim of this section is to provide a smooth refinement of the massive scalar field. More precisely, our goal is, harnessing Example 4.4.12, to build a stack morphism $\mathfrak{W} : \mathbf{Loc}^\infty_1 \to \mathsf{Map}(\underline{\tilde{U}}, \mathbf{PoVec}^\infty_\mathbb{R})$ that, post-composed with $\mathsf{Map}(\underline{\tilde{U}}, \mathfrak{CCR}) : \mathsf{Map}(\underline{\tilde{U}}, \mathbf{PoVec}^\infty_\mathbb{R}) \to \mathsf{Map}(\underline{\tilde{U}}, {}^*\mathbf{Alg}^\infty_\mathbb{C})$ from Subsection 4.3.1, will provide us the stack morphism

$$\begin{tikzcd} \mathbf{Loc}^\infty_1 \arrow[rr, "\mathfrak{B}"] \arrow[dr, "\mathfrak{W}"'] & & \mathsf{Map}\big(\underline{\tilde{U}}, {}^*\mathbf{Alg}^\infty_\mathbb{C}\big) \\ & \mathsf{Map}\big(\underline{\tilde{U}}, \mathbf{PoVec}^\infty_\mathbb{R}\big) \arrow[ur, "\mathsf{Map}(\underline{\tilde{U}}, \mathfrak{CCR})"'] & \end{tikzcd} \quad , \tag{4.4.21}$$

representing a smooth $\tilde{U}$-family of 1-dimensional scalar fields with smoothly varying mass parameter $m \in C^\infty(U, \mathbb{R}^{>0})$.

Let $\tilde{P}_{(\pi,E)} : C^\infty_{\pi\times\mathrm{id}} \to C^\infty_{\pi\times\mathrm{id}}$ be the vertical differential operator from Example 4.4.12. Then, the following data defines a stack morphism $\mathfrak{W} : \mathbf{Loc}^\infty_1 \to \mathsf{Map}(\underline{\tilde{U}}, \mathbf{PoVec}^\infty_\mathbb{R})$:

(a) *Functors*: For every manifold $U \in \mathbf{Man}$, functors

$$\mathfrak{W}_U \;:\; \mathbf{Loc}^\infty_1(U) \;\longrightarrow\; \mathbf{PoVec}^\infty_\mathbb{R}(U \times \tilde{U}) \tag{4.4.22}$$

assigning to any smooth U-family of 1-dimensional spacetimes $(\pi : M \to U, E) \in \mathbf{Loc}^\infty_1(U)$ the object

$$\mathfrak{W}_U\big(\pi : M \to U, E\big) \;:=\; \Big(\frac{C^\infty_{\pi\times\mathrm{id}\,\mathrm{vc}}}{\tilde{P}_{(\pi,E)} C^\infty_{\pi\times\mathrm{id}\,\mathrm{vc}}}, \tau_{(\pi\times\mathrm{id}, \mathrm{pr}^*_M(E))}\Big) \;\in\; \mathbf{PoVec}^\infty_\mathbb{R}(U \times \tilde{U}) \tag{4.4.23a}$$

with Poisson structure given by

$$\tau_{(\pi\times\mathrm{id}, \mathrm{pr}^*_M(E))} \;=\; \langle \cdot, \tilde{G}_{(\pi,E)}(\cdot) \rangle_{(\pi\times\mathrm{id}, \mathrm{pr}^*_M(E))} \quad , \tag{4.4.23b}$$

where $\tilde{G}_{(\pi,E)}$ is the causal propagator for $\tilde{P}_{(\pi,E)}$, see Example 4.4.12, and where

$$\langle \cdot, \cdot \rangle_{(\pi\times\mathrm{id}, \mathrm{pr}^*_M(E))} \;:\; C^\infty_{\pi\times\mathrm{id}\,\mathrm{vc}} \otimes_{C^\infty_{\mathbb{R}, U\times\tilde{U}}} C^\infty_{\pi\times\mathrm{id}\,\mathrm{vc}} \xrightarrow{\;\mu\;} C^\infty_{\pi\times\mathrm{id}\,\mathrm{vc}} \xrightarrow{\int_{\pi\times\mathrm{id}}(-)\mathrm{pr}^*_M(E)} C^\infty_{\mathbb{R}, U\times\tilde{U}} \tag{4.4.23c}$$

is the $\mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U \times \tilde{U})$-morphism given by post-composition of the fiber integration on $(\pi \times \mathrm{id} : M \times \tilde{U} \to U \times \tilde{U}, \mathrm{pr}_M^*(E)) \in \mathbf{Loc}_1^\infty(U \times \tilde{U})$ with the multiplication map of functions μ, i.e.

$$\tau_{(\pi\times\mathrm{id},\mathrm{pr}_M^*(E))}\big(\varphi,\varphi'\big) \;=\; \int_\pi \varphi\, \tilde{G}_{(\pi,E)}(\varphi')\, \mathrm{pr}_M^*(E) \quad . \tag{4.4.24}$$

(Notice that the Poisson structure is well-defined on the quotient in (4.4.23a) because $\tilde{P}_{(\pi,E)}$ is formally self-adjoint with respect to the pairing (4.4.23c) and $\tilde{G}_{(\pi,E)} \circ \tilde{P}_{(\pi,E)} = 0 = \tilde{P}_{(\pi,E)} \circ \tilde{G}_{(\pi,E)}$ due to the definition of Green operators, see Definition 4.4.7.)

Moreover, the functors $\mathfrak{W}_U$ are defined on $\mathbf{Loc}_1^\infty(U)$-morphisms $f : (\pi : M \to U, E) \to (\pi' : M' \to U, E')$ (keep in mind that f is an open embedding) by

$$\mathfrak{W}_U(f) \;:=\; (f\times\mathrm{id})_* \,:\, \mathfrak{W}_U\big(\pi : M \to U, E\big) \;\longrightarrow\; \mathfrak{W}_U\big(\pi' : M' \to U, E'\big) \quad . \tag{4.4.25}$$

where $(f \times \mathrm{id})_* : C^\infty_{\pi\times\mathrm{id}\,\mathrm{vc}} \to C^\infty_{\pi'\times\mathrm{id}\,\mathrm{vc}}$ is the pushforward (of compactly supported functions, i.e. extension by zero, see Definition 4.2.4) $\mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U \times \tilde{U})$-morphism associated to the $\mathbf{Loc}_1^\infty(U \times \tilde{U})$-morphism

$$\begin{aligned} f\times\mathrm{id} : (\pi\times\mathrm{id} : M\times\tilde{U} \to U\times\tilde{U}, \mathrm{pr}_M^*(E)) \\ \to (\pi'\times\mathrm{id} : M'\times\tilde{U} \to U\times\tilde{U}, \mathrm{pr}_{M'}^*(E')) \end{aligned}$$

To conclude, notice that $(f \times \mathrm{id})_* : C^\infty_{\pi\times\mathrm{id}\,\mathrm{vc}} \to C^\infty_{\pi'\times\mathrm{id}\,\mathrm{vc}}$ intertwines the equation of motion operators, i.e. $\tilde{P}_{(\pi',E')}\,(f\times\mathrm{id})_* = (f\times\mathrm{id})_*\,\tilde{P}_{(\pi,E)}$ and that the preservation of Poisson structures follows from the uniqueness of retarded/advanced Green operators.

(b) *Coherence isomorphisms*: For each morphism $h : U \to U'$ in $\mathbf{Man}$, natural isomorphisms (see Definition 4.1.2)

$$\begin{tikzcd} \mathbf{Loc}_1^\infty(U') \arrow[r, "\mathfrak{W}_{U'}"] \arrow[d, "h^*"'] & \mathbf{PoVec}_{\mathbb{R}}^\infty(U'\times\tilde{U}) \arrow[d, "(h\times\mathrm{id})^*"] \arrow[dl, Rightarrow, "\mathfrak{W}_h"'] \\ \mathbf{Loc}_1^\infty(U) \arrow[r, "\mathfrak{W}_U"'] & \mathbf{PoVec}_{\mathbb{R}}^\infty(U\times\tilde{U}) \end{tikzcd} \tag{4.4.26a}$$

defined on each component $(\pi : M \to U', E) \in \mathbf{Loc}_1^\infty(U')$ by the morphisms

$$\mathfrak{W}_h \;:\; (h\times\mathrm{id})^*\mathfrak{W}_{U'}\big(\pi : M \to U', E\big) \;\longrightarrow\; \mathfrak{W}_U\big(h^*(\pi : M \to U', E)\big) \tag{4.4.26b}$$

given on the underlying $\mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U \times \tilde{U})$-modules by

$$(\overline{h}^M \times \mathrm{id})^* \;:\; (h\times\mathrm{id})^*\Big(\frac{C^\infty_{\pi\times\mathrm{id}\,\mathrm{vc}}}{\tilde{P}_{(\pi,E)}C^\infty_{\pi\times\mathrm{id}\,\mathrm{vc}}}\Big) \;\longrightarrow\; \frac{C^\infty_{\pi_h\times\mathrm{id}\,\mathrm{vc}}}{\tilde{P}_{h^*(\pi,E)}C^\infty_{\pi_h\times\mathrm{id}\,\mathrm{vc}}} \quad . \tag{4.4.27}$$

Equation (4.4.27) needs some explaining: Recall the pullback bundle construction in (4.2.3) and (4.2.4), and consider the $\mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U \times \tilde{U})$-morphism

$$(\overline{h}^M \times \mathrm{id})^* \,:\, (h \times \mathrm{id})^* C^\infty_{\pi \times \mathrm{id}\,\mathrm{vc}} \longrightarrow C^\infty_{\pi_h \times \mathrm{id}\,\mathrm{vc}} \tag{4.4.28}$$

defined through its adjunct under $(h \times \mathrm{id})^* : \mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U \times \tilde{U}) \rightleftarrows \mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U' \times \tilde{U})$ by the components (denoted with abuse of notation by the same symbol)

$$(\overline{h}^M \times \mathrm{id})^* \,:\, C^\infty_{\mathbb{R}}\big((M \times \tilde{U})|_{U''}\big) \longrightarrow C^\infty_{\mathbb{R}}\big((h^*M \times \tilde{U})|_{(h\times\mathrm{id})^{-1}(U'')}\big) \quad , \tag{4.4.29}$$

for all open subsets $U'' \subseteq U' \times \tilde{U}$, which describe the pullback of functions along the map of total spaces. Due to the universal property of pullback bundles, one easily checks that each section $\sigma : U' \times \tilde{U} \to M \times \tilde{U}$ induces a section $\sigma_h : U \times \tilde{U} \to h^*M \times \tilde{U}$ of the pullback bundle that satisfies $\sigma\,(h \times \mathrm{id}) = (\overline{h}^M \times \mathrm{id})\,\sigma_h$, hence the maps in (4.4.29) preserve vertically compact support. Due to naturality of the vertical differential operators $\tilde{P}$ in (4.4.13), we obtain the commutative diagram

$$\begin{tikzcd}
(h \times \mathrm{id})^* C^\infty_{\pi \times \mathrm{id}\,\mathrm{vc}} \arrow[r, "(\overline{h}^M \times \mathrm{id})^*"] \arrow[d, "(h\times\mathrm{id})^*\tilde{P}_{(\pi,E)}"'] & C^\infty_{\pi_h \times \mathrm{id}\,\mathrm{vc}} \arrow[d, "\tilde{P}_{h^*(\pi,E)}"] \\
(h \times \mathrm{id})^* C^\infty_{\pi \times \mathrm{id}\,\mathrm{vc}} \arrow[r, "(\overline{h}^M \times \mathrm{id})^*"'] & C^\infty_{\pi_h \times \mathrm{id}\,\mathrm{vc}}
\end{tikzcd} \tag{4.4.30}$$

in $\mathbf{Sh}_{C^\infty_{\mathbb{R}}}(U \times \tilde{U})$, which allows us to induce (4.4.28) to the quotients

$$(\overline{h}^M \times \mathrm{id})^* \,:\, (h \times \mathrm{id})^* \bigg(\frac{C^\infty_{\pi \times \mathrm{id}\,\mathrm{vc}}}{\tilde{P}_{(\pi,E)} C^\infty_{\pi \times \mathrm{id}\,\mathrm{vc}}}\bigg) \longrightarrow \frac{C^\infty_{\pi_h \times \mathrm{id}\,\mathrm{vc}}}{\tilde{P}_{h^*(\pi,E)} C^\infty_{\pi_h \times \mathrm{id}\,\mathrm{vc}}} \quad .$$

Here we also used that $(h \times \mathrm{id})^*$ is a left adjoint functor, hence it commutes with the colimit defining these quotients. From the explicit expression (4.4.29) for (the adjunct of) this morphism and observing that a diagram similar to (4.4.30) involving retarded/advanced Green operators commutes due to their uniqueness, one checks that (4.4.27) preserves the relevant Poisson structures and thereby defines the desired $\mathbf{PoVec}^\infty_{\mathbb{R}}(U' \times \tilde{U})$-morphism $\mathfrak{W}_h := (\overline{h}^M \times \mathrm{id})^*$ in (4.4.26b).

It remains to confirm that (4.4.27) is an isomorphism in $\mathbf{PoVec}^\infty_{\mathbb{R}}(U' \times \tilde{U})$. Using the causal propagators (4.4.10) and the initial data morphisms (4.4.18) corresponding to any choice of section $\sigma : U' \times \tilde{U} \to M \times \tilde{U}$ and its induced section

$\sigma_h : U \times \tilde{U} \to h^*M \times \tilde{U}$ of the pullback bundle, we obtain the commutative diagram

$$\begin{tikzcd}
(h \times \mathrm{id})^* \Big(\frac{C^\infty_{\pi \times \mathrm{id}\,\mathrm{vc}}}{\tilde{P}_{(\pi,E)} C^\infty_{\pi \times \mathrm{id}\,\mathrm{vc}}} \Big) \arrow[r, "(\bar{h}^M \times \mathrm{id})^*"] \arrow[d, "(h\times \mathrm{id})^* \tilde{G}_{(\pi,E)}"'] & \frac{C^\infty_{\pi_h \times \mathrm{id}\,\mathrm{vc}}}{\tilde{P}_{h^*(\pi,E)} C^\infty_{\pi_h \times \mathrm{id}\,\mathrm{vc}}} \arrow[d, "\tilde{G}_{h^*(\pi,E)}"] \\
(h \times \mathrm{id})^* \mathsf{Sol}_{\pi \times \mathrm{id}} \arrow[r, "(\bar{h}^M \times \mathrm{id})^*"] \arrow[d, "(h\times \mathrm{id})^* \mathrm{data}^{2\mathrm{nd}}_\sigma"'] & \mathsf{Sol}_{\pi_h \times \mathrm{id}} \arrow[d, "\mathrm{data}^{2\mathrm{nd}}_{\sigma_h}"] \\
(h \times \mathrm{id})^* \big(C^\infty_{\mathbb{R}, U' \times \tilde{U}} \big)^{\oplus 2} \arrow[r, "\cong"'] & \big(C^\infty_{\mathbb{R}, U \times \tilde{U}} \big)^{\oplus 2}
\end{tikzcd} \tag{4.4.31}$$

in $\mathbf{Sh}_{C^\infty_\mathbb{R}}(U \times \tilde{U})$, where the bottom horizontal isomorphism uses that $(h \times \mathrm{id})^*$ preserves coproducts (as it is a left adjoint functor) and the symmetric monoidal coherence isomorphism for the monoidal unit in (4.2.7). By Remark 4.4.10 and Examples 4.4.12 and 4.4.16, all vertical arrows in this diagram are isomorphisms, hence the top horizontal arrow is an isomorphism too. This implies that $\mathfrak{W}_h$ is an isomorphism in $\mathbf{PoVec}^\infty_\mathbb{R}(U \times \tilde{U})$.

Post-composing the stack morphism $\mathfrak{W} : \mathbf{Loc}^\infty_1 \to \mathsf{Map}(\underline{\tilde{U}}, \mathbf{PoVec}^\infty_\mathbb{R})$ with the stack morphism $\mathsf{Map}(\underline{\tilde{U}}, \mathfrak{CCR}) : \mathsf{Map}(\underline{\tilde{U}}, \mathbf{PoVec}^\infty_\mathbb{R}) \to \mathsf{Map}(\underline{\tilde{U}}, {}^*\mathbf{Alg}^\infty_\mathbb{C})$ (see Proposition 4.3.7), we obtain a smooth $\tilde{U}$-family of smooth 1-dimensional scalar field theories $\mathfrak{B} : \mathbf{Loc}^\infty_1 \to \mathsf{Map}(\underline{\tilde{U}}, {}^*\mathbf{Alg}^\infty_\mathbb{C})$.

4.4.3 1-*dimensional Dirac field*

As mentioned in the introduction to this section, the aim of this subsection is to leverage Example 4.4.13 in order to define a stack morphism $\mathfrak{L}^\mathsf{f} : \mathbf{Loc}^\infty_1 \to \mathbf{IPVec}^\infty_\mathbb{C}$ that post-composed with the CAR stack morphism $\mathfrak{CAR} : \mathbf{IPVec}^\infty_\mathbb{C} \to {}^*\mathbf{Alg}^\infty_\mathbb{C}$ will give us a smooth 1-dimensional algebraic quantum field theory $\mathfrak{A}^\mathsf{f} := \mathfrak{CAR} \circ \mathfrak{L}^\mathsf{f}$ that should be thought as a smooth refinement of the massless 1-dimensional Dirac field.

Moreover, we will show that such smooth 1-dimensional algebraic quantum field theory can be lifted to a $U(1)$-equivariant smooth algebraic quantum field theory, i.e the smooth automorphism group $\mathrm{Aut}(\mathfrak{A}^\mathsf{f})$ (see (4.2.19)) includes the global $U(1)$-symmetry of the Dirac field (see the discussion at the end of Subsection 4.2.3).

Let $D_{(\pi,E)}$ be the vertical differential operator from Example 4.4.13. Then, the following data defines a stack morphism $\mathfrak{L}^\mathsf{f} : \mathbf{Loc}^\infty_1 \to \mathbf{IPVec}^\infty_\mathbb{C}$:

(a) For each manifold $U \in \mathbf{Man}$, functors

$$\mathfrak{L}^\mathsf{f}_U : \mathbf{Loc}^\infty_1(U) \longrightarrow \mathbf{IPVec}^\infty_\mathbb{C}(U) \tag{4.4.32}$$

assigning to each $(\pi : M \to U, E) \in \mathbf{Loc}^\infty_1(U)$ the object

$$\mathfrak{L}^\mathsf{f}_U(\pi : M \to U, E) := \Big(\frac{C^\infty_{\pi\,\mathrm{vc}} \otimes \mathbb{C}^2}{D_{(\pi,E)}(C^\infty_{\pi\,\mathrm{vc}} \otimes \mathbb{C}^2)}, *_{(\pi,E)}, \langle \cdot, \cdot \rangle_{(\pi,E)} \Big) \in \mathbf{IPVec}^\infty_\mathbb{C}(U) \quad , \tag{4.4.33a}$$

with $*$-involution

$$*_{(\pi,E)}\begin{pmatrix}\psi\\ \overline{\psi}\end{pmatrix} := \begin{pmatrix}\overline{\psi}^*\\ \psi^*\end{pmatrix} \tag{4.4.33b}$$

given by swapping the components followed by complex conjugation (which descends to the quotient since $*_{(\pi,E)} \circ D_{(\pi,E)} = D_{(\pi,E)} \circ *_{(\pi,E)}$) and symmetric pairing

$$\left\langle \begin{pmatrix}\psi\\ \overline{\psi}\end{pmatrix}, \begin{pmatrix}\psi'\\ \overline{\psi}'\end{pmatrix} \right\rangle_{(\pi,E)} := \int_\pi \begin{pmatrix}\psi & \overline{\psi}\end{pmatrix} \begin{pmatrix}0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix} S_{(\pi,E)} \begin{pmatrix}\psi'\\ \overline{\psi}'\end{pmatrix} E \tag{4.4.33c}$$

given by fiber integration, by the causal propagator $S_{(\pi,E)}$ for $D_{(\pi,E)}$ and by the displayed matrix multiplications. It is easy to check that $\langle\cdot,\cdot\rangle_{(\pi,E)}$ descends to the quotient in (4.4.33a) and that it satisfies the compatibility condition (4.3.2) for $*$-involutions.

The definition of the functor (4.4.32) on morphisms $f : (\pi : M \to U, E) \to (\pi' : M' \to U, E')$ is as in (4.4.25) via pushforward of vertically compactly supported functions.

(b) For **Man**-morphisms $h : U \to U'$, coherence isomorphisms constructed in complete analogy to (4.4.26).

Post-composing the stack morphism $\mathfrak{L}^{\mathrm{f}} : \mathbf{Loc}_1^\infty \to \mathbf{IPVec}_{\mathbb{C}}^\infty$ with the CAR stack morphism $\mathfrak{CAR} : \mathbf{IPVec}_{\mathbb{C}}^\infty \to {}^*\mathbf{Alg}_{\mathbb{C}}^\infty$ (see Proposition 4.3.7), we obtain the smooth 1-dimensional massless Dirac field $\mathfrak{A}^{\mathrm{f}} : \mathbf{Loc}_1^\infty \to {}^*\mathbf{Alg}_{\mathbb{C}}^\infty$.

To conclude this section, we prove that $\mathfrak{L}^{\mathrm{f}} : \mathbf{Loc}_1^\infty \to \mathbf{IPVec}_{\mathbb{C}}^\infty$ lifts to a stack morphism $\tilde{\mathfrak{L}}^{\mathrm{f}} : \mathbf{Loc}_1^\infty \times [\{*\}/U(1)]_{\mathrm{pre}} \to \mathbf{IPVec}_{\mathbb{C}}^\infty$. In particular, this will imply that the smooth 1-dimensional Dirac field $\mathfrak{A}^{\mathrm{f}} = \mathfrak{CAR} \circ \mathfrak{L}^{\mathrm{f}} : \mathbf{Loc}_1^\infty \to {}^*\mathbf{Alg}_{\mathbb{C}}^\infty$ is endowed with a smooth action of the unitary group $U(1)$

$$\begin{tikzcd}
\mathbf{Loc}_1^\infty \times [\{*\}/U(1)]_{\mathrm{pre}} \arrow[rr, "\tilde{\mathfrak{A}}^{\mathrm{f}}"] \arrow[dr, "\tilde{\mathfrak{L}}^{\mathrm{f}}"'] & & {}^*\mathbf{Alg}_{\mathbb{C}}^\infty \\
& \mathbf{IPVec}_{\mathbb{C}}^\infty \arrow[ur, "\mathfrak{CAR}"'] &
\end{tikzcd} \tag{4.4.34}$$

(see the final discussion in Subsection 4.2.3 and diagram (4.2.23)).

The following data defines a stack $\tilde{\mathfrak{L}}^{\mathrm{f}} : \mathbf{Loc}_1^\infty \times [\{*\}/U(1)]_{\mathrm{pre}} \to \mathbf{IPVec}_{\mathbb{C}}^\infty$:

(a) *Functors*: For each manifold $U \in \mathbf{Man}$, functors

$$\tilde{\mathfrak{L}}^{\mathrm{f}}_U : \mathbf{Loc}_1^\infty(U) \times [\{*\}/U(1)]_{\mathrm{pre}}(U) \longrightarrow \mathbf{IPVec}_{\mathbb{C}}^\infty(U) \tag{4.4.35}$$

that act on objects $(\pi : M \to U, E) \in \mathbf{Loc}_1^\infty(U) \times [\{*\}/U(1)]_{\mathrm{pre}}(U)$ precisely as in (4.4.33), i.e.

$$\tilde{\mathfrak{L}}^{\mathrm{f}}_U(\pi : M \to U, E) := \mathfrak{L}^{\mathrm{f}}_U(\pi : M \to U, E) = \left(\tfrac{C^\infty_{\pi\,\mathrm{vc}}\otimes\mathbb{C}^2}{D_{(\pi,E)}(C^\infty_{\pi\,\mathrm{vc}}\otimes\mathbb{C}^2)}, *_{(\pi,E)}, \langle\cdot,\cdot\rangle_{(\pi,E)}\right) \tag{4.4.36}$$

(Notice that the objects of $\mathbf{Loc}_1^\infty(U) \times [\{*\}/U(1)]_{\mathrm{pre}}(U)$ are canonically identified with the objects of $\mathbf{Loc}_1^\infty(U)$ because $[\{*\}/U(1)]_{\mathrm{pre}}(U)$ has only a single object, see (4.2.24).). On $\mathbf{Loc}_1^\infty(U) \times [\{*\}/U(1)]_{\mathrm{pre}}(U)$-morphisms (f,g), i.e. pairs consisting of a $\mathbf{Loc}_1^\infty(U)$-morphism $f : (\pi : M \to U, E) \to (\pi' : M' \to U, E')$ and a $U(1)$-valued smooth function $g \in C^\infty(U, U(1))$, the functor acts by a combination of the pushforward of vertically compactly supported functions and a complex phase rotation

$$\tilde{\mathfrak{L}}^{\mathrm{f}}_U(f,g)\begin{pmatrix}\psi\\ \overline{\psi}\end{pmatrix} := \begin{pmatrix} f_*\big(\pi^*(g)\,\psi\big)\\ f_*\big(\pi^*(g)^{-1}\,\overline{\psi}\big)\end{pmatrix} = \begin{pmatrix}\pi^*(g)\,f_*(\psi)\\ \pi^*(g)^{-1}\,f_*(\overline{\psi})\end{pmatrix} \quad , \tag{4.4.37}$$

where $\pi^*(g)$ denotes the pullback of $g \in C^\infty(U, U(1))$ along the projection map $\pi : M \to U$. (The second equality in (4.4.37) follows from the fact that f_* only acts along the fibers where $\pi^*(g)$ is constant.) These maps clearly preserve the quotient in (4.4.36), the $*$-involution (4.4.33b) and the pairing (4.4.33c), hence they define $\mathbf{IPVec}^\infty_{\mathbb{C}}(U)$-morphisms.

(b) Coherence isomorphisms: For every **Man**-morphisms $h : U \to U'$, coherence isomorphisms constructed in complete analogy to our previous examples.

Post-composing the stack morphism $\tilde{\mathfrak{L}}^{\mathrm{f}} : \mathbf{Loc}_1^\infty \times [\{*\}/U(1)]_{\mathrm{pre}} \to \mathbf{IPVec}^\infty_{\mathbb{C}}$ with the CAR stack morphism $\mathfrak{CAR} : \mathbf{IPVec}^\infty_{\mathbb{C}} \to {}^*\mathbf{Alg}^\infty_{\mathbb{C}}$ defines the smooth $U(1)$-equivariant smooth 1-dimensional massless Dirac field $\tilde{\mathfrak{A}}^{\mathrm{f}} := \mathfrak{CAR} \circ \tilde{\mathfrak{L}}^{\mathrm{f}} : \mathbf{Loc}_1^\infty \times [\{*\}/U(1)]_{\mathrm{pre}} \to {}^*\mathbf{Alg}^\infty_{\mathbb{C}}$.

5

CONCLUSIONS AND OUTLOOK

The aim of this chapter is to gather the conclusions and outlooks of this thesis. In Section 5.1 we "take stock of the situation" and briefly summarize the main results discussed in this work, while we reserve Section 5.2 to propose some outlooks for our work and discuss some possible future developments.

5.1 CONCLUSIONS

In this work we have presented in a (hopefully) self-contained way the results contained in our papers [BPS19, BPS20, BPSW21]. The central theme surrounding this thesis has been the use of Category Theory as a unifying and powerful language to build bridges between subjects (see Chapter 2), but also as a tool to bring new ideas and features to Algebraic Quantum Field Theory (see Chapter 3 and Chapter 4).

In Chapter 2 we have seen that Factorization Algebras and Algebraic Quantum Field Theory are intrinsically related. In particular, we have proven that the category of time-orderable additive Cauchy constant prefactorization algebras is equivalent to the category of additive Cauchy constant algebraic quantum field theories. Moreover, we have seen how involutive $*$-structures can be passed from the latter category to the former.

The relevance of these results lies in their model-independent (categorical) nature and their potential to lead to a prolific exchange of techniques between Factorization Algebras and Algebraic Quantum Field Theory. Notice however, that our work is probably far from being the last attempt toward a comparison between FAs and AQFTs, since further investigations will be needed to obtain fully general results in the context of gauge theories (see 5.2.1 for a more thorough discussion of this point).

In Chapter 3, motivated by gauge theory, we have introduced, via suitable multicategorical tools, a 2-categorical analogue of ordinary algebraic quantum field theories, namely 2-algebraic quantum field theories, and we have proven that ordinary algebraic quantum field theories can be equivalently studied in the 2-category of 2-algebraic quantum field theories. Moreover, we have provided a gauging construction assigning to each G-equivariant AQFT its categorified orbifold construction, which we leveraged to produce simple toy-models of non-truncated 2-algebraic quantum field theories. To conclude we have discussed a categorification of Fredenhagen's universal algebra, i.e. Fredenhagen's universal category.

These results are promising since they show that our 2AQFTs are more sensitive to global aspects of gauge theories, such as disconnected components of the gauge group, than previous approaches to higher categorical AQFT and can be considered as a first step toward the use of n-categories to study higher categorical aspects of algebraic quantum field theories as opposed to the use of model-categories. This work cannot be considered as the end point of the discussion, but as the beginning. In fact, there is a lot of room for work and further developments as better explained in Subsection 5.2.2.

In Chapter 4 we have made a first step toward the formalization of a notion of "smoothness" for algebraic quantum field theories. More precisely, we have defined smooth generalizations $\mathbf{Loc}_1^\infty$ and ${}^*\mathbf{Alg}_\mathbb{C}^\infty$ of the category $\mathbf{Loc}_1$ of 1-dimensional Lorentzian manifolds and of the category ${}^*\mathbf{Alg}_\mathbb{C}^\infty$ of associative and unital $*$-algebras, and we have introduced smooth 1-dimensional algebraic quantum field theories as stack morphisms $\mathfrak{A} : \mathbf{Loc}_1^\infty \to {}^*\mathbf{Alg}_\mathbb{C}^\infty$. Moreover, we have defined a stack $\mathbf{AQFT}_1^\infty$ of smooth 1-dimensional algebraic quantum field theories, introduced the smooth automorphism group of a smooth 1-dimensional algebraic quantum field theory and showed examples of such constructions.

These results represent, hopefully, a starting point of a much broader discussion regarding "smoothness", as hoped for by many researchers in the community. Their relevance, in fact, relies on their role as foundation and template for further generalizations. Although we decided to relegate a more in-depth discussion of such aspects to Subsection 5.2.3, we would like to point out that a suitable generalization of our results that considers m-dimensional globally hyperbolic Lorentzian manifolds, for a generic $m \geq 1$, will probably require contributions of researcher from several distinct areas of mathematics, therefore representing a potential source of collaborations and interactions within different communities.

5.2 OUTLOOKS

The outlooks of this thesis are various and heterogeneous in nature, since the topics presented in our work ascribe to slightly different domains and can capture the attention of researchers with distinct backgrounds. Therefore, for the sake of clarity and order, we will present a section of outlooks for each of the papers presented in this work ([BPS19, BPS20, BPSW21]).

5.2.1 *Outlook: FA vs AQFT in model/∞-categories*

In Chapter 2 we have proven that the category of time-orderable additive Cauchy constant prefactorization algebras and the category of additive Cauchy constant algebraic quantum field theories are equivalent (see Theorem 2.4.1), and that, remarkably, this equivalence can be interpreted in multicategorical terms (see Remark 2.4.2). Although this result is by itself pretty pleasing it does not represent the end of the story. In fact, in the context of gauge theories, as it is customary in situations involving higher structures, the correct way to compare two objects is not via isomorphisms, but via *weak equivalences*. Therefore, the target monoidal categories where the values of algebraic quantum field theories and prefactorization algebras are assigned, are, in general, model-categories (or higher-categories). Even though in this context the adjunction in Remark 2.4.2 becomes a Quillen adjunction between model-categories, it is not clear, at least to us, which analogues of Cauchy constancy and additivity would induce desirable Quillen-equivalent restrictions of these higher categories of AQFTs and tPFAs. We believe that this problem could be of interest to different communities of topologists/category theorists working in model-category theory (notice in particular [CFM21], in which a model structure for locally constant factorization algebras is constructed, and also [Car21], in which model structures presenting the homotopy theory of algebraic quantum field theories satisfying the time-slice and additivity axioms are described) and mathematical physicists working either in algebraic quantum field theory or factorization algebras.

5.2.2 *Outlook: Examples of genuine non-truncated 2-AQFTs*

In Chapter 3 we introduced a 2-categorical analogue of ordinary algebraic quantum field theories in order to capture some of the higher categorical features appearing in gauge theories and we produced some simple examples. It is precisely the simplicity of those examples that naturally brings new outlooks for our work. In fact, in Example 3.4.14 we showed an instance of a non-truncated classical orbifold field theory that is quantized to a truncated orbifold quantum field theory and the question whether more sophisticated non-truncated examples of quantum field theories exist is still open.

5.2.3 *Outlook: Smooth algebraic quantum field theories in higher dimensions*

In Chapter 4, while introducing smooth refinements of 1-dimensional (ordinary) algebraic quantum field theories, namely smooth 1-dimensional algebraic quantum field theories, we have mentioned several times that our choice of focusing on 1-dimensional globally hyperbolic Lorentzian manifolds was dictated by complications of both categorical and analytical nature arising with spacetimes of dimension $m \geq 2$, but we have not been precise on what these difficulties are. We will use this subsection to briefly mention them and to discuss possible avenues for solutions.

Let us first discuss possible generalizations of the stack $\mathbf{Loc}_1^\infty$ of smooth 1-dimensional spacetimes. The issue in trying to generalize this notion relies on determining how "global hyperbolicity" should be enforced. Let us be more precise. Given $U \in \mathbf{Man}$, we can define a smooth U-family of m-dimensional manifolds to be fiber bundles $(\pi : M \to U)$ with typical fiber an m-dimensional manifold N. Moreover, one can easily define notions of vertical orientation $\mathfrak{o}$ and vertical time-orientation $\mathfrak{t}$ and introduce notions of smooth U-families of oriented and time-oriented m-dimensional Lorentzian manifolds $(\pi : M \to U, g, \mathfrak{o}, \mathfrak{t})$. The problem now is that we are unsure about how a good definition of "smooth global hyperbolicity" should look like. Should it be implemented fiber-wise, i.e. asking each fiber $(M|_x, g|_x)$ to be globally hyperbolic? Should it be "U-uniform" in some suitable sense? However one decides to implement it, this condition should guarantee that *vertical* normally hyperbolic operators admit retarded and advanced Green operators and a well-posed initial value problem. Finding a suitable candidate would enable us to generalize most of the constructions in Section 4.4, with one exception: since in higher dimensions the space of initial data is infinite dimensional, we could not proceed as in (4.4.31) to conclude that the assignment of linear observables $\mathfrak{L} : \mathbf{Loc}^{\infty}_{m} \to \mathbf{PoVec}^{\infty}_{\mathbb{R}}$ is a stack morphism. Therefore, we are presented with the following questions: should we consider weaker notions of stack morphisms (*lax* stack morphisms)? Should we instead consider replacing the sheaf $\mathbf{Sh}_{C^{\infty}_{\mathbb{K}}}$ of right $C^{\infty}_{\mathbb{K},U}$-modules with other candidates?

Another issue of categorical nature is the following: we have seen that algebraic quantum field theories can be equivalently introduced using multicategorical tools and how this is convenient to handle "intrinsically" Einstein causality (which in dimensions higher $m \geq 2$ enters the picture). Should we then consider "stacks of multicategories" in any appropriate sense?

We believe that to approaching these questions will require the contributions of researchers with distinct mathematical backgrounds and interests.

Let us wrap-up the possible challenges:

Open Problem 5.2.1. *Find a suitable generalization of global hyperbolicity to smooth U-families of m-dimensional oriented and time-oriented Lorentzian manifolds* $(\pi : M \to U, g, \mathfrak{o}, \mathfrak{t})$ *such that vertical normally hyperbolic operators admit smoothly parametrized retarded and advanced Green operators and a well-posed smoothly parametrized initial value problem.*

Open Problem 5.2.2. *Find a suitable framework such that the assignment* $\mathfrak{L} : \mathbf{Loc}^{\infty}_{m} \to \mathbf{PoVec}^{\infty}_{\mathbb{R}}$ *of linear observables for a smooth m-dimensional free AQFT is a morphism between stacks. Possible options could be enlarging the 2-category* $\mathbf{St}$ *of stacks to allow for lax morphisms or replacing the stack* $\mathbf{Sh}_{C^{\infty}_{\mathbb{R}}}$ *of sheaves of* $C^{\infty}_{\mathbb{R}}$*-modules by a stack describing sheaves of topological (or bornological) modules.*

Open Problem 5.2.3. *Develop a theory of stacks of multicategories in order to define the stack* $\mathbf{AQFT}^{\infty}_{m}$ *of smooth m-dimensional AQFTs in terms of a suitable mapping stack between stacks of multicategories.*

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