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# POSITIVITY AND HIERARCHICAL STRUCTURE OF FOUR GREEN FUNCTIONS CORRESPONDING TO A BENDING PROBLEM OF A BEAM ON A HALF LINE

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Abstract. We consider the boundary value problem for fourth order linear ordinary differential equation in a half line $(0,\infty)$, which represents bending of a beam on an elastic foundation under a tension. A tension is relatively stronger than a spring constant of elastic foundation. We here treat four self-adjoint boundary conditions, clamped, Dirichlet, Neumann and free edges, at $x=0$. We show the positivity and the hierarchical structure of four Green functions.

## 1. Introduction

A beam is supported by uniformly distributed springs with spring constant $q>0$ on a fixed floor and is exerted a tension $p>0$ on both sides as Figure 1.

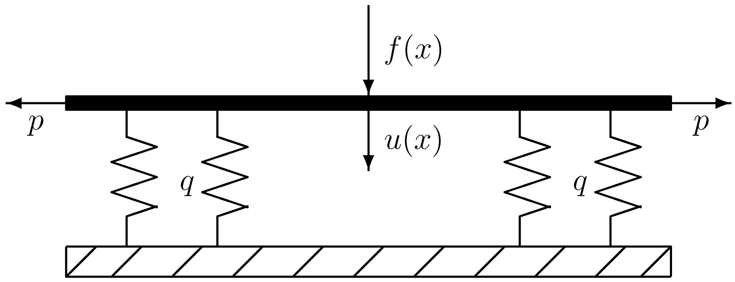


**Figure 1.** *Bending of a beam.*

Under a density of a load $f(x)$, a bending of a beam $u(x)$ on a half line satisfies the following boundary value problem [7, 8]:

BVP($m$)

$$\begin{cases} P(d/dx)u := u^{(4)} - pu'' + qu = f(x) & (0<x<\infty) \\ u^{(m_i)}(0) = 0 & (i=0,1) \\ u, u', u'', u''' : \text{bounded} & (0<x<\infty) \end{cases}$$

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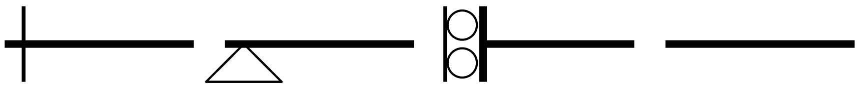
where $m = (m_0, m_1)$ take four different values as follows:

$$
\left\{
\begin{array}{lll}
u(0) = u'(0) = 0 & m = (0,1) & \text{Clamped} \\
u(0) = u''(0) = 0 & m = (0,2) & \text{Dirichlet} \\
u'(0) = u'''(0) = 0 & m = (1,3) & \text{Neumann} \\
u''(0) = u'''(0) - pu'(0) = 0 & m = (2,\widetilde{3}) & \text{Free}
\end{array}
\right. . \tag{1.1}
$$

We here treat only self-adjoint cases $m = (0,1)$, $(0,2)$, $(1,3)$, $(2,\widetilde{3})$, which have engineering importance and correspond to clamped, Dirichlet (simply-supported), Neumann (sliding) and free edge, respectively [4, Chap. 2] as Figure 2. We show that BVP$(m)$ is the self-adjoint boundary value problem in section 2.



$(0,1)$ Clamped  $(0,2)$ Dirichlet  $(1,3)$ Neumann  $(2,\widetilde{3})$ Free

**Figure 2.** *Boundary conditions.*

We assume that a tension is relatively stronger than a spring constant. That is to say, we impose the following two equivalent assumptions:

$$
\begin{array}{ll}
(p/2)^2 > q > 0, \quad 0 < p < \infty & \Leftrightarrow \\
p = a^2 + b^2, \quad q = a^2 b^2 \qquad (0 < b < a). &
\end{array} \tag{1.2}
$$

We state these assumptions (1.2) in section 2. Because the relationship between $a, b$ and $p, q$ is

$$
a = \sqrt{\frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 - q}}, \qquad b = \sqrt{\frac{p}{2} - \sqrt{\left(\frac{p}{2}\right)^2 - q}}
$$
$$
(0 < p < \infty,\ 0 < q < (p/2)^2),
$$

we describe the conclusion by using the parameters $a$ and $b$ for the sake of conciseness. Under these assumptions (1.2), for any bounded continuous function $f(x)$, BVP$(m)$ has a unique solution $u(x)$ given as

$$
u(x) = \int_0^\infty G(m; x, y)\, f(y)\, dy \qquad (0 < x < \infty),
$$

where $G(m; x, y)$ is Green function (an impulse response). The purpose of this paper is to show the positivity and the hierarchical structure of four Green functions.

**Theorem 1.1.** *Four Green functions $G(m;x,y)$ are positive and satisfy the hierarchical structure shown as*

$$0 < G(0,1;x,y) < G(0,2;x,y) < \left\{ \begin{array}{l} G(1,3;x,y) \\ G(2,\widetilde{3};x,y) \end{array} \right. \qquad (0 < x,y < \infty).$$

*In particular, we have*

$$\begin{aligned} &0 < G(0,1;x,y) < G(0,2;x,y) < G(1,3;x,y) < G(2,\widetilde{3};x,y) \\ &(x,y) \in D_0 = \Big\{ 0 < x,y < x_0 \quad or \quad x_0 < x,y < \infty \Big\} \end{aligned}$$

*and*

$$\begin{aligned} &0 < G(0,1;x,y) < G(0,2;x,y) < G(2,\widetilde{3};x,y) < G(1,3;x,y) \\ &(x,y) \in D_1 = \Big\{ 0 < x < x_0 < y < \infty \quad or \quad 0 < y < x_0 < x < \infty \Big\}, \end{aligned}$$

*where* $x_0 = \dfrac{1}{a-b} \log \dfrac{a}{b}$.

This theorem shows that if boundary condition becomes looser as $(0,1) \to (0,2) \to (1,3)$ or $(2,\widetilde{3})$, the impulse response gets larger in $0 < x,y < \infty$.

We state our previous study. Under a density of a load $f(x)$, a bending of a beam $u(x)$ on an interval satisfies the following boundary value problem [2, 3, 6, 9, 10]:

$$\mathrm{BVP}(m,n)$$
$$\left\{ \begin{array}{ll} u^{(4)} - pu'' + qu = f(x) & (0 < x < L) \\ u^{(m_i)}(0) = u^{(n_i)}(L) = 0 & (i = 0,1) \end{array} \right. ,$$

where $m = (m_0, m_1)$, $n = (n_0, n_1)$ take four different values as (1.1). We set hyperbolic functions $\mathrm{ch}(x) = \cosh(x)$ and $\mathrm{sh}(x) = \sinh(x)$ for short. We introduce

$$K_0(x) = \frac{1}{a^2 - b^2} \Big[ a^{-1}\mathrm{sh}(ax) - b^{-1}\mathrm{sh}(bx) \Big] \qquad (0 < x < \infty)$$

and its successive derivatives $K_j(x) = K_0^{(j)}(x)$ and $K_{\widetilde{j}}(x) = K_j(x) - pK_{j-2}(x)$. In particular, we put $K_j = K_j(L)$. Under the assumption (1.2), for any bounded continuous function $f(x)$, $\mathrm{BVP}(m,n)$ has a unique solution $u(x)$ given as

$$\begin{aligned} &u(x) = \int_0^L G(m,n;x,y) f(y) dy \qquad (0 < x < L), \\ &G(m,n;x,y) = \end{aligned}$$

$$\left|\begin{array}{cc|c} K_{m_0+n_0} & K_{m_0+n_1} & K_{m_0}(x\wedge y) \\ K_{m_1+n_0} & K_{m_1+n_1} & K_{m_1}(x\wedge y) \\ \hline K_{n_0}(L-x\vee y) & K_{n_1}(L-x\vee y) & 0 \end{array}\right| \Big/ \left|\begin{array}{cc} K_{m_0+n_0} & K_{m_0+n_1} \\ K_{m_1+n_0} & K_{m_1+n_1} \end{array}\right|,$$

where $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$. $G(m, n; x, y)$ is Green function of BVP$(m, n)$. BVP$(m, n)$ is the self-adjoint boundary value problem. In the previous paper [2], we derived nine self-adjoint Green functions $G(m, n; x, y)$ $(0 < x, y < L)$, where $m = (m_0, m_1)$ and $n = (n_0, n_1)$ take three sets of values $(0, 1)$, $(0, 2)$, $(1, 3)$ [2, Theorem 4.1], and showed their positivity and hierarchical structure on an interval [2, Theorem 7.1] as

$$\begin{aligned}
&0 < G(0,1,0,1;x,y) < \\
&\left\{\begin{array}{l} G(0,1,0,2;x,y) < \left\{\begin{array}{c} G(0,1,1,3;x,y) \\ G(0,2,0,2;x,y) \end{array}\right\} < G(0,2,1,3;x,y) \\[2ex] G(0,2,0,1;x,y) < \left\{\begin{array}{c} G(0,2,0,2;x,y) \\ G(1,3,0,1;x,y) \end{array}\right\} < G(1,3,0,2;x,y) \end{array}\right\} < \\
&G(1,3,1,3;x,y) \qquad (0 < x, y < L).
\end{aligned}$$

In [3], we have one more condition $m = (2, \widetilde{3})$ or $n = (2, \widetilde{3})$ and have obtained a more detailed hierarchical structure among sixteen self-adjoint Green functions. As an application, we obtained the best constants of Sobolev inequality corresponding to bending problem of a beam on a half line [7, 8] and a finite interval [9, 10].

This paper is composed of six sections. In section 2, we show the positivity and hierarchical structure of Green function corresponding to the bending problem of a string, which is based study of this paper. In section 3, we derive Green functions $G(m; x, y)$ corresponding to BVP$(m)$. In section 4, we show one more expression of $G(m; x, y)$. In section 5, we prove Theorem 1.1. In section 6, as an appendix, we show the best constants of Sobolev inequality corresponding BVP$(m)$.

The bending problem of a beam BVP$(m)$ is important in the field of classical mechanics of materials. This paper gives a mathematical foundation of this bending problem of a beam BVP$(m)$.

## 2. Bending problem of a string

A string is supported by uniformly distributed springs with spring constant $q = a^2$ $(0 < a < \infty)$ on a fixed floor. Under a density of a load $f(x)$, a bending of a string $u(x)$ is shown as Figure 3.

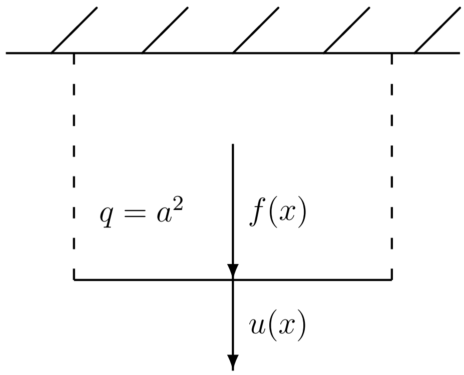



**Figure 3.** *Bending of a string.*

First, we consider the case of a bending of a string on a line. For any bounded continuous function $f(x)$ $(-\infty < x < \infty)$, the boundary value problem

$$\mathrm{BVP}_0(\mathbf{R})$$
$$\begin{cases} -u'' + qu = f(x) & (-\infty < x < \infty) \\ u, u' : \text{bounded} & (-\infty < x < \infty) \end{cases}$$

has a unique solution $u(x)$ given as

$$u(x) = \int_{-\infty}^{\infty} G(x,y)\, f(y)\, dy \qquad (-\infty < x < \infty),$$
$$G(x,y) = \frac{1}{2a} e^{-a|x-y|} \qquad (-\infty < x, y < \infty),$$

where $G(x,y)$ is Green function.

Second, we consider the case of a bending of a string on a half line. For any bounded continuous function $f(x)$ $(0 < x < \infty)$, the boundary value problem

$$\mathrm{BVP}_0(m)$$
$$\begin{cases} -u'' + qu = f(x) & (0 < x < \infty) \\ u^{(m)}(0) = 0 & (m = 0, 1) \\ u, u' : \text{bounded} & (0 < x < \infty) \end{cases}$$

has a unique solution $u(x)$ given as

$$u(x) = \int_0^{\infty} G(m; x, y)\, f(y)\, dy \qquad (0 < x < \infty),$$

$$G(m;x,y) = G(x,y) - (-1)^m G(x,-y) =$$
$$\frac{1}{2a}e^{-a|x-y|} - (-1)^m \frac{1}{2a}e^{-a(x+y)} \qquad (0 < x, y < \infty).$$

$G(m;x,y)$ is Green function of $\mathrm{BVP}_0(m)$ and $G(x,y)$ is Green function of $\mathrm{BVP}_0(\mathbf{R})$. The boundary condition $m$ take two different values. $m = 0$ is Clamped or Dirichlet boundary condition. $m = 1$ is Neumann or Free boundary condition. It is very easy to see the positivity and hierarchical structure as

$$0 < G(0;x,y) < G(1;x,y) \qquad (0 < x, y < \infty).$$

Finally, we consider the case of a bending of a string on an interval. We introduce $H_0(x) = a^{-1}\mathrm{sh}(ax)$ and its successive derivatives $H_j(x) = H_0^{(j)}(x)$. In particular, we put $H_j = H_j(L)$. For any bounded continuous function $f(x)$ $(0 < x < L)$, the boundary value problem

$$\mathrm{BVP}_0(m,n)$$
$$\begin{cases} -u'' + qu = f(x) & (0 < x < L) \\ u^{(m)}(0) = u^{(n)}(0) = 0 & (m, n = 0, 1) \end{cases}$$

has a unique solution $u(x)$ given as

$$u(x) = \int_0^L G(m,n;x,y)\, f(y)\, dy \qquad (0 < x < L),$$
$$G(m,n;x,y) = H_{m+n}^{-1} H_m(x \wedge y)\, H_n(L - x \vee y) \qquad (0 < x, y < L).$$

$G(m,n;x,y)$ is Green function of $\mathrm{BVP}_0(m,n)$. The boundary condition $m, n$ take two different values. $m = 0$ or $n = 0$ is Clamped or Dirichlet boundary condition. $m = 1$ or $n = 1$ is Neumann or Free boundary condition. In [5, 6], Green function $G(m,n;x,y)$ has the positivity and the hierarchical structure as

$$0 < G(0,0;x,y) < \left\{ \begin{array}{c} G(0,1;x,y) \\ G(1,0;x,y) \end{array} \right\} < G(1,1;x,y)$$
$$(0 < x, y < L).$$

The discrete version is given by [11].

## 3. Boundary value problem

We consider the assumptions (1.2). Using characteristic coefficients $p, q$ and characteristic roots $a_j$ $(j = 0, 1, 2, 3)$, we have the characteristic polynomial of the differential equation of $\mathrm{BVP}(m)$ as

$$P(z) = z^4 - pz^2 + q = \prod_{j=0}^{3}(z - a_j).$$

For

$$P(z) = z^4 - pz^2 + q = \left(z^2 - \frac{p}{2}\right)^2 - \left(\left(\frac{p}{2}\right)^2 - q\right),$$

if $P(z) = 0$, then we have

$$z^2 = \frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q} \qquad \Leftrightarrow \qquad z = \pm\sqrt{\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}}. \tag{3.1}$$

We show the range of values $p$ and $q$ in Figure 4.

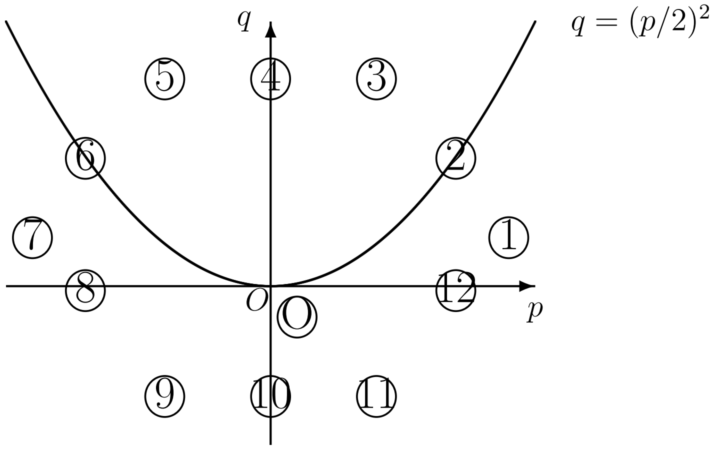



**Figure 4.** *The range of values $p$ and $q$.*

We rewrite ⓪ $\sim$ ⑫ in Figure 4 as

⓪ $p = 0, \quad q = 0,$
① $p > 0, \quad 0 < q < (p/2)^2,$
② $p > 0, \quad q = (p/2)^2,$
③ $p > 0, \quad 0 < (p/2)^2 < q,$
④ $p = 0, \quad 0 < q,$
⑤ $p < 0, \quad 0 < (p/2)^2 < q,$
⑥ $p < 0, \quad q = (p/2)^2,$
⑦ $p < 0, \quad 0 < q < (p/2)^2,$
⑧ $p < 0, \quad q = 0,$
⑨ $p < 0, \quad q < 0,$
⑩ $p = 0, \quad q < 0,$
⑪ $p > 0, \quad q < 0,$
⑫ $p > 0, \quad q = 0.$

If $p > 0$, then $p$ means tension. If $p < 0$, then $|p|$ means pressure. If $q > 0$, then $q$ means spring constant. If $q < 0$, then $q$ means extended spring constant. In this paper, we treat ① because we can come to exhaustive conclusion of the positivity and the hierarchical structure of four Green functions. ② $\sim$ ⑫ are interesting as a engineering phenomenon, but the positivity and the hierarchical structure of four Green functions cannot exist. So we do not treat ② $\sim$ ⑫ in this paper. In ①, a tension is relatively stronger than a spring constant which is the obedient engineering phenomenon. If we assume ①, then $z^2$ in (3.1) is positive value. When we introduce parameters $a$ and $b$ as

$$\left.\begin{matrix} a^2 \\ b^2 \end{matrix}\right\} = \frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q} \qquad \Leftrightarrow \qquad \left\{\begin{matrix} p = a^2 + b^2 \\ q = a^2 b^2 \end{matrix}\right. \qquad (0 < b < a).$$

① is equivalent to the assumptions (1.2). Then, characteristic roots $a_j$ $(j = 0, 1, 2, 3)$ are

$$a_0 = a, \qquad a_1 = -a, \qquad a_2 = b, \qquad a_3 = -b. \tag{3.2}$$

**Lemma 3.1.** *Four kinds of* BVP$(m)$ *are the self-adjoint boundary value problems.*

**Proof of Lemma 3.1** If $u$ and $v$ satisfy the homogeneous boundary condition and the bounded conditions of BVP$(m)$, the following relation

$$\int_0^\infty \left(u^{(4)} - pu'' + qu\right)\overline{v}dx - \int_0^\infty u\left(\overline{v}^{(4)} - p\overline{v}'' + q\overline{v}\right)dx =$$
$$\left[(u''' - pu')\overline{v} - u''\overline{v}' + u'\overline{v}'' - u(\overline{v}''' - p\overline{v}')\right]\Big|_{x=0}^{x=\infty}$$

is equal to 0. This completes the proof of Lemma 3.1. ■

We start with the boundary value problem on whole line:

BVP$(\mathbf{R})$

$$\left\{\begin{array}{ll} u^{(4)} - pu'' + qu = f(x) & (-\infty < x < \infty) \\ u, u', u'', u''' \ : \ \text{bounded} & (-\infty < x < \infty) \end{array}\right. ,$$

which possesses a unique solution given by

$$u(x) = \int_{-\infty}^{\infty} G(x,y)f(y)dy \qquad (-\infty < x < \infty),$$

where $G(x,y)$ is Green function on whole line [1]. Introducing the function

$$G(x) = G(0)\frac{1}{a-b}\left[a\,e^{-b|x|} - b\,e^{-a|x|}\right] > 0 \qquad (-\infty < x < \infty), \tag{3.3}$$
$$G(0) = \frac{1}{2ab(a+b)},$$

we have $G(x,y) = G(x-y)$ as

$$G(x-y) = G(0)\frac{1}{a-b}\left[a\,e^{-b|x-y|} - b\,e^{-a|x-y|}\right] = \tag{3.4}$$
$$\frac{1}{a^2-b^2}\left[\frac{1}{2b}e^{-b|x-y|} - \frac{1}{2a}e^{-a|x-y|}\right] > 0 \qquad (-\infty < x, y < \infty)$$

satisfying the following properties:

$$(3.5)\qquad (\partial_x^4 - p\partial_x^2 + q)G(x,y) = 0 \qquad (-\infty < x, y < \infty,\ x \neq y),$$

$$(3.6)\qquad \partial_x^i G(x,y)\ :\ \text{bounded} \quad (i = 0,1,2,3, \quad -\infty < x, y < \infty, \quad x \neq y),$$

$$(3.7)\qquad \partial_x^i G(x,y)\Big|_{y=x-0} - \partial_x^i G(x,y)\Big|_{y=x+0} = \begin{cases} 0 & (i = 0,1,2) \\ 1 & (i = 3) \end{cases}$$
$$(-\infty < x < \infty).$$

In order to give concrete forms of fundamental functions $A_i(m;x)$ and Green functions $G(m;x,y)$ of BVP$(m)$, we introduce matrices defined by

$$(3.8)\qquad \boldsymbol{\Phi}_{\pm}(m) = \begin{pmatrix} (\pm a)^{m_0} & (\pm b)^{m_0} \\ (\pm a)^{m_1} & (\pm b)^{m_1} \end{pmatrix},$$

$$(3.9)\qquad \boldsymbol{E} = G(0)\frac{1}{a-b}\begin{pmatrix} b & 0 \\ 0 & -a \end{pmatrix},$$

$$(3.10)\quad \boldsymbol{C}(m) = \boldsymbol{\Phi}_{-}(m)^{-1}\boldsymbol{\Phi}_{+}(m)\boldsymbol{E},$$

where

$$(\pm a)^{\widetilde{3}} = (\pm a)^3 - p(\pm a) = \mp ab^2, \qquad (\pm b)^{\widetilde{3}} = (\pm b)^3 - p(\pm b) = \mp a^2 b.$$

We list the concrete forms of $\boldsymbol{\Phi}_{\pm}(m)$ as

$$\boldsymbol{\Phi}_{\pm}(0,1) = \begin{pmatrix} 1 & 1 \\ \pm a & \pm b \end{pmatrix}, \qquad \boldsymbol{\Phi}_{\pm}(0,2) = \begin{pmatrix} 1 & 1 \\ a^2 & b^2 \end{pmatrix},$$
$$\boldsymbol{\Phi}_{\pm}(1,3) = \pm\begin{pmatrix} a & b \\ a^3 & b^3 \end{pmatrix}, \qquad \boldsymbol{\Phi}_{\pm}(2,\widetilde{3}) = \begin{pmatrix} a^2 & b^2 \\ \mp ab^2 & \mp a^2 b \end{pmatrix}$$

and $\boldsymbol{C}(m)$ as

$$\boldsymbol{C}(0,1) = -G(0)\frac{1}{(a-b)^2}\begin{pmatrix} (a+b)b & -2ab \\ -2ab & a(a+b) \end{pmatrix},$$
$$\boldsymbol{C}(0,2) = \boldsymbol{E}, \qquad \boldsymbol{C}(1,3) = -\boldsymbol{E},$$
$$\boldsymbol{C}(2,\widetilde{3}) = G(0)\frac{1}{(a-b)(a^3-b^3)}\begin{pmatrix} (a^3+b^3)b & -2a^2b^2 \\ -2a^2b^2 & a(a^3+b^3) \end{pmatrix}.$$

We remark that $\boldsymbol{C}(m)$ is a symmetric matrix.

Lemma 3.2 shows the uniqueness of the solution of BVP$(m)$. For later convenience sake, we consider the inhomogeneous boundary condition of BVP$(m)$.

**Lemma 3.2.** *For any bounded continuous function $f(x)$ on a half line $0 < x < \infty$ and any set of complex numbers $(\alpha_0, \alpha_1)$, the boundary value problem*

BVP$'(m)$

$$\begin{cases} u^{(4)} - pu'' + qu = f(x) & (0 < x < \infty) \\ u^{(m_i)}(0) = \alpha_i & (i = 0, 1) \\ u, u', u'', u''' : \textit{bounded} & (0 < x < \infty) \end{cases}$$

*has a unique classical solution*

$$(3.11)\quad u(x) = \sum_{i=0}^{1} A_i(m; x)\alpha_i + \int_0^\infty G(m; x, y)\, f(y)\, dy \qquad (0 < x < \infty),$$

*where the fundamental solutions satisfying the prescribed boundary conditions are given by*

$$(3.12)\quad \begin{pmatrix} A_0 & A_1 \end{pmatrix}(m; x) = \begin{pmatrix} e^{-ax} & e^{-bx} \end{pmatrix} \boldsymbol{\Phi}_-(m)^{-1} \qquad (0 < x < \infty)$$

*and Green functions are given by*

$$(3.13)\quad G(m; x, y) = G(x - y) + G_c(m; x, y) \qquad (0 < x, y < \infty),$$

$$(3.14)\quad G(x - y) = -\begin{pmatrix} e^{-a(x\vee y)} & e^{-b(x\vee y)} \end{pmatrix} \boldsymbol{E} \begin{pmatrix} e^{a(x\wedge y)} \\ e^{b(x\wedge y)} \end{pmatrix},$$

$$(3.15)\quad G_c(m; x, y) = \begin{pmatrix} e^{-a(x\vee y)} & e^{-b(x\vee y)} \end{pmatrix} \boldsymbol{C}(m) \begin{pmatrix} e^{-a(x\wedge y)} \\ e^{-b(x\wedge y)} \end{pmatrix}.$$

(3.14) *is another important expression of* (3.4). *In particular, using* (3.3), $G_c(0, 2; x, y)$ *and* $G_c(1, 3; x, y)$ *are given as*

$$(3.16)\quad \left.\begin{matrix} G_c(0, 2; x, y) \\ G_c(1, 3; x, y) \end{matrix}\right\} = \mp G(x + y) =$$

$$\mp \frac{G(0)}{a - b}\Big[a\, e^{-b(x+y)} - b\, e^{-a(x+y)}\Big] \begin{cases} < 0 \\ > 0 \end{cases} \quad (0 < x, y < \infty).$$

Lemma 3.2 shows that Green function $G(m; x, y)$ is expressed as a sum of principal solution $G(x - y)$ and a compensating function $G_c(m; x, y)$. We show the concrete forms of $A_i(m; x)$ and $G(m; x, y)$.

**Corollary 3.1.** *We show the concrete forms and the positivity of* $A_i(m; x)$ *as follows:*

$$(3.17)\quad A_0(0, 1; x) = \frac{1}{a - b}\Big[a\, e^{-bx} - b\, e^{-ax}\Big] > 0 \qquad (0 < x < \infty),$$

$$(3.18)\quad A_1(0, 1; x) = \frac{1}{a - b}\Big[e^{-bx} - e^{-ax}\Big] > 0 \qquad (0 < x < \infty),$$

$$A_0(0, 2; x) = \frac{1}{a^2 - b^2}\Big[a^2\, e^{-bx} - b^2\, e^{-ax}\Big] > 0 \qquad (0 < x < \infty),$$

$$A_1(0, 2; x) = -\frac{1}{a^2 - b^2}\Big[e^{-bx} - e^{-ax}\Big] < 0 \qquad (0 < x < \infty),$$

$$A_0(1, 3; x) = -\frac{1}{ab(a^2 - b^2)}\Big[a^3\, e^{-bx} - b^3\, e^{-ax}\Big] < 0 \qquad (0 < x < \infty),$$

$$A_1(1,3;x) = \frac{1}{ab(a^2-b^2)}\Big[a\,e^{-bx} - b\,e^{-ax}\Big] > 0 \qquad (0<x<\infty),$$

$$A_0(2,\widetilde{3};x) =$$

$$\frac{1}{a^3-b^3}\Big[a\,e^{-ax} - b\,e^{-bx}\Big]\begin{cases} >0 & \left(0<x<\dfrac{1}{a-b}\log\dfrac{a}{b}\right) \\ =0 & \left(x=\dfrac{1}{a-b}\log\dfrac{a}{b}\right) \\ <0 & \left(\dfrac{1}{a-b}\log\dfrac{a}{b}<x<\infty\right)\end{cases},$$

$$A_1(2,\widetilde{3};x) = \frac{1}{ab(a^3-b^3)}\Big[a^2\,e^{-bx} - b^2\,e^{-ax}\Big] > 0 \qquad (0<x<\infty).$$

**Corollary 3.2.** *We show the concrete forms of* $G(m;x,y)$ $(0<x,y<\infty)$ *as follows:*

$$G(m;x,y) = G(x-y) + G_c(m;x,y),$$

$$G(x-y) = \frac{2ab\,G(0)}{a-b}\left[\frac{1}{2b}e^{-b|x-y|} - \frac{1}{2a}e^{-a|x-y|}\right],$$

$$G_c(0,1;x,y) = -\frac{G(0)}{(a-b)^2}\Big[$$

$$b(a+b)e^{-a(x+y)} - 2ab\left(e^{-ax-by} + e^{-bx-ay}\right) + a(a+b)e^{-b(x+y)}\Big],$$

$$\left.\begin{matrix} G_c(0,2;x,y) \\ G_c(1,3;x,y)\end{matrix}\right\} =$$

$$\mp G(x+y) = \mp\frac{2ab\,G(0)}{a-b}\left[\frac{1}{2b}e^{-b(x+y)} - \frac{1}{2a}e^{-a(x+y)}\right],$$

$$G_c(2,\widetilde{3};x,y) = \frac{G(0)}{(a-b)(a^3-b^3)}\Big[$$

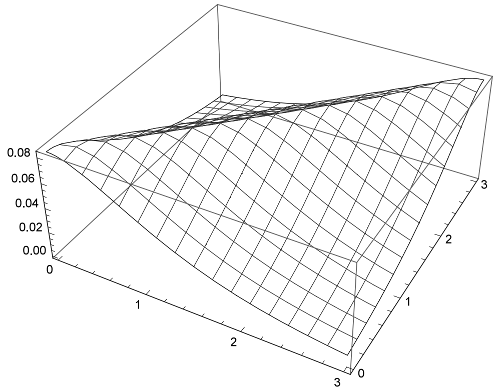
$$b(a^3+b^3)e^{-a(x+y)} - 2a^2b^2\left(e^{-ax-by} + e^{-bx-ay}\right) + a(a^3+b^3)e^{-b(x+y)}\Big].$$

*We show the 3D graph of Green functions in Figure 5.*

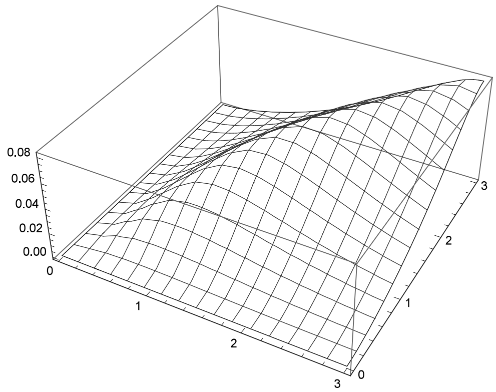
**Proof of Lemma 3.2** The new functions

$$u_i = u_i(x) = u^{(i)}(x) \qquad (i=0,1,2,3,\ 0<x<\infty),$$

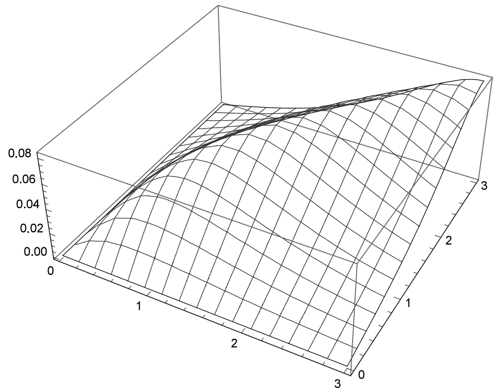
$$u_{\widetilde{3}} = u_{\widetilde{3}}(x) = u^{(\widetilde{3})}(x) = u'''(x) - pu'(x) \qquad (0<x<\infty)$$



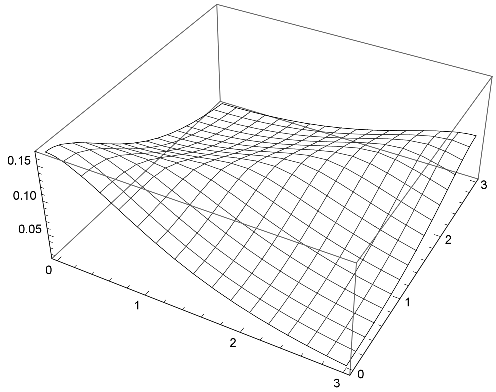

$G(x-y)$



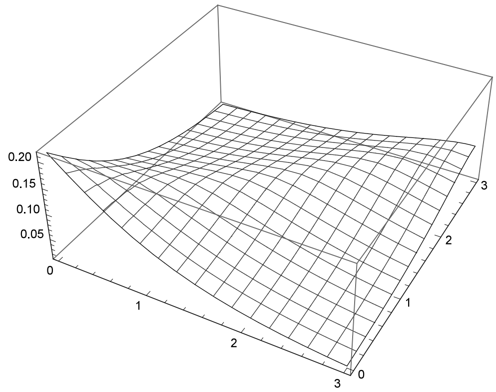

$G(0,1;x,y)$




$G(0,2;x,y)$




$G(1,3;x,y)$




$G(2,\widetilde{3};x,y)$

**Figure 5.** *Green functions* $(a=2, b=1, L=3)$.

satisfy

$$\begin{cases} u_0' = u_1 \\ u_1' = u_2 \\ u_2' = u_3 \\ u_3' = -qu_0 + pu_2 + f(x) \end{cases} \quad \Leftrightarrow$$

$$\begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -q & 0 & p & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} f(x).$$

If we introduce the matrices

$$\boldsymbol{u} = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}, \qquad \boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -q & 0 & p & 0 \end{pmatrix}, \qquad \boldsymbol{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

then we have

$$\boldsymbol{u}' = \boldsymbol{A}\boldsymbol{u} + \boldsymbol{e}_3 f(x) \qquad (0 < x < \infty). \tag{3.19}$$

Using the characteristic roots $a_j$ $(0 \le j \le 3)$ and matrices $\boldsymbol{W}$ and $\widehat{\boldsymbol{A}}$ as

$$a_0 = a, \quad a_1 = b, \quad a_2 = -a_0, \quad a_3 = -a_1,$$

$$\boldsymbol{W} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ a_0 & a_1 & a_2 & a_3 \\ a_0^2 & a_1^2 & a_2^2 & a_3^2 \\ a_0^3 & a_1^3 & a_2^3 & a_3^3 \end{pmatrix}, \qquad \widehat{\boldsymbol{A}} = \begin{pmatrix} a_0 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_3 \end{pmatrix},$$

we have the Jordan canonical form $\boldsymbol{A} = \boldsymbol{W}\widehat{\boldsymbol{A}}\boldsymbol{W}^{-1}$. We introduce new vector $\boldsymbol{v}$ as

$$\boldsymbol{u} = \boldsymbol{W}\boldsymbol{v} \qquad \Leftrightarrow \qquad \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ a_0 & a_1 & a_2 & a_3 \\ a_0^2 & a_1^2 & a_2^2 & a_3^2 \\ a_0^3 & a_1^3 & a_2^3 & a_3^3 \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

and $\boldsymbol{e}$ as

$$\boldsymbol{e} = \boldsymbol{W}^{-1}\boldsymbol{e}_3 \qquad \Leftrightarrow \qquad \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix} = \left( \frac{1}{P'(a_i)} \right)_{0 \le i \le 3},$$

$$e_0 = \frac{1}{2a(a^2 - b^2)}, \quad e_1 = -\frac{1}{2b(a^2 - b^2)}, \quad e_2 = -e_0, \quad e_3 = -e_1.$$

The differential equation (3.19) is rewritten equivalently as

$$\begin{aligned} &\boldsymbol{v}' = \widehat{\boldsymbol{A}}\boldsymbol{v} + \boldsymbol{e}f(x) \qquad (0 < x < \infty), \\ \Leftrightarrow\quad & v_i' = a_i v_i + e_i f(x) \qquad (i = 0,1,2,3,\ 0 < x < \infty). \end{aligned}$$

Solving this differential equation, we have

$$(3.20)\quad v_i(x) = v_i(0)e^{a_i x} + \int_0^x e_i e^{a_i(x-y)} f(y)dy \\ (i = 0,1,2,3,\ 0 < x < \infty).$$

Here, because $u_i$ $(i = 0,1,2,3)$ are bounded, $v_i$ $(i = 0,1,2,3)$ are also bounded from $\boldsymbol{v} = \boldsymbol{W}^{-1}\boldsymbol{u}$.

We consider the case $i = 0,1$ in (3.20). Noting $a_i > 0$ and taking the limit $x \to +\infty$ on both sides of

$$e^{-a_i x} v_i(x) = v_i(0) + \int_0^x e_i e^{-a_i y} f(y)dy \qquad (i = 0,1,\ 0 < x < \infty),$$

we have

$$(3.21)\quad 0 = v_i(0) + \int_0^\infty e_i e^{-a_i y} f(y)dy \qquad (i = 0,1).$$

Subtracting the above two relations, we have

$$v_i(x) = -\int_x^\infty e_i e^{a_i(x-y)} f(y)dy \qquad (i = 0,1,\ 0 < x < \infty).$$

Thus we have

$$(3.22)\quad v_0(x) + v_1(x) = -\int_x^\infty \sum_{i=0}^{1} e_i e^{-a_i|x-y|} f(y)dy \qquad (0 < x < \infty).$$

We consider the case of $i = 2,3$ in (3.20). Noting $a_{i+2} = -a_i$, $e_{i+2} = -e_i$ $(i = 0,1)$, we have

$$(3.23)\quad v_2(x) + v_3(x) = \\ \begin{pmatrix} e^{-a_0 x} & e^{-a_1 x} \end{pmatrix} \begin{pmatrix} v_2 \\ v_3 \end{pmatrix}(0) - \int_0^x \sum_{i=0}^{1} e_i e^{-a_i|x-y|} f(y)dy \qquad (0 < x < \infty).$$

Taking a sum of (3.22) and (3.23), we have

$$(3.24)\quad u(x) = u_0(x) = \sum_{i=0}^{3} v_i(x) = \\ \begin{pmatrix} e^{-a_0 x} & e^{-a_1 x} \end{pmatrix} \begin{pmatrix} v_2 \\ v_3 \end{pmatrix}(0) + \int_0^\infty G(x-y)f(y)dy \qquad (0 < x < \infty),$$

where

$$G(x-y) = -\sum_{i=0}^{1} e_i e^{-a_i|x-y|} = -\sum_{i=0}^{1} e_i e^{-a_i(x\vee y)} e^{a_i(x\wedge y)} =$$

$$-\begin{pmatrix} e^{-a_0(x\vee y)} & e^{-a_1(x\vee y)} \end{pmatrix} \boldsymbol{E} \begin{pmatrix} e^{a_0(x\wedge y)} \\ e^{a_1(x\wedge y)} \end{pmatrix},$$

$$\boldsymbol{E} = \begin{pmatrix} e_0 & 0 \\ 0 & e_1 \end{pmatrix} = \frac{1}{2ab(a^2-b^2)} \begin{pmatrix} b & 0 \\ 0 & -a \end{pmatrix} = G(0)\frac{1}{a-b} \begin{pmatrix} b & 0 \\ 0 & -a \end{pmatrix}.$$

The above $\boldsymbol{E}$ is equal to (3.9). Putting $a_0 = a, a_1 = b$ for $G(x-y)$, we have (3.14). From the relation $\boldsymbol{u} = \boldsymbol{W}\boldsymbol{v}$, we have

$$\begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}(0) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ a_0 & a_1 & a_2 & a_3 \\ a_0^2 & a_1^2 & a_2^2 & a_3^2 \\ a_0^3 & a_1^3 & a_2^3 & a_3^3 \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}(0).$$

If we replace the last row by $u_{\widetilde{3}} = u_3 - p\,u_1$, then we have

$$\begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_{\widetilde{3}} \end{pmatrix}(0) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ a_0 & a_1 & a_2 & a_3 \\ a_0^2 & a_1^2 & a_2^2 & a_3^2 \\ a_0^{\widetilde{3}} & a_1^{\widetilde{3}} & a_2^{\widetilde{3}} & a_3^{\widetilde{3}} \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}(0),$$

where we use $a_i^{\widetilde{3}} = a_i^3 - pa_i$ $(i = 0, 1, 2, 3)$. In the case of $m = (1, 3)$, $u_{\widetilde{3}}(0) = (u_3 - p\,u_1)(0) = u_3(0)$ follows from $u_1(0) = 0$. Using the boundary condition of BVP$(m)$, we have

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} u_{m_0} \\ u_{m_1} \end{pmatrix}(0) =$$

$$\begin{pmatrix} a_0^{m_0} & a_1^{m_0} \\ a_0^{m_1} & a_1^{m_1} \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}(0) + \begin{pmatrix} a_2^{m_0} & a_3^{m_0} \\ a_2^{m_1} & a_3^{m_1} \end{pmatrix} \begin{pmatrix} v_2 \\ v_3 \end{pmatrix}(0).$$

From

$$a_0 = a, \qquad a_1 = b, \qquad a_2 = -a_0 = -a, \qquad a_3 = -a_1 = -b,$$

$$(\pm a)^{\widetilde{3}} = \mp ab^2, \qquad (\pm b)^{\widetilde{3}} = \mp a^2 b,$$

we have

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} a^{m_0} & b^{m_0} \\ a^{m_1} & b^{m_1} \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}(0) + \begin{pmatrix} (-a)^{m_0} & (-b)^{m_0} \\ (-a)^{m_1} & (-b)^{m_1} \end{pmatrix} \begin{pmatrix} v_2 \\ v_3 \end{pmatrix}(0).$$

Using $\boldsymbol{\Phi}_{\pm}(m)$ in (3.8), we have

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \boldsymbol{\Phi}_{+}(m) \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}(0) + \boldsymbol{\Phi}_{-}(m) \begin{pmatrix} v_2 \\ v_3 \end{pmatrix}(0). \tag{3.25}$$

On the other hand, from (3.21), we have

$$(3.26)\quad \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}(0) = -\int_0^\infty \begin{pmatrix} e_0 e^{-a_0 y} \\ e_1 e^{-a_1 y} \end{pmatrix} f(y)dy = -\int_0^\infty \boldsymbol{E} \begin{pmatrix} e^{-a_0 y} \\ e^{-a_1 y} \end{pmatrix} f(y)dy.$$

Inserting (3.25) into (3.26) and using $\boldsymbol{C}(m)$ in (3.10), we have

$$(3.27)\quad \begin{pmatrix} v_2 \\ v_3 \end{pmatrix}(0) = \boldsymbol{\Phi}_-(m)^{-1} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} + \int_0^\infty \boldsymbol{C}(m) \begin{pmatrix} e^{-ay} \\ e^{-by} \end{pmatrix} f(y)dy.$$

Applying (3.27) to (3.24), we have

$$u(x) = \begin{pmatrix} e^{-ax} & e^{-bx} \end{pmatrix} \boldsymbol{\Phi}_-(m)^{-1} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} +$$
$$\int_0^\infty \begin{pmatrix} e^{-ax} & e^{-bx} \end{pmatrix} \boldsymbol{C}(m) \begin{pmatrix} e^{-ay} \\ e^{-by} \end{pmatrix} f(y)dy + \int_0^\infty G(x-y)f(y)dy$$
$$(0 < x < \infty).$$

Here, if we put

$$G_c(m;x,y) = \begin{pmatrix} e^{-ax} & e^{-bx} \end{pmatrix} \boldsymbol{C}(m) \begin{pmatrix} e^{-ay} \\ e^{-by} \end{pmatrix},$$

then $G_c(m;x,y) = G_c(m;y,x)$ follows from ${}^t\boldsymbol{C}(m) = \boldsymbol{C}(m)$. (3.15) follows from $G_c(m;x,y) = G_c(m;y,x)$. Thus we have (3.11), (3.12) and (3.13). This completes the proof of Lemma 3.2. ■

**Lemma 3.3.** *The fundamental solutions satisfying the prescribed boundary conditions $A_i(m;x)$ satisfy the following properties:*

(1) Differential equation

$$A_j^{(4)} - p\,A_j'' + q\,A_j = 0 \qquad (j = 0, 1, \quad 0 < x < \infty)$$

(2) Boundary condition

$$A_j^{(m_i)}(m;0) = \delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases} \qquad (i, j = 0, 1),$$

*where* $A_j^{(\widetilde{3})}(m;0) = A_j'''(m;0) - pA_j'(m;0)$.

**Proof of Lemma 3.3** (1) is obvious since $A_j(m;x)\quad (j = 0,1)$ are linear combinations of $\{e^{-ax},\ e^{-bx}\}$. Differentiating $A_j(m;x)$ $i$ times, we have

$$\begin{pmatrix} A_0^{(i)} & A_1^{(i)} \end{pmatrix}(m;x) =$$
$$\begin{pmatrix} (-a)^i e^{-ax} & (-b)^i e^{-bx} \end{pmatrix} \boldsymbol{\Phi}_-(m)^{-1} \qquad (0 < x < \infty).$$

(2) follows from

$$\begin{pmatrix} A_0^{(m_0)} & A_1^{(m_0)} \\ A_0^{(m_1)} & A_1^{(m_1)} \end{pmatrix}(m;0) = \begin{pmatrix} (-a)^{m_0} & (-b)^{m_0} \\ (-a)^{m_1} & (-b)^{m_1} \end{pmatrix} \boldsymbol{\Phi}_-(m)^{-1} =$$

$$\boldsymbol{\Phi}_-(m)\boldsymbol{\Phi}_-(m)^{-1} = \boldsymbol{I},$$

where $\boldsymbol{I}$ is $2 \times 2$ identity matrix. Thus we have Lemma 3.3. ■

**Lemma 3.4.** *Green function $G(m;x,y)$ satisfies the following properties:*

(1) Symmetry property

$$G(m;x,y) = G(m;y,x) \qquad (0 < x, y < \infty).$$

(2) Differential equation

$$(\partial_x^4 - p\partial_x^2 + q)G(m;x,y) = 0 \qquad (0 < x, y < \infty,\ x \neq y).$$

(3) Boundary condition

$$\partial_x^{m_i} G(m;x,y)\Big|_{x=0} = 0 \qquad (i = 0, 1, \quad 0 < y < \infty).$$

(4) Jumping condition

$$\partial_x^i G(m;x,y)\Big|_{y=x-0} - \partial_x^i G(m;x,y)\Big|_{y=x+0} = \begin{cases} 0 & (i = 0,1,2) \\ 1 & (i = 3) \end{cases}$$

$$(0 < x < \infty),$$

*where* $\widetilde{\partial_x^3} = \partial_x^3 - p\partial_x$.

**Proof of Lemma 3.4** (1) is obvious. (2) follows from

$$P(\partial_x)G(m;x,y) = P(\partial_x)G(x-y) + P(\partial_x)G_c(m;x,y) = 0,$$

where we use $P(\pm a) = P(\pm b) = 0$. We treat (3). For the expression of Green function (3.13) as

$$G(m;x,y) = G(x-y) + G_c(m;x,y) \qquad (0 < x < y < \infty),$$

$$G(x-y) = \begin{pmatrix} e^{-ay} & e^{-by} \end{pmatrix} (-\boldsymbol{E}) \begin{pmatrix} e^{ax} \\ e^{bx} \end{pmatrix},$$

$$G_c(m;x,y) = \begin{pmatrix} e^{-ay} & e^{-by} \end{pmatrix} \boldsymbol{C}(m) \begin{pmatrix} e^{-ax} \\ e^{-bx} \end{pmatrix},$$

differentiating $m_i$ $(i = 0, 1)$ times with respect to $x$ and taking limit $x \to 0$, we have

$$\partial_x^{m_i} G(m;x,y)\Big|_{x=0} = \partial_x^{m_i} \Big[ G(x-y) + G_c(m;x,y) \Big]\Big|_{x=0} =$$

$$\begin{pmatrix} e^{-ay} & e^{-by} \end{pmatrix} \Big( -\boldsymbol{E} + (-1)^{m_i} \boldsymbol{C}(m) \Big) \begin{pmatrix} a^{m_i} \\ b^{m_i} \end{pmatrix}.$$

Applying each boundary condition $m = (0,1)$, $(0,2)$, $(1,3)$ and $(2,\widetilde{3})$ to the above relation, we have (3). (4) follows from (3.7). Thus we have Lemma 3.4. ■

**Lemma 3.5.** *The classical solution* (3.11) *satisfies* BVP$(m)$.

**Proof of Lemma 3.5** Lemma 3.5 follows from Lemma 3.3 and Lemma 3.4. Lemma 3.5 shows the existence of the solution of BVP$(m)$. ■

## 4. Another standard expression of Green function

In this section we derive another standard expression of Green function. Green function is expressed using the successive derivatives of the fundamental solution of special initial value problem IVP [2, 3, 6, 7, 8, 10] and fundamental solutions which satisfy the special boundary data.

The initial value problem

IVP

$$\begin{cases} u^{(4)} - pu'' + qu = 0 \qquad (0 < x < \infty) \\ u(0) = u'(0) = u''(0) = 0, \quad u'''(0) = 1 \end{cases}$$

has a unique solution $u(x) = K_0(x)$ as

$$K_0(x) = \frac{1}{a^2 - b^2}\left[a^{-1}\mathrm{sh}(ax) - b^{-1}\mathrm{sh}(bx)\right] \qquad (0 < x < \infty).$$

From $0 < b < a$ in (1.2), all the coefficients of Taylor series of $K_0(x)$

$$K_0(x) = \sum_{j=1}^{\infty} \frac{1}{(2j+1)!}\frac{a^{2j} - b^{2j}}{a^2 - b^2}x^{2j+1} > 0 \qquad (0 < x < \infty)$$

are positive, therefore $K_0(x)$ is completely monotone. If we introduce successive derivatives

$$K_j(x) = \left(\frac{d}{dx}\right)^j K_0(x) > 0 \qquad (j = 0, 1, 2, \cdots,\ 0 < x < \infty), \tag{4.1}$$

then we have the recurrence relation

$$K_{j+4}(x) - pK_{j+2}(x) + qK_j(x) = 0 \qquad (j = 0, 1, 2, \cdots,\ 0 < x < \infty).$$

Through straightforward calculations, we can show

$$(K_1(x))^2 - K_0(x)K_2(x) =$$
$$\frac{1}{ab}\left[\left(\frac{\mathrm{sh}((a+b)x/2)}{a+b}\right)^2 - \left(\frac{\mathrm{sh}((a-b)x/2)}{a-b}\right)^2\right] > 0 \qquad (0 < x < \infty).$$

Using this relation, we have

$$\left(\frac{K_1(x)}{K_0(x)}\right)' = -\frac{(K_1(x))^2 - K_0(x)K_2(x)}{(K_0(x))^2} < 0 \qquad (0 < x < \infty). \tag{4.2}$$

Using fundamental solutions satisfying the prescribed boundary conditions $A_i(m;x)$ and fundamental solutions to the initial value problem $K_j(x)$, we have another expression of Green function $G(m;x,y)$.

**Lemma 4.1.** *Green function possesses another expression given as follows:*

$$G(m;x,y) = -\begin{pmatrix}(-1)^{m_0}A_0 & (-1)^{m_1}A_1\end{pmatrix}(m;x\vee y)\begin{pmatrix}K_{m_0}\\ K_{m_1}\end{pmatrix}(x\wedge y) \tag{4.3}$$
$$(0 < x,y < \infty),$$

*where* $(-1)^{\widetilde{3}} = -1$ *and* $K_{\widetilde{3}}(x) = K_3(x) - pK_1(x)$.

**Proof of Lemma 4.1** Applying (3.12) to (3.14) and (3.15), Green function (3.13) is given as

$$G(m;x,y) = G(x-y) + G_c(m;x,y) = \tag{4.4}$$
$$\begin{pmatrix}A_0 & A_1\end{pmatrix}(m;x\vee y)\Big[-\boldsymbol{\Phi}_-(m)\boldsymbol{E}\begin{pmatrix}e^{a(x\wedge y)}\\ e^{b(x\wedge y)}\end{pmatrix} + \boldsymbol{\Phi}_+(m)\boldsymbol{E}\begin{pmatrix}e^{-a(x\wedge y)}\\ e^{-b(x\wedge y)}\end{pmatrix}\Big].$$

On the other hand, the exponential function expression of $K_0(x)$ is given as

$$K_0(x) = \begin{pmatrix}1 & 1\end{pmatrix}\boldsymbol{E}\begin{pmatrix}e^{ax}\\ e^{bx}\end{pmatrix} - \begin{pmatrix}1 & 1\end{pmatrix}\boldsymbol{E}\begin{pmatrix}e^{-ax}\\ e^{-bx}\end{pmatrix}.$$

Differentiating $K_0(x)$ $m_i$ $(i = 0,1)$ times, we have

$$\begin{pmatrix}K_{m_0}\\ K_{m_1}\end{pmatrix}(x) =$$
$$\begin{pmatrix}a^{m_0} & b^{m_0}\\ a^{m_1} & b^{m_1}\end{pmatrix}\boldsymbol{E}\begin{pmatrix}e^{ax}\\ e^{bx}\end{pmatrix} - \begin{pmatrix}(-a)^{m_0} & (-b)^{m_0}\\ (-a)^{m_1} & (-b)^{m_1}\end{pmatrix}\boldsymbol{E}\begin{pmatrix}e^{-ax}\\ e^{-bx}\end{pmatrix} =$$
$$-\begin{pmatrix}(-1)^{m_0} & 0\\ 0 & (-1)^{m_1}\end{pmatrix}\Big\{-\boldsymbol{\Phi}_-(m)\,\boldsymbol{E}\begin{pmatrix}e^{ax}\\ e^{bx}\end{pmatrix} + \boldsymbol{\Phi}_+(m)\,\boldsymbol{E}\begin{pmatrix}e^{-ax}\\ e^{-bx}\end{pmatrix}\Big\}.$$

Inserting the above relation into (4.4), we have (4.3). This shows Lemma 4.1. ■

## 5. Proof of Theorem 1.1

This section is devoted to a proof of the positivity and the hierarchical structure among four Green functions in Theorem 3.2.

First, we show the positivity of $G(0,1;x,y)$. To prove this fact, we show the positivity of the diagonal value $G(0,1;y,y)$.

**Lemma 5.1.** *The following relation holds:*

$$G(0,1;y,y) > 0 \qquad (0 < y < \infty).$$

**Proof of Lemma 5.1** From (3.13), (3.14) and (3.15), we have

$$G(0,1;y,y) = G(0) + G_c(0,1;y,y) \qquad (0 < y < \infty),$$

$$G_c(0,1;y,y) = \begin{pmatrix} e^{-ay} & e^{-by} \end{pmatrix} \boldsymbol{C}(0,1) \begin{pmatrix} e^{-ay} \\ e^{-by} \end{pmatrix} \qquad (0 < y < \infty).$$

Since

$$G_c(0,1;0,0) = \begin{pmatrix} 1 & 1 \end{pmatrix} \boldsymbol{C}(0,1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -G(0),$$

we have the initial value

$$G(0,1;0,0) = G(0) + G_c(0,1;0,0) = 0.$$

Differentiating $G(0,1;y,y)$ and noting ${}^t\boldsymbol{C}(m) = \boldsymbol{C}(m)$, we have

$$\frac{d}{dy} G(0,1;y,y) =$$

$$2 \begin{pmatrix} e^{-ay} & e^{-by} \end{pmatrix} \begin{pmatrix} -a & 0 \\ 0 & -b \end{pmatrix} \boldsymbol{C}(0,1) \begin{pmatrix} e^{-ay} \\ e^{-by} \end{pmatrix} \qquad (0 < y < \infty).$$

Using

$$\begin{pmatrix} -a & 0 \\ 0 & -b \end{pmatrix} \boldsymbol{C}(0,1) = G(0) \frac{ab}{(a-b)^2} \begin{pmatrix} a+b & -2a \\ -2b & a+b \end{pmatrix},$$

we have

$$\frac{d}{dy} G(0,1;y,y) =$$

$$G(0) \frac{2ab(a+b)}{(a-b)^2} \left( e^{-by} - e^{-ay} \right)^2 > 0 \qquad (0 < y < \infty).$$

This proves Lemma 5.1. ■

Finally, we prove the main theorem.

**Proof of Theorem 1.1** We first prove

$$(5.1) \qquad G(0,1;x,y) > 0 \qquad (0 < x, y < \infty).$$

We use the abbreviation $A_i(x) = A_i(0,1;x)$. From Lemma 4.1, we have

$$G(0,1;x,y) = -A_0(x \vee y) K_0(x \wedge y) + A_1(x \vee y) K_1(x \wedge y) =$$

$$K_0(x \wedge y) \left[ A_1(x \vee y) \left( \frac{K_1(x \wedge y)}{K_0(x \wedge y)} \right) - A_0(x \vee y) \right]$$

$$(0 < (x \wedge y) < (x \vee y) < \infty).$$

From (3.17), (3.18), (4.1) and (4.2), the relations

$$A_0(x) > 0, \quad A_1(x) > 0,$$

$$K_0(x) > 0, \quad K_1(x) > 0, \quad -\left(\frac{K_1(x)}{K_0(x)}\right)' > 0 \qquad (0 < x < \infty)$$

hold. Hence, we have

$$\begin{aligned}
&G(0,1;x,y) = \\
&K_0(x\wedge y)\left[A_1(x\vee y)\left(\frac{K_1(x\wedge y)}{K_0(x\wedge y)}\right) - A_0(x\vee y)\right] \geq \\
&K_0(x\wedge y)\left[A_1(x\vee y)\left(\frac{K_1(x\vee y)}{K_0(x\vee y)}\right) - A_0(x\vee y)\right] = \\
&\frac{K_0(x\wedge y)}{K_0(x\vee y)}\Big[A_1(x\vee y)K_1(x\vee y) - A_0(x\vee y)K_0(x\vee y)\Big] = \\
&\frac{K_0(x\wedge y)}{K_0(x\vee y)}G(0,1;x\vee y, x\vee y) \qquad (0 < (x\wedge y) < (x\vee y) < \infty).
\end{aligned}$$

Using Lemma 5.1, we have the inequality (5.1).

Second, we show the hierarchical structure of Green functions or equivalently, the following inequalities:

$$\text{(5.2)} \qquad \begin{cases} G(0,2;x,y) - G(0,1;x,y) > 0 & (0 < x,y < \infty) \\ G(1,3;x,y) - G(0,2;x,y) > 0 & (0 < x,y < \infty) \\ G(2,\widetilde{3};x,y) - G(0,2;x,y) > 0 & (0 < x,y < \infty) \\ G(2,\widetilde{3};x,y) - G(1,3;x,y) \begin{cases} > 0 & (x,y) \in D_0 \\ < 0 & (x,y) \in D_1 \end{cases} \end{cases},$$

where $D_0$ and $D_1$ are defined by Theorem 1.1. Using the expression of

$$\left.\begin{array}{l} G(0,2;x,y) \\ G(1,3;x,y) \end{array}\right\} = G(x-y) \mp G(x+y) \qquad (0 < x,y < \infty),$$

we have

$$G(1,3;x,y) - G(0,2;x,y) = 2\,G(x+y) > 0 \qquad (0 < x,y < \infty),$$

where we use $G(x) > 0$ $(0 < x < \infty)$ in (3.3). Using the expression of Green functions in (3.13), we have

$$\begin{aligned}
&G(0,2;x,y) - G(0,1;x,y) = G_c(0,2;x,y) - G_c(0,1;x,y) = \\
&\begin{pmatrix} e^{-a(x\vee y)} & e^{-b(x\vee y)} \end{pmatrix} \Big(\boldsymbol{C}(0,2) - \boldsymbol{C}(0,1)\Big) \begin{pmatrix} e^{-a(x\wedge y)} \\ e^{-b(x\wedge y)} \end{pmatrix} = \\
&\frac{2ab\,G(0)}{(a-b)^2}\left(e^{-b(x\vee y)} - e^{-a(x\vee y)}\right)\left(e^{-b(x\wedge y)} - e^{-a(x\wedge y)}\right) > 0
\end{aligned}$$

$$(0 < x, y < \infty),$$

$$G(2,\widetilde{3};x,y) - G(0,2;x,y) = G_c(2,\widetilde{3};x,y) - G_c(0,2;x,y) =$$
$$\begin{pmatrix} e^{-a(x\vee y)} & e^{-b(x\vee y)} \end{pmatrix} \left( \boldsymbol{C}(2,\widetilde{3}) - \boldsymbol{C}(0,2) \right) \begin{pmatrix} e^{-a(x\wedge y)} \\ e^{-b(x\wedge y)} \end{pmatrix} =$$
$$\frac{2\,G(0)}{(a-b)(a^3-b^3)} \left( a^2 e^{-b(x\vee y)} - b^2 e^{-a(x\vee y)} \right) \left( a^2 e^{-b(x\wedge y)} - b^2 e^{-a(x\wedge y)} \right)$$
$$> 0 \qquad\qquad (0 < x, y < \infty),$$

$$G(2,\widetilde{3};x,y) - G(1,3;x,y) = G_c(2,\widetilde{3};x,y) - G_c(1,3;x,y) =$$
$$\begin{pmatrix} e^{-a(x\vee y)} & e^{-b(x\vee y)} \end{pmatrix} \left( \boldsymbol{C}(2,\widetilde{3}) - \boldsymbol{C}(1,3) \right) \begin{pmatrix} e^{-a(x\wedge y)} \\ e^{-b(x\wedge y)} \end{pmatrix} =$$
$$\frac{2ab\,G(0)}{(a-b)(a^3-b^3)} \left( ae^{-a(x\vee y)} - be^{-b(x\vee y)} \right) \left( ae^{-a(x\wedge y)} - be^{-b(x\wedge y)} \right)$$
$$\begin{cases} > 0 & (x,y) \in D_0 \\ < 0 & (x,y) \in D_1 \end{cases},$$

where we use the relation

$$ae^{-ax} - be^{-bx} = ae^{-ax}\left(1 - \frac{b}{a}e^{(a-b)x}\right) \begin{cases} > 0 & (0 < x < x_0) \\ = 0 & (x = x_0) \\ < 0 & (x_0 < x < \infty) \end{cases},$$

$$x_0 = \frac{1}{a-b}\log\frac{a}{b}.$$

Thus we have obtained the hierarchical structures in (5.2).
Theorem 1.1 follows from (5.1) and (5.2). ■

In the proof of Theorem 1.1, we have the inequalities:

$$\begin{cases} G_c(0,2;x,y) - G_c(0,1;x,y) > 0 & (0 < x, y < \infty) \\ G_c(1,3;x,y) - G_c(0,2;x,y) > 0 & (0 < x, y < \infty) \\ G_c(2,\widetilde{3};x,y) - G_c(0,2;x,y) > 0 & (0 < x, y < \infty) \\ G_c(2,\widetilde{3};x,y) - G_c(1,3;x,y) \begin{cases} > 0 & (x,y) \in D_0 \\ < 0 & (x,y) \in D_1 \end{cases} \end{cases}.$$

From (3.16), we have

$$G_c(0,2;x,y) < 0 < G_c(1,3;x,y) \qquad (0 < x, y < \infty).$$

So we have the following corollary.

**Corollary 5.1.** *Four kinds of compensating functions $G_c(m;x,y)$ satisfy the hierarchical structure shown as*

$$G_c(0,1;x,y) < G_c(0,2;x,y) < 0 < \left\{ \begin{array}{l} G_c(1,3;x,y) \\ G_c(2,\widetilde{3};x,y) \end{array} \right.$$

$$(0 < x,y < \infty).$$

*In particular, we have*

$$G_c(0,1;x,y) < G_c(0,2;x,y) < 0 < G_c(1,3;x,y) < G_c(2,\widetilde{3};x,y)$$

$$(x,y) \in D_0 = \Big\{ 0 < x,y < x_0 \quad \mathit{or} \quad x_0 < x,y < \infty \Big\}$$

*and*

$$G_c(0,1;x,y) < G_c(0,2;x,y) < 0 < G_c(2,\widetilde{3};x,y) < G_c(1,3;x,y)$$

$$(x,y) \in D_1 = \Big\{ 0 < x < x_0 < y < \infty \quad \mathit{or} \quad 0 < y < x_0 < x < \infty \Big\},$$

*where* $x_0 = \dfrac{1}{a-b} \log \dfrac{a}{b}$.

We remark that Green functions of BVP($m$) are $G(m;x,y) = G(x-y) + G_c(m;x,y) > 0$ $(0 < x,y < \infty)$, and Green function of BVP($\mathbf{R}$) is $G(x-y) > 0$ $(-\infty < x,y < \infty)$. Hence, we have another corollary.

**Corollary 5.2.** *Five Green functions $G(x-y)$ and $G(m;x,y)$ are positive and satisfy the hierarchical structure shown as*

$$0 < G(0,1;x,y) < G(0,2;x,y) < G(x-y) < \left\{ \begin{array}{l} G(1,3;x,y) \\ G(2,\widetilde{3};x,y) \end{array} \right.$$

$$(0 < x,y < \infty).$$

*In particular, we have*

$$0 < G(0,1;x,y) < G(0,2;x,y) <$$

$$G(x-y) < G(1,3;x,y) < G(2,\widetilde{3};x,y)$$

$$(x,y) \in D_0 = \Big\{ 0 < x,y < x_0 \quad \mathit{or} \quad x_0 < x,y < \infty \Big\}$$

*and*

$$0 < G(0,1;x,y) < G(0,2;x,y) <$$

$$G(x-y) < G(2,\widetilde{3};x,y) < G(1,3;x,y)$$

$$(x,y) \in D_1 = \Big\{ 0 < x < x_0 < y < \infty \quad \mathit{or} \quad 0 < y < x_0 < x < \infty \Big\},$$

*where* $x_0 = \dfrac{1}{a-b} \log \dfrac{a}{b}$.

## 6. Appendix : Sobolev inequality

In this section, we show the best constants of Sobolev inequality. Although this study is shown in [8], we also treat them for the sake of self-containedness.

We introduce Sobolev space

$$H = \left\{ u \,\middle|\, u, u', u'' \in L^2(0,\infty), \right.$$
$$\left. \left\{ \begin{array}{ll} u(0) = u'(0) = 0 & (\mathrm{X}) = (0,1) \\ u(0) = 0 & (\mathrm{X}) = (0,2) \\ u'(0) = 0 & (\mathrm{X}) = (1,3) \\ \text{none} & (\mathrm{X}) = (2,\widetilde{3}) \end{array} \right\} \right\}$$

and Sobolev inner product

$$(u,v)_H = \int_0^\infty \Big[ u''(x)\overline{v}''(x) + p\,u'(x)\overline{v}'(x) + q\,u(x)\overline{v}(x) \Big]\, dx,$$
$$\|u\|_H^2 = (u,u)_H.$$

**Lemma 6.1.** *For any $u \in H$ and fixed $y$ $(0 \le y < \infty)$, we have the reproducing relation*

(6.1) $\quad u(y) = (u(\cdot), G(m;\cdot,y))_H.$

*Putting $u(x) = G(m;x,y) \in H$, we have*

(6.2) $\quad G(m;y,y) = \|G(m;\cdot,y)\|_H^2.$

**Proof of Lemma 6.1** We put $v = v(x) = G(m;x,y) \in H$ with $y$ arbitrarily fixed in $y \in [0,\infty)$. Noting $\overline{v} = v$ and integrating the identity

$$u''v'' + p\,u'v' + q\,uv =$$
$$\Big[ u'v'' - uv''' + p\,uv' \Big]' + uv^{(4)} - p\,uv'' + q\,uv$$

with respect to $x$ on $0 < x < y$ and $y < x < \infty$, we have

$$(u,v)_H =$$
$$\left\{ \int_0^y + \int_y^\infty \right\} \left[ \Big( u'v'' - u(v''' - p\,v') \Big)' + u\left( v^{(4)} - p\,v'' + q\,v \right) \right] dx.$$

From Lemma 3.4 (2) and (4), we have

$$(u,v)_H = u(y) - u'(0)v''(0) + u(0)\left(v''' - p\,v'\right)(0).$$

Setting $u = u(x) \in H$ and using Lemma 3.4 (3), we have (6.1). Putting $u = u(x) = G(m;x,y) \in H$ with $y$ arbitrarily fixed in $y \in [0,\infty)$, we have (6.2). This completes the proof of Lemma 6.1. ■

**Theorem 6.1.** *There exists a positive constant $C$ such that for any $u \in H$ the Sobolev inequality*

$$\left( \sup_{0 \le y < \infty} |u(y)| \right)^2 \le C \|u\|_H^2 = C \int_0^\infty \left[ \left|u''(x)\right|^2 + p \left|u'(x)\right|^2 + q \left|u(x)\right|^2 \right] dx \tag{6.3}$$

*holds. Among such $C$, the best constant $C_0 = C_0(m)$ is*

$$C_0(m) = \max_{0 \le y < \infty} G(m; y, y) = \begin{cases} G(0,1;y,y)\Big|_{y=\infty} & = G(0) \\ G(0,2;y,y)\Big|_{y=\infty} & = G(0) \\ G(1,3;0,0) & = 2G(0) \\ G(2,\widetilde{3};0,0) & = 2G(0)\dfrac{(a+b)^2}{a^2+ab+b^2} \end{cases}. \tag{6.4}$$

*If we replace $C$ by $C_0$ in the above inequality, the equality holds if and only if the constant multiple of*

$$u(x) = \begin{cases} \textit{no existence} & m = (0,1),\ (0,2) \\ G(m;x,0) & m = (1,3),\ (2,\widetilde{3}) \end{cases} \qquad (0 < x < \infty). \tag{6.5}$$

The engineering meaning of Sobolev inequality is that the square of the maximum bending of a beam is estimated from above by the constant multiple of the potential energy. Among these constants, the best constant is the maximum of the diagonal value of the Green function.

**Corollary 6.1.** *The hierarchical structure of four best constant of Sobolev inequality $C(m)$ is*

$$C(0,1) = C(0,2) < C(1,3) < C(2,\widetilde{3}).$$

**Proof of Theorem 6.1** Applying Schwarz inequality to (6.1) and using (6.2), we have

$$|u(y)|^2 = |(u(\cdot), G(m;\cdot,y))_H|^2 \le \|u\|_H^2 \|G(m;\cdot,y)\|_H^2 = G(m;y,y)\|u\|_H^2. \tag{6.6}$$

Taking the maximum with respect to $y$ $(0 \le y < \infty)$ on both sides, we have Sobolev inequality

$$\left( \sup_{0 \le y < \infty} |u(y)| \right)^2 \le C_0 \|u\|_H^2, \tag{6.7}$$

where

$$C_0 = \max_{0\le y<\infty} G(m;y,y) = G(m;y_0,y_0).$$

The inequality (6.7) implies that $\|u\|_H = 0$ holds if and only if $u(x) \equiv 0$, which shows the positive definiteness of inner product $(\cdot,\cdot)_H$. If we take $u(x) = G(m;x,y_0) \in H$ in (6.7), then we have

$$\left(\sup_{0\le y<\infty} |G(m;y,y_0)|\right)^2 \le C_0 \|G(m;\cdot,y_0)\|_H^2 = C_0 G(m;y_0,y_0) = C_0^2.$$

Combining this with the trivial inequality

$$C_0^2 = G(m;y_0,y_0)^2 \le \left(\sup_{0\le y<\infty} |G(m;y,y_0)|\right)^2,$$

we have

$$\left(\sup_{0\le y<\infty} |G(m;y,y_0)|\right)^2 = C_0 \|G(m;\cdot,y_0)\|_H^2.$$

This shows that $C_0$ is the best constant of (6.7) and the equality holds for $G(m;x,y_0)$.

Next, we search for the concrete value of the maximum of $G(m;y,y)$. From (3.13), the diagonal value of Green function $G(m;x,y)$ is

$$G(m;y,y) = G(0) + G_c(m;y,y) \qquad (0<y<\infty),$$
$$G_c(m;y,y) = \begin{pmatrix} e^{-ay} & e^{-by} \end{pmatrix} \boldsymbol{C}(m) \begin{pmatrix} e^{-ay} \\ e^{-by} \end{pmatrix} \qquad (0<y<\infty).$$

We prepare

$$\text{(6.8)} \qquad G(m;0,0) = \begin{cases} 0 & m=(0,1),(0,2) \\ 2G(0) & m=(1,3) \\ 2G(0)\dfrac{(a+b)^2}{a^2+ab+b^2} & m=(2,\widetilde{3}) \end{cases},$$

$$\text{(6.9)} \qquad G(m;y,y)\Big|_{y=\infty} = G(0) \qquad m=(0,1),(0,2),(1,3),(2,\widetilde{3}).$$

The derivative of $G(m;y,y)$ $(0<y<\infty)$ as

$$\frac{d}{dy}G(m;y,y) = \frac{d}{dy}G_c(m;y,y) =$$
$$\begin{pmatrix} e^{-ay} & e^{-by} \end{pmatrix} \begin{pmatrix} -a & 0 \\ 0 & -b \end{pmatrix} \boldsymbol{C}(m) \begin{pmatrix} e^{-ay} \\ e^{-by} \end{pmatrix} +$$
$$\begin{pmatrix} e^{-ay} & e^{-by} \end{pmatrix} \boldsymbol{C}(m) \begin{pmatrix} -a & 0 \\ 0 & -b \end{pmatrix} \begin{pmatrix} e^{-ay} \\ e^{-by} \end{pmatrix} \qquad (0<y<\infty).$$

Because of ${}^t\boldsymbol{C}(m) = \boldsymbol{C}(m)$, we have

$$\frac{d}{dy}G(m;y,y) =$$
$$2\begin{pmatrix} e^{-ay} & e^{-by} \end{pmatrix} \begin{pmatrix} -a & 0 \\ 0 & -b \end{pmatrix} \boldsymbol{C}(m) \begin{pmatrix} e^{-ay} \\ e^{-by} \end{pmatrix} \qquad (0 < y < \infty).$$

Here

$$\begin{pmatrix} -a & 0 \\ 0 & -b \end{pmatrix} \boldsymbol{C}(0,1) = G(0)\frac{ab}{(a-b)^2}\begin{pmatrix} a+b & -2a \\ -2b & a+b \end{pmatrix},$$
$$\begin{pmatrix} -a & 0 \\ 0 & -b \end{pmatrix} \left\{ \begin{array}{c} \boldsymbol{C}(0,2) \\ \boldsymbol{C}(1,3) \end{array} \right\} = \mp G(0)\frac{ab}{a-b}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\begin{pmatrix} -a & 0 \\ 0 & -b \end{pmatrix} \boldsymbol{C}(2,\widetilde{3}) = -G(0)\frac{ab}{(a-b)(a^3-b^3)}\begin{pmatrix} a^3+b^3 & -2a^2b \\ -2ab^2 & a^3+b^3 \end{pmatrix},$$

we have

$$\frac{d}{dy}G(0,1;y,y) = G(0)\frac{2ab(a+b)}{(a-b)^2}\left(e^{-by} - e^{-ay}\right)^2 > 0,$$
$$\frac{d}{dy}G(\left\{ \begin{array}{c} 0,2 \\ 1,3 \end{array} \right\};y,y) = \pm G(0)\frac{2ab}{a-b}\left(e^{-2by} - e^{-2ay}\right)\left\{ \begin{array}{c} >0 \\ <0 \end{array} \right.,$$
$$\frac{d}{dy}G(2,\widetilde{3};y,y) =$$
$$-G(0)\frac{4ab(a+b)}{(a-b)(a^3-b^3)}e^{-(a+b)y}\Big[(a^2-ab+b^2)\mathrm{ch}((a-b)y) - ab\Big] < 0$$
$$(0 < y < \infty),$$

where we note $a^2 - ab + b^2 = (a-b)^2 + ab > ab$. So we have the behavior of $G(m;y,y)$ $(0 < y < \infty)$ as

$$\text{(6.10)} \quad \frac{d}{dy}G(m;y,y) = \left\{ \begin{array}{ll} >0 & (m) = (0,1),(0,2) \\ <0 & (m) = (1,3),(2,\widetilde{3}) \end{array} \right. \qquad (0 < y < \infty).$$

Since (6.8)$\sim$(6.10), we have the concrete value of (6.4).

This completes of the proof of Theorem 6.1. ■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