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When You Come at the King You Best Not Miss

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Abstract

A tournament is an orientation of a complete graph. We say that a vertex x in a tournament $\vec{T}$ *controls* another vertex y if there exists a directed path of length at most two from x to y. A vertex is called a *king* if it controls every vertex of the tournament. It is well known that every tournament has a king. We follow Shen, Sheng, and Wu [8] in investigating the *query complexity* of finding a king, that is, the number of arcs in $\vec{T}$ one has to know in order to surely identify at least one vertex as a king.

The aforementioned authors showed that one always has to query at least $\Omega(n^{4/3})$ arcs and provided a strategy that queries at most $O(n^{3/2})$. While this upper bound has not yet been improved for the original problem, Biswas et al. [3] proved that with $O(n^{4/3})$ queries one can identify a *semi*-king, meaning a vertex which controls at least half of all vertices.

Our contribution is a novel strategy which improves upon the number of controlled vertices: using $O(n^{4/3} \operatorname{polylog} n)$ queries, we can identify a $(\frac{1}{2} + \frac{2}{17})$-king. To achieve this goal we use a novel structural result for tournaments.

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1 Introduction and Related Work

A *tournament* is a directed graph in which there is exactly one directed edge between every pair of vertices. Due to their usefulness in modelling many real world scenarios such as game tournaments, voting strategies and many more, tournaments are a very well studies concept in structural as well as algorithmic graph theory. The early monograph of Moon [7] has been followed by extensive research on the topic. For example, Dey [4] studied the identification of the "best subset of vertices" in a tournament motivated by the high cost of comparing a pair of drugs for a specific disease. Goyal et al. [5] studied the identification of vertices with specific in- or out-degrees.

In this work we investigate the *query complexity* of finding a *king* in a tournament graph, that is, a vertex from which we can reach every other vertex of the tournament via a directed path of length at most two. It is well known that every tournament has such a vertex.

The study of *query complexity* problems in tournaments has the following general shape: Initially, we are only given the vertex set of the tournament while the directions of its arcs are hidden from us. For each pair of vertices u, v we can, at unit cost, learn whether the arc uv or vu is in the tournament. Our goal is to use the fewest possible queries in order to reveal some combinatorial object in the tournament. The motivation for our paper is found in Shen, Sheng, and Wu's work [8] on the query complexity of identifying a king. They

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showed that $\Omega(n^{4/3})$ queries are always necessary and provided an algorithm which reveals a king using $O(n^{3/2})$ queries. Ajtai et al. [1] independently proved the same upper bound in the context of imprecise comparison.

One of the enticing aspects of this setting is its game-theoretic nature: we can alternatively think of it as an adversarial game where one player, the *seeker*, wants to identify a combinatorial structure by querying arcs of the tournament while an adversary, the *obscurer*, tries to delay the seeker for as long as possible by choosing the orientation of queried arcs.

When reading Shen, Sheng, and Wu [8], one may be tempted to conjecture that a better analysis of their obscurer-strategy for finding a king can lead to a better lower bound. However, Biswas et al. [3] showed that against this strategy, the seeker can find a king with $O(n^{4/3})$ queries. They also showed that there exists a seeker strategy with $O(n^{4/3})$ queries for identifying a *semi-king*, that is, a vertex which controls at least half of all vertices. This result is optimal by Lemma 6, Biswas et al. [3]. In fact, one needs to make $\Omega(t^{4/3})$ queries for identifying a vertex which controls at least $t \leq n$ vertices against the obscurer-strategy of Shen, Sheng and Wu. (See Lemma 6 of Biswas et al. [3] and Ajtai et al. [1] for more details.) Therefore, if there exists an obscurer-strategy that proves a stronger than $\Omega(n^{4/3})$ lower bound for the king problem, then this strategy must rely on some factors which distinguish the king problem from the semi-king problem. In our eyes, this means that such a lower bound is much more difficult to find than one might think at first.

Proceeding from the above, it is tempting to try to improve the upper bound by using a variation of the seeker-strategy from Shen, Sheng, and Wu [8] and we can interpret the Biswas et al. [3]'s seeker-strategy for finding a semi-king as such an attempt. These strategies both rely on repeatedly selecting a set of vertices and then querying all the edges between them to find a maximum out-degree (MOD) vertex in this sub-tournament[1]. Balasubramanian, Raman and Srinivasaragavan [2] showed that identifying an MOD vertex in a tournament of size k requires $\Omega(k^2)$ queries in the worst case, which may explain the limits of the existing seeker strategies.

Our Result

In this paper, we proceed along the line of research just described. On the one hand, we show that with $\tilde{O}(n^{4/3})$ queries[2], it is possible to identify a $(\frac{1}{2} + \frac{2}{17})$-king, which indicates that improving upon the $\Omega(n^{4/3})$ lower bound is probably even harder than indicated by the semi-king results. On the other hand, our technique does not rely on finding MOD vertices of sub-tournaments which circumvents the inherent high cost of this operation.

Technical Overview

Our result is based on the combinatorial structure of tournaments, which may be of independent interest. We believe that this paper provides a novel toolkit which could lead towards resolving the query complexity of finding a king. Specifically, we design a seeker-strategy which consists of two main stages:

(i) The seeker queries the orientation of a set of edges defined by a so-called *template-graph*. These queries are *non-adaptive* in the sense that the queries do not change as a result of the answers provided by the obscurer.

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[1] The relationship between MOD vertices and kings is well-established: Landau [6], while studying the structure of animal societies, showed that every MOD vertex is a king, but non-MOD kings can exist as well.

[2] The big-$\tilde{O}$ notation hides constants and polylogarithmic factors

(ii) The seeker analyses the answer to the queries of (i) in order to select queries that lead to the revelation of a $(\frac{1}{2} + \frac{2}{17})$-king.

The template-graph is an undirected graph over the tournament's vertices that has $\tilde{O}(n^{4/3})$ edges, with the property that every set of vertices of size around $n^{2/3}$ or more has edges to almost all the graph. In Section 3, we use the probabilistic method to prove that such a graph exists. Given the template-graph, the seeker's queries in the first stage are simply given by its edges, i. e. if there exists an edge uv in the template-graph, then the seeker asks the obscurer about the orientation of the edge uv in the tournament. The sparsity of the template-graph ensures that the seeker does not make too many queries and the connectivity of every sufficiently large set ensures that we do not miss any relevant information.

The second stage of the seeker-strategy is built on showing that when the obscurer chooses how the edges of the template graph are oriented they have a trade-off. The trade-off is either to reveal an *ultra-set* or not. We show that if the obscurer reveals an *ultra-set*, then the seeker can reveals a $(\frac{1}{2} + \frac{2}{17})$-king with $\tilde{O}(n^{4/3})$ extra queries. If the obscurer does not do this, then the seeker can use this to find a partition of the vertex set of the tournament into sets of size $O(n^{2/3})$ each (which we refer to as *tiles*), so that the edges of the template-graph that are incident to the tiles satisfy a certain property. We obtain this combinatorial object by showing that if such a partition does not exist, then a simple set of queries already reveals a $(\frac{1}{2} + \frac{2}{17})$-king.

The tiles are analysed by the construction of what we refer to as the *free matrix* which contains a row for every tile and a column for every vertex of the tournament. An entry of the matrix indexed by a given tile-vertex pair is 1 if every edge between the vertex and a vertex in the tile is directed towards the tile, otherwise the entry is 0. We then use this free matrix to guide the seeker-strategy.

Given that the first part of the seeker-strategy is non-adaptive and against any adversary, this approach also reveals a combinatorial property of tournaments: for any fixed tournament and template graph with the same set of vertices, knowing only the direction of the arcs of the tournament that correspond to the arcs of the template graph is sufficient for finding a set of vertices S of size $O(n^{2/3})$ such that querying all edges inside of S necessarily reveals a $(\frac{1}{2} + \frac{2}{17})$-king. We note that fraction $\frac{2}{17}$ is the result of balancing the various trade-offs in the seeker strategy.

The rest of the paper is organised as follows. In Section 2, we provide necessary definitions and prove some basic lemmas about tournaments that are used in the rest of the paper. Section 3 is dedicated to the formal definition and the proof of existence of template graphs. In Section 4, we describe our seeker-strategy and prove that it leads to the discovery of a $(\frac{1}{2} + \frac{2}{17})$-king. In Section 5, we give concluding remarks and open problems.

2 Preliminaries

For simplicity, we assume that the vertices of any n-vertex graph are the numbers $[n] := \{1, \ldots, n\}$.

An orientation of a graph G is a directed graph obtained from G by replacing every one of its edges by a directed arc.

$d(v, X)$,$N(v)$ Given an undirected graph G, a vertex $v \in V(G)$, and a vertex set $X \subseteq V(G)$ we define the *relative degree* $d(v, X) := |N(v) \cap X|$, where $N(v)$ is the neighbourhood of v in G.

$d^+(X)$,$N^+(v)$ For a vertex v in a directed graph $\vec{G}$, a vertex u in $\vec{G}$ is an out-neighbour of v, if the edge between u and v is oriented from v to u. For a directed graph $\vec{G}$, we denote the out-neighbourhood of a vertex $v \in \vec{G}$ by $N^+(v)$ and its out-degree by $d^+(v)$. For a vertex set $X \subseteq V(\vec{G})$, we let $d^+(X)$ be the number of arcs from a vertex in X to a vertex not in X. $d^+(v, X)$ Given additionally a vertex subset $X \subseteq V(G)$, we define the *relative out-degree* $d^+(v, X) := |N^+(v) \cap X|$.

$N^{++}[v]$ The (closed) *second-out-neighbourhood* $N^{++}[v]$ of v is the set of vertices $u \in V(\vec{G})$ for which there exists a directed path from v to u of length at most two.

control, direct – For simplicity we will adopt the following vocabulary for digraphs. We say that a vertex x *controls* a vertex y if $y \in N^{++}[x]$. We say that x *directly controls* a vertex y if $y \in N^+(x) \cup \{x\}$. We extend both of these terms to vertex sets U, for example, we will often write statements like "x controls at least half of the vertices in U".

$\vec{T}$, $\vec{T}[S]$ A tournament is a digraph $\vec{T}$ obtained from a complete graph by replacing each edge with a directed arc. As done usually, we denote the subgraph induced by a vertex set $S \subseteq V(\vec{T})$, with $\vec{T}[S]$. Note that an induced subgraph of a tournament is necessarily also a tournament. We will need the following basics facts about tournaments in the following.

▶ **Lemma 1.** *Let $\vec{T}$ be a tournament with m vertices and $\alpha \in [0, 1]$ such that αm is even. Then $\vec{T}$ has at least $(1 - \alpha)m$ vertices of out-degree at least $\alpha m/2$.*

Proof. Let S initially be the vertices of $\vec{T}$ and proceed according to the following process: find a vertex of out-degree at least $\alpha m/2$ and remove it from S; and repeat until no such vertex exists in S. From here on we focus on set S after the vertex removal process ended.

Let $r = |S|$ be the size of the final set and consider the sub-tournament $\vec{T}[S]$. We know that by averaging considerations, every tournament of size r has at least one vertex of out-degree at least $r/2 - 1/2$. We also know that S does not contain any vertex of out-degree at least $\alpha m/2$. Hence, we conclude that $r \leq \alpha m$.

Consequently, our process discovered $m - r \geq (1 - \alpha)m$ vertices of out-degree at least $\alpha m/2$ in $\vec{T}$. ◀

▶ **Lemma 2.** *Let $\vec{G}$ be an orientation of a complete bipartite graph (V_0, V_1, E), where $|V_0| = |V_1| = m$, and m is divisible by 4 . Then, there exists $i \in \{0, 1\}$, such that V_i has at least $m/2 + 1$ vertices v, where $d^+(v, V_{1-i}) \geq m/4$.*

Proof. Let S initially contain all the vertices in $V_0 \cup V_1$ and proceed according to the following process: We find a vertex of out-degree at least $m/4$ and remove it from S. Repeat until no such vertex exists and we are left with $S' \subseteq S$.

Every orientation of the complete bipartite graph $K_{t,t}$ must contain, by a simple averaging argument, a vertex of out-degree at least $t/2$. Therefore the induced subgraph $\vec{G}[S']$ must have at least one partite set of size strictly less than $m/2$ or we could continue the process. Consequently, our process discovered at least $m/2 + 1$ vertices of out-degree at least $m/4$ in that partite set. ◀

▶ **Lemma 3.** *Let $\vec{T}$ be a tournament on $2m$ vertices, where m is divisible by 4. Let further sets S_0, S_1 be a partition of the vertices of $\vec{T}$ into sets of equal size. Then there exists a vertex v such that both $d^+(v, S_0) \geq m/4$ and $d^+(v, S_1) \geq m/4$.*

Proof. By Lemma 1, for both $i \in \{0, 1\}$, there exist $m/2$ vertices v in S_i such that $|N^+(v) \cap S_i| \geq m/4$. By Lemma 2 for *one* of $i \in \{0, 1\}$, there exist $m/2 + 1$ vertices v such that $|N^+(v) \cap S_{1-i}| \geq m/4$. Then by the pigeonhole principle, there exists $i \in \{0, 1\}$ and a vertex $v \in S_i$, such that $|N^+(v) \cap S_i| \geq m/4$, for every $i \in \{0, 1\}$, as claimed. ◀

3 Constructing the template-graph

▶ **Definition 4** (κ-template-graph)**.** *Let $\kappa \in (0,1)$ and G be an undirected graph over the vertex set $[n]$. The graph G is a* κ-template-graph*, if for every pair of disjoint sets $H_1, H_2 \subseteq [n]$ both of size at least $\kappa n^{2/3}$, there exists at least one edge between them, that is, $|E(H_1,H_2)| \geq 1$.*

For the remainder of this section, we fix $\kappa \in (0,1)$ and set $p = \frac{2\log n+2}{\kappa n^{2/3}}$. We next show that with strictly positive probability the Erdős–Renyi random graph $G(n,p)$ is a *κ-template-graph*, with $O(n^{4/3}\log n)$ edges, where the O notation hides a dependence on κ. By the probabilistic method, this implies that there actually exists such a graph.

All probabilities in the following are with respect to the probability space of this random graph.

▶ **Lemma 5.** *Let $\kappa \in (0,1)$. With probability at least $3/4$, the graph $G(n,p)$, where p is defined as above, is a κ-template-graph.*

Proof. Note that if we prove the statement of the lemma for sets of size *exactly* (up to rounding errors) $\kappa n^{2/3}$, then the claim follows for all larger sets as well. To prove this, we next show, with the help of the union bound, that the probability that $G(n,p)$ has two disjoint subsets of vertices, each of size $\kappa n^{2/3}$, with no edge between them is strictly less than $1/4$.

Let H_1 and H_2 be any pair of disjoint subsets of $[n]$ of size $\kappa n^{2/3}$, then the total number of vertex pairs between them is $(\kappa n^{2/3})^2$. The probability that none of these pairs is an edge in the template-graph G is accordingly $(1-p)^{(\kappa n^{2/3})^2}$.

We apply the exponential bound $(1-p)^k \leq e^{-pk}$ for a k-round Bernoulli trial and obtain

$$(1-p)^{(\kappa n^{\frac{2}{3}})^2} \leq e^{-p(\kappa n^{2/3})^2} = e^{-(2\log n+2)\kappa n^{2/3}}$$
$$= n^{-2\kappa n^{2/3}} e^{-2\kappa n^{2/3}} \leq \frac{1}{4} n^{-2\kappa n^{2/3}},$$

where the last inequality holds when n is large enough so that $\kappa n^{\frac{2}{3}} \geq 1$ and hence $e^{-2\kappa n^{2/3}} < \frac{1}{4}$. Since the total number of pairs of sets H_1, H_2 of size $\kappa n^{2/3}$ is bounded above by $n^{2\kappa n^{2/3}}$, the claim now follows from the union bound. ◀

▶ **Theorem 6.** *Let $\kappa \in (0,1)$, there exists a κ-template-graph G with at most $O(n^{4/3}\log n/\kappa)$ edges.*

Proof. The expected number of edges of $G(n,p)$ for our choice of $p = (2\log n+2)/(\kappa n^{2/3})$ is less than $m := (2\log n+2)n^{4/3}/\kappa$. Since every edge of the graph is selected independently, by the Chernoff bound the probability that the number of edges in $G(n,p)$ exceeds $2m$ is at most

$$e^{-p\binom{n}{2}/3} \leq e^{-\frac{(2\log n+2)}{\kappa n^{2/3}}\frac{n(n-1)}{6}}$$
$$\leq e^{-(2\log n+2)n/6} \leq \frac{1}{4}$$

where the last inequality holds for $n \geq 4$.

Together with Lemma 5 this implies that with probability at least $1/2$, $G(n,p)$ is a κ-template-graph G with at most $O(n^{4/3}\log n/\kappa)$ edges. The claim follows by the probabilistic method. ◀

4 The seeker strategy

Having proved the existence of a κ-template-graph, we next examine the properties of an arbitrary orientation of such a graph. Given a κ-template-graph G_κ on a vertex set $[n]$, we use $\vec{G}_\kappa$ (*henceforth*) to refer to a directed graph obtained by replacing every edge of G_κ by a directed arc. Note that we assume nothing about $\vec{G}_\kappa$ and analyze as if its arcs were arbitrarily oriented by an adversary.

▶ **Definition 7** (η-weak, η-strong, η-ultra). *η-weak, η-strong, η-ultra For an oriented template-graph $\vec{G}_\kappa$ and any $\eta > 0$, a set $H \subseteq [n]$ is η-*weak *if $d^+(H) < (1/2+\eta)n$, is η-*strong *if $d^+(H) \geq (1/2+\eta)n$. We call a set η-*ultra *if every subset $H' \subset H$, of size at least $|H|/2$ is η-strong.*

To understand why η-ultra sets are important, it is useful to think of the seeker as trying to force the obscurer to reveal enough information (in the form of query answers) so that the seeker can achieve their goal. This is done by first querying the orientation of all the edges of a template graph and nothing else. The observation below implies that if the orientation of the edges of the template graphs reveals an *eta*-ultra set, of size $\tilde{O}(n^{2/3})$, then the seeker can achieve its goal with an additional $\tilde{O}(n^{4/3})$ queries. Thus, the obscurer cannot reveal such an ultra-set. However, as we show further on, by doing this the obscurer reveals enough information for the seeker to achieve their goal.

▶ **Observation 8.** *Let $H \subseteq V(\vec{G}_\kappa)$ be an η-ultra set. Then we can find a $(1/2+\eta)$-king using $\leq |H|^2$ additional queries.*

Proof. Query all $\leq |H|^2$ edges inside H. Let $v \in H$ be a vertex such that $d^+(v, H) \geq |H|/2$. Since H is η-ultra, the set $H' := N^+(v) \cap H$ is η-strong, meaning $d^+(H') \geq (1/2+\eta)n$. Therefore $|N^{++}[v]| \geq (1/2+\eta)n$ and v is a $(1/2+\eta)$-king. ◀

▶ **Definition 9** (Free set). *Free set, $F(W)$ Let $W \subseteq V(\vec{G}_\kappa)$ be an η-weak set. Then the* free set *of W is the vertex set $F(W) := V(\vec{G}_\kappa) \setminus (N^+(W) \cup W)$, that is, all vertices that lie neither in W nor in $N^+(W)$.*

▶ **Observation 10.** *Let W be an η-weak set. Then $|F(W)| > (\frac{1}{2} - \eta)n - |W|$.*

By the properties of template-graphs, namely that each pair of large enough sets must have an edge between them, and by the definition of free sets it follows that all the arcs of $\vec{G}_\kappa$ between a sufficiently large set W and its free set $F(W)$ must point towards W. Let us formalize this intuition:

▶ **Definition 11** (α-covers). *For $\alpha \in [0,1]$ we say that a set S α-*covers *a set W if $|N^+(S) \cap W| \geq \alpha|W|$.*

▶ **Lemma 12.** *In the template graph, for every set $W \subset [n]$ of size $n^{2/3}$ and every subset $S \subseteq F(W)$ of size at least $\kappa n^{2/3}$, it holds that S $(1-\kappa)$-covers W.*

Proof. Consider a set $S \subseteq F(W)$ of size $\kappa n^{2/3}$. Then, by Lemma 5, there is at least one edge $s_1 w_1$ between some $s_1 \in S$ and $w_1 \in W$. Remove w_1 from W and apply the argument to the remainder. In this way, we construct a sequence $w_1, \ldots, w_t$ such that each w_i has at least one neighbour in S.

The application of Lemma 5 is possible until the remainder of W has size less than $\kappa n^{2/3}$, hence the process works for at least $t = n^{2/3} - \kappa n^{2/3} = (1-\kappa)n^{2/3}$ steps. Now simply note that each edge sw_i for $s \in S$ must be oriented from s to w_i since S is a subset of $F(W)$. It follows that $|N^+(S) \cap W| \geq (1-\kappa)|W|$, as claimed. ◀

In the previous lemma lies the inherent usefulness of free sets. If, for some set W of size $n^{2/3}$, we find a vertex v that has at least $\kappa n^{2/3}$ out-neighbours in the free set $F(W)$, then v controls almost all of W. As observed above, η-weak sets have necessarily large free sets which makes them "easy targets" for our strategy.

We now show that in case no η-ultra set exists (in which case we already win as per Observation 8), we can instead partition most of the vertices of $V(\vec{G}_\kappa)$ into weak sets.

▶ **Definition 13.** *An* η-weak tiling *of* $\vec{G}_\kappa$ *is a vertex partition* $W_1, \dots, W_m, R$ *where* $|W_i| = n^{2/3}$, $|R| < 2n^{2/3}$ *and every set* W_i *is* η*-weak. We call the sets* W_i *the* tiles *and* R *the* remainder*.*

By definition, the number of tiles m in an η-weak tiling is at least $n^{1/3} - 2$.

▶ **Lemma 14.** *Fix* $\eta > 0$*. For large enough* n*,* $\vec{G}_\kappa$ *either contains an* η*-ultra set of size* $2n^{2/3}$ *or an* η*-weak tiling.*

Proof. We construct the tiling iteratively. Assume we have constructed $W_1, \dots, W_j$ so far. Let $R := V(\vec{G}_\kappa) \setminus \bigcup_{i \le j} W_i$ be all the vertices of $\vec{G}_\kappa$ which are not yet part of the tiling. If $|R| < 2n^{2/3}$ we are done, so assume otherwise. Let $H \subseteq R$ be an arbitrary vertex set of size $2n^{2/3}$. If H is η-ultra, then by Observation 8, we are done. Otherwise there exists an η-weak set $W_{j+1} \subseteq H$, $|W_{j+1}| = |H|/2$. Add this set to the tiling and repeat the construction. At the end of this procedure, we will either find an η-ultra set or an η-weak tiling. ◀

Our goal is now to find a vertex whose out-neighbourhood has large intersections with many free sets. To organise this search, we define the following auxiliary structure:

▶ **Definition 15** (Free matrix). *Let* $W_1, \dots, W_m, R$ *be an* η*-weak tiling of* $\vec{G}_\kappa$ *and let* $\mathcal{W} = \{W_1, \dots, W_m\}$*. The* free matrix M *of the tiling* $\mathcal{W}, R$ *is a binary matrix with* m *rows indexed by* $\mathcal{W}$ *and* n *columns indexed by* $[n]$*. The entry at position* $(W_i, v) \in \mathcal{W} \times V$ *is* 1 *if* $v \in F(W_i)$ *and* 0 *otherwise.*

weight, $\sum M$ We will use the following notation in the rest of this section. Given a free matrix M of an η-weak tiling $\mathcal{W}, R$ let $M[\mathcal{W}', U]$ denote a sub-matrix of M induced by a subset $\mathcal{W}' \subseteq \mathcal{W}$ of the tile set and a subset $U \subseteq [n]$ of the vertex set. For example, a column of M corresponding to a vertex $v \in [n]$ can be written as $M[\mathcal{W}, \{v\}]$ in this notation. Analogously a row of M corresponding to a tile $W_i \in \mathcal{W}$ can be written as $M[\{W_i\}, [n]]$. Given a sub-matrix M' of the free matrix M, we call the number of 1's in M' the *weight* of M' and denote it by $\sum M'$.

The following is a direct consequence of the construction of the free matrix and Observation 10.

▶ **Observation 16.** *Every row of the free matrix* M *has a weight of at least* $(\frac{1}{2} - \eta - n^{-1/3})n$*.*

▶ **Definition 17** (Good Sub-Matrix). *A sub-matrix* $M[\mathcal{W}, U]$*, for some* $U \subset [n]$*, is* η-good *if, each one of its rows has weight at least* $(\frac{1}{2} - \eta - 2n^{-1/3} \log^{1/2} n)|U|$*.*

We next show that a good sub-matrix with $2n^{2/3}$ columns exists, by using the probabilistic method. Specifically, we show that if we randomly pick $2n^{2/3}$ columns from the matrix $M[\mathcal{W}, [n]]$ then with strictly positive probability the matrix that includes exactly these columns is good.

▶ **Lemma 18.** *Let* $\eta \in (0, \frac{1}{2})$*. For large enough* n *the free matrix* M *has an* η*-good sub-matrix with* $2n^{2/3}$ *columns.*

Proof. Select $K \subset [n]$ of size $2n^{2/3}$ uniformly at random. Let $M' = M[\mathcal{W}, K]$.

We set $p = 1/2 - \eta - n^{-1/3}$ and $t = n^{-1/3} \log^{1/2} n$. By Observation 16, every row of M has weight at least pn. By the Hypergeometric tail bound the probability that a specific row of M' has weight less than $(p-t)n$ is at most $e^{-2t^2 2n^{2/3}} \le 1/n$, where the inequality follows from our choice of t. Then by the union bound the probability that *every* row of M' has weight at least $(p-t)n$ is strictly positive.

Note that by our choice of p and t, we get that $(p-t)n$, for large enough n, is at least as large as $(1/2 - \eta - 2n^{-1/3} \log^{1/2} n)n$, therefore a good sub-matrix M' exists with strictly positive probability. By the probabilistic methods the claim therefore holds. ◀

Next we show that the only way that the adversary does not provide us with a $(1/2+\delta)$-king once we have identified a δ-good sub-matrix is if the distribution of 1's in that matrix is very restricted. We will use this additional structure to find a $(1/2+\delta)$-king in the sequel. For simplicity, we first query all the edges between the vertices associated with the columns of the δ-good sub-matrix, but note that this is not strictly necessary: we can instead inspect all potential partitions (with properties as stated in the lemma) and only if no such partition exists query said edges which then surely identifies a $(1/2+\delta)$-king. With this change the lemma is consistent with the structural claim from the introduction.

▶ **Lemma 19.** *Let $M' = M[\mathcal{W}, V]$ be a δ-good sub-matrix of M with $2n^{2/3}$ columns. Let further $\kappa + \delta \le 1/2$. If we query each one of the $O(n^{4/3})$ edges in V, then either we find a $(\frac{1}{2}+\delta)$-king, or we find partitions $V_1 \uplus V_2 = V$ and $\mathcal{W}_1 \uplus \mathcal{W}_2 = \mathcal{W}$ with the following properties:*

- $|V_1| = |V_2| = n^{2/3}$
- $|\mathcal{W}_1| \ge (\frac{1}{2} - \delta - \kappa)n^{1/3} - 2$ *and* $|\mathcal{W}_2| < (\frac{1}{2} + \delta + \kappa)n^{1/3}$
- *Every row in $M'[\mathcal{W}_1, V_1]$ has weight at most $\kappa n^{2/3}$*
- *Every row in $M'[\mathcal{W}_2, V_1]$ has weight at least $\kappa n^{2/3}$*

Proof. We query all the edges in $V \times V$ and select a vertex $y \in V$ such that $d^+(y, V) \ge n^{2/3}$. Let V_1 be an arbitrary subset of $N^+(y) \cap V$ of size $n^{2/3}$ and let $V_2 = V \setminus V_1$. Partition the rows of M' into $\mathcal{W}_1 \cup \mathcal{W}_2$ so that $\mathcal{W}_1$ contains all rows with weight less than $\kappa n^{2/3}$ in the sub-matrix $M'[\mathcal{W}, V_1]$.

We claim that if $|\mathcal{W}_1| < (\frac{1}{2} - \delta - \kappa)n^{1/3} - 2$ then y is a $(\frac{1}{2}+\delta)$-king. By construction, every row in $\mathcal{W}_2$ has weight at least $\kappa n^{2/3}$ in $M'[\mathcal{W}, V_1]$ and if the condition of the claim holds then $|\mathcal{W}_2| \ge (\frac{1}{2} + \delta + \kappa)n^{1/3}$. By Lemma 12, the set V_1 must $(1-\kappa)$-cover every tile in $\mathcal{W}_2$. It follows that

$$\begin{aligned}
|N^{++}[y]| &\ge (1-\kappa)\big|\bigcup \mathcal{W}_2\big| \ge (1-\kappa)\left(\Big(\frac{1}{2} + \delta + \kappa\Big)n^{1/3}\right) n^{2/3} \\
&= (1-\kappa)\Big(\frac{1}{2} + \delta + \kappa\Big)n = \Big(\frac{1}{2} + \delta + \frac{\kappa}{2} - \kappa\delta - \kappa^2\Big)n \\
&= \left(\frac{1}{2} + \delta + \kappa\Big(\frac{1}{2} - \delta - \kappa\Big)\right) n \\
&\ge \Big(\frac{1}{2} + \delta\Big)n
\end{aligned}$$

where the last inequality holds since $\delta + \kappa \le 1/2$. ◀

Lemma 19 implies the following about good sub-matrices.

▶ **Lemma 20.** *Let $M' = M[\mathcal{W}, V]$ be a δ-good sub-matrix with $|V| = 2n^{2/3}$ and $\mathcal{W}_1 \uplus \mathcal{W}_2$, $V_1 \uplus V_2$ be partitions as in Lemma 19. Then every row in $M'[\mathcal{W}_1, V_2]$ has weight at least $(1 - 2\delta - 2\kappa)n^{2/3}$.*

Proof. Since M' is a δ-good sub-matrix, by Definition 17, every row in $M'[\mathcal{W}_1, V]$ has weight at least $(1/2 - \delta - 2n^{-1/3}\log^{1/2} n)2n^{2/3}$. By Lemma 19, every row in $M'[\mathcal{W}_1, V_1]$ has weight at most $\kappa n^{2/3}$. Therefore, the weight of every row in $M'[\mathcal{W}_1, V_2]$ is

$$\begin{aligned}&\geq (1/2 - \delta - 2n^{-1/3}\log^{1/2} n)2n^{2/3} - \kappa n^{2/3}\\&= (1 - 2\delta - 4\frac{\log^{1/2} n}{n^{1/3}} - \kappa)n^{2/3}\\&\geq (1 - 2\delta - 2\kappa)n^{2/3}\end{aligned}$$

where we assume that n is large enough so that $4\frac{\log^{1/2} n}{n^{1/3}} \leq \kappa$. ◀

Our final technical lemma lets us, for a given set of rows of M, identify a set of columns with high enough weight when restricted to those rows.

▶ **Lemma 21.** *Let $U \subset [n]$ be of size $2n^{2/3}$ and $\mathcal{W}' \subset \mathcal{W}$. Then there exists a set $V' \subset [n] \setminus U$, of size $n^{2/3}$ such that every column in $M[\mathcal{W}', V']$ has weight at least $(1/2 - \delta - 3n^{-1/3})|\mathcal{W}'|$.*

Proof. Let $\bar{U} := [n] \setminus U$ and let V' be an arbitrary subset of $n^{2/3}$ columns in $M[\mathcal{W}', \bar{U}]$ with the largest column-weight. Let t be the smallest weight among these columns when restricted to $M[\mathcal{W}', V']$. We bound the weight of $M[\mathcal{W}', \bar{U}]$ first from below and then from above, then we use these bounds to show that $t > (\frac{1}{2} - \delta - 3n^{-1/3}) \cdot |\mathcal{W}'|$, which implies that V' is the claimed set.

For the lower bound on the weight of $M[\mathcal{W}', \bar{U}]$, we use the simple fact that

$$\sum M[\mathcal{W}', \bar{U}] = \sum M[\mathcal{W}', [n]] - \sum M[\mathcal{W}', U]. \tag{1}$$

By Observation 16, every row of the matrix M has weight least $(\frac{1}{2} - \delta - n^{-1/3}) \cdot n$. It follows that $\sum M[\mathcal{W}', [n]]$ is at least $(\frac{1}{2} - \delta - n^{-1/3}) \cdot n \cdot |\mathcal{W}'|$. For the second term, we have the trivial bound $\sum M[\mathcal{W}', U] \leq |U| \cdot |\mathcal{W}'| = 2n^{2/3} \cdot |\mathcal{W}'|$. Plugging these values into (1) we obtain

$$\begin{aligned}\sum M[\mathcal{W}', \bar{U}] &\geq (\frac{1}{2} - \delta - n^{-1/3}) \cdot n \cdot |\mathcal{W}'| - 2n^{2/3} \cdot |\mathcal{W}'|\\&= (\frac{1}{2} - \delta - 3n^{-1/3}) \cdot n \cdot |\mathcal{W}'|.\end{aligned} \tag{2}$$

For the upper bound on the total weight of $M[\mathcal{W}', \bar{U}]$ we use that

$$\sum M[\mathcal{W}', \bar{U}] = \sum M[\mathcal{W}', V'] + \sum M[\mathcal{W}', \bar{U} \setminus V']. \tag{3}$$

We use the trivial bound $\sum M[\mathcal{W}', V'] \leq |V'| \cdot |\mathcal{W}'| = n^{2/3} \cdot |\mathcal{W}'|$ for the first term. By definition of the value t, we have that every column in $M[\mathcal{W}', \bar{U} \setminus V']$ has weight at most t. Accordingly, $\sum M[\mathcal{W}', \bar{U} \setminus V'] \leq t \cdot |\bar{U} \setminus V'| = t \cdot (n - 3n^{2/3})$. Plugging in these values into (3) we obtain

$$\sum M[\mathcal{W}', \bar{U}] \leq n^{2/3} \cdot |\mathcal{W}'| + t \cdot (n - 3n^{2/3}). \tag{4}$$

Finally, (2) and (4) taken together give us that

$$t \cdot (n - 3n^{2/3}) + n^{2/3} \cdot |\mathcal{W}'| \geq (\frac{1}{2} - \delta - 2n^{-1/3}) \cdot n \cdot |\mathcal{W}'|.$$

Consequently, $t > (\frac{1}{2} - \delta - 3n^{-1/3}) \cdot |\mathcal{W}'|$ and we conclude that V' has the claimed property. ◀

We are finally ready to prove our seeker-strategy for finding a $1/2+\delta$-king using $\tilde{O}(n^{4/3})$ queries. For readability, we will state our main result in terms of concrete and simple values for κ and δ, however, note that smaller values of κ allow δ to be slightly larger than the stated bound of $\frac{2}{17}$.

▶ **Theorem 22.** *Fix $\delta = \frac{2}{17}$, let $\kappa = \frac{1}{4000}$. For large enough n, there exists a seeker strategy for finding a $(1/2+\delta)$-king using $\tilde{O}(n^{4/3})$ edge queries.*

Proof. We construct the template-graph G_κ and query all $\tilde{O}(n^{4/3})$ of its edges to obtain $\vec{G}_\kappa$.

By Lemma 14, we either obtain a δ-ultra set of size $2 \cdot n^{2/3}$ or a δ-weak tiling of $\vec{G}_\kappa$. If we find the former, by Observation 8 we can find a $(\frac{1}{2}+\delta)$-king using $O(n^{4/3})$ additional queries. Therefore assume that we obtained a δ-weak tiling $\mathcal{W}, R$ of $\vec{G}_\kappa$.

Let M be the free matrix of $\mathcal{W}, R$. By Lemma 18, M has a δ-good sub-matrix $M[\mathcal{W}, V]$ with $|V| = 2n^{2/3}$. We query all $O(n^{4/3})$ edges in $V \times V$ and by Lemma 19 either identify a $(\frac{1}{2}+\delta)$-king, or obtain partitions $V_1 \uplus V_2 = V$, $\mathcal{W}_1 \uplus \mathcal{W}_2$ with properties as listed in Lemma 19. Importantly, by Lemma 20, every row in the sub-matrix $M[\mathcal{W}_1, V_2]$ has weight at least $(1-2\delta-2\kappa)n^{2/3}$.

We now apply Lemma 21 with $\mathcal{W}' = \mathcal{W}_2$ and find a set of columns $V_3 \subseteq [n] \setminus V$ of size $n^{2/3}$ such that every column in $M[\mathcal{W}_2, V_3]$ has weight at least $(\frac{1}{2} - \delta - 3n^{-1/3})|\mathcal{W}_2|$. We now query all edges in $V_3 \times V_3$ and $V_2 \times V_3$, since $|V_2| = |V_3| = n^{2/3}$ this amounts to $O(n^{4/3})$ additional queries.

Since $\vec{G}[V_2 \cup V_3]$ is completely revealed, it is a tournament of size $2n^{2/3}$ and we apply Lemma 3 using the bipartition (V_2, V_3) to find a vertex $v \in V_2 \cup V_3$ such that $d^+(v, V_2)$ and $d^+(v, V_3)$ are both at least $n^{2/3}/4$. We claim that v is a $(\frac{1}{2}+\delta)$-king. Let in the following $V_2' = N^+(v) \cap V_2$ and $V_3' = N^+(v) \cap V_3$. We first prove the following two claims about these two sets:

▷ Claim 23. Every row in $M[\mathcal{W}_1, V_2']$ has weight at least $\kappa n^{2/3}$.

Proof of the claim. According to Lemma 20, every row in $M[\mathcal{W}_1, V_2]$ has weight at least $(1-2\delta-2\kappa)n^{2/3}$. Since $|V_2'| = |V_2|/4 = n^{2/3}/4$, we have that each row in $M[\mathcal{W}_1, V_2']$ has weight at least

$$(1-2\delta-2\kappa)n^{2/3} - \frac{3}{4}n^{2/3}$$

which is larger than $\kappa n^{2/3}$ for $\delta \leq \frac{1}{8} - \frac{3\kappa}{2}$ which holds true for our choices of δ and κ. ◁

▷ Claim 24. At least $(\frac{1}{2} - \delta - 4\kappa - 3n^{-1/3})\frac{|\mathcal{W}_2|}{1-4\kappa}$ rows in $M[\mathcal{W}_2, V_3']$ have weight at least $\kappa n^{2/3}$.

Proof of the claim. Recall that by choice of V_3, every column in $M[\mathcal{W}_2, V_3]$ and therefore also $M[\mathcal{W}_2, V_3']$ has weight at least $(\frac{1}{2} - \delta - 3n^{-1/3})|\mathcal{W}_2|$. Accordingly,

$$\begin{aligned}\sum M[\mathcal{W}_2, V_3'] &\geq (\frac{1}{2} - \delta - 3n^{-1/3})|\mathcal{W}_2| \cdot |V_3'| \\ &\geq (\frac{1}{2} - \delta - 3n^{-1/3})|\mathcal{W}_2| \cdot \frac{1}{4}n^{2/3}.\end{aligned} \tag{5}$$

Let t denote the number of rows in $M[\mathcal{W}_2, V_3']$ with weight at least $\kappa n^{2/3}$. Our goal is to find a lower bound for t. Since t is minimized if every row that has weigth at least $\kappa n^{2/3}$ has in fact the maximum possible weight $|V_3'| = n^{2/3}/4$, we can lower-bound t using

$$\frac{t}{4}n^{2/3} + (|\mathcal{W}_2| - t)\kappa n^{2/3} \geq \sum M[\mathcal{W}_2, V_3'].$$

Combining this inequality with (5), we obtain

$$t\frac{n^{2/3}}{4} + (|\mathcal{W}_2| - t)\kappa n^{2/3} \geq (\frac{1}{2} - \delta - 3n^{-1/3})|\mathcal{W}_2| \cdot \frac{1}{4}n^{2/3}$$
$$\iff t(1-4\kappa) \geq (\frac{1}{2} - \delta - 4\kappa - 3n^{-1/3})|\mathcal{W}_2|$$
$$\iff t \geq (\frac{1}{2} - \delta - 4\kappa - 3n^{-1/3})\frac{|\mathcal{W}_2|}{1-4\kappa}. \blacktriangleleft$$

Now note that for every tile $W \in \mathcal{W}$ for which the row $M[\{W\}, V_2' \cup V_3']$ has weight at least $\kappa n^{2/3}$ we have that $d^+(v, F(W)) \geq \kappa n^{2/3}$, therefore by Lemma 12 the set $N^+(v) \cap F(W)$ $(1-\kappa)$-covers W. In other words, v controls at least $(1-\kappa)n^{2/3}$ vertices in W.

Our goal is now to lower-bound the total number of such tiles, hence let s denote the total number of rows in $M[\mathcal{W}, V_2' \cup V_3']$ with weight at least $\kappa n^{2/3}$. By the previous two observations and by plugging in the concrete values of $\delta = \frac{2}{17}$ and $\kappa = \frac{1}{4000}$, we have that

$$s \geq |\mathcal{W}_1| + (\frac{1}{2} - \delta - 4\kappa - 3n^{-1/3})\frac{|\mathcal{W}_2|}{1-4\kappa}$$
$$= |\mathcal{W}_1| + (\frac{6483}{17000} - 3n^{-1/3})|\mathcal{W}_2|\frac{1000}{999}.$$

Again we are aiming to prove a lower-bound, thus we assume that $\mathcal{W}_1$ is as small as possible. By Lemma 19, this means that

$$|\mathcal{W}_1| = (\frac{1}{2} - \delta - \kappa)n^{1/3} - 2 = \frac{25983}{68000}n^{1/3} - 2 \quad \text{and}$$
$$|\mathcal{W}_2| = (\frac{1}{2} + \delta + \kappa)n^{1/3} = \frac{42017}{68000}n^{1/3}.$$

Plugging in the sizes of $\mathcal{W}_1, \mathcal{W}_2$ we obtain

$$s \geq |\mathcal{W}_1| + (\frac{6483}{17000} - 3n^{-1/3})|\mathcal{W}_2|\frac{1000}{999}$$
$$\geq \frac{25983}{68000}n^{1/3} + (\frac{6483}{17000} - 3n^{-1/3})\frac{42017}{67932}n^{1/3} - 2$$
$$\geq \frac{475777}{769896}n^{1/3} - 4.$$

Since v controls a $(1-\kappa) = \frac{3999}{4000}$ fraction of each tile counted by s and each tile has a size of $n^{2/3}$, we finally have the following lower bound on the second out-neighbourhood of v:

$$|N^{++}[v]| \geq \frac{3999}{4000}sn^{2/3} \geq \frac{3999}{4000} \cdot \frac{475777}{769896}n - 4n^{2/3}$$
$$= 0.61782\ldots n - 4n^{2/3}.$$

This value lies, for large enough n, above our target value of $(\frac{1}{2} + \delta)n = 0.61764\ldots n$. ◀

5 Conclusion

We have shown how the usage of a *template-graph* helped us devise a seeker strategy that reveals a $(\frac{1}{2} + \frac{2}{17})$-king in a tournament using $\tilde{O}(n^{4/3})$ queries, shedding light on a long-standing open problem. Our approach begins with a non-adaptive querying strategy based on what we called a *template graph*, which then helps to guide the seeker to identify a small set of queries which necessarily lead to the discovery of a $(\frac{1}{2} + \frac{2}{17})$-king.

Naturally, we ask whether it is possible to find an improved strategy which reveals a $(\frac{1}{2} + \delta)$-king with δ substantially larger than $\frac{2}{17}$ using a similar amount of queries.

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