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Output consensus control for linear port-Hamiltonian systems

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Abstract: In this paper, we study output consensus of coupled linear port-Hamiltonian systems on graphs in the presence of constant disturbances, where couplings are allowed to be both static and dynamic. Utilizing port-Hamiltonian structures, we present dynamic controllers achieving output consensus where the consensus values are determined by the disturbances. Finally, the utility of the proposed controller is illustrated by applying it to current sharing of DC microgrids.

Keywords: port-Hamiltonian systems, networked systems, output consensus

1. INTRODUCTION

Port-Hamiltonian structures naturally arise in networked interconnections of physical systems such as electrical circuits and networked mechanical systems (van der Schaft, 2006). Taking advantages of passivity properties, various control methodologies have been developed for stabilization or trajectory tracking of port-Hamiltonian systems, e.g. Vos et al. (2014, 2015). Motivated by power systems and multi-agent systems, consensus control as a new trend has also attracted research attentions, e.g., (Monshizadeh and De Persis, 2017; Olfati-Saber et al., 2007; Li et al., 2009; Ran and Xie, 2021). However, there is no general control framework for consensus of port-Hamiltonian systems.

The consensus problem considered in this paper is motivated by current sharing of power systems in which the consensus value of generated currents as outputs are determined by power loads as disturbances. That is, we consider an output consensus problem driven by "disturbances". Such a consensus property is sometimes referred to as output agreement (Monshizadeh and De Persis, 2017) to distinguish it from ones typically appeared in the context of multi-agent systems (Olfati-Saber et al., 2007) in which consensus values are determined by "the initial states". In the paper Monshizadeh and De Persis (2017), the authors consider the problem of output agreement for port-Hamiltonian systems, in which dynamic interactions (or dynamic edges) among subsystems are considered. As will be shown later, in our paper, we are interested in a consensus problem in which the model of the coupled port-Hamiltonian systems and the edges can be more general.

In this paper, we first introduce the model of interconnected port-Hamiltonian systems, in which disturbance is considered. The model allows the interaction to be static and/or dynamic. Examples of static interaction and dynamic interactions are provided later in Subsection 2. Actually, for a single subsystem, the static and dynamic interactions can be considered as the interaction ports, defining the interaction of the system with (the rest of) its environment (van der Schaft, 2006). For static interaction, two typical types are considered and they correspond to power conserving and power dissipating types of interconnections. The dynamic interactions acts as a power conserving and also dissipating parts. At last, the "large-scale" system composed by the interconnected port-Hamiltonian systems is still a port-Hamiltonian system. Compared with the paper (Monshizadeh and De Persis, 2017) where only the dynamic interactions among subsystems are considered, the model in our paper can incorporate the cases of static and dynamic couplings, and the dynamics of interactions can be expressed as a function of subsystems' state, i.e., the dynamics of interactions do not necessarily have to be a function of the systems' outputs. For more details, we refer the readers to Remark 1.

The main control objects of this paper are output weighted consensus of the interconnected/coupled port-Hamiltonian systems and stability. By control system design, the steady state of the output of each subsystem can be proportional to a common value, in which the proportion is a constant design parameter. The computation of the common value depending on disturbance and controller reference signal is also provided. If there is no interaction among the

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port-Hamiltonian systems and let the proportion value identical, then the consensus problem reduces to the nominal output consensus problems of multi-agent systems in the form of port-Hamiltonian systems. In the absence of disturbance, the port-Hamiltonian systems can track the average of the reference signals in the controllers with a uniform gain.

The contributions of this paper are as follows.

- Our method is general in the sense that we can handle weighted output consensus of general linear port-Hamiltonian systems regardless of static or dynamics interconnections. To the best of our knowledge, this is the first attempt to solve output consensus problems for such general interconnected port-Hamiltonian systems.
- Technical analysis is general in the sense of that we compute the consensus value and corresponding equilibrium point explicitly for the general port-Hamiltonian systems mentioned above.

The remainder of this paper is organized as follows. In Section 2, we provide a motivating example to explain the whole picture of this paper such as the model of interconnected port-Hamiltonian systems and control objectives. In Section 3, we show the coupled port-Hamiltonian systems on graphs in the presence of disturbances. Section 4 presents distributed controllers for achieving output consensus of general port-Hamiltonian systems, where the consensus value is determined by the disturbances. In Section 5, the proposed method is illustrated by a current sharing problem of a DC microgrid.

Notation: We denote by $\mathbb{R}$ the set of real numbers. Given $y \in \mathbb{R}$, $\mathbb{R}_{\geq y}$ denotes the set of real numbers that are not smaller than y. Let $\mathbf{0}$ denote the column vector or the matrix with the appropriate dimensions having all 0 elements. Let $\mathbf{1}$ denote the column vector with the appropriate dimension having all 1 elements. For a symmetric matrix Ψ, $\Psi > 0$ and $\Psi \geq 0$ mean that Ψ is positive definite and positive semi-definite, respectively. Let Ψ^+ denote the Moore-Penrose inverse of Ψ. Let $\mathbf{I}$ denote the identity matrix with the appropriate dimension.

Preliminaries of graph theory: We let the graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ denote the topology between subsystems, where $\mathcal{N} = \{1, 2, \cdots, N\}$ denotes the set of subsystems and $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ denotes the set of edges (interconnecting the N subsystems) with cardinality E. Let $\mathcal{N}_i$ denote the set of the neighbors of subsystem i, where $i = 1, 2, \cdots, N$. In this paper, we assume that the graph $\mathcal{G}$ is undirected and connected, i.e., if $j \in \mathcal{N}_i$, then $i \in \mathcal{N}_j$. Let $A_{\mathcal{G}} = [a_{ij}] \in \mathbb{R}^{N\times N}$ denote the adjacency matrix of the graph $\mathcal{G}$, where $a_{ij} = 1$ if and only if $j \in \mathcal{N}_i$ and $a_{ii} = 0$. Accordingly, define the Laplacian matrix $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N\times N}$, in which $l_{ii} = \sum_{j=1}^{N} a_{ij}$ and $l_{ij} = -a_{ij}$ if $i \neq j$.

2. MOTIVATING EXAMPLE: A DC MICRO GRID

In this paper, we are interested in coupled port-Hamiltonian systems on graphs. Such system structures can be widely found in practice as exemplified by DC micro grids in the left picture of Fig. 1, e.g., (Ferguson et al., 2021; Trip et al., 2018), where the voltage sourced converters (VSCs) and power lines are modeled as vertices and edges, respectively as in the right picture of Fig. 1. For the ease of exposition, we sometimes use the terminologies "interconnection", "coupuling", and "line" instead of an edge; they all have the same meaning.

Table 1. Tables of coefficients and variables in Section 2

Table of coefficients		Table of variables	
L_{si}	Filter inductance	ϕ_i	Flux (node)
R_{si}	Filter resistance	q_i	Charge
L_k	Line inductance	ψ_k	Flux (line)
R_k	Line resistance	u_{si}	Control input
C_{si}	Filter capacitor	I_{Li}	Current load
Z_i	Load resistance	y_i	Output

We see that the DC micro grids can be represented as port-Hamiltonian systems on graphs. As node dynamics, for instance, we consider the model of a distributed generation (storage) unit including a VSC and a current load (Ferguson et al., 2021). With the notation in Table 1, the dynamics of the i-th VSC can be expressed as

$$\dot{\phi}_i = -\frac{R_{si}}{L_{si}}\phi_i - \frac{1}{C_{si}}q_i + u_{si} \tag{1a}$$

$$\dot{q}_i = -\frac{1}{Z_i C_{si}}q_i + \frac{1}{L_{si}}\phi_i - I_{Li} - \sum_{k\in\mathcal{E}_i}\frac{1}{L_k}\psi_k \tag{1b}$$

$$y_i = \frac{\phi_i}{L_{si}}. \tag{1c}$$

Next, we consider models of edges. In fact, depending on problems, edges can be static and dynamic.

Case i) Static edges: By neglecting the inductance of lines, the flux in the resistive line k connecting VSCs i and j is expressed by

$$\frac{\psi_k}{L_k} = \frac{1}{R_k}\left(\frac{q_i}{C_{si}} - \frac{q_j}{C_{sj}}\right). \tag{2}$$

Substituting (2) into (1b), one can rewrite (1) into the form of port-Hamiltonian system as follows

$$\underbrace{\begin{bmatrix}\dot{\phi}_i\\ \dot{q}_i\end{bmatrix}}_{\dot{x}_i} = \underbrace{\begin{bmatrix}-R_{si} & -1\\ 1 & -Z_i^{-1}\end{bmatrix}}_{J_i-R_i}\underbrace{\begin{bmatrix}L_{si}^{-1} & 0\\ 0 & C_{si}^{-1}\end{bmatrix}}_{Q_i}\underbrace{\begin{bmatrix}\phi_i\\ q_i\end{bmatrix}}_{x_i} + \underbrace{\begin{bmatrix}1\\ 0\end{bmatrix}}_{B_i}u_{si}$$
$$+\underbrace{\begin{bmatrix}0\\ -1\end{bmatrix}}_{M_i}I_{Li} - \sum_{k\in\mathcal{E}_i}\underbrace{\begin{bmatrix}0 & 0\\ 0 & R_k^{-1}\end{bmatrix}}_{S_{ij}}(Q_i x_i - Q_j x_j) \tag{3a}$$

$$y_i = \underbrace{\begin{bmatrix}1 & 0\end{bmatrix}}_{B_i^T}\underbrace{\begin{bmatrix}L_{si}^{-1} & 0\\ 0 & C_{si}^{-1}\end{bmatrix}}_{Q_i}\underbrace{\begin{bmatrix}\phi_i\\ q_i\end{bmatrix}}_{x_i} \tag{3b}$$

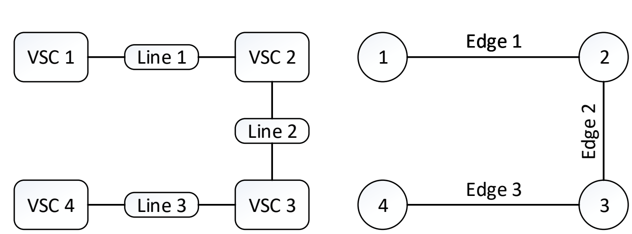

Fig. 1. Left: A DC micro grid; Right: The corresponding graph.

where $R_i = \mathrm{diag}(R_{si}, Z_i^{-1})$ and $J_i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Case ii) Dynamic edges: When lines are resistive and inductive as in (Trip et al., 2018), the dynamics of the flux in the resistive-inductive line k connecting VSCs i and j is expressed as

$$\dot{\psi}_k = \left(\frac{q_i}{C_{si}} - \frac{q_j}{C_{sj}}\right) - \frac{R_k}{L_k}\psi_k. \tag{4}$$

Similarly, the dynamics in (1) can be written in the form of port-Hamiltonian system as follows

$$\underbrace{\begin{bmatrix} \dot{\phi}_i \\ \dot{q}_i \end{bmatrix}}_{\dot{x}_i} = \underbrace{\begin{bmatrix} -R_{si} & -1 \\ 1 & -Z_i^{-1} \end{bmatrix}}_{J_i - R_i} \underbrace{\begin{bmatrix} L_{si}^{-1} & 0 \\ 0 & C_{si}^{-1} \end{bmatrix}}_{Q_i} \underbrace{\begin{bmatrix} \phi_i \\ q_i \end{bmatrix}}_{x_i} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B_i} u_{si} + \underbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}_{M_i} I_{Li} - \sum_{k \in \mathcal{E}_i} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{H_k} \underbrace{\frac{1}{L_k}}_{T_k^{-1}} \psi_k \tag{5a}$$

$$y_i = \underbrace{[1\ 0]}_{B_i^T} \underbrace{\begin{bmatrix} L_i^{-1} & 0 \\ 0 & C_{si}^{-1} \end{bmatrix}}_{Q_i} \underbrace{\begin{bmatrix} \phi_i \\ q_i \end{bmatrix}}_{x_i}, \tag{5b}$$

and the dynamics of ψ_k can be described by

$$\dot{\psi}_k = -\underbrace{R_k}_{D_k} \underbrace{\frac{1}{L_k}}_{T_k^{-1}} \psi_k + \underbrace{[0\ 1]}_{H_k^T} (Q_i x_i - Q_j x_j), \tag{6}$$

where J_i and R_i are the same as those in Case i).

In both Cases i) and ii), the dynamics of VSC i are written in the form of a port-Hamiltonian system (3) and (5), respectively. Load I_{Li} can be considered as disturbance entering the port-Hamiltonian system. In particular, the terms $\sum_{k\in\mathcal{E}_i} S_{ij}(Q_i x_i - Q_j x_j)$ in (3) and $\sum_{k\in\mathcal{E}} H_k\psi_k$ in (5) represent the overall flux exchange of VSC i with its neighbors through the lines. However, one can observe that the dynamics of the port-Hamiltonian systems in the two cases, i.e., (3) and (5), are different due to the cases of dynamic and static models of lines.

In both cases, output consensus of the systems are important in practices. Note that $y_i = \phi_i/L_{si}$ equals to the generated current of VSC i. An approach to realizing current sharing among the VSCs ($y_i = y_j$) is to achieve output consensus for the systems in (3) or (5)–(6). However, there is still no uniform framework of control design for output consensus, which can handle both static and dynamic couplings. Consequently, we are interested in the following theoretical-oriented question: Given a "general" model of distributed port-Hamiltonian systems (including (3) and (5)), whether we can design distributed controllers to achieve output consensus.

3. PORT-HAMILTONIAN SYSTEMS ON GRAPHS

3.1 Node Dynamics and Interconnection Structures

In this subsection, we consider general linear port-Hamiltonian systems on graphs. Each node dynamics is given by

$$\dot{x}_i = (J_i - R_i)Q_i x_i + B_i u_i + d_i + C_i(z_k, x_i, x_j) \tag{7a}$$

$$y_i = B_i^T Q_i x_i, \tag{7b}$$

where $x_i \in \mathbb{R}^{n_i}$ and $u_i \in \mathbb{R}^m$ denote the state and input of node i, respectively, and $d_i \in \mathbb{R}^{n_i}$ represents the constant disturbance injected to node i. The matrices $J_i, R_i, Q_i \in \mathbb{R}^{n_i\times n_i}$ are $J_i = -J_i^T$, $R_i \geq 0$, and $Q_i > 0$. In this paper, as a stabilizability assumption, we assume that $B_i \in \mathbb{R}^{n_i\times m}$ is of full column rank, and $(R_i|B_i)$ is of full row rank.

The term $C_i(z_k, x_i, x_j)$ accounts for the (physical) couplings of subsystem i with other subsystems $j \in \mathcal{N}$ ($j \neq i$), given by

$$C_i(z_k, x_i, x_j) = -\underbrace{\sum_{k\in\mathcal{E}} H_k T_k^{-1} z_k}_{\text{dynamic}} + \underbrace{\sum_{j\in\mathcal{N}} G_{ij} Q_j x_j}_{\text{static}} + \underbrace{\sum_{j\in\mathcal{N}} S_{ij}(Q_j x_j - Q_i x_i)}_{\text{static}} \tag{8}$$

$$\dot{z}_k = -D_k T_k^{-1} z_k + H_k^T(Q_i x_i - Q_j x_j), \tag{9}$$

where $z_k \in \mathbb{R}^k$ represents the state of the dynamic interaction between subsystems i and $j \neq i$, and D_k and $T_k \in \mathbb{R}^{k\times k}$ are positive definite matrices with D_k specifying the energy dissipation associated with the (physical) interaction, and $H_k \in \mathbb{R}^{n_i\times k}$. For the sake latter analysis, the incidence matrix of the interconnection between subsystems i and j in (9) is given by $\mathcal{B}_{i,k} = +1$ if subsystem i is the positive end of the labeled interaction k. Or $\mathcal{B}_{i,k} = -1$, if subsystem i is the negative end. Otherwise, $\mathcal{B}_{i,k} = 0$. In the static interaction in (8), $G_{ij} = -G_{ij}^T$ and $S_{ij} \geq 0$ if $j \in \mathcal{N}_i$. Otherwise, $G_{ij} = \mathbf{0}$ and $S_{ij} = \mathbf{0}$. The term $\sum_{j\in\mathcal{N}} G_{ij}Q_j x_j$ accounts for the static interactions without energy losses, and the term $\sum_{j\in\mathcal{N}} S_{ij}(Q_j x_j - Q_i x_i)$ can model the interactions with energy losses, which will be observed later when we represent the overall system in compact form. Also, it will become clearer that the whole system consisting of the subsystems (7) and interconnections (8) and (9) is again a port-Hamiltonian system.

Remark 1. Compared with the model in (Monshizadeh and De Persis, 2017) in which only the dynamic interactions among subsystems are considered, the model (7)-(9) in this paper can incorporate the cases of static and dynamic couplings. Moreover, dynamics of interactions (or called edges) in (Monshizadeh and De Persis, 2017) are $\dot{z}_k = y_j - y_i$. Therefore, output consensus is automatically achieved when the system is stabilized. This is not always true in this paper, since dynamics of interactions z_k in this paper also depend on D_k and H_k. In case H_k, G_{ij} and S_{ij} are matrices with all zero entries, the model in (7) reduces to multi-agent systems as the one in (Vos et al., 2015, 2014). ■

3.2 Compact Forms

Combining (7)–(9) leads to the compact form for the whole networked systems as in

$$\dot{x} = (J + G - R - S)Qx + Bu + d + (\mathcal{B}\otimes I_n)HT^{-1}z \tag{10a}$$

$$y = B^T Qx \tag{10b}$$

$$\dot{z} = -DT^{-1}z - H^T(\mathcal{B}^T\otimes I_n)Qx, \tag{10c}$$

where $x := [x_1^T \cdots x_N^T]^T$, $u := [u_1 \cdots u_N]^T$, $z = [z_1^T \cdots z_E^T]^T$, and $d = [d_1^T \cdots d_N^T]^T$. The block diagonal matrices are $Q = \text{diag}(Q_1, \cdots, Q_N)$, $J = \text{diag}(J_1, \cdots, J_N)$, $R = \text{diag}(R_1, \cdots, R_N)$, $B = \text{diag}(B_1, \cdots, B_N)$, $D = \text{diag}(D_1, \cdots, D_E)$, $H = \text{diag}(H_1, \cdots, H_E)$, and $T = \text{diag}(T_1, \cdots, T_E)$. The matrices G and S are given by

$$G = \begin{bmatrix} \mathbf{0} & G_{12} & G_{13} & \cdots & G_{1N} \\ G_{21} & \mathbf{0} & G_{23} & \cdots & G_{2N} \\ G_{31} & G_{32} & \mathbf{0} & \cdots & G_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G_{N1} & G_{N2} & G_{N3} & \cdots & \mathbf{0} \end{bmatrix}$$

$$S = \begin{bmatrix} \sum_{p=1}^{N} S_{1p} & -S_{12} & \cdots & -S_{1N} \\ -S_{21} & \sum_{p=1}^{N} S_{2p} & \cdots & -S_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -S_{N1} & -S_{N2} & \cdots & \sum_{p=1}^{N} S_{Np} \end{bmatrix}.$$

The description of the interconnected systems can be further simplified by defining $\tilde{x} = [x^T \ z^T]^T$. Indeed, it follows that

$$\dot{\tilde{x}} = (\tilde{J} - \tilde{R})\tilde{Q}\tilde{x} + \tilde{B}u + \tilde{d} \tag{11a}$$

$$y = \tilde{B}^T\tilde{Q}\tilde{x}, \tag{11b}$$

where $\tilde{Q} = \text{diag}(Q, T^{-1})$ and

$$\tilde{J} = \begin{bmatrix} J + G & (\mathcal{B} \otimes I_n)H \\ -H^T(\mathcal{B}^T \otimes I_n) & \mathbf{0} \end{bmatrix}, \tag{12}$$

$$\tilde{R} = \begin{bmatrix} R + S & \mathbf{0} \\ \mathbf{0} & D \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} B \\ \mathbf{0} \end{bmatrix}, \tilde{d} = \begin{bmatrix} d \\ \mathbf{0} \end{bmatrix}. \tag{13}$$

It is worth emphasizing that this compact form is a port-Hamiltonian system because of the structure of node dynamics (7) and interconnections (8) and (9). Namely, $\tilde{J} = -\tilde{J}^T$, $\tilde{R} \geq 0$, and $\tilde{Q} > 0$. Moreover, $\tilde{B}$ is of full column rank, and $(\tilde{R}|\tilde{B})$ is of full row rank. The aforementioned DC microgrid can be represented in the form of (11).

4. CONTROL DESIGN FOR WEIGHTED OUTPUT CONSENSUS

4.1 Control Objective

In this paper, the control objective is to achieve the following two goals simultaneously.

i) The state of the port-Hamiltonian systems (7) asymptotically converges to some value $\bar{x}$ determined by the constant disturbance d;
ii) The port-Hamiltonian systems achieve weighted output consensus, i.e.,

$$\lim_{t\to\infty} (w_j y_j(t) - w_i y_i(t)) = 0, \ i, j = 1, \ldots, N, \tag{14}$$

where $w_i \in \mathbb{R}^{m\times m}$, $i = 1, \ldots, N$ are non-singular.

In the above, the parameters w_i, $i = 1, \ldots, m$ are for making the consensus problem more general. Selecting $w_i = I$, $i = 1, \ldots, m$ recovers the standard (proportional) output consensus $\lim_{t\to\infty}(y_j(t) - y_i(t)) = 0$.

4.2 Distributed controller design

To achieve the control objectives i) and ii), we design the distributed controller as follows

$$u_i = -K_i y_i + w_i \sum_{j\in\mathcal{N}_i} (\xi_j - \xi_i) + r_i \tag{15a}$$

$$\dot{\xi}_i = \sum_{j\in\mathcal{N}_i} (w_j y_j - w_i y_i), \tag{15b}$$

where $\xi_i \in \mathbb{R}^m$, and $K_i > 0$. As will be shown later, the constant $r_i \in \mathbb{R}^m$ is used to shift the steady state. For example, when one wants to achieve current sharing, it is also desirable to regulate the voltage towards the reference value.

The compact form of the controller dynamics is given by

$$u = -Ky + W\mathcal{L}\xi + r \tag{16a}$$

$$\dot{\xi} = -\mathcal{L}Wy \tag{16b}$$

where $u = [u_1 \cdots u_N]^T$, $\xi = [\xi_1 \cdots \xi_N]^T$, and $r = [r_1 \cdots r_N]^T$. For the matrices, $K = \text{diag}(K_1, \cdots, K_N)$, $W = \text{diag}(w_1, \cdots, w_N)$, and $\mathcal{L} = \mathcal{B}\mathcal{B}^T$ is the Laplacian matrix for the topology of distributed systems.

4.3 Analysis of the closed-loop systems

Substituting (16) into (11), one obtains the compact form of the closed-loop system as follows

$$\dot{\tilde{x}} = (\tilde{J} - \tilde{R} - \tilde{B}K\tilde{B}^T)\tilde{Q}\tilde{x} + \tilde{B}W\mathcal{L}\xi + \tilde{d} + \tilde{B}r \tag{17a}$$

$$\dot{\xi} = -\mathcal{L}W\tilde{B}^T\tilde{Q}\tilde{x}. \tag{17b}$$

One of the most important fact is that this is a port-Hamiltonian system. In other words, the controller (15) is designed to preserve the port-Hamiltonian structure.

In the rest of this section, we show that this closed-loop system satisfies the objectives i) and ii). First, we show that its equilibrium point can be computed explicitly.

Proposition 1. Given constants d and r, the following $(\bar{x}, \bar{\xi})$ is an equilibrium point of the closed-loop system (17):

$$\bar{x} = \tilde{Q}^{-1}A^{-1}(\tilde{B}Wv - c) \tag{18}$$

$$\bar{\xi} = \begin{bmatrix} \mathbf{1}^T \\ \mathcal{L} \end{bmatrix}^+ \begin{bmatrix} \mathbf{1}^T\xi(0) \\ -v \end{bmatrix}, \tag{19}$$

$$v := (W^T\tilde{B}^TA^{-1}\tilde{B}W)^{-1}(\alpha\mathbf{1} + W^T\tilde{B}^TA^{-1}c), \tag{20}$$

$$\alpha := -\frac{\mathbf{1}^T(W^T\tilde{B}^TA^{-1}\tilde{B}W)^{-1}\tilde{B}^TA^{-1}c}{\mathbf{1}^T(W^T\tilde{B}^TA^{-1}\tilde{B}W)^{-1}\mathbf{1}} \tag{21}$$

$$A := \tilde{J} - \tilde{R} - \tilde{B}K\tilde{B}^T \tag{22}$$

$$c := \tilde{d} + \tilde{B}r, \tag{23}$$

where $\xi(0)$ is the initial state of $\xi(t)$. Moreover, the output $\bar{y} := \tilde{B}^T\tilde{Q}\bar{x}$ satisfies

$$W\bar{y} = \alpha\mathbf{1}. \tag{24}$$

Proof. First, we confirm that $(\bar{x}, \bar{\xi})$ is well defined by showing the non-singularity of A and $W^T\tilde{B}^TA^{-1}\tilde{B}W$. Since $(\tilde{R}|\tilde{B})$ is of full row rank, $R \geq 0$, and $K > 0$, we have $\tilde{R} + \tilde{B}K\tilde{B}^T > 0$. Therefore, A is non-singular.

Recall that W is non-singular. Then, it suffices to show the non-singularity of $\tilde{B}^TA^{-1}\tilde{B}$. Consider a vector p such that $\tilde{B}^TA^{-1}\tilde{B}p = 0$. Using this p, define $q := A^{-1}\tilde{B}p$. Then, it follows that

$$\tilde{B}^Tq = \tilde{B}^TA^{-1}\tilde{B}p = 0, \tag{25}$$

i.e., $q^T\tilde{B}=0$. Also, it holds that $Aq=\tilde{B}p$, and thus

$$q^TAq=q^T\tilde{B}p=0. \tag{26}$$

From the definition of A in (22), we have $q=0$. Since $\tilde{B}$ is of full column rank, $p=0$. Therefore, $\tilde{B}^TA^{-1}\tilde{B}$ is non-singular.

Next, we verify that $(\bar{x},\bar{\xi})$ is an equilibrium point. Substituting $(\bar{x},\bar{\xi})$ into the right-hand side of (17a) yields

$$\begin{aligned}&(\tilde{J}-\tilde{R}-\tilde{B}K\tilde{B}^T)\tilde{Q}\bar{x}+\tilde{B}W\mathcal{L}\bar{\xi}+\tilde{d}+\tilde{B}r\\&=\tilde{B}Wv-c+\tilde{B}W(-v)+\tilde{d}+\tilde{B}r=0.\end{aligned}$$

Substituting $\bar{x}$ into the right-hand side of (17b) yields

$$\begin{aligned}-\mathcal{L}W\tilde{B}^T\tilde{Q}\bar{x}&=-\mathcal{L}W\tilde{B}^TA^{-1}\tilde{B}Wv+\mathcal{L}W\tilde{B}^TA^{-1}c\\&=-\mathcal{L}(\alpha\mathbf{1}+W^T\tilde{B}^TA^{-1}c)+\mathcal{L}W\tilde{B}^TA^{-1}c\\&=0\end{aligned}$$

Finally, we compute $W\bar{y}$ at $\bar{x}$ as follows:

$$\begin{aligned}W\bar{y}&=W\tilde{B}^T\tilde{Q}\bar{x}\\&=W\tilde{B}^TA^{-1}\tilde{B}Wv-W\tilde{B}^TA^{-1}c\\&=\alpha\mathbf{1}+W^T\tilde{B}^TA^{-1}c-W\tilde{B}^TA^{-1}c=\alpha\mathbf{1}.\end{aligned}$$

That completes the proof. ∎

The consensus value α depends on $c=\tilde{d}+\tilde{B}r$, where $\tilde{d}$ contains an (uncontrollable) constant disturbance, but r is a design parameter that can be used to specify α. Proposition 1 implies that if x converges to $\bar{x}$, then the weighted output consensus (14) is automatically achieved, where the consensus value α is in (21). In other words, control objective i) also implies objective ii). Indeed, it is possible to show the convergence of x.

Theorem 1. Given constants d and r, the closed-loop system (17) satisfies

$$\lim_{t\to\infty}Wy(t)=\alpha\mathbf{1}$$

for α in (21). That is, the distributed controller (15) achieves weighted output consensus (14) for the port-Hamiltonian system (11). Moreover, the state $\tilde{x}$ converges to the equilibrium $\bar{x}$ asymptotically.

Proof. According to Proposition 1, it only suffices to show $\tilde{x}\to\bar{x}$ as $t\to\infty$. Consider the Lyapunov candidate,

$$V=\frac{(\tilde{x}-\bar{x})^T\tilde{Q}(\tilde{x}-\bar{x})}{2}+\frac{(\xi-\bar{\xi})^T(\xi-\bar{\xi})}{2}.$$

Its time derivative along the solution to (17) satisfies

$$\begin{aligned}\dot{V}&=(\tilde{x}-\bar{x})^T\tilde{Q}(\tilde{J}-\tilde{R}-\tilde{B}K\tilde{B}^T)\tilde{Q}(\tilde{x}-\bar{x})\\&\quad+(\tilde{x}-\bar{x})^T\tilde{Q}\tilde{B}W\mathcal{L}(\xi-\bar{\xi})\\&\quad-(\xi-\bar{\xi})^T\mathcal{L}W\tilde{B}^T\tilde{Q}(\tilde{x}-\bar{x})\\&=-(\tilde{x}-\bar{x})^T\tilde{Q}(\tilde{R}+\tilde{B}K\tilde{B}^T)\tilde{Q}(\tilde{x}-\bar{x}).\end{aligned}$$

It follows from $\tilde{R}+\tilde{B}K\tilde{B}^T>0$ and $\tilde{Q}>0$ that $\tilde{x}\to\bar{x}$ as $t\to\infty$. ∎

5. SIMULATION

We revisit Case ii) in Section 2 and consider four prosumers in the DC micro grid (Ferguson et al., 2021). The incidence matrix for the topology of the DC micro grid is given by

Table 2. Coefficients of VSCs and lines

	VSC 1	VSC 2	VSC. 3	VSC 4
L_{si} (mH)	1.8	2.0	3.0	2.2
C_{si} (mF)	2.2	1.9	2.5	1.7
R_{si} (mΩ)	2.0	3.0	1.5	1.0
I_{Li} (A)	30	15	30	26
Z_i (Ω)	16.7	20	16.7	20
	Line 1	Line 2	Line 3	Line 4
R_k (mΩ)	70	50	80	60
L_k (μH)	2.1	2.0	3.0	2.2

$$\mathcal{B}=\begin{bmatrix}-1&0&0&-1\\1&-1&0&0\\0&1&-1&0\\0&0&1&1\end{bmatrix}. \tag{27}$$

The coefficients of the VSCs and lines are listed in Table 2. Moreover, we select $K_i=0.01$, $w_i=1$ and $r_i=380$. r_i is the nominal value of voltage in this example. The current load d_i of the four VSCs is written compactly as $[30\ 15\ 30\ 26]^T$ (A) at the initial time and has a variant $[1.1\ 1.9\ 1.6\ -1]^T$ (A) at 1s. Note that the output $y_i=\phi_i/L_{si}$ is actually the generated current of prosumer i, denoted by I_{si}. Therefore, from Theorem 1, current sharing, i.e., $I_{si}=I_{sj}$ for $i,j\in\mathcal{N}$ is achieved.

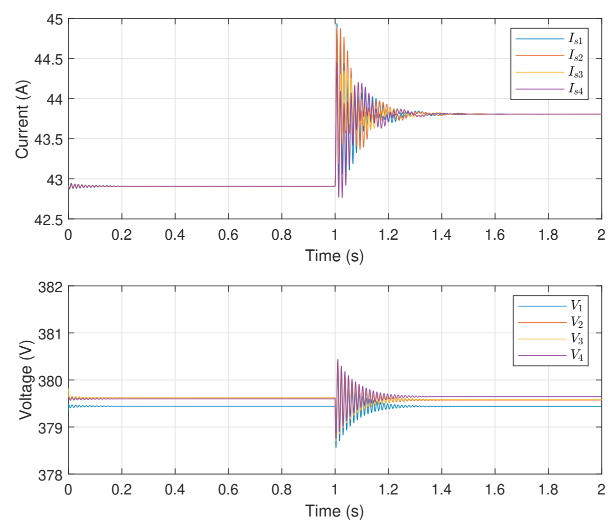

Fig. 2. Top: Generated current of VSCs; Bottom: Voltage of VSCs

Figure 2 shows the simulation results. From the top, one can see that the current sharing is achieved after the variant of loads, where the consensus value $\alpha=43.8073$ after 1s matches the one theoretically computed in Proposition 1. Next, the bottom plots the time response of the voltage $V_i=q_i/C_{si}$ of VSC i. One can see that the voltage is closely attached to 380V as the consequence of imposing r_i in the controller.

6. CONCLUSIONS

In the paper, we designed a controller achieving output consensus for coupled port-Hamiltonian systems on graphs in the presence of constant disturbances. The considered class of port-Hamiltonian systems is fairly general as it can incorporate with static and dynamic couplings. We explicitly computed the consensus value of the output as well as the equilibrium point of the closed-loop system;

both depend on disturbances. Then, we proved closed-loop stability, which implies output consensus. For more details, we refer the readers to Kawano et al. (2022), which generalizes the output consensus problem to nonlinear systems and includes the result of linear systems as a special case.

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