COLUMN BUCKLING INVESTIGATION OF PLANE FRAMES

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Abstract. This paper deals with elastic stability analysis of the plane frame structures. The main aim is to investigate the accuracy of related parts of European and domestic codes for steel and concrete structures. Numerical analysis of frame structures is performed using the self-developed Matlab computer program. Matrix analysis of the whole structure according to the second order theory, based on application of trigonometric shape functions, is applied. As opposed to that, the dominant approach given in most structural codes is based upon the stability analysis of compressed structural elements isolated from the structure as a whole. Several numerical examples are given in the paper and comparative analyses presented herein show that, in some cases, solutions given in domestic and European codes are rather inaccurate. For example, the error in determination of the effective buckling length of frame columns can sometimes exceed even 100%. Finally, it is concluded that innovation of actual codes should be done in the part where the effective buckling length of frame columns is considered. Improvement of this calculation could be achieved using the global stability approach and the corresponding calculation the critical load for complete structure, as it is presented in the paper.

1. Introduction

There are numerous civil engineering structures which are subjected to the compression forces. Calculation of compressed structural elements requires investigation of stability phenomena. Dominant compressive stresses may cause structural instability, loss of load-bearing capacity and collapse of structure, even in the case when allowed stresses are not reached. It means that modern regulations have to take into account all knowledge about buckling of such kind of structures.
The subject of this paper is the analysis of codes related to stability problem of plane frame structures. The main aim of this analysis is to investigate what is the accuracy of existing codes.
Euler first studied stability problems of compressed bar and his "stability cases" are well known. Critical load in all these cases can be obtained by same expression, as it is presented in Figure1.

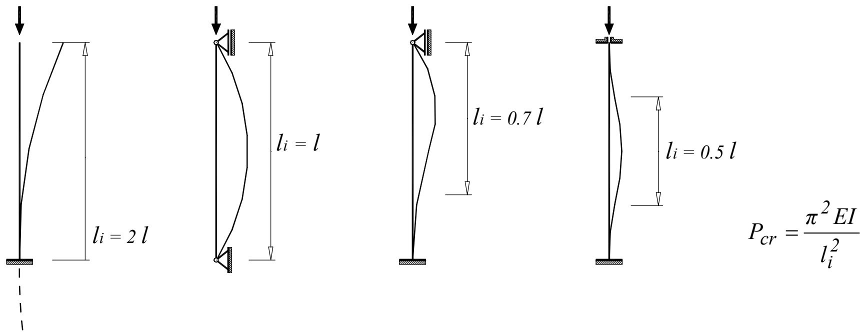


Figure 1. Euler's cases

As it is shown in the Figure 1, definition of "effective length" is given as the distance between inflection points of the bended member. For the stability analysis of plane frame columns, codes use simplified static scheme, as presented in Figure 2.

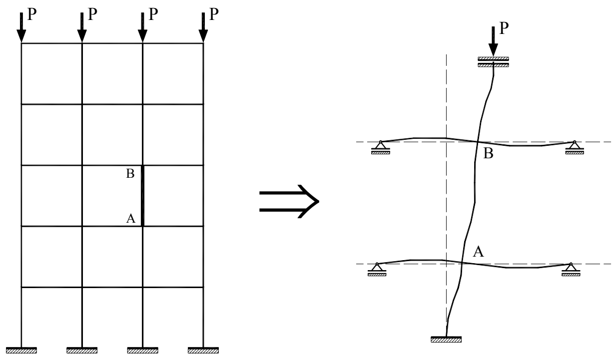


Figure 2. Simplified static scheme of the frames

As it is given in Figure 2, codes consider only columns which are separated from the frame structure. These separated columns are supported only by the adjacent columns and beams. So, stability analysis of columns is simplified and the results can be obtained very easily. They are shown by the corresponding diagrams and approximate formulas. Of course, very important question is to what extent these approximate solutions are correct and how they can be used in engineering practice. In the last decade there is a fast development of computer programs in general and also for the plane frame stability analysis, using the second order theory. Global stability analysis of whole frame structures becomes a routine calculation. So, the current concept of the stability analysis of structures in engineering practice must be correspondingly changed and it has to be also reflected in the codes. This paper is analyzing what is the accuracy obtained by use of the approximate solutions as given in the structural codes, or rather, how large are the obtained errors and should they be tolerated.

2. Matrix stability analysis of plane frame structures

Matrix analysis according to the second order theory is applied to solve the problem of the frame structure instability.
It is well known that during the member buckling, axial force produces the bending of the member. The basic differential equation of this stability problem is:

$$v^{iv} + k^2 v'' = 0 , \qquad (1)$$

where are: $k = \sqrt{\frac{P}{EI}}$, P – axial force, EI – member bending stiffness.

In order to formulate the exact matrix stability analysis, shape functions are used in the trigonometric form:

$$v(x) = \alpha_1 + \alpha_2 kx + \alpha_3 \sin(kx) + \alpha_4 \cos(kx) \qquad (2)$$

Stiffness matrix, obtained by trigonometric shape functions based on (2), is given in (3):

$$\boldsymbol{K} = \frac{EI}{l^3 \Delta} \begin{bmatrix} \omega^3 \sin\omega & \omega^2 l(1-\cos\omega) & -\omega^3 \sin\omega & \omega^2 l(1-\cos\omega) \\ & \omega l^2 (\sin\omega - \omega\cos\omega) & -\omega^2 l(1-\cos\omega) & \omega l^2 (\omega - \sin\omega) \\ & & \omega^3 \sin\omega & -\omega^2 l(1-\cos\omega) \\ & symmetr. & & \omega l^2 (\sin\omega - \omega\cos\omega) \end{bmatrix} \qquad (3)$$

where: $\Delta = 2 \cdot (1-\cos\omega) - \omega \cdot \sin\omega$; $\omega = k\, l$.

Equation (3) is the solution for the member of the so-called type "k" (i.e. clamped at both ends), subjected to compressive force. Stiffness matrix for the member of the type "g" (i.e. hinged at one end), is given by:

$$\boldsymbol{K}_g = \frac{EI}{l^3(\sin\omega - \omega\cos\omega)} \begin{bmatrix} \omega^3 \cos\omega & \omega^2 l \sin\omega & -\omega^3 \cos\omega \\ & \omega^2 l^2 \sin\omega & -\omega^2 l \sin\omega \\ symm. & & \omega^3 \cos\omega \end{bmatrix} \qquad (4)$$

Using the standard displacement-based finite element approach, where each column and beam of the plane frame is treated as a single finite element, since the stiffness matrix is adopted according to (3) or (4), the finite element discretization of the governing equation of the stability analysis is obtained as

$$\mathbf{K}(\omega)\ \boldsymbol{\delta} = 0 \qquad (5)$$

K(ω) is the global stiffness matrix for the whole frame, including the corresponding boundary conditions. Of course, all loads acting upon the frame are expressed as the corresponding multiple of a single load parameter P, so also all parameters ω for each element with the axial force are also expressed through one representative ω. The critical load for the whole frame is then obtained as the solution of the corresponding stability equation:

$$\det \mathbf{K}(\omega) = 0 \tag{6}$$

According to this theoretical approach, numerical analysis was performed using the self-developed MATLAB computer program.

3. Concrete structures

3.1 Codes

Serbian and European codes are analyzed in this paper. In Serbian codes, stability analysis of concrete structures is presented in "Codes for concrete structures (BAB'87)". The effective length of column, according to BAB'87, can be described by

$$h_i = k \cdot l \tag{7}$$

where are: h_i – effective length; l – column length; k – effective length coefficient.

Effective length coefficient can be calculated using:

- nomograms,
- formulas (according to British standard),
- average slenderness ratio of the columns (by the first order theory).

In the first two methods, stability analysis is applied to "separated column". Ratio between total stiffness of all columns and stiffness off all beams (fixed in the observed joints at the ends of the column) has to be calculated. Coefficient value can be obtained using nomograms, or from formulas. These coefficients are given for sway or non-sway system, separately. In the third method, the whole structure is taken into account, but the analysis is performed according to the first order theory.

European codes for concrete structures (EC2) in Chapter 5.8.3.2 give expressions for the effective length value:

$$l_0 = 0.5 \cdot l \cdot \sqrt{\left(1 + \frac{k_1}{0.45 + k_1}\right) \cdot \left(1 + \frac{k_2}{0.45 + k_2}\right)} \text{, for braced members} \tag{8}$$

$$l_0 = l \cdot \max\left\{\sqrt{\left(1 + 10 \cdot \frac{k_1 \cdot k_2}{k_1 + k_2}\right)};\ \left(1 + \frac{k_1}{1 + k_1}\right) \cdot \left(1 + \frac{k_2}{1 + k_2}\right)\right\} \text{, for unbraced members,} \tag{9}$$

where: k_1 , k_2 are the flexibilities of the rotational restraints at ends 1 and 2 respectively relative to flexural stiffness of the member itself : $k = (\theta/M) \cdot (EI/l)$,
θ - the rotation of the restraint for bending moment M; EI - bending stiffness of the compressed member; and l is the clear height of the compressed member between end restraints.

It has to be mentioned that calculation of the rotation of the restraint for bending moment has to be done using the first order (i.e. the linear) theory.

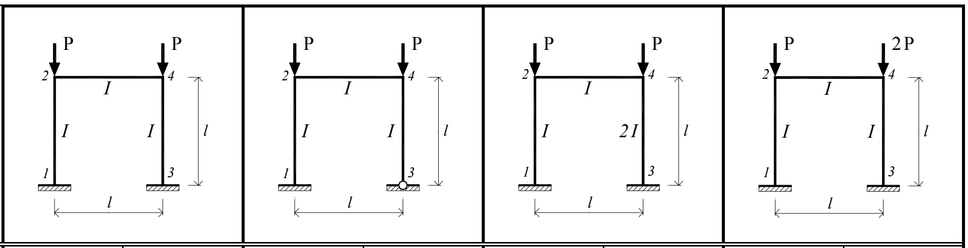
3.2 Numerical analysis

Accuracy of obtained results according to code methods is checked by the numerical analysis of frame structures. Stability analysis was performed according to the second order theory which gives, conditionally speaking, the exact solution.

In this analysis, it was investigated how the complexity of the structural system and load can influence the solution accuracy.

Sway and non-sway systems were analyzed separately.

In the case of sway systems one-floor, two-floor and multi-floor frames were considered.



	exact	BAB / EC	exact	BAB / EC	exact	BAB / EC	exact	BAB / EC
k_{12}	1.16	1.15 (1.19)	1.49	1.15 (1.54)	1.01	1.05 (1.15)	1.42	1.15 (1.20)
		1.14		1.15		1.13		1.13
k_{34}	1.16	1.15 (1.19)	1.49	1.54 (2.30)	1.43	1.27 (1.48)	1.00	1.15 (1.20)
		1.14		1.51		1.20		1.13
	max.error	2-3 %	max.error	3-30%	max.error	13-19%	max.error	18-25%

Figure 3. One-floor concrete frames

Comparative numerical analysis was done for the following examples:

- Symmetric frames,
- Frames with nonsymmetrical supports (where one column is clamped and other is pinned),
- Frames with nonsymmetrical rigidity of columns,
- Frames with different axial force in the columns.

Results obtained according to the codes are compared with the exact solution obtained by presented matrix analysis. Critical load for the whole frame was calculated firstly, and after that the effective length coefficient was determined according to the formula

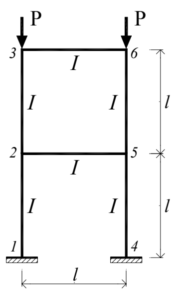
$$k = \sqrt{\frac{\pi^2 \cdot EI}{P_{cr} \cdot l^2}} \tag{10}$$

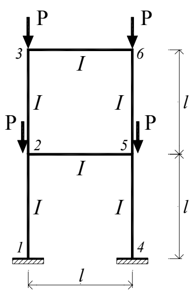
Summary of obtained results and errors, are shown in Figure 3.

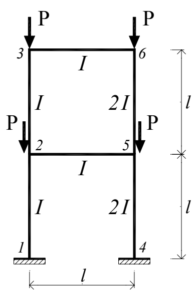
Only in the case of symmetric frames, calculation according to codes, gives almost exact solution.

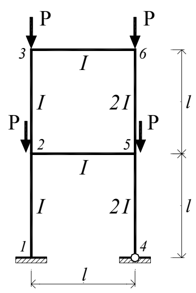
In the other numerical examples, where the unsymmetrical frames were calculated, it can be noticed that error reaches the level of about 30%.

Results and errors in the case of two and multi floor frames are presented in the Figures 4 and 5. Obviously, errors are greater then in the case of one-floor frames.









	exact	BAB / EC	exact	BAB / EC	exact	BAB / EC	exact	BAB / EC
k_{12}	1.38	1.25 (1.35)	1.19	1.25 (1.35)	1.06	1.20 (1.30)	1.47	1.25 (1.67)
		1.19		1.19		1.18		1.21
k_{23}	1.38	1.45 (1.54)	1.69	1.45 (1.54)	1.50	1.43 (1.45)	2.08	1.45 (1.58)
		1.41		1.41		1.38		1.42
k_{45}	1.38	1.25 (1.35)	1.19	1.25 (1.35)	1.50	1.42 (1.70)	1.47	1.67 (2.70)
		1.19		1.19		1.27		1.60
k_{56}	1.38	1.45 (1.54)	1.69	1.45 (1.54)	2.12	1.80 (2.02)	2.08	1.45 (1.58)
		1.41		1.41		1.60		1.41
	max.error	4-16%	max.error	13-20%	max.error	13-32%	max.error	31-84%

Figure 4. Two-floor concrete frames

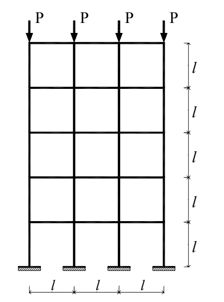
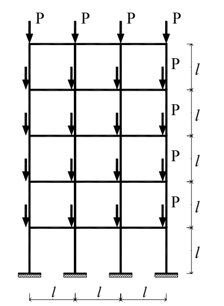
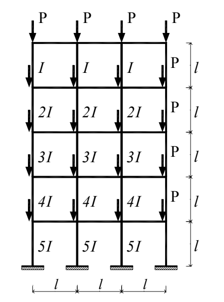
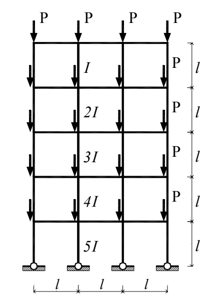
								
	exact	BAB / EC	exact	BAB / EC	exact	BAB / EC	exact	BAB / EC
I FLOOR	1.40	1.15 (1.28) -1.25 (1.30)	1.21	1.15 (1.28) -1.25 (1.30)	1.95	1.45 (1.84) -1.60 (2.00)	2.99	3.40 (3.66) -3.66 (4.70)
		1.13 – 1.17		1.13 – 1.17		1.22 – 1.25		1.40 – 1.44
II FLOOR	1.40	1.30 (1.55) -1.55 (1.60)	1.36	1.30 (1.54) -1.54 (1.60)	1.95	2.00 (2.33) -2.33 (3.10)	2.99	2.00 (2.78) -2.70 (3.10)
		1.31 – 1.43		1.31 – 1.43		1.55 – 1.64		1.61 – 1.71
III FLOOR	1.40	1.30 (1.55) -1.55 (1.60)	1.57	1.30 (1.55) -1.55 (1.60)	1.95	1.85 (2.21) -2.21 (2.50)	2.99	1.85 (2.31) -2.31 (2.50)
		1.32 – 1.43		1.32 – 1.43		1.51 – 1.61		1.52 – 1.61
IV FLOOR	1.40	1.30 (1.55) -1.55 (1.60)	1.92	1.30 (1.54) -1.54 (1.60)	1.95	1.60 (1.92) -1.92 (2.20)	2.99	1.60 (1.93) -1.93 (2.20)
		1.32 – 1.44		1.32 – 1.44		1.43 – 1.54		1.43 – 1.54
V FLOOR	1.40	1.22 (1.44) -1.44 (1.45)	2.71	1.22 (1.44) -1.44 (1.45)	1.95	1.30 (1.44) -1.44 (1.60)	2.99	1.30 (1.44) -1.44 (1.60)
		1.26 – 1.38		1.26 – 1.38		1.28 – 1.40		1.28 – 1.40
	max. error	10-24%	max. error	88-117%	max. error	35-59%	max. error	86-130%

Figure 5. Multi-floor concrete frames

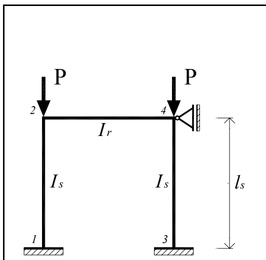
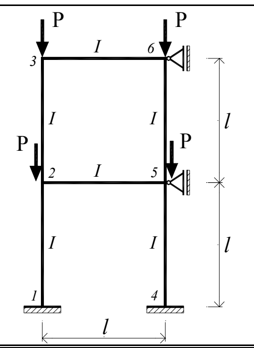
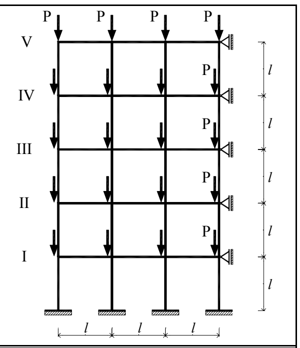
		
max.error 20-54%	max.error 32-81%	max.error 43-146%

Figure 6. Non-sway concrete systems

From these numerical examples it can be concluded that effective buckling length of columns does not have any physical meaning, as it was in the case of Euler's stability solution. According to codes, each single member has different critical load. But, it should be noticed that when the whole structure starts buckling, critical load is of the same value for all members. That is the reason, why effective length does not describe, anymore, deflection shape of the bended member (as it was in Euler's cases). Now, effective length is just a length which can be calculated by the expression

$$P^{g}_{cr,memb} = \pi^2 \left(\frac{EI}{(kl)^2} \right)_{memb} \tag{11}$$

4. Steel structures

4.1 Codes

This paper deals with European and Serbian codes, same as it was in a case of concrete structures. Stability analysis in Serbian codes is presented in JUS Standard, where effective length of column can be calculated according to equation:

$$l_{s,i} = \beta \cdot h_s \tag{12}$$

where: $l_{s,i}$ is the effective length of the column, h_s is the column length and β is the coefficient of effective length.
Coefficient of effective length can be calculated for the sway and non-sway systems separately, using given formulas (13) and (14) or diagrams.

$$\beta = \frac{1.6 + 1.9 \cdot (\eta_A + \eta_B) + 2.1 \cdot \eta_A \cdot \eta_B}{3.2 + 1.8 \cdot (\eta_A + \eta_B) + 0.7 \cdot \eta_A \cdot \eta_B} \quad \text{, for non-sway systems} \tag{13}$$

$$\beta_0 = \sqrt{\frac{1.5 - 0.70 \cdot (\eta_A + \eta_B) + 0.22 \cdot \eta_A \cdot \eta_B}{1.5 - 1.30 \cdot (\eta_A + \eta_B) + 1.10 \cdot \eta_A \cdot \eta_B}} \quad \text{, for sway systems} \tag{14}$$

This code considers only separated member and ratio between total stiffness of all columns and stiffness off all beams, fixed in the observed joint.

In the last published version of EUROCODE 3 (from 2005.), two ways of calculations are suggested. In the case of "typical" structures, which are not too deformable, the classical method of analysis has to be used. This method implicates structural calculation according to the first order theory, and stability check of structural elements taking into account their effective buckling length. In the case of complex deformable structures, calculation according to the second order theory is recommended, and global and local imperfections have to be taken into account. But, in this case final solution for the effective length calculation is not presented.
In this paper analysis was performed using expressions and diagrams for the first case of "typical" structures, given in EC3 - annex E (version ENV 1993-1-1:1992).

$$l/L = \left[\frac{1 + 0.145 \cdot (\eta_1 + \eta_2) - 0.265 \cdot \eta_1 \cdot \eta_2}{2 - 0.364 \cdot (\eta_1 + \eta_2) - 0.247 \cdot \eta_1 \cdot \eta_2} \right] \text{, for non-sway frames} \qquad (15)$$

$$l/L = \left[\frac{1 - 0.2 \cdot (\eta_1 + \eta_2) - 0.12 \cdot \eta_1 \cdot \eta_2}{1 - 0.8 \cdot (\eta_1 + \eta_2) - 0.6 \cdot \eta_1 \cdot \eta_2} \right]^{0,5} \text{, for sway frames} \qquad (16)$$

where η_1 and η_2 are coefficients of member rigidity distributions.

4.2 Numerical analysis

The same numerical examples, as it was in the case of concrete structures, were used. Comparative numerical analysis between exact and the code results is presented in Figures 7, 8, 9 and 10.

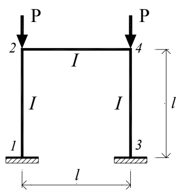
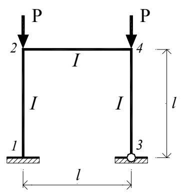
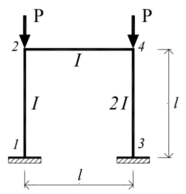
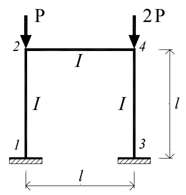
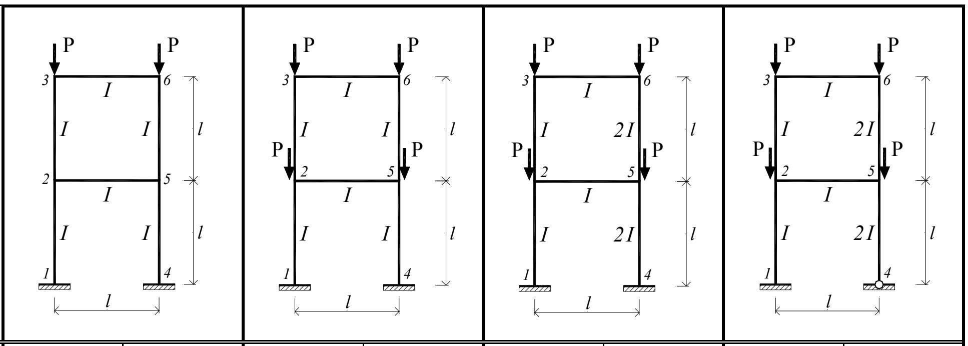
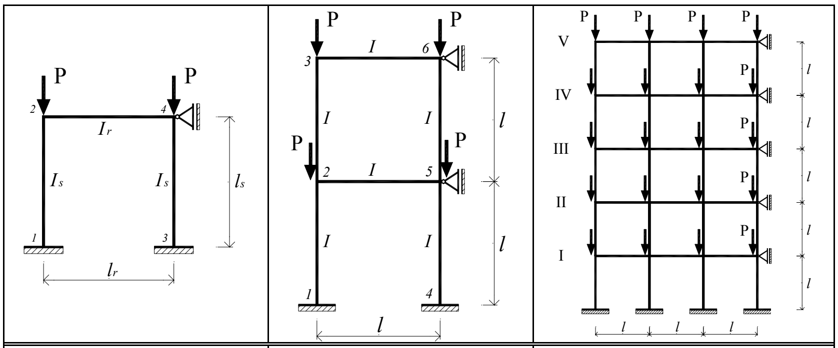
								
	exact	JUS	exact	JUS	exact	JUS	exact	JUS
		EC		EC		EC		EC
k_{12}	1.16	1.22	1.49	1.22	1.01	1.00	1.42	1.50
		1.16		1.16		1.16		1.16
k_{34}	1.16	1.22	1.49	2.53	1.43	1.42	1.00	1.06
		1.16		2.37		1.16		1.16
	max.error	1-6 %	max.error	28-70%	max.error	14-23%	max.error	16-21%

Figure 7. One-floor steel frames



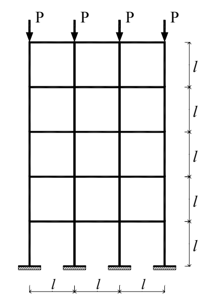
	exact	JUS / EC	exact	JUS / EC	exact	JUS / EC	exact	JUS / EC
k_{12}	1.38	1.36	1.19	1.36	1.06	1.11	1.47	1.36
		1.28		1.28		1.28		1.28
k_{23}	1.38	1.65	1.69	1.65	1.50	1.35	2.08	1.65
		1.47		1.47		1.47		1.47
k_{45}	1.38	1.36	1.19	1.36	1.50	1.76	1.47	2.97
		1.28		1.28		1.43		2.68
k_{56}	1.38	1.65	1.69	1.65	2.12	2.42	2.08	1.65
		1.47		1.47		1.81		1.47
	max.error	8-20%	max.error	14-15%	max.error	17-20%	max.error	14-102%

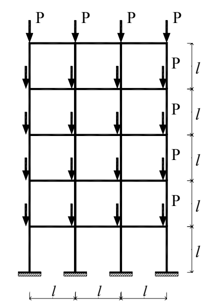
Figure 8. Two-floor steel frames

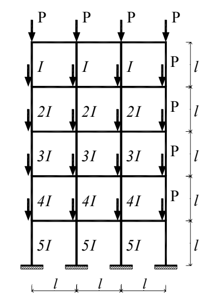


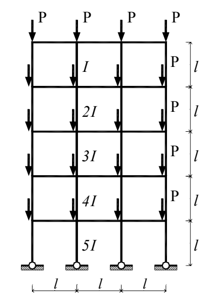
max.error 12-51%	max.error 10-41%	max.error 27-104%

Figure 9. Non-sway steel systems









	exact	JUS / EC	exact	JUS / EC	exact	JUS / EC	exact	JUS / EC
I FLOOR	1.40	1.22 - 1.36	1.21	1.22 - 1.36	1.95	1.56 - 1.71	2.99	3.85 - 5.06
		1.16 – 1.28		1.16 – 1.28		1.46 – 1.62		3.35 – 4.29
II FLOOR	1.40	1.48 - 1.84	1.36	1.48 - 1.84	1.95	2.40 - 3.24	2.99	2.40 - 3.24
		1.34 – 1.61		1.34 – 1.61		2.04 – 2.70		2.04 – 2.70
III FLOOR	1.40	1.48 - 1.84	1.57	1.48 - 1.84	1.95	2.13 - 2.84	2.99	2.13 - 2.84
		1.34 – 1.61		1.34 – 1.61		1.84 – 2.39		1.84 – 2.39
IV FLOOR	1.40	1.48 - 1.84	1.92	1.48 - 1.84	1.95	1.82 - 2.37	2.99	1.82 - 2.37
		1.34 – 1.61		1.34 – 1.61		1.60 – 2.02		1.60 – 2.02
V FLOOR	1.40	1.37 - 1.65	2.71	1.37 - 1.65	1.95	1.45 - 1.77	2.99	1.45 - 1.77
		1.26 – 1.47		1.26 – 1.47		1.33 – 1.57		1.33 – 1.57
	max. error	20-31%	max. error	64-115%	max. error	34-47%	max. error	86-125%

Figure 10. Multi-floor steel frames

5. Conclusions of the analysis

Considering results shown in the numerical analysis, it may be concluded:

- Serbian and European codes for concrete and steel structures give errors which are of the same order,
- Calculations of sway and non-sway structures give errors which are similar,
- Geometric interpretation of effective length is valid only for symmetric one-floor frames,
- Calculation according to presented codes gives the best results in the case of one-floor frames,
- In the case when frames are not symmetrically loaded, or columns are not of the same rigidity, errors are higher,
- Maximum errors are obtained in the case of multi-floor frames, and exceeded 100%.

The final conclusion is that innovation of codes, both for concrete and steel structures, is recommended in the part where the effective length (h_i) of frame columns is considered. Improvement of this calculation could be done using the global stability analysis and applying calculation of the critical load for the complete structure (P_{cr}^g). Possibility of such calculation is only suggested in EC3, but it is not explained.

When the critical load is calculated for the whole structure, the critical load for each column member can be obtained, as it is presented in this paper. In that case, the effective buckling length of the member could be found by the formula:

$$P_{cr,mem}^{g} = \pi^2 \cdot \left(\frac{EI}{h_i^2} \right)_{,mem} \quad \Rightarrow \quad h_i \tag{17}$$

where $P_{cr}{}^{g}{}_{,mem}$ is the critical load in the column member when the complete structure is buckled (P_{cr}^g).

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