García, A. G. & Muñoz-Bouzo, M. J. (2015). Sampling-related frames in finite U-invariant subspaces. Applied and Computational Harmonic Analysis, 39(1), 173-184.

DOI: https://doi.org/10.1016/j.acha.2014.09.008

Sampling-related frames in finite U-invariant subspaces

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Abstract

Recently, a sampling theory for infinite dimensional U-invariant subspaces of a separable Hilbert space $\mathcal{H}$ where U denotes a unitary operator on $\mathcal{H}$ has been obtained. Thus, uniform average sampling for shift-invariant subspaces of $L^2(\mathbb{R})$ becomes a particular example. As in the general case it is possible to have finite dimensional U-invariant subspaces, the main aim of this paper is to derive a sampling theory for finite dimensional U-invariant subspaces of a separable Hilbert space $\mathcal{H}$. Since the used samples are frame coefficients in a suitable euclidean space $\mathbb{C}^N$, the problem reduces to obtain dual frames with a U-invariance property.

Keywords: Stationary sequences; U-invariant subspaces; Finite frame; Dual frames; Moore-Penrose pseudo-inverse.
AMS: 42C15; 94A20; 15A09.

1 Statement of the problem

The frame concept was introduced by Duffin and Shaeffer in [7] while studying some problems in nonharmonic Fourier series; some years later it was revived by Daubechies, Grossman and Meyer in [6]. Nowadays, frames have become a tool in pure and applied mathematics, computer science, physics and engineering used to derive redundant, yet stable decompositions of a signal for analysis or transmission, while also promoting sparse expansions. Recall that a sequence $\{x_k\}$ is a frame for a separable Hilbert space $\mathcal{H}$ if there exist two constants $A, B > 0$ (frame bounds) such that

$$A\|x\|^2 \leq \sum_k |\langle x, x_k\rangle|^2 \leq B\|x\|^2 \quad \text{for all } x \in \mathcal{H}\,.$$

Given a frame $\{x_k\}$ for $\mathcal{H}$ the representation property of any vector $x \in \mathcal{H}$ as a series $x = \sum_k c_k x_k$ is retained, but, unlike the case of Riesz bases, the uniqueness of this

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representation (for overcomplete frames) is sacrificed. Suitable frame coefficients c_k which depend continuously and linearly on x are obtained by using the dual frames $\{y_k\}$ of $\{x_k\}$, i.e., $\{y_k\}$ is another frame for $\mathcal{H}$ such that $x = \sum_k \langle x, y_k\rangle\, x_k = \sum_k \langle x, x_k\rangle\, y_k$ for each $x \in \mathcal{H}$. For more details on the frame theory see, for instance, the monograph [5] and references therein; see also Ref. [4] for finite frames.

Traditionally, frames were used in signal and image processing, nonharmonic analysis, data compression, and sampling theory, but nowadays frame theory plays also a fundamental role in a wide variety of problems in both pure and applied mathematics, computer science, physics and engineering. The redundancy of frames, which gives flexibility and robustness, is the key to their significance for applications; see, for instance, the nice introduction in Chapter1 of Ref. [4] and the references therein.

In particular, the use of frames in sampling theory has become very fruitful; see, for instance, Refs. [1, 2, 3, 10, 16, 17]. Recently, in Refs. [8, 9, 15] a generalization of the sampling theory for shift-invariant subspaces $V_\varphi^2 := \big\{ \sum_{n\in\mathbb{Z}} \alpha_n\, \varphi(t-n) \,:\, \{\alpha_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) \big\}$ of $L^2(\mathbb{R})$ with generator φ has been obtained in the following sense: Let $U : \mathcal{H} \to \mathcal{H}$ be a unitary operator in a separable Hilbert space $\mathcal{H}$; for a fixed $a \in \mathcal{H}$, consider the closed subspace given by $\mathcal{A}_a := \overline{\operatorname{span}}\big\{U^n a,\ n \in \mathbb{Z}\big\}$. In case that the (infinite) sequence $\{U^n a\}_{n\in\mathbb{Z}}$ is a Riesz sequence in $\mathcal{H}$ we have

$$\mathcal{A}_a = \Big\{ \sum_{n\in\mathbb{Z}} \alpha_n\, U^n a : \{\alpha_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) \Big\}.$$

Here, the sequence of generalized samples $\{(\mathcal{L}_j x)(rm)\}_{m\in\mathbb{Z};\, j=1,2,\ldots,s}$ of $x \in \mathcal{A}_a$ is obtained from s elements $b_j \in \mathcal{H}$ as

$$(\mathcal{L}_j x)(rm) := \langle x, U^{rm} b_j\rangle_{\mathcal{H}}\,, \quad m \in \mathbb{Z}\,;\ j = 1, 2, \ldots, s\,. \tag{1}$$

Thus, under appropriate hypotheses it was proved in [9, 15], by using different techniques, the existence of frames in $\mathcal{A}_a$, having the form $\big\{U^{rm} c_j\big\}_{m\in\mathbb{Z};\, j=1,2,\ldots,s}$, where $c_j \in \mathcal{A}_a$ for $j = 1, 2, \ldots, s$, such that for each $x \in \mathcal{A}_a$ we get the sampling expansion

$$x = \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} \mathcal{L}_j x(rm)\, U^{rm} c_j \quad \text{in } \mathcal{H}\,. \tag{2}$$

The U-sampling problem was introduced, for the first time, in Refs. [13, 15]. A particular case is the shift-invariant subspace V_φ^2 where $U : f(t) \mapsto f(t-1)$ is the shift operator in $L^2(\mathbb{R})$. For any $f \in V_\varphi^2$ the samples are

$$(\mathcal{L}_j f)(rm) = \langle x, U^{rm} b_j\rangle_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} f(t)\, \overline{b_j(t-rm)}\, dt = \big(f * h_j\big)(rm)\,, \quad m \in \mathbb{Z}\,,$$

where $h_j(t) := \overline{b_j(-t)}$ for each $j = 1, 2, \ldots, s$. Besides, sampling formula (2) for $f \in V_\varphi^2$ reads as

$$f(t) = \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} \big(\mathcal{L}_j f\big)(rm)\, S_j(t - rm)\,, \quad t \in \mathbb{R}\,,$$

where the sequence of reconstruction functions $\{S_j(\cdot - rm)\}_{m\in\mathbb{Z};\, j=1,2,\ldots,s}$ is a frame for V_φ^2 (see, for instance, Ref. [10] for the details).

In [9, 15] it was implicitely assumed that the stationary sequence $\{U^n a\}_{n\in\mathbb{Z}}$ in $\mathcal{H}$ has infinite different elements. It could happen that for some $a \in \mathcal{H}$ there exists $N \in \mathbb{N}$ such that $U^N a = a$, i.e., 1 is an eigenvalue of the unitary operator U^N with eigenvector a. In this case, $\mathcal{A}_a$ is just the finite dimensional subspace of $\mathcal{H}$ spanned by the set $\{a, Ua, U^2a, \dots, U^{N-1}a\}$. The main aim in this work is to derive a sampling theory in $\mathcal{A}_a$ involving dual finite frames.

In applications, frames in finite dimensional spaces are required. Since in finite dimension, a frame is nothing but a spanning set of vectors, finite frames require control of certain condition numbers, and over the spectrum of certain matrices. Thus, frame theory, firstly considered applied harmonic analysis, meets matrix analysis and numerical linear algebra.

Concretely in this work, whenever $\dim \mathcal{A}_a = N$, we consider a positive integer r such that $r|N$ and $\ell = N/r$; the goal is to obtain finite frames in $\mathcal{A}_a$ having the form $\{U^{rn}c_j\}_{\substack{j=1,2,\dots,s\\ n=0,1,\dots,\ell-1}}$, where $c_j \in \mathcal{A}_a$, $j = 1, 2, \dots, s$, such that any $x \in \mathcal{A}_a$ can be recovered from the samples $\{\mathcal{L}_j x(rn)\}_{\substack{j=1,2,\dots,s\\ n=0,1,\dots,\ell-1}}$ given in (1) by means of the expansion

$$x = \sum_{j=1}^{s} \sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, U^{rn} c_j \,.$$

In so doing, we express the given samples as frame coefficients with respect to a frame in $\mathbb{C}^N$. The challenge problem is to obtain its dual frames in $\mathbb{C}^N$ yielding, via an isomorphism $\mathcal{T}_{N,a}$ between $\mathbb{C}^N$ and $\mathcal{A}_a$ (see (3) infra), the desired frames $\{U^{rn}c_j\}_{\substack{j=1,2,\dots,s\\ n=0,1,\dots,\ell-1}}$ for $\mathcal{A}_a$. All these steps will be carried out throughout the remaining sections.

2 The mathematical setting

For a fixed $a \in \mathcal{H}$, assume that there exists a nonnegative integer N such that $U^N a = a$; let N be the smallest index with this property. Next, we consider the finite dimensional subspace $\mathcal{A}_a := \operatorname{span}\{a, Ua, U^2a, \dots, U^{N-1}a\}$ in $\mathcal{H}$. The *auto-covariance* R_a of the stationary sequence $\{U^n a\}_{n\in\mathbb{Z}}$ defined in [12] as

$$R_a(k) := \langle U^k a, a\rangle_{\mathcal{H}}\,, \quad k \in \mathbb{Z}\,,$$

inherits its N-periodic character. The $N \times N$ auto-covariance matrix is defined by

$$\mathbf{R_a} := \begin{pmatrix} R_a(0) & R_a(1) & \dots & R_a(N-1) \\ R_a(1) & R_a(2) & \dots & R_a(0) \\ \vdots & \vdots & \ddots & \vdots \\ R_a(N-1) & R_a(0) & \dots & R_a(N-2) \end{pmatrix}.$$

Proposition 1. *The set of vectors $\{a, Ua, U^2a, \dots, U^{N-1}a\}$ is linearly independent if and only if* $\det \mathbf{R_a} \neq 0$.

Proof. If $\det \mathbf{R_a} = 0$ then there exists $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \dots, \lambda_{N-1})^\top \in \mathbb{C}^N$ such that $\lambda \neq 0$ and $\mathbf{R_a}\boldsymbol{\lambda} = 0$. Thus $\sum_{k=0}^{N-1} \lambda_k U^k a$ is orthogonal to $a, Ua, U^2a, \dots, U^{N-1}a$ so that $\sum_{k=0}^{N-1} \lambda_k U^k a = 0$. Conversely, if $\sum_{k=0}^{N-1} \lambda_k U^k a = 0$ for some $\boldsymbol{\lambda} \neq 0$ then the inner product in the above expression with each $a, Ua, U^2a, \dots, U^{N-1}a$ yields $\mathbf{R_a}\boldsymbol{\lambda} = 0$. □

In the sequel we assume that $\det \mathbf{R_a} \neq 0$; thus, $\dim \mathcal{A}_a = N$ and the set of vectors $\{a, Ua, U^2a, \dots, U^{N-1}a\}$ forms a basis for $\mathcal{A}_a$.

The isomorphism $\mathcal{T}_{N,a}$

Next we consider the following isomorphism $\mathcal{T}_{N,a}$ between $\mathbb{C}^N$ and $\mathcal{A}_a$:

$$\begin{aligned} \mathcal{T}_{N,a}: \quad & \mathbb{C}^N \longrightarrow \mathcal{A}_a \\ & \boldsymbol{\alpha} = \sum_{k=0}^{N-1} \alpha_k \mathbf{e_k} \longmapsto x = \sum_{k=0}^{N-1} \alpha_k U^k a\,. \end{aligned} \tag{3}$$

where $\{\mathbf{e_1}, \mathbf{e_2}, \dots, \mathbf{e_N}\}$ denotes the canonical basis for $\mathbb{C}^N$. The isomorphism $\mathcal{T}_{N,a}$ has the following shifting property:

Proposition 2. *Let $\{T(k)\}_{k\in\mathbb{Z}}$ be an N-periodic sequence in $\mathbb{C}$. For $1 \le m \le N-1$ consider the vectors in $\mathbb{C}^N$*

$$\mathbf{T}_0 := \big(T(0), T(1), \dots, T(N-1)\big)^\top \quad \text{and}$$
$$\mathbf{T}_{N-m} := \big(T(N-m), T(N-m+1), \dots, T(N-m+N-1)\big)^\top .$$

Then, the following shifting property holds

$$\mathcal{T}_{N,a}(\mathbf{T}_{N-m}) = U^m\big(\mathcal{T}_{N,a}(\mathbf{T}_0)\big) \quad \text{for any} \;\; 1 \le m \le N-1\,.$$

Proof. The change of index $p = N-m+k$ and the N-periodic character of the sequences $\{T(k)\}$ and $\{U^k a\}$ give

$$\begin{aligned} \mathcal{T}_{N,a}(\mathbf{T}_{N-m}) &= \sum_{k=0}^{N-1} T(n-m+k)\, U^k a = \sum_{p=N-m}^{2N-m-1} T(p)\, U^{p-N+m} a = \sum_{p=N-m}^{2N-m-1} T(p)\, U^{p+m} a \\ &= \sum_{q=0}^{N-1} T(q)\, U^{q+m} a = U^m\Big(\sum_{q=0}^{N-1} T(q)\, U^q a\Big) = U^m\big(\mathcal{T}_{N,a}(\mathbf{T}_0)\big)\,. \end{aligned}$$

□

Generalized samples: a suitable expression

Let r be a positive integer such that $r|N$, and consider $\ell = N/r$. Fixed s elements $b_j \in \mathcal{H}$, $j = 1, 2, \dots, s$, for each $x \in \mathcal{A}$ we consider its generalized samples $\{\mathcal{L}_j x(rn)\}_{\substack{j=1,2,\dots,s \\ n=0,1,\dots,\ell-1}}$ with sampling period r defined by

$$\mathcal{L}_j x(rn) := \langle x, U^{rn} b_j \rangle_{\mathcal{H}}\,, \quad n = 0, 1, \dots, \ell-1 \text{ and } j = 1, 2, \dots, s\,.$$

The goal in this paper is to recover any $x \in \mathcal{A}_a$ from its finite sequence of samples $\big\{\mathcal{L}_j x(rn)\big\}_{\substack{j=1,2,\dots,s\\ n=0,1,\dots,\ell-1}}$ by means of suitable frames in $\mathcal{A}_a$. First, we obtain a more convenient expression for $\mathcal{L}_j x(rn)$; namely

$$\begin{aligned}\mathcal{L}_j x(rn) &= \langle x, U^{rn} b_j\rangle_{\mathcal{H}} = \Big\langle \sum_{k=0}^{N-1} \alpha_k\, U^k a, U^{rn} b_j\Big\rangle_{\mathcal{H}} = \sum_{k=0}^{N-1} \alpha_k\, \langle U^k a, U^{rn} b_j\rangle_{\mathcal{H}} \\ &= \Big\langle \sum_{k=0}^{N-1} \alpha_k\, \mathbf{e_k}, \sum_{k=0}^{N-1} \overline{\langle U^k a, U^{rn} b_j\rangle_{\mathcal{H}}}\, \mathbf{e_k}\Big\rangle_{\mathbb{C}^N} = \big\langle \boldsymbol{\alpha}, \mathbf{G}_{j,n}\big\rangle_{\mathbb{C}^N}\,,\end{aligned} \tag{4}$$

where $\mathbf{G}_{j,n} = \sum_{k=0}^{N-1} \overline{\langle U^k a, U^{rn} b_j\rangle_{\mathcal{H}}}\, \mathbf{e_k}$. The *cross-covariance* between the sequences $\{U^n a\}$ and $\{U^n b_j\}$, i.e., the N-periodic sequence defined in [12] as

$$R_{a,b_j}(m) := \langle U^m a, b_j\rangle_{\mathcal{H}}\,, \quad m \in \mathbb{Z}\,,$$

allows to write

$$\begin{aligned}\mathbf{G}_{j,n} &= \sum_{k=0}^{N-1} \overline{\langle U^{k-rn} a, b_j\rangle_{\mathcal{H}}}\, \mathbf{e_k} = \sum_{k=0}^{N-1} \overline{\langle U^{N+k-rn} a, b_j\rangle_{\mathcal{H}}}\, \mathbf{e_k} \\ &= \sum_{k=0}^{N-1} \overline{R_{a,b_j}(N+k-rn)}\, \mathbf{e_k}\,.\end{aligned} \tag{5}$$

Having in mind the expression (4) for the samples $\mathcal{L}_j x(rn)$, $n = 0, 1, \dots, \ell-1$ and $j = 1, 2, \dots, s$, and the isomorphism $\mathcal{T}_{N,a}$ given in (3), any $x \in \mathcal{A}_a$ can be recovered from its samples if and only if the set of vectors $\big\{\mathbf{G}_{j,n}\big\}_{\substack{j=1,2,\dots,s\\ n=0,1,\dots,\ell-1}}$ in $\mathbb{C}^N$ forms a spanning set for $\mathbb{C}^N$. In other words, it is a frame for $\mathbb{C}^N$ (see, for instance, Refs. [4, 5]). This is equivalent to the condition $\operatorname{rank} \mathbf{G_{a,b}} = N$, where $\mathbf{G_{a,b}}$ denotes the $N \times s\ell$ matrix

$$\mathbf{G_{a,b}} := \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{G}_{1,0} & \dots & \mathbf{G}_{1,\ell-1} & \mathbf{G}_{2,0} & \dots & \mathbf{G}_{2,\ell-1} & \dots & \mathbf{G}_{s,0} & \dots & \mathbf{G}_{s,\ell-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Thus we have that $N \le s\ell$, that is, $s \ge r$.

Having in mind (5) and the N-periodic character of the cross-covariance we obtain that

$$\mathbf{G_{a,b}} = \begin{pmatrix} \mathbf{R^*_{a,b_1}} & \mathbf{R^*_{a,b_2}} & \dots & \mathbf{R^*_{a,b_s}} \end{pmatrix}\,,$$

where each $\ell \times N$ block $\mathbf{R_{a,b_j}}$, $j = 1, 2, \dots, s$, is given by

$$\mathbf{R_{a,b_j}} = \begin{pmatrix} R_{a,b_j}(0) & R_{a,b_j}(1) & \dots & R_{a,b_j}(N-1) \\ R_{a,b_j}(N-r) & R_{a,b_j}(N-r+1) & \dots & R_{a,b_j}(2N-r-1) \\ \vdots & \vdots & \ddots & \vdots \\ R_{a,b_j}(N-r(\ell-1)) & R_{a,b_j}(N-r(\ell-1)+1) & \dots & R_{a,b_j}(2N-1-r(\ell-1)) \end{pmatrix}.$$

Since $N = r\ell$ we have

$$\mathbf{R_{a,b_j}} = \begin{pmatrix} R_{a,b_j}(0) & R_{a,b_j}(1) & \dots & R_{a,b_j}(N-1) \\ R_{a,b_j}(N-r) & R_{a,b_j}(N-r+1) & \dots & R_{a,b_j}(2N-r-1) \\ \vdots & \vdots & \ddots & \vdots \\ R_{a,b_j}(r) & R_{a,b_j}(r+1) & \dots & R_{a,b_j}(2r-1) \end{pmatrix}.$$

As usual, the symbol $*$ denotes the transpose conjugate matrix. Given the $s\ell \times N$ matrix of cross-covariances

$$\mathbf{R_{a,b}} := \begin{pmatrix} \mathbf{R_{a,b_1}} \\ \mathbf{R_{a,b_2}} \\ \vdots \\ \mathbf{R_{a,b_s}} \end{pmatrix}, \tag{6}$$

where each $\ell \times N$ block is given by $\mathbf{R_{a,b_j}}$, $j = 1, 2, \dots, s$, we deduce that the matrix $\mathbf{R_{a,b}}$ coincides with the matrix $\mathbf{G}^*_{\mathbf{a,b}}$.

Lemma 1. *Let $x = \sum_{k=0}^{N-1} \alpha_k U^k a \in \mathcal{A}_a$. For each $j = 1, 2, \dots, s$ we have the following expression for the samples $\{\mathcal{L}_j x(rn)\}_{n=0}^{\ell-1}$*

$$\begin{pmatrix} \mathcal{L}_j x(0) \\ \mathcal{L}_j x(r) \\ \vdots \\ \mathcal{L}_j x(r(\ell-1)) \end{pmatrix} = \mathbf{R_{a,b_j}} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{N-1} \end{pmatrix}.$$

In other words, denoting the vectors $\boldsymbol{\alpha} := (\alpha_0, \alpha_1, \dots, \alpha_{N-1})^\top \in \mathbb{C}^N$ and

$$\boldsymbol{\mathcal{L}}_{sam} := \big(\mathcal{L}_1 x(0), \mathcal{L}_1 x(r), \dots, \mathcal{L}_1 x(r(\ell-1)), \dots, \mathcal{L}_s x(0), \dots, \mathcal{L}_s x(r(\ell-1))\big)^\top \in \mathbb{C}^{s\ell}, \tag{7}$$

the matrix relationship

$$\boldsymbol{\mathcal{L}}_{sam} = \mathbf{R_{a,b}}\, \boldsymbol{\alpha}$$

holds where $\mathbf{R_{a,b}}$ is the $s\ell \times N$ matrix of cross-covariances given in (6).

Proof. Given $x = \sum_{k=0}^{N-1} \alpha_k U^k a \in \mathcal{A}_a$ let consider $\boldsymbol{\alpha} = \sum_{k=0}^{N-1} \alpha_k \mathbf{e_k} \in \mathbb{C}^N$. By using (4) and (5) we obtain

$$\mathcal{L}_j x(rn) = \langle \boldsymbol{\alpha}, \mathbf{G}_{j,n} \rangle_{\mathbb{C}^N} = \sum_{k=0}^{N-1} \alpha_k\, R_{a,b_j}(N + k - rn), \quad \text{for } n = 0, 1, \dots, \ell - 1.$$

$\square$

As $\operatorname{rank} \mathbf{R_{a,b}} = \operatorname{rank} \mathbf{G_{a,b}} = N$, the Moore-Penrose pseudo-inverse of $\mathbf{R_{a,b}}$ is the $N \times s\ell$ matrix $\mathbf{R}^\dagger_{\mathbf{a,b}} = \big[\mathbf{R}^*_{\mathbf{a,b}}\, \mathbf{R_{a,b}}\big]^{-1} \mathbf{R}^*_{\mathbf{a,b}}$.

Writing the columns of $\mathbf{R}^\dagger_{\mathbf{a,b}}$ as

$$\mathbf{R}^\dagger_{\mathbf{a,b}} = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{R}^\dagger_{1,0} & \dots & \mathbf{R}^\dagger_{1,\ell-1} & \mathbf{R}^\dagger_{2,0} & \dots & \mathbf{R}^\dagger_{2,\ell-1} & \dots & \mathbf{R}^\dagger_{s,0} & \dots & \mathbf{R}^\dagger_{s,\ell-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

given $x = \sum_{k=0}^{N-1} \alpha_k\, U^k a \in \mathcal{A}_a$, for $\boldsymbol{\alpha} = \sum_{k=0}^{N-1} \alpha_k\, \mathbf{e_k} \in \mathbb{C}^N$, from the matrix relationship $\boldsymbol{\mathcal{L}}_{\text{sam}} = \mathbf{R_{a,b}}\, \boldsymbol{\alpha}$ we obtain

$$\boldsymbol{\alpha} = \big(\alpha_0, \alpha_1, \dots, \alpha_{N-1}\big)^\top = \mathbf{R}^\dagger_{\mathbf{a,b}}\, \boldsymbol{\mathcal{L}}_{\text{sam}} = \sum_{j=1}^{s} \sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, \mathbf{R}^\dagger_{j,n} \,. \tag{8}$$

In other words, the columns $\big\{\mathbf{R}^\dagger_{j,n}\big\}_{\substack{j=1,2,\dots,s \\ n=0,1,\dots,\ell-1}}$ of the Moore-Penrose pseudo-inverse $\mathbf{R}^\dagger_{\mathbf{a,b}}$ are a dual frame of $\big\{\mathbf{G}_{j,n}\big\}_{\substack{j=1,2,\dots,s \\ n=0,1,\dots,\ell-1}}$ in $\mathbb{C}^N$; that is for each $\boldsymbol{\alpha} \in \mathbb{C}^N$

$$\boldsymbol{\alpha} = \sum_{j=1}^{s} \sum_{n=0}^{\ell-1} \big\langle \boldsymbol{\alpha}, \mathbf{G}_{j,n} \big\rangle_{\mathbb{C}^N}\, \mathbf{R}^\dagger_{j,n} \,.$$

In particular, we derive that rank $\mathbf{R}^\dagger_{\mathbf{a,b}} = N$. Any other dual frame of $\big\{\mathbf{G}_{j,n}\big\}_{\substack{j=1,2,\dots,s \\ n=0,1,\dots,\ell-1}}$ in $\mathbb{C}^N$ is given by the columns of any left-inverse $\mathbf{H}$ of the matrix $\mathbf{R_{a,b}}$; i.e., $\mathbf{H}\,\mathbf{R_{a,b}} = \mathbf{I}_N$. All these matrices are expressed as (see [14]):

$$\mathbf{H} = \mathbf{R}^\dagger_{\mathbf{a,b}} + \mathbf{U}\big[\mathbf{I}_{s\ell} - \mathbf{R_{a,b}}\,\mathbf{R}^\dagger_{\mathbf{a,b}}\big] \,, \tag{9}$$

where $\mathbf{U}$ denotes any arbitrary $N \times s\ell$ matrix.

The pseudo-inverse $\mathbf{R}^\dagger_{\mathbf{a,b}}$ is computed by using the singular value decomposition of $\mathbf{R_{a,b}}$; the singular values are the square root of the eigenvalues of the $N \times N$ invertible and positive definite matrix $\mathbf{R}^*_{\mathbf{a,b}}\,\mathbf{R_{a,b}}$ (see, for instance, [5, 11]). Note that the singular value decomposition of $\mathbf{R_{a,b}}$ is the most reliable method to reveal its rank in practice.

3 The result

Applying the isomorphism $\mathcal{T}_{N,a}$ in (8), for any $x = \sum_{k=0}^{N-1} \alpha_k\, U^k a \in \mathcal{A}_a$ we obtain the sampling formula:

$$x = \mathcal{T}_{N,a}\big(\boldsymbol{\alpha}\big) = \sum_{j=1}^{s} \sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, \mathcal{T}_{N,a}\big(\mathbf{R}^\dagger_{j,n}\big) \,.$$

The sampling functions $\mathcal{T}_{N,a}\big(\mathbf{R}^\dagger_{j,n}\big)$ in the above formula do not have, in principle, any special structure since $\mathbf{R}^\dagger_{\mathbf{a,b}}$ does not it. However, having in mind the structure of the matrix $\mathbf{R_{a,b}}$ we construct left-inverses of $\mathbf{R_{a,b}}$ with the same structure.

Left-inverses of $\mathbf{R_{a,b}}$ with its same structure

We construct a specific left-inverse $\mathbf{H_S}$ of $\mathbf{R_{a,b}}$ from $\mathbf{R}^\dagger_{\mathbf{a,b}}$ (or from another left-inverse of $\mathbf{R_{a,b}}$ given in (9)) in the following way: We denote as $\mathbf{S}$ the first r rows of the matrix $\mathbf{R}^\dagger_{\mathbf{a,b}}$, i.e., $\mathbf{S}\,\mathbf{R_{a,b}} = \big[\mathbf{I}_r, \mathbf{O}_{r\times(N-r)}\big]$, where $\mathbf{I}_r$ and $\mathbf{O}_{r\times(N-r)}$ denote, respectively, the identity matrix of order r and the zero matrix of order $r \times (N-r)$. According to the structure of the matrix $\mathbf{R_{a,b}}$ (see (6)), we write the $r \times s\ell$ matrix $\mathbf{S}$ as

$$\mathbf{S} = \big(\mathbf{S}_1\ \mathbf{S}_2\ \dots\ \mathbf{S}_s\big) \,,$$

where each $r \times \ell$ block $\mathbf{S}_j$, $j = 1, 2, \ldots, s$ is denoted by

$$\mathbf{S}_j := \begin{pmatrix} S_j(0) & S_j(N-r) & S_j(N-2r) & \ldots & S_j(N-r(\ell-1)) \\ S_j(1) & S_j(N-r+1) & S_j(N-2r+1) & \ldots & S_j(N-r(\ell-1)+1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_j(r-1) & S_j(N-1) & S_j(N-2r+r-1) & \ldots & S_j(N-r(\ell-1)+r-1) \end{pmatrix}$$

or, using that $N = r\ell$, as

$$\mathbf{S}_j = \begin{pmatrix} S_j(0) & S_j(N-r) & S_j(N-2r) & \ldots & S_j(r) \\ S_j(1) & S_j(N-r+1) & S_j(N-2r+1) & \ldots & S_j(r+1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_j(r-1) & S_j(N-1) & S_j(N-r-1) & \ldots & S_j(2r-1) \end{pmatrix}.$$

Now, we form the $N \times s\ell$ matrix

$$\mathbf{H_S} := \big(\widetilde{\mathbf{S}}_1 \; \widetilde{\mathbf{S}}_2 \; \ldots \; \widetilde{\mathbf{S}}_s\big) \tag{10}$$

by using the columns of $\mathbf{S}_j$, $j = 1, 2, \ldots, s$ in the following manner

$$\widetilde{\mathbf{S}}_j := \begin{pmatrix} S_j(0) & S_j(N-r) & S_j(N-2r) & \ldots & S_j(r) \\ S_j(1) & S_j(N-r+1) & S_j(N-2r+1) & \ldots & S_j(r+1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_j(r-1) & S_j(N-1) & S_j(N-r-1) & \ldots & S_j(2r-1) \\ S_j(r) & S_j(0) & S_j(N-r) & \ldots & S_j(2r) \\ S_j(r+1) & S_j(1) & S_j(N-r+1) & \ldots & S_j(2r+1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_j(N-1) & S_j(N-r-1) & S_j(N-2r-1) & \ldots & S_j(r-1) \end{pmatrix}.$$

In other words:

- The first column of $\widetilde{\mathbf{S}}_j$ is a concatenation of the columns 1, ℓ, $\ell-1$, …, and 2 of $\mathbf{S}_j$;
- The second column of $\widetilde{\mathbf{S}}_j$ is a concatenation of the columns 2, 1, ℓ, …, and 3 of $\mathbf{S}_j$;
- The third column of $\widetilde{\mathbf{S}}_j$ is a concatenation of the columns 3, 2, 1, …, and 4 of $\mathbf{S}_j$;

Repeating the process, finally,

- The column ℓ of $\widetilde{\mathbf{S}}_j$ is a concatenation of the columns ℓ, $\ell-1$, $\ell-2$, …, and 1 of $\mathbf{S}_j$.

Thus, with this procedure we have obtained a left-inverse matrix $\mathbf{H_S}$ for $\mathbf{R_{a,b}}$:

Lemma 2. *Let* $\mathbf{R_{a,b}}$ *and* $\mathbf{H_S}$ *be the matrices defined in* (6) *and* (10) *respectively. Then we have that* $\mathbf{H_S}\,\mathbf{R_{a,b}} = \mathbf{I}_N$.

Proof. Having in mind the N-periodic character of the entries in matrices $\mathbf{H_S}$ and $\mathbf{R_{a,b}}$, the product $\alpha_{m,k}$ of row $m+1$ of $\mathbf{H_S}$, $m = 0, 1, \ldots, N-1$, with column $k+1$ of $\mathbf{R_{a,b}}$, $k = 0, 1, \ldots, N-1$, can be written as

$$\alpha_{m,k} = \sum_{j=1}^{s} \sum_{i=0}^{\ell-1} S_j(N-ir+m) R_{a,b_j}(N-ir+k)\,.$$

Since $\mathbf{S}\,\mathbf{R}_{\mathbf{a},\mathbf{b}} = \big[\mathbf{I}_r, \mathbf{O}_{r\times(N-r)}\big]$, we have that $\alpha_{m,k} = \delta_{m,k}$ if $m = 0, 1, \dots, r-1$ and $k = 0, 1, \dots, N-1$. Whenever $m \geq r$, we write $m = m_0 + qr$ where $m_0 = 0, 1, \dots, r-1$, and $q = 1, 2, \dots \ell - 1$. Then

$$\begin{aligned}
\alpha_{m,k} &= \sum_{j=1}^{s}\sum_{i=0}^{\ell-1} S_j(N-(i-q)r+m_0)R_{a,b_j}(N-(i-q)r+k-qr) \\
&= \begin{cases} \alpha_{m_0,k-qr} & \text{if } k-qr \geq 0 \\ \alpha_{m_0,N+k-qr} & \text{if } k-qr < 0 \end{cases}.
\end{aligned}$$

Now, the result follows from the fact that $m_0 = k - qr$ if and only if $m = k$ and the failure of the equality $m_0 = N + k - qr$. $\square$

Next we denote the columns of $\mathbf{H}_{\mathbf{S}}$ as

$$\mathbf{H}_{\mathbf{S}} = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{H}_{1,0} & \dots & \mathbf{H}_{1,\ell-1} & \mathbf{H}_{2,0} & \dots & \mathbf{H}_{2,\ell-1} & \dots & \mathbf{H}_{s,0} & \dots & \mathbf{H}_{s,\ell-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}. \tag{11}$$

Since the matrix $\mathbf{H}_{\mathbf{S}}$ is a left-inverse matrix of $\mathbf{R}_{\mathbf{a},\mathbf{b}}$, using $\mathbf{H}_{\mathbf{S}}$ instead of $\mathbf{R}^{\dagger}_{\mathbf{a},\mathbf{b}}$ in (8) we obtain

$$\boldsymbol{\alpha} = \big(\alpha_0, \alpha_1, \dots, \alpha_{N-1}\big)^{\top} = \mathbf{H}_{\mathbf{S}}\,\boldsymbol{\mathcal{L}}_{\text{sam}} = \sum_{j=1}^{s}\sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\,\mathbf{H}_{j,n}\,. \tag{12}$$

Then, by using the isomorphism $\mathcal{T}_{N,a}$ in (12) and Proposition 2 we obtain, for each $x \in \mathcal{A}_a$, the sampling formula:

$$\begin{aligned}
x = \mathcal{T}_{N,a}\big(\boldsymbol{\alpha}\big) &= \sum_{j=1}^{s}\sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\,\mathcal{T}_{N,a}\big(\mathbf{H}_{j,n}\big) \\
&= \sum_{j=1}^{s}\sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\,U^{rn}\big(\mathcal{T}_{N,a}\big(\mathbf{H}_{j,0}\big)\big) = \sum_{j=1}^{s}\sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\,U^{rn} c_j\,,
\end{aligned} \tag{13}$$

where $c_j = \mathcal{T}_{N,a}\big(\mathbf{H}_{j,0}\big) \in \mathcal{A}_a$, $j = 1, 2, \dots, s$.

Collecting all the pieces that we have obtained until now we prove the following result:

Theorem 3. *Given the $s\ell \times N$ matrix of cross-covariances $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ defined in (6), the following statements are equivalents:*

(a) rank $\mathbf{R}_{\mathbf{a},\mathbf{b}} = N$

(b) There exists an $r \times s\ell$ matrix $\mathbf{S}$ such that

$$\mathbf{S}\,\mathbf{R}_{\mathbf{a},\mathbf{b}} = \big[\mathbf{I}_r, \mathbf{O}_{r\times(N-r)}\big]\,, \tag{14}$$

where $\mathbf{I}_r$ and $\mathbf{O}_{r\times(N-r)}$ denote, respectively, the identity matrix of order r and the zero matrix of order $r \times (N-r)$.

(c) *There exist $c_j \in \mathcal{A}_a$, $j = 1, 2, \dots, s$ such that the sequence $\{U^{rn} c_j\}_{\substack{j=1,2,\dots,s \\ n=0,1,\dots,\ell-1}}$ is a frame for $\mathcal{A}_a$, and for any $x \in \mathcal{A}_a$ the expansion*

$$x = \sum_{j=1}^{s} \sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, U^{rn} c_j \tag{15}$$

holds.

(d) *There exist a frame $\{C_{j,n}\}_{\substack{j=1,2,\dots,s \\ n=0,1,\dots,\ell-1}}$ for $\mathcal{A}_a$ such that, for each $x \in \mathcal{A}_a$ the expansion*

$$x = \sum_{j=1}^{s} \sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, C_{j,n}$$

holds.

Proof. Condition (a) implies condition (b); it is enough to take as the matrix $\mathbf{S}$ the first r rows of the Moore-Penrose pseudo-inverse $\mathbf{R}^{\dagger}_{\mathbf{a},\mathbf{b}}$. Condition (b) implies condition (c); it is just the proof of the sampling formula (13). Condition (c) implies condition (d); it is obvious, take $C_{j,n} = U^{rn} c_j$, $j = 1, 2, \dots, s$ and $n = 0, 1, \dots, \ell - 1$. Finally, we prove that condition (d) implies condition (a). Indeed, let $x = \sum_{k=0}^{N-1} \alpha_k\, U^k a$ be an arbitrary element in $\mathcal{A}_a$, and let define $\mathbf{K}_{j,n} := \mathcal{T}^{-1}_{N,a}\big(C_{j,n}\big)$ for $j = 1, 2, \dots, s$ and $n = 0, 1, \dots, \ell - 1$. Applying $\mathcal{T}^{-1}_{N,a}$ in $x = \sum_{j=1}^{s} \big(\sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, C_{j,n}\big)$ we get

$$\boldsymbol{\alpha} = \big(\alpha_0, \alpha_1, \dots, \alpha_{N-1}\big)^{\top} = \sum_{j=1}^{s} \sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, \mathbf{K}_{j,n} = \mathbf{K}\, \boldsymbol{\mathcal{L}}_{\text{sam}}\,,$$

where $\boldsymbol{\mathcal{L}}_{\text{sam}}$ is defined in (7) and $\mathbf{K}$ is the $N \times s\ell$ matrix having $\mathbf{K}_{j,n}$ as columns as in (10). Now, by using Lemma 1 we have $\boldsymbol{\mathcal{L}}_{\text{sam}} = \mathbf{R}_{\mathbf{a},\mathbf{b}}\, \boldsymbol{\alpha}$; therefore, $\boldsymbol{\alpha} = \mathbf{K}\, \mathbf{R}_{\mathbf{a},\mathbf{b}}\, \boldsymbol{\alpha}$ for all $\boldsymbol{\alpha} \in \mathbb{C}^N$, i.e., $\mathbf{K}\, \mathbf{R}_{\mathbf{a},\mathbf{b}} = \mathbf{I}_N$. This implies that rank $\mathbf{R}_{\mathbf{a},\mathbf{b}} = N$ which completes the proof. □

For the particular case where the number of systems $\mathcal{L}_j$ and the sampling period r coincides, i.e., $s = r$ we obtain:

Corollary 4. *Assume that $s = r$ and consider the $N \times N$ matrix of cross-covariances $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ defined in (6). The following statements are equivalents:*

(i) *The matrix $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ is invertible.*

(ii) *There exist r unique elements $c_j \in \mathcal{A}_a$, $j = 1, 2, \dots, r$, such that the sequence $\{U^{rn} c_j\}_{\substack{j=1,2,\dots,r \\ n=0,1,\dots,\ell-1}}$ is a basis for $\mathcal{A}_a$, and the expansion of any $x \in \mathcal{A}_a$ with respect to this basis is*

$$x = \sum_{j=1}^{r} \sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, U^{rn} c_j\,.$$

In case the equivalent conditions are satisfied, the interpolation property $\mathcal{L}_j c_{j'}(rn) = \delta_{j,j'}\, \delta_{n,0}$, whenever $n = 0, 1, \dots, \ell - 1$ and $j, j' = 1, 2, \dots, r$, holds.

Proof. Notice that the inverse matrix $\mathbf{R}_{\mathbf{a},\mathbf{b}}^{-1}$ has necessarily the structure of the matrix $\mathbf{H}_{\mathbf{S}}$ in (11). The uniqueness of the expansion with respect to a basis gives the interpolation property. □

A filter-bank interpretation

Assume that the rank of $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ is N, and let $\mathbf{S}$ be an $r \times s\ell$ matrix satisfying (14). Proceeding as before we construct a left-inverse $\mathbf{H}_{\mathbf{S}}$ of $\mathbf{R}_{\mathbf{a},\mathbf{b}}$, with columns $\mathbf{H}_{j,n}$, $j = 1, 2, \dots, s$ and $n = 0, 1, \dots, \ell - 1$ (see (11)). In the corresponding sampling formula (15) we have $c_j = \mathcal{T}_{N,a}\big(\mathbf{H}_{j,0}\big)$, $j = 1, 2, \dots, s$; suppose that

$$\mathbf{H}_{j,0} = \big(\beta_j(0), \beta_j(1), \dots, \beta_j(N-1)\big)^\top, \quad j = 1, 2, \dots, s\,.$$

Substituting in (15), for $x \in \mathcal{A}_a$ we get

$$x = \sum_{j=1}^{s}\sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, U^{rn}\big(\sum_{m=0}^{N-1} \beta_j(m)\, U^m a\big) = \sum_{j=1}^{s}\sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, \big(\sum_{m=0}^{N-1} \beta_j(m)\, U^{rn+m} a\big).$$

The change of index $k := rn + m$ and the N-periodicity gives

$$\begin{aligned} x &= \sum_{j=1}^{s}\sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, \big(\sum_{k=rn}^{rn+N-1} \beta_j(k-rn)\, U^k a\big) \\ &= \sum_{j=1}^{s}\sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, \big(\sum_{k=0}^{N-1} \beta_j(k-rn)\, U^k a\big) \\ &= \sum_{k=0}^{N-1} \Big\{ \sum_{j=1}^{s}\sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, \beta_j(k-rn) \Big\} U^k a\,. \end{aligned}$$

In other words, for $x = \sum_{k=0}^{N-1} \alpha_k\, U^k a$, the coefficients α_k, $k = 0, 1, \dots, N-1$, are the output of a filter-bank

$$\alpha_k = \sum_{j=1}^{s}\sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, \beta_j(k-rn)\,, \quad k = 0, 1, \dots, N-1\,,$$

involving the data $\big\{\mathcal{L}_j x(rn)\big\}_{\substack{j=1,2,\dots,s \\ n=0,1,\dots,\ell-1}}$ and the columns $\mathbf{H}_{j,0}$, $j = 1, 2, \dots, s$, of $\mathbf{H}_{\mathbf{S}}$.

A toy model involving periodic sequences

Let $\ell^2_N(\mathbb{Z})$ be the Hilbert space of N-periodic sequences of complex numbers

$$\mathbf{x} = \big(\dots, x(N-1), \underbrace{x(0), x(1), \dots, x(N-1)}_{N}, x(0), \dots\big)$$

endowed with the inner product $\langle \mathbf{x}, \mathbf{y} \rangle_{\ell^2_N} = \sum_{m=0}^{N-1} x(m)\,\overline{y(m)}$. We consider the cyclic shift $T : \ell^2_N(\mathbb{Z}) \to \ell^2_N(\mathbb{Z})$ defined as

$$\begin{aligned} T\mathbf{x} &= T\big(\dots, x(N-1), \underbrace{x(0), x(1), \dots, x(N-1)}_{N}, x(0), \dots\big) \\ &= \big(\dots, x(N-2), \underbrace{x(N-1), x(0), \dots, x(N-2)}_{N}, x(N-1), \dots\big)\,, \end{aligned}$$

and the N-periodic sequence $\mathbf{a} := (\dots, \underbrace{1, 0, \dots, 0}_{N}, \dots)$; obviously, $T^N\mathbf{a} = \mathbf{a}$ and $\mathcal{A}_\mathbf{a} = \ell^2_N(\mathbb{Z})$.

Given $\mathbf{b}_j \in \ell^2_N$, $j = 1, 2, \dots, s$, each sample from $\mathbf{x} \in \ell^2_N$ in $\{\mathcal{L}_j\mathbf{x}(rn)\}_{\substack{j=1,2,\dots,s \\ n=0,1,\dots,\ell-1}}$ is obtained from the N-periodic convolution

$$\mathcal{L}_j\mathbf{x}(rn) = \langle \mathbf{x}, T^{rn}\mathbf{b}_j \rangle_{\ell^2_N} = \sum_{m=0}^{N-1} x(m)\,\overline{\mathbf{b}_j(m-rn)} = \big(\mathbf{x} * \mathbf{h}_j\big)(rn)\,,$$

where $\mathbf{h}_j(m) = \overline{\mathbf{b}_j(-m)}$, $m = 0, 1, \dots, N-1$.

As the cross-covariance $R_{a,b_j}(m) = \langle T^m\mathbf{a}, \mathbf{b}_j \rangle_{\ell^2_N} = \overline{b_j(m)}$, each block $\mathbf{R}_{\mathbf{a},\mathbf{b_j}}$, $j = 1, 2, \dots, s$, of $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ in (6) takes the form

$$\mathbf{R}_{\mathbf{a},\mathbf{b_j}} = \begin{pmatrix} \overline{b_j(0)} & \overline{b_j(1)} & \dots & \overline{b_j(N-1)} \\ \overline{b_j(N-r)} & \overline{b_j(N-r+1)} & \dots & \overline{b_j(2N-r-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{b_j(N-r(\ell-1))} & \overline{b_j(N-r(\ell-1)+1)} & \dots & \overline{b_j(2N-1-r(\ell-1))} \end{pmatrix}.$$

In case the rank of $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ is N, from Theorem 3 we obtain in $\ell^2_N(\mathbb{Z})$ the sampling formula

$$x(m) = \sum_{j=1}^{s}\sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, c_j(m-rn)\,, \quad m = 0, 1, \dots, N-1\,,$$

which coincides with the output of a filter-bank. The sampling sequences in $\ell^2_N(\mathbb{Z})$ are $\mathbf{c}_j = \mathcal{T}_{N,a}\big(\mathbf{H}_{j,0}\big)$, $j = 1, 2, \dots, s$, where $\mathbf{H}_{j,0}$ are columns (see (11)) of a left-inverse $\mathbf{H}_\mathbf{S}$ of $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ constructed, from a matrix $\mathbf{S}$ satisfying (14), as in (10).

Acknowledgments: This work has been supported by the grant MTM2009–08345 from the Spanish *Ministerio de Ciencia e Innovación* (MICINN).

References

[1] A. Aldroubi. Non-uniform weighted average sampling and reconstruction in shift-invariant and wavelet spaces. *Appl. Comput. Harmon. Anal.*, 13:151–161, 2002.

[2] A. Aldroubi and K. Gröchenig. Non-uniform sampling and reconstruction in shift-invariant spaces. *SIAM Rev.*, 43:585–620, 2001.

[3] J. J. Benedetto. Irregular Sampling and Frames. In *Wavelets: A Tutorial in Theory and Applications*, C. K. Chui (ed.), pp. 445–507. Academic Press, San Diego, 1992.

[4] P. G. Casazza and G. Kutyniok (Eds.) *Finite Frames: Theory and Applications*. Birkhäuser, Boston, 2014.

[5] O. Christensen. *An Introduction to Frames and Riesz Bases*. Birkhäuser, Boston, 2003.

[6] I. Daubechies, A. Grossman and Y. Meyer. Painless nonorthogonal expansions. *J. Math. Phys.*, 27: 1271–1283, 1985.

[7] R. J. Duffin and A. C. Shaeffer. A class of nonharmonic Fourier series. *Trans. Amer. Math. Soc.*, 72: 341–366, 1952.

[8] H. R. Fernández-Morales, A. G. García, M. A. Hernández-Medina and M. J. Muñoz-Bouzo. On some sampling-related frames in U-invariant spaces. *Abstr. Appl. Anal.*, Vol. 2013, Article ID 761620, 2013.

[9] H. R. Fernández-Morales, A. G. García, M. A. Hernández-Medina and M. J. Muñoz-Bouzo. Generalized sampling: from shift-invariant to U-invariant spaces. Accepted in *Anal. Appl.*, DOI: 10.1142/S0219530514500213, 2014.

[10] A. G. García and G. Pérez-Villalón. Dual frames in $L^2(0,1)$ connected with generalized sampling in shift-invariant spaces. *Appl. Comput. Harmon. Anal.*, 20(3):422–433, 2006.

[11] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, 1999.

[12] A. N. Kolmogorov. Stationary sequences in Hilbert space. *Boll. Moskow. Gos. Univ. Mat.*, 2:1–40, 1941.

[13] T. Michaeli, V. Pohl and Y. C. Eldar. U-invariant sampling: extrapolation and causal interpolation from generalized samples. *IEEE Trans. Signal Process.*, 59(5):2085–2100, 2011.

[14] R. Penrose. A generalized inverse for matrices. *Math. Proc. Cambridge Philos. Soc.*, 51:406–413, 1955.

[15] V. Pohl and H. Boche. U-invariant sampling and reconstruction in atomic spaces with multiple generators. *IEEE Trans. Signal Process.*, 60(7):3506–3519, 2012.

[16] W. Sun and X. Zhou. Average sampling in shift-invariant subspaces with symmetric averaging functions. *J. Math. Anal. Appl.*, 287:279–295, 2003.

[17] X. Zhou and W. Sun. On the Sampling Theorem for Wavelet Subspaces. *J. Fourier Anal. Appl.*, 5(4):347–354, 1999.