IDEALS WITH MAXIMAL LOCAL COHOMOLOGY MODULES

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INTRODUCTION AND NOTATIONS

This paper finds its motivation in the pursuit of ideals whose local cohomology modules have maximal Hilbert functions. In [11], [12] we proved that the lexicographic (resp. squarefree lexicographic) ideal of the family of graded (resp. squarefree) ideals with assigned Hilbert function provides sharp upper bounds for the local cohomology modules of any ideal of the family. More precisely, the Hilbert series of the local cohomology modules of any ideal of the family are smaller than or equal to those of the lexicographic ideal. Moreover these bounds are determined explicitly in terms of the Hilbert function, which is the specified starting data. In the present paper a characterization of the class of ideals with the property of having maximal local cohomology modules in the sense explained above is accomplished.

Let us set some notation to be used henceforth.

Let $R \doteq K[X_1, \ldots, X_n]$ denote the polynomial ring in n variables over a field K of characteristic 0 with its standard grading, $\mathfrak{m} \doteq (X_1, \ldots, X_n)$ the maximal homogeneous ideal of R. We set $X_1 > X_2 > \ldots > X_n$ and consider the lexicographic order induced by this assignment on M_d, the set of all monomials of R of degree d, for all d. A *lex-segment of degree d* is thus a set $L = \{u \in M_d \colon u \geq v\}$ for some $v \in M_d$, and a graded ideal is called *lexicographic* if every graded component of I is generated as a K-vector space by a lex-segment. It is well known that given a homogeneous ideal $I \subseteq R$ there exists a unique lexicographic ideal which has the same Hilbert function as I. We shall denote it by I^{lex} and call it the lexicographic ideal associated with I.

The canonical module of R will be denoted by $\omega_R \simeq R(-n)$. If M stands for a graded R-module, then $\mathrm{Hilb}(M, t)$ will denote its Hilbert series in terms of t. The local cohomology modules $H^i_{\mathfrak{m}}(M)$ of M will be considered with support on the maximal graded ideal $\mathfrak{m}$ and with their natural grading. We write $h^i(M)_j$ for the dimension as K-vector space of $H^i_{\mathfrak{m}}(M)_j$. The dual of the local cohomology modules according to the Local Duality Theorem will be denoted with $E^i(M) \doteq \mathrm{Ext}^i_R(M, \omega_R)$.

We write I^{sat} for the saturation of an ideal I with respect to $\mathfrak{m}$.

A well known theorem of Bigatti-Hulett-Pardue states that in the family of graded ideals with a given Hilbert function the lexicographic ideal has the greatest Betti numbers. The class of ideals with the same resolution as that (i.e. with all of the Betti numbers equal to those) of the lexicographic ideal is characterized in [7]. These ideals are the so-called *Gotzmann ideals*. We recall that an ideal is called Gotzmann iff in each degree it has the same number of generators as its associated lexicographic ideal. Note that the definition can be re-read as follows: Gotzmann ideals have the same 0th graded Betti numbers as those of their associated lexicographic ideal. The result of [7] shows then that the maximality of all the other graded Betti numbers is forced by that of the 0th ones.
It is worth to point out that a similar behaviour underlies our situation as well, where Gotzmann ideals will play again some important role. We state now the main result of this paper.

Theorem 0.1. *For any graded module I, the following are equivalent conditions:*

   (i) $(I^{sat})^{lex} = (I^{lex})^{sat}$;
  (ii) $h^0(R/I)_j = h^0(R/I^{lex})_j$, *for all j;*
 (iii) $h^i(R/I)_j = h^i(R/I^{lex})_j$, *for all i, j.*

Thus, the theorem states that, as it happens in the context of Betti numbers, the equality of the 0th local cohomology forces the equality of any other. One can wonder if this sort of rigidity behaviour is to be expected more generally, i.e. is it true that if $h^i(R/I)_k = h^i(R/I^{lex})_k$ for some i and all k, then $h^j(R/I)_k = h^j(R/I^{lex})_k$ for all $j \geq i$ and all k? There is some computational evidence that this might be true. The analogous question for the graded Betti numbers has been investigated in [5], where it has been answered positively.

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1. GOTZMANN IDEALS AND SEQUENTIALLY CM MODULES

Let us start by proving the equivalence of Conditions (i) and (ii) of Theorem 0.1.

Lemma 1.1. *For any ideal I, we have that*

$$(I^{sat})^{lex} = (I^{lex})^{sat} \text{ iff } h^0(R/I)_j = h^0(R/I^{lex})_j, \text{ for all } j.$$

Proof. Observe that the inclusion $(I^{sat})^{lex} \subseteq (I^{lex})^{sat}$ holds true in general. In fact, one shows first that $(I : \mathfrak{m})^{lex} \subseteq I^{lex} : \mathfrak{m}$, which is easy and descends from the property which defines lexicographic ideals. Secondly, one observes that $(I : \mathfrak{m}^2)^{lex} = ((I : \mathfrak{m}) : \mathfrak{m})^{lex}$ which is contained in $(I : \mathfrak{m})^{lex} : \mathfrak{m} \subseteq I^{lex} : \mathfrak{m}^2$. Proceeding in this manner will eventually lead to the desired inclusion. Now one notices that Condition (ii) provides an information about some Hilbert functions: Since $H^0_{\mathfrak{m}}(R/I) \simeq I^{sat}/I$, one has that $h^0(R/I)_j = \dim_K I^{sat}_j - \dim_K I_j = \dim_K (I^{sat})^{lex}_j - \dim_K I^{lex}_j$, which is less than or equal to $\dim_K (I^{lex})^{sat}_j - \dim_K I^{lex}_j = h^0(R/I^{lex})_j$ for what we said before. One sees immediately that equality holds iff $(I^{sat})^{lex} = (I^{lex})^{sat}$, and this completes the proof. ▲

Let us denote by $\text{Gin}(I)$ the *generic initial ideal of I* with respect to the reverse lexicographical order induced by the assignment $X_1 > X_2 > \ldots > X_n$. In [9] we studied the problem of characterizing those ideals such that $h^i(R/I)_j = h^i(R/\text{Gin}(I))_j$. For the reader's sake we recall the main theorem here and recall the definition of sequentially Cohen-Macaulay after Proposition 1.8 when it is really needed.

Theorem 1.2. *Let M be a finitely generated graded R-module with graded free presentation $M = F/U$. The following conditions are equivalent.*

(a) *F/U is sequentially CM;*
(b) *for all $i \geq 0$ and all j one has $h^i(F/U)_j = h^i(F/\text{Gin}(U))_j$.*

In general it holds that $h^i(R/I)_j \leq h^i(R/\text{Gin}(I))_j \leq h^i(R/I^{lex})_j$ (see [11]). Thus, the class of ideals we are searching for must have the property that R/I is sequentially CM. Therefore one may state a fourth condition, wondering if this is equivalent to those of Theorem 0.1:

(iv) R/I is sequentially CM and $\text{Gin}(I) = I^{lex}$.

By virtue of the above theorem one sees that $(iv) \Rightarrow (iii)$. In fact, if R/I is sequentially CM, for all i, j one has $h^i(R/I)_j = h^i(R/\text{Gin}(I))_j$, where the latter is equal to $h^i(R/I^{lex})_j$, since $\text{Gin}(I) = I^{lex}$.
On the other hand, one sees that in general (iv) needs not to be implied by (i). First let us prove an easy lemma.

Lemma 1.3. *Let I be a homogeneous ideal and $\text{Gin}(I)$ its generic initial ideal. If $(I^{sat})^{lex} = (I^{lex})^{sat}$ then $(\text{Gin}(I)^{sat})^{lex} = (\text{Gin}(I)^{lex})^{sat}$.*

Proof. As observed at the beginning of this section, one inclusion is trivially true: $(\text{Gin}(I)^{sat})^{lex} \subseteq (\text{Gin}(I)^{lex})^{sat}$. Thus, it is enough to show that the two ideals have the same Hilbert function. Since I and

$\text{Gin}(I)$ have the same Hilbert function and therefore same lexicographic ideal, one has $(\text{Gin}(I)^{lex})^{sat} = (I^{lex})^{sat} = (I^{sat})^{lex}$, by hypothesis. Recall now that $\text{Gin}(I^{sat}) = \text{Gin}(I)^{sat}$ (see for instance [6]). Therefore, $H(I^{sat}, t) = H(\text{Gin}(I^{sat}), t) = H(\text{Gin}(I)^{sat}, t)$, and as a consequence $H(I^{sat}, t) = H((\text{Gin}(I)^{sat})^{lex}, t)$. Finally, we deduce that

$$\begin{aligned} H((\text{Gin}(I)^{lex})^{sat}, t) &= H((I^{sat})^{lex}, t) = H(I^{sat}, t) \\ &= H((\text{Gin}(I)^{sat})^{lex}, t), \end{aligned}$$

which yields the desired conclusion. ▲

By virtue of the above lemma, we need now an example of a strongly stable ideal, which is not lexicographic, but satisfies (i). This is provided in what follows.

Example 1.4. Let $I = (x^2, xy, y^2, xz^2, yz^2)$ be an ideal of $K[x, y, z]$. It is easy to verify that I is strongly stable and therefore $I = \text{Gin}(I)$. An easy computation provides that the associated lex-ideal is $I^{lex} = (x^2, xy, xz, y^3, y^2z, yz^2)$. Thus, $I \neq I^{lex}$ and $(I^{sat})^{lex} = (I^{lex})^{sat} = (x, y)$. This shows that Condition (i) is not equivalent to Condition (iv).

Next, some lemmata which illustrate our hypothesis and characterize it. We shall prove that an ideal with the exchange property is sequentially CM.

Lemma 1.5. *Let I be a homogeneous ideal. Then*

$$I = I^{sat} \ and \ (I^{sat})^{lex} = (I^{lex})^{sat} \Leftrightarrow I^{lex} = (I^{lex})^{sat}.$$

Proof. "$\Rightarrow$": It is immediately seen.
"$\Leftarrow$": Since $I^{lex} = (I^{lex})^{sat}$, one has that $H^0_\mathfrak{m}(R/I^{lex}) = 0$ and, by virtue of [11], Theorem 5.4, also $H^0_\mathfrak{m}(R/I) = 0$, i.e. $I = I^{sat}$. Now the conclusion follows immediately. ▲

Lemma 1.6. *Let I be a homogeneous ideal. If $I = I^{sat}$ is Gotzmann then $(I^{sat})^{lex} = (I^{lex})^{sat}$.*

Proof. By hypothesis depth $R/I > 0$ and since I is a Gotzmann ideal, it has the same resolution as I^{lex}. Therefore, also R/I^{lex} has positive depth. This implies that $I^{lex} = (I^{lex})^{sat}$ and we are done. ▲

The stronger counterpart of the above lemma is the following: Let I be a homogeneous ideal such that $(I^{sat})^{lex} = (I^{lex})^{sat}$. Then I^{sat} is Gotzmann. One can see that this last statement is equivalent to that of the following lemma.

Lemma 1.7. *Any ideal I such that I^{lex} is saturated is a Gotzmann ideal.*

Before we start to prove the latter fact, it is worth to underline that if the lexicographic ideal is saturated, the Hilbert function has a very rigid behaviour. In fact, saying that any ideal associated with that lexicographic ideal is Gotzmann implies that any of these ideals has the same resolution as the lex-ideal.

A saturated lexicographic ideal has indeed a very special structure and its generators can be described explicitly in terms of the Hilbert polynomial of R/I by means of a vector v of integers in a way that we are going to recall for the reader's sake.

Let $P_{R/I}(X) = \binom{X+a_1}{a_1} + \binom{X+a_2-1}{a_2} + \ldots + \binom{X+a_l-(l-1)}{a_l}$, with $a_1 \geq a_2 \geq \ldots a_l \geq 0$ be a representation of the Hilbert polynomial, also referred to as its *Gotzmann representation*. Let now $v_i \doteq |\{j \colon n - a_j - 1 = i\}|$, for $i = 1, \ldots, n-1$ and order the monomials of the minimal set of generators of I^{lex}, call it $G(I^{lex})$, lexicographically. It is not difficult to see that v_i represents the exponent of the variable X_i in the least monomial of highest degree in $G(I^{lex})$. Let us also set h to be the maximum index of a non-zero v_i. Then, the minimal set of generators of $(I^{lex})^{sat}$ is the set

$$\{X_1^{v_1+1}, X_1^{v_1} X_2^{v_2+1}, \ldots, X_1^{v_1} \cdot \ldots \cdot X_{h-1}^{v_{h-1}+1}, X_1^{v_1} \cdot \ldots \cdot X_{h-1}^{v_{h-1}} X_h^{v_h}\},$$

where, according to our settings, $v_i \geq 0$, for $i = 1, \ldots, n-1$ and $v_h > 0$. We also recall that the vanishing of the local cohomology modules $H_{\mathfrak{m}}^i(R/I^{lex})$ is determined by the vanishing of the $(n-i)^{th}$ entry of the vector $v = (v_1, \ldots, v_h)$ (cf. [11], Proposition 6.6).

Proof of Lemma 1.7. We make use of an induction argument on h. If $h = 1$, then I^{lex} is simply the principal ideal (X_1^a), for some $a \in \mathbb{N}_{>0}$, and there is nothing to prove. Suppose now the thesis proven for any ideal such that the length of the vector v is $h-1$. There are two possible cases. If $v_1 = 0$, then the ideal I^{lex} contains the linear form X_1, and consequently I contains a linear form, let us say l. Thus $I^{lex} = (X_1, J)$ and $I = (l, I')$, for some ideals $J = J'R$, where J' is the saturated lexicographic ideal in $R' \doteq K[X_2, \ldots, X_n]$ represented by the vector $(v_2, \ldots, v_h)$, and $I' \subset R$. Clearly X_1 is R/J-regular, and we also may assume that l is R/I'-regular. From this fact one deduces that J' is the lexicographic ideal associated with $I'R'$, and one can use the inductive hypothesis to reach the conclusion.

Otherwise, if $v_1 > 0$, we observe that $\mathrm{grade}_{I^{lex}}(R) = 1$ since $E^1(R/I^{lex}) \neq 0$, and that $I^{lex} = X_1^{v_1} J$, where J is a saturated lexicographic ideal represented by the vector $(0, v_2, \ldots, v_h)$ in $K[X_1, \ldots, X_n]$. Moreover v_1 equals the multiplicity e of R/I^{lex}, which is the same as that of R/I.

We also have that $\mathrm{grade}_I(R) = 1$, since the dimension of R/I is the same as that of R/I^{lex}, therefore I can be written as a product of a polynomial f times an ideal I', whose grade is bigger than 1. Observe that $\deg f$ equals the multiplicity of R/I, which is v_1. For showing this simple fact, let us look at the short exact sequence $0 \to fR/I \to R/I \to R/fR \to 0$. It is easy to see that the multiplicity of R/fR is $\deg f$, since the h-vector of R/fR is $\sum_{i=1}^{\deg f} t^i$, while its dimension is $n-1$. On the other hand the Hilbert function of the left-hand side module is up to a shift that of the module R/J, whose dimension is less than $n-1$ and therefore cannot contribute to the multiplicity of R/I.

Since $t^{\deg f} \mathrm{Hilb}(I',t) = \mathrm{Hilb}(I,t) = \mathrm{Hilb}(I^{lex},t) = t^{v_1}\mathrm{Hilb}(J,t)$, we deduce that J is the lexicographic ideal associated with I', and the proof of the lemma is complete by the use of the previous case. ▲

Proposition 1.8. *Let I be an ideal such that $(I^{sat})^{lex} = (I^{lex})^{sat}$. Then R/I^{sat} is sequentially CM.*

It is now convenient to recall the definition of sequentially Cohen-Macaulay modules. A finitely generated graded R-module M is said to be *sequentially Cohen-Macaulay* if there exists a finite filtration $0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_r = M$ of M by graded submodules of M such that each quotient M_i/M_{i-1} is CM, and $\dim(M_1/M_0) < \dim(M_2/M_1) < \ldots \dim(M_r/M_{r-1})$.

Sequentially CM modules have an interesting characterization in terms of their homological properties, as illustrated by a theorem of Peskine (cf. [9], Theorem 1.4). Peskine's result asserts that a module M is sequentially CM iff for all $0 \leq i \leq \dim M$ the modules $E^{n-i}(M) \doteq \mathrm{Ext}_R^{n-i}(M, \omega_R)$ are either 0 or CM of dimension i. For a more complete overview of the properties of sequentially CM modules we refer to [9], and we proceed by proving Proposition 1.8.

Proof. Let us assume that I^{lex} is saturated and let us prove that R/I is sequentially CM. We shall use the same notation as above. In particular the vector that determines I^{lex} will be denoted by v, and v_i, for $i = 1, \ldots, h$, will denote its entries.

The idea is the same as that of the proof of the above lemma, by induction on the index h. If $h = 1$, then I and I^{lex} are principal, therefore R/I and R/I^{lex} are Cohen-Macaulay, and Cohen-Macaulay modules are obviously sequentially CM.

If $h > 1$ we may assume without loss of generality that v_1 is 0 and therefore that I and I^{lex} contain the linear form X_1. The following is an application of the graded version of the Rees' Lemma. Let $I = (X_1, I')$

and $I^{lex} = (X_1, J)$, where $J = J'R$ and J' is the (saturated) lexicographic ideal associated with $I'(R/(X_1))$ determined by the vector $(v_2, \ldots, v_h)$ in $K[X_2, \ldots, X_n]$. Thus we have graded isomorphisms for all $i \geq 1$

$$\begin{aligned}
\mathrm{Ext}^i_R(R/I, R) &\simeq \mathrm{Ext}^i_R(R/(X_1, I'), R) \\
&\simeq \mathrm{Ext}^{i-1}_{R/(X_1)}((R/(X_1))/(I'(R/(X_1))), R/(X_1))(1),
\end{aligned}$$

which, by induction, is either 0 or Cohen-Macaulay of dimension $(n - 1) - (i - 1) = n - i$. By virtue of the aforementioned homological characterization of sequentially CM modules this is equivalent to the thesis. ▲

As an immediate consequence of the above observations, we deduce the property we were interested in.

Proposition 1.9. *Let I be an ideal such that $(I^{sat})^{lex} = (I^{lex})^{sat}$. Then R/I is sequentially CM.*

Proof. By virtue of the above proposition it is enough to notice that an R-module M is sequentially CM iff $M/H^0_{\mathfrak{m}}(M)$ is sequentially CM. In our case $M = R/I$ and $M/H^0_{\mathfrak{m}}(M) \simeq (R/I)/(I^{sat}/I) \simeq R/I^{sat}$. ▲

Observe that if I is a proper cyclic ideal, its lexicographic ideal is (X_1^d) for some positive integer d, which is saturated. Moreover, R/I is CM of dimension $n - 1$, and its only non-vanishing Ext-group is the first one. This is isomorphic to a shifted copy of R/I itself, and therefore cyclic. Applying this observation to the inductive argument of the proof of Proposition 1.8, one deduces the following.

Proposition 1.10. *Let I be an ideal such that $(I^{sat})^{lex} = (I^{lex})^{sat}$, then all of its Ext-groups except possibly the nth-one are cyclic.*

2. Proof of Theorem 0.1

We prove first the result for strongly stable ideals. Recall that a monomial ideal I is said to be *strongly stable* if, for every $u \in I$, one has $X_i u / X_j \in I$ for all $X_j | u$ and $i < j$. For a strongly stable ideal one has that $I^{sat} = I : (X_n)^\infty$. In particular, one knows that R/I has positive depth iff X_n does not appear in any of the monomials which minimally generate I. Suppose now that I is strongly stable and that X_n is R/I^{lex}-regular. Then, if we denote by a $\bar{\cdot}$ the equivalence classes in the quotient of the polynomial ring by the last variable, one has $\overline{I^{lex}} = \overline{I}^{lex}$. Let us give a quick explanation for this fact. Since I^{lex} is strongly stable as well and X_n is R/I^{lex}-regular, none of the monomial of the minimal set of generators of I^{lex} contain the last variable, thus $\overline{I^{lex}}$ is a lexicographic

ideal in the variables $X_1, \ldots, X_{n-1}$. Since $\operatorname{depth} R/I^{lex} \leq \operatorname{depth} R/I$, and I is a strongly stable ideal, for the reason explained above X_n is also R/I-regular. Therefore we can control the behaviour of the Hilbert function when passing to the quotient by the last variable, and the conclusion follows easily.

The following is a technical lemma about local cohomology and we refer to [2] or [3] for more details about the Local Duality Theorem and the properties of the canonical module.

Lemma 2.1. *Let $I \subset R$ an ideal of R and let $S \doteq R[X]$, with maximal graded ideal $\mathfrak{n}$. The following graded isomorphism of R-modules holds.*

$$H^i_{\mathfrak{n}}(S/IS) \simeq \operatorname{Hom}_R(S, H^{i-1}_{\mathfrak{m}}(R/I))(1).$$

Proof. The relation $\omega_S = (\omega_R \otimes_R S)(-1)$ between the canonical modules of S and R is well known. By the Local Duality Theorem, one has that $H^i_{\mathfrak{n}}(S/IS)$ is the dual of $\operatorname{Ext}^{n+1-i}_S(S/IS, \omega_S)$, which is defined to be $\operatorname{Hom}_K(\operatorname{Ext}^{n+1-i}_S(S/IS, \omega_S), K)$. After writing $S = S \otimes_R R$ and substituting ω_S by means of the formula written above, one obtains that the latter is isomorphic to $\operatorname{Hom}_K(\operatorname{Ext}^{n+1-i}_{S \otimes_R R}(R/I \otimes_R S, (\omega_R \otimes_R S)(-1)), K) \simeq \operatorname{Hom}_K(\operatorname{Ext}^{n+1-i}_{S \otimes_R R}(R/I \otimes_R S, \omega_R \otimes_R S), K)(1)$ and, since S is R-flat, this is isomorphic to $\operatorname{Hom}_K(S \otimes_R \operatorname{Ext}^{n+1-i}_R(R/I, \omega_R), K)(1)$. Using a well-known formula in homological algebra and applying the Local Duality Theorem again, one finally deduces

$$
\begin{aligned}
H^i_{\mathfrak{n}}(S/IS) &\simeq \operatorname{Hom}_R(S, \operatorname{Hom}_K(\operatorname{Ext}^{n+1-i}_R(R/I, \omega_R), K))(1) \\
&\simeq \operatorname{Hom}_R(S, H^{i-1}_{\mathfrak{m}}(R/I))(1),
\end{aligned}
$$

as required. ▲

Lemma 2.2. *Let $S = R[X]$ be a polynomial ring in one indeterminate over R with graded maximal ideal $\mathfrak{n}$. Let I be an ideal of R. Then, for all i, j,*

$$\dim_K H^i_{\mathfrak{n}}(S/IS)_j = \sum_{h \geq j} \dim_K H^{i-1}_{\mathfrak{m}}(R/I)_{h+1}.$$

Proof. The proof is based on the above lemma and on some considerations about the S-module structure of $\operatorname{Hom}_R(S, N)$, where N is an arbitrary R-module. Let us consider the R-linear application $\cdot : S \times \operatorname{Hom}_R(S, N) \to \operatorname{Hom}_R(S, N)$ defined by $\cdot (s, \varphi)(t) \doteq s \cdot \varphi(t) \doteq \varphi(st)$. This map endows $\operatorname{Hom}_R(S, N)$ with an S-module structure.

Let now N and S be graded. One defines a graded structure of S-module on $\operatorname{Hom}_R(S, N)$ as follows. We set

$$\operatorname{Hom}_R(S, N)_i \doteq \{\varphi \in \operatorname{Hom}_R(S, N) \mid \varphi(S_j) \subset N_{i+j}, \text{for all } j\}$$

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to be the ith graded component of $\mathrm{Hom}_R(S, N)$.

Observe that, if $\varphi \in \mathrm{Hom}_R(S, N)_i \cap \mathrm{Hom}_R(S, N)_j$ and $i \neq j$, then $\varphi(S_k) \subset N_{i+k} \cap N_{j+k} = (0)$, i.e. $\varphi = 0$. If $\varphi \in \mathrm{Hom}_R(S, N)_i$, s is an element of S_j and $t \in S_k$ then $s\varphi(t) = \varphi(st) \in N_{i+(k+j)}$, i.e. $s\varphi \in \mathrm{Hom}_R(S, N)_{i+j}$. Finally, one can verify that $\mathrm{Hom}_R(S, N) \subset \oplus_i \mathrm{Hom}_R(S, N)_i$.

Let now S be as in the hypothesis and consider the homogeneous isomorphism of graded S-modules α

$$\mathrm{Hom}_R(S, N) \overset{\alpha}{\to} \Pi_{j \geq 0} N x^{-j}$$

that maps an element φ of $\mathrm{Hom}_R(S, N)$ into $(\ldots, \varphi(x^j)x^{-j}, \ldots)$.

Thus, $\mathrm{Hom}_R(S, N)_i \simeq (\Pi_{j \geq 0} N x^{-j})_i \simeq \oplus_{h \geq i} N_h$, and if N is Artinian one can deduce that the dimension as a K-vector space of $\mathrm{Hom}_S(R, N)$ is just the sum of the dimensions of the graded components N_h of N with $h \geq i$. Now the conclusion follows from Lemma 2.1. ▲

Proposition 2.3. *Let I be a strongly stable ideal such that $(I^{sat})^{lex} = (I^{lex})^{sat}$. Then*

$$h^i(R/I)_j = h^i(R/I^{lex})_j, \; \text{for all } i, j.$$

Proof. As we noticed already several times, the hypothesis is equivalent to saying that $h^0(R/I)_j = h^0(R/I^{lex})_j$ for all j. We may then assume that I is saturated, i.e. that depth R/I is positive. By induction on n we also suppose the thesis to be true for any strongly stable ideal on a polynomial ring with less than n variables. Bearing in mind the remarks before Lemma 2.2, the conclusion is a straightforward application of the latter. ▲

Proof of Theorem 0.1, $(i) \Rightarrow (iii)$. By Proposition 1.9, we know that R/I is sequentially CM. If $\mathrm{Gin}(I) = I^{lex}$, we achieve the conclusion immediately for what we said before Lemma 1.3. Otherwise, by virtue of Lemma 1.3 we may assume without loss of generality that I is strongly stable, by substituting it with its generic initial, since $h^i(R/I)_j = h^i(R/\mathrm{Gin}(I)_j$. The thesis is thus an application of Proposition 2.3. ▲

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