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A note on regularity of weak solutions of the Navier-Stokes equations in $\mathbf{R}^n$

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Abstract. In this paper we consider the n dimensional Navier-Stokes equations and we prove a new regularity criterion for weak solutions. More precisely, if $n = 3, 4$ we show that the "smallness" of at least $n-1$ components of the velocity in $L^\infty(0,T;L^n_w(\mathbf{R}^n))$ is sufficient to ensure regularity of the weak solutions.

1. Introduction

The aim of this paper is to improve a criterion for the regularity of weak solutions of the Navier-Stokes equations in $\mathbf{R}^n$ and to give a simpler proof of some known results. We shall consider the system of the Navier-Stokes equations below

$$(1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} + (u\cdot\nabla)\,u - \nu\Delta u + \nabla p = f & \text{in } \mathbf{R}^n\times(0,T),\\ \nabla\cdot u = 0 & \text{in } \mathbf{R}^n\times(0,T),\\ u(x,0) = u_0(x) & \text{in } \mathbf{R}^n, \end{cases}$$

and to avoid inessential calculation we assume that the external force f vanishes, even if it is easy to include nonzero smooth forces. Here $L^p := (L^p(\mathbf{R}^n))^n$, for $1 \le p \le \infty$ with norm $\|\,.\,\|_p$, is the usual Lebesgue space. It is well-known that if a weak solution belongs to

$$(2)\qquad L^p(0,T;L^q) \quad \text{with} \quad \frac{2}{p}+\frac{n}{q}\le 1, \quad \text{for} \quad q>n,$$

then it is regular, see Prodi [18] and Serrin [19] if $2/p+n/q<1$. For the case with $2/p+n/q=1$ see for instance Sohr [20]. A recent monograph on the existence and regularity theory for the Navier-Stokes equations, that collects almost all the results cited and used in this paper, is the book of Galdi [7]. The classical result

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above, involving the condition (2) (which is present just in the 1934 Leray's paper [13]), does not cover the limit case $p = \infty$ and $q = n$, but the remarkable result of non-existence of *Leray's self similar solutions* in L^3 proved in Nečas, Růžička, and Šverák [17], may also suggest that $L^\infty(0,T;L^3)$ is a regularity class. Recall also that the study of L^3 solutions is interesting by itself, since it is a starting point in the dimensionfree estimates of Caffarelli, Kohn, and Nirenberg [6].

In this direction there are some partial results, in particular it is known that

$$L^\infty(0,T;L^n) \tag{3}$$

is a uniqueness class, see Kozono and Sohr [11]. See also P.-L. Lions and Masmoudi [15]. A little bit stronger condition on the time variable, for instance continuity (Sohr and von Whal [21]), or left continuity (Masuda [16]), or bounded variation (Kozono and Sohr [12]) ensures the regularity. For a review on these results see Kozono [10].

Another condition, ensuring the regularity, is the one involving the so called "Hypothesis A", introduced by Beirão da Veiga [3]. In reference [3] it is proved (for the problem with $n = 3, 4$) that if $u \in L^\infty(0,T;L^n)$ satisfies the "Hypothesis A," then u is a strong solution of (1) in $(0,T)$. Observe that, as quite particular cases, Hypothesis A is satisfied if the solution is left-continuous or of bounded variation with values in L^n or even if the jumps of the L^n-norm are small enough. In Beirão da Veiga [4] this result is improved by showing that, if at least $n-1$ components of u satisfy Hypothesis A, then the solution is regular.

On the other hand it is well-known that if the norm of u in $L^\infty(0,T;L^n)$ is small enough, then the solution is smooth. For an elementary proof see [3]. By requiring some smallness it is possible to weaken the condition in the space variables. In Kozono [10] it is also shown that the smallness in

$$L^\infty(0,T;L^n_w) \tag{4}$$

is enough to have regularity, where L^n_w is the weak-L^n space.

In this paper we give a simple proof of this result and we improve it by showing that only $n-1$ components of the velocity satisfying the latter condition is sufficient to have smoothness of weak solutions.

Our calculations are true at a formal level for any n, but the result will follow for $n \leq 4$. In particular, our result encompasses the case $n = 3$, which is more interesting from the physical point of view. For $n \geq 5$ it is necessary to resort to more technical methods, see for example Struwe [22]. Recall also (see § 4 in Kato [9]) that $n = 4$ is a limit case, just for the problem of existence of weak solutions.

We recall that the divergence-free vector u is a *weak* (*or Leray-Hopf*) *solution* of the Navier-Stokes equations if it has the following regularity properties

$$u \in C_w(0,T;L^2) \cap L^2(0,T;H^1), \quad \text{with} \quad \frac{\partial u}{\partial t} \in L^1_{loc}(0,T;H^{-1}); \tag{5}$$

if it satisfies

(6)
$$\int_0^T \int_{\mathbf{R}^n} \left[u\frac{\partial \phi}{\partial t} - \nu \nabla u \cdot \nabla \phi - (u \cdot \nabla)\, u\, \phi \right] dx\, dt = \int_{\mathbf{R}^n} [u(T)\phi(T) - u_0\phi(0)] dx,$$

for all divergence-free $\phi \in C^1(0,T; H^1 \cap L^n)$ and furthermore if the so called "energy inequality" holds:

$$\frac{1}{2}\|u(t)\|_2 + \int_0^t \|\nabla u(s)\|_2\, ds \leq \frac{1}{2}\|u_0\|_2, \quad \text{for any } t \geq 0. \tag{7}$$

Here $W^{k,p} := (W^{k,p}(\mathbf{R}^n))^n$ and $H^s := W^{s,2}$ are the customary Sobolev spaces. Recall that $C_w(0,T;L^2)$ is the space of weakly continuous functions on $(0,T)$ with values in L^2.

We say that a weak solution u is *strong* if it also satisfies:

$$u \in L^2(0,T;H^2) \cap L^\infty(0,T;H^1), \quad \text{with } \frac{\partial u}{\partial t} \in L^2(0,T;L^2). \tag{8}$$

In particular it is well-known that strong solutions are unique in the class of weak solutions. Furthermore, a strong solution is also regular (say a classical solution). Note that, by a standard result of functional analysis, condition (8) implies that $u \in C(0,T;H^1)$ and, if $n \leq 4$, we have that $H^1 \subset L^n$. Then strong solutions belong to $C(0,T;L^n)$, that is a regularity class.

We denote by $[\,.\,]_n$, the quasi-norm of or weak-L^n space, see Section 2 for the precise definition of this space and some of its properties. In the sequel we shall need the following definition.

DEFINITION 1. Given a vector $b \in \mathbf{R}^n$, we define $\widehat{b}$ as the projection of b onto the hyperplane generated by the first $n-1$ vectors of an orthonormal basis of $\mathbf{R}^n$.

The main result we shall prove is the following.

THEOREM 1.1. *Let $u_0 \in H^1$, with $\nabla \cdot u_0 = 0$ and let u be a weak solution of the Navier-Stokes equations* (1) *in $\mathbf{R}^n \times (0,T)$, for $n \leq 4$. There exists a positive constant C_0 such that if*

$$\sup_{0<t<T} [\widehat{u}(t)]_n < C_0, \tag{9}$$

then the solution u is strong in $(0,T)$.

REMARK 1.2. The positive constant C_0 depends only on the space dimension n and on the viscosity coefficient ν. Moreover, C_0 is independent of the initial datum and of the time T, see the inequality (13) below.

2. Some results regarding Lorentz spaces

In the sequel we shall use some results related to Lorentz spaces. We briefly recall the definition of such spaces. Given a measurable function $f : \mathbf{R}^n \to \mathbf{R}$ we say that $f \in L^{p,q}$, for $1 \le p \le \infty$ and $1 \le q < \infty$, if

$$\|f\|_{p,q} := \left[\int_0^\infty \left[\sigma\left(\mu\{x \in \mathbf{R}^n : \; |f(x)| > \sigma\}\right)^{1/p}\right]^q \frac{d\sigma}{\sigma}\right]^{1/q} < \infty,$$

where $\mu\{\,.\,\}$ denotes the Lebesgue measure. With this definition the spaces $L^{p,q}$ are not Banach spaces, since $\|\,.\,\|_{p,q}$ is just a quasi-norm. If $p > 1$ it is possible to define $L^{p,q}$ in an equivalent way so as to make it a Banach space, see Ch. 1 in Bergh and Löfström [5]. We observe that $L^{p,\infty} = L^p_w$ the Marcinkiewicz (or weak) L^p space of measurable functions such that

$$\|f\|_{L^p_w} := [f]_p := \sup_{\sigma > 0} \sigma\left(\mu\{x \in \mathbf{R}^n : \; |f(x)| > \sigma\}\right)^{1/p} < \infty.$$

We also recall that, if $r_1 \le r_2$, then $L^{p,r_1} \subset L^{p,r_2}$; furthermore $L^{p,p}$ is isomorphic to the usual Lebesgue space L^p, for $1 \le p < \infty$.

For our purposes we need the following lemma, which is proved in Kozono [10], Section 2. For the reader's convenience we sketch its proof.

LEMMA 2.1. *Let v be in $\mathcal{D}^{1,2} := \left\{v \in L^1_{loc} : \nabla v \in L^2\right\}$, w be in L^2 and z be in L^n_w. Then*

$$\left|\int_{\mathbf{R}^n} v\, w\, z\, dx\right| \le C_1(n)\|\nabla v\|_2 \|w\|_2 [z]_n,$$

with the constant $C_1(n)$ depending only on the space dimension n.

PROOF. We recall that if $f \in L^{p_1,q_1}$ and $g \in L^{p_2,q_2}$ (for $1 < p_1, p_2 < \infty$) with $1/p := 1/p_1 + 1/p_2 < 1$, then $f\,g \in L^{p,q}$ with

$$\|f\,g\|_{p,q} \le C\|f\|_{p_1,q_1}\|g\|_{p_2,q_2} \quad \text{for} \;\; q := \min\{q_1, q_2\}.$$

By using the above result (a generalization of the classical Hölder inequality) we obtain

$$\left|\int_{\mathbf{R}^n} v\, w\, z\, dx\right| \le C\|v\|_{\frac{2n}{n-2},2}\|w\|_{2,2}\|z\|_{n,\infty}.$$

We now observe that if $f \in \mathcal{D}^{1,2}$, then $f \in L^{\frac{2n}{n-2},2}$ with

$$\|f\|_{\frac{2n}{n-2},2} \le C(n)\|\nabla f\|_2. \tag{10}$$

This last inequality can be easily shown by a density argument. In fact, if $f \in C_0^\infty(\mathbf{R}^n)$, then

$$f(x) = \Gamma\,[\nabla f](x) := \frac{1}{n\,\omega_n} \int_{\mathbf{R}^n} \frac{x-y}{|x-y|^n} \cdot \nabla f(y)\,dy,$$

where ω_n is the volume of the unit ball in $\mathbf{R}^n$. By using the Hardy-Littlewood-Sobolev inequality it follows that the operator Γ belongs to $\mathcal{L}(L^p, L^q)$ (the space of bounded linear operators from L^p into L^q) for $1 < p < n$, $1 < q < \infty$ and $1/q = 1/p - 1/n$. Then by real interpolation (the General Marcinkiewicz Interpolation Theorem 5.3.2 in Bergh and Löfström [5]) we obtain

$$\Gamma \in \mathcal{L}\left(L^{2,2}, L^{\frac{2n}{n-2},2}\right).$$

Finally, by using (10) and by recalling that $L^{2,2} = L^2$ we complete the proof of Lemma 2.1. $\square$

3. Proof of Theorem 1.1

We can now prove the main result of this paper. The proof is based on classical manipulations (see for instance J.-L. Lions [14], Ch. I) together with a new result, namely Eq. (11) below, that is obtained in Bae and Choe [1]. By a continuation argument we shall show that to prove Theorem 1.1 it is sufficient to prove the following lemma.

LEMMA 3.1. *Given a divergence-free $a_0 \in H^1$, consider the Cauchy problem for the Navier-Stokes equations in $(\tau - \delta, \tau) \subseteq (0, T)$, with a_0 as initial value. Let u be a weak solution in $(\tau - \delta, \tau)$, as well as a strong solution in $(\tau - \delta, \tau')$ for each $\tau' < \tau$. Assume moreover that $\widehat{u}$ satisfies* (9) *for a small enough positive constant C_0. Under the above hypotheses u is a strong (hence regular) solution in $(\tau - \delta, \tau)$.*

PROOF. We differentiate the first equation in (1) with respect to x_k, we take the scalar product with $\partial u_i/\partial x_k$ and we integrate over $\mathbf{R}^n$. We get

$$\frac{1}{2}\frac{d}{dt} \int_{\mathbf{R}^n} |\nabla u|^2\,dx + \nu \int_{\mathbf{R}^n} |\nabla^2 u|^2\,dx = -\int_{\mathbf{R}^n} \nabla[(u\cdot\nabla)\,u]\cdot\nabla u\,dx,$$

where

$$|\nabla u|^2 = \sum_{i,j=1}^{n} \left|\frac{\partial u_i}{\partial x_j}\right|^2 \quad \text{and} \quad |\nabla^2 u|^2 = \sum_{i,j,k=1}^{n} \left|\frac{\partial^2 u_i}{\partial x_j \partial x_k}\right|^2.$$

By a careful inspection of the right hand side (see Eq. (2.3)–(2.4)–(2.5) in Beirão da Veiga [4]) it follows that

$$\left|\int_{\mathbf{R}^n} \nabla[(u\cdot\nabla)\,u]\cdot\nabla u\,dx\right| \le C_2(n) \int_{\mathbf{R}^n} |\widehat{u}|\,|\nabla u|\,|\nabla^2 u|\,dx, \tag{11}$$

where $C_2(n)$ is a positive constant, depending only on the space dimension n. To prove (11) note that, by the incompressibility condition, we have

$$\int_{\mathbf{R}^n} \nabla[(u\cdot\nabla)\,u]\cdot\nabla u\,dx = \int_{\mathbf{R}^3} \sum_{i,j,k=1}^{n} \frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_i}\frac{\partial u_j}{\partial x_k}\,dx. \tag{12}$$

If $i = j = n$, and by recalling that

$$\frac{\partial u_n}{\partial x_n} = -\sum_{l=1}^{n-1}\frac{\partial u_l}{\partial x_l},$$

we obtain that the right hand side of (12) is equal to

$$-\int_{\mathbf{R}^3}\sum_{l=1}^{n-1}\sum_{k=1}^{n}\frac{\partial u_n}{\partial x_k}\frac{\partial u_l}{\partial x_l}\frac{\partial u_n}{\partial x_k}\,dx = \int_{\mathbf{R}^3}\sum_{l=1}^{n-1}u_l\sum_{k=1}^{n}\frac{\partial}{\partial x_l}\left[\frac{\partial u_n}{\partial x_k}\frac{\partial u_n}{\partial x_k}\right]\,dx.$$

We then consider the case $i \neq n$

$$\int_{\mathbf{R}^3}\sum_{i=1}^{n-1}\sum_{j,k=1}^{n}\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_i}\frac{\partial u_j}{\partial x_k}\,dx = -\int_{\mathbf{R}^n}\sum_{i=1}^{n-1}u_i\sum_{j,k=1}^{n}\frac{\partial}{\partial x_k}\left[\frac{\partial u_j}{\partial x_k}\frac{\partial u_j}{\partial x_i}\right]\,dx.$$

Furthermore, if $i = n$ and $j \neq n$

$$\int_{\mathbf{R}^n}\sum_{j=1}^{n-1}\sum_{k=1}^{n}\frac{\partial u_n}{\partial x_k}\frac{\partial u_j}{\partial x_n}\frac{\partial u_j}{\partial x_k}\,dx = -\int_{\mathbf{R}^n}\sum_{j=1}^{n-1}u_j\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left[\frac{\partial u_n}{\partial x_k}\frac{\partial u_j}{\partial x_n}\right]\,dx,$$

and (11) easily follows. By using (11) together with Lemma 2.1, we get the following estimate.

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^n}|\nabla u|^2\,dx + \nu\int_{\mathbf{R}^n}|\nabla^2 u|^2\,dx \\ \leq C_1(n)C_2(n)\sup_{\tau-\delta<t<\tau}[\widehat{u}(t)]_n\int_{\mathbf{R}^n}|\nabla^2 u|^2\,dx.\end{aligned}$$

Finally, if

$$\sup_{\tau-\delta<t<\tau}[\widehat{u}(t)]_n \leq \sup_{0<t<T}[\widehat{u}(t)]_n < \frac{\nu}{C_1(n)C_2(n)} := C_0 \tag{13}$$

a standard application of the Gronwall lemma implies that

$$\frac{1}{2}\int_{\mathbf{R}^n}|\nabla u|^2(t)\,dx \leq \frac{1}{2}\int_{\mathbf{R}^n}|\nabla a_0|^2\,dx, \quad t\in[\tau-\delta,\tau].$$

This proves Lemma 3.1 if $n = 3$, because $L^\infty(\tau - \delta, \tau; H^1)$ is a regularity class. Observe that the Gronwall lemma implies also that

$$\nu \int_{\tau-\delta}^{\tau} \int_{\mathbf{R}^n} |\nabla^2 u|^2 \, dx \, dt \leq \frac{1}{2} \int_{\mathbf{R}^n} |\nabla a_0|^2 \, dx$$

and, if $n = 4$,

$$u \in L^2(\tau - \delta, \tau; H^2(\mathbf{R}^4)) \subset L^2(\tau - \delta, \tau; W^{1,4}(\mathbf{R}^4)).$$

Since $W^{1,4}(\mathbf{R}^4) \not\subset L^\infty(\mathbf{R}^4)$, to conclude it is necessary to resort to the criterion proved in Beirão da Veiga [2]. In fact, in the last reference it is proved that if

$$u \in L^p(\tau - \delta, \tau; W^{1,q}(\mathbf{R}^n)) \quad \text{for} \quad \frac{2}{p} + \frac{n}{q} \leq 2 \quad \text{with} \quad < p \leq 2,$$

then the weak solution u is regular in $(\tau - \delta, \tau)$. □

REMARK 3.2. If the external force f does not vanishes, but if it is smooth enough (say for instance $f \in L^2(0, T; L^2)$ and $\nabla \cdot f = 0$), then Lemma 3.1 is still true. In fact, in this case the same manipulations as above imply that, for all $\eta > 0$,

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbf{R}^n} |\nabla u|^2 \, dx + \left[\nu - \frac{\eta}{2} - C_1(n) C_2(n) \sup_{\tau-\delta<t<\tau} [\widehat{u}(t)]_n \right] \leq \frac{1}{2\eta} \int_{\mathbf{R}^n} |f|^2 \, dx.$$

By using Lemma 3.1 we can now easily prove Theorem 1.1.

PROOF OF THEOREM 1.1. The proof of Theorem 1.1 follows by a continuation argument. In fact, the local existence theorem for strong solutions (see Leray [13] Chap. 4, § 23, for the three-dimensional case and Kato [9] for the Cauchy problem in $\mathbf{R}^4$) implies that the Cauchy problem (1) has a unique strong solution in some interval $[0, T_1)$, for some strictly positive T_1. For convenience, let us suppose that this interval is the *maximal* interval of existence of the strong solution starting from u_0 at time $t = 0$. Let us suppose that $T_1 < T$. Then Lemma 3.1 with $a_0 = u_0$ and $(\tau - \delta, \tau) = (0, T_1)$ shows that $u(T_1)$ belongs to H^1. A further continuation of u is then possible. Absurd since $[0, T_1)$ was the maximal existence interval. □

REMARK 3.3. We recall again that a strong solution is unique also in the larger class of Leray-Hopf weak solutions. In particular, to identify a weak solution on $[0, T]$ and a strong solution on the same interval, just the classical energy inequality (7) is necessary. We also observe that a more restrictive form of this inequality (roughly speaking the same for almost all $s \in [0, T]$ as initial values) is used in the proof of uniqueness of weak solutions with arbitrary norm in $L^\infty(0, T; L^n)$.

REMARK 3.4. Observe that if u_0 belongs just to L^2 and if condition (9) is satisfied, then the solution is strong in (t, T), for each $t > 0$.

4. Some remarks on the problem in a general domain

We observe that to obtain (11) it is essential to study the Navier-Stokes equations in the whole space, or at least in the periodic setting, to avoid the presence of boundary conditions. On the other hand, we may consider the Navier-Stokes equations in Ω, where $\Omega \subset \mathbf{R}^3$ is a smooth (say $C^{2+\mu}$, for $\mu > 0$) bounded domain.

We can prove in a different way the regularity-criterion involving the smallness of u in the class (4), proved in Kozono [10]. We shall prove the following theorem, by assuming a condition that involves *all* the components of the velocity field. Recall that, since the initial-boundary value problem for the Navier-Stokes equations is generally supplemented with the homogeneous Dirichlet conditions

$$u = 0 \quad \text{on} \quad \partial\Omega,$$

a weak solution must satisfy the following condition

$$u \in C_w(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)),$$

instead of (5). Furthermore, in (6) the test-functions must vanish on the boundary. For the existence of weak solution of the Navier-Stokes equations in such a domain Ω, see for instance J.-L. Lions [14] or Masuda [16]. We have the following result.

THEOREM 4.1 (Kozono). *Let $u_0 \in H_0^1(\Omega)$, with $\nabla \cdot u_0 = 0$ and let u be a weak solution of the Navier-Stokes equations in $\Omega \times (0, T)$, for $\Omega \subset \mathbf{R}^3$ as above. There exists a positive constant C_1 such that if*

$$\sup_{0<t<T} [u(t)]_3 < C_1, \tag{14}$$

then the solution u is strong in $(0, T)$.

REMARK 4.2. By using the same techniques of Galdi and Maremonti [8] it is possible to prove the same result as Theorem 4.1 in a more general domain. In particular, it is sufficient that Ω is an arbitrary domain, uniformly of class C^2. It means that Ω lies on one part of its boundary $\partial\Omega$ and, for each $x_0 \in \partial\Omega$, there exists a ball centered at x_0 and of radius independent of x_0, such that $\partial\Omega \cap B$ admits a Cartesian representation $x_n = \gamma(x_1, \dots, x_n)$, where γ is a function of class C^2 with derivatives up to the second bounded independently of x_0.

Reasoning as in the proof of Theorem 1.1, it is clear that to prove Theorem 4.1 it is enough to prove the following lemma.

LEMMA 4.3. *Given a divergence-free $a_0 \in H_0^1(\Omega)$, consider the initial value-problem for the Navier-Stokes equations in $\Omega \times (\tau - \delta, \tau) \subset \Omega \times (0, T)$, with a_0 as the initial value. Let u be a weak solution of (1) in $(\tau - \delta, \tau)$, as well as a strong* *solution in* $(\tau - \delta, \tau')$ *for each* $\tau' < \tau$. *Assume moreover that* u *satisfies* (14) *for a small enough positive constant* C_1. *Under the above hypotheses* u *is a strong solution in* $(\tau - \delta, \tau)$.

Proof. We use the classical manipulations introduced in Prodi [18], namely multiplying the equation by Au (where A is the well-known Stokes operator) and doing suitable integration by parts. Observe that the same estimate can be obtained by using the technique of the previous section.

It follows that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx + \nu \int_\Omega |Au|^2\,dx \leq \int_\Omega |u||\nabla u||Au|\,dx. \tag{15}$$

To estimate the right hand side of (15), we observe that if we work in a domain Ω as above of $\mathbf{R}^3$ the result of Lemma 2.1 is still true. In this case it is necessary to require, as additional hypothesis, that $v \in H_0^1(\Omega)$, $w \in L^2(\Omega)$ and $z \in L_w^3(\Omega)$ to obtain (the norms and quasi-norms are obviously in $L^{p,q}(\Omega)$)

$$\left|\int_\Omega v\,w\,z\,dx\right| \leq C\|\nabla v\|_2\|w\|_2[z]_3,$$

see again Kozono [10]. We finally obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx + \nu \int_\Omega |Au|^2\,dx \leq C \sup_{\tau-\delta<t<\tau} [u(t)]_n \int_\Omega |Au|^2\,dx$$

and the same argument of Lemma 3.1 holds, since the Gronwall lemma implies that

$$u \in L^\infty(\tau - \delta, \tau; H_0^1(\Omega)),$$

provided that

$$\sup_{\tau-\delta<t<\tau} [u(t)]_n < C_1 = \nu/C.$$

This concludes the proof of Lemma 4.3. □

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