On the splitting principle for cohomoligical invariants of reflection groups

Gille, Stefan; Hirsch, Christian

Published in:
Transformation Groups

DOI:
10.1007/s00031-020-09637-6

Transformation Groups
Vol. 27, No. 4, 2022, pp. 1261–1285

ON THE SPLITTING PRINCIPLE FOR COHOMOLOGICAL INVARIANTS OF REFLECTION GROUPS

STEFAN GILLE*

Department of Mathematical and Statistical Sciences
University of Alberta
Edmonton T6G 2G1
Canada
gille@ualberta.ca

CHRISTIAN HIRSCH**

Bernoulli-Institute
University of Groningen
Nijenborgh 9
9747 AG Groningen
The Netherlands
c.p.hirsch@rug.nl

Abstract. Let k_0 be a field of characteristic not two, (V, b) a finite-dimensional regular bilinear space over k_0, and W a subgroup of the orthogonal group of (V, b) with the property that the subring of W-invariants of the symmetric algebra of V is a polynomial algebra over k_0. We prove that Serre's splitting principle holds for cohomological invariants of W with values in Rost's cycle modules.

1. Introduction

Let $\mathfrak{F}_{\mathrm{k}_0}$ be the category of finitely generated field extensions of a field k_0, and M_* a cycle module in the sense of Rost [27] over k_0. A *cohomological invariant* of degree n of an algebraic group G over k_0 with values in M_* is a natural transformation

$$a\colon\ \mathrm{H}^1(\,-, G) \to \mathrm{M}_n(\,-\,)$$

of functors on $\mathfrak{F}_{\mathrm{k}_0}$. Here $\mathrm{H}^1(\,-, G)$ denotes the first non-abelian Galois cohomology set of G. Cohomological invariants are an old topic. For instance the discriminant, or the Clifford invariant of a quadratic form can be interpreted as a cohomological invariant of an orthogonal group. However the formalization of this concept has been done only some 20 years ago by Serre, see his lectures [31] for a thorough account and some information on the history of the subject.

In general the cohomological invariants of an algebraic group with values in a given cycle module are hard (if not impossible) to compute. For most groups we

DOI: 10.1007/s00031-020-09637-6
*Supported by an NSERC grant.
**Supported by The Danish Council for Independent Research — Natural Sciences, grant DFF–7014-00074 *Statistics for point processes in space and beyond*, and by the *Centre for Stochastic Geometry and Advanced Bioimaging*, funded by grant 8721 from the Villum Foundation.
Received March 20, 2020. Accepted October 24, 2020.
Published online February 1, 2021.
Corresponding Author: Stefan Gille, e-mail: gille@ualberta.ca

know only some of the invariants, and even finding new ones can be quite a task, as is exemplified in the construction of the Rost invariant, see, e.g., Merkurjev [19]. Besides (the natural) applications to the classification of algebraic groups and their torsors there are further applications of the theory of cohomological invariants, as for instance to rationality questions around Noether's problem, see Serre's lectures [31, Sects. 33 and 34].

In [16] the second named author has computed the invariants of a Weyl group W with values in a cycle module that is annihilated by 2 over a field k_0, whose characteristic does not divide the cardinality of W. Crucial for these computations is the so-called *splitting principle* for invariants of orthogonal reflection groups. The proof of this principle is the content of this article. We show:

Theorem. *Let* k_0 *be a field of characteristic not* 2*,* (V,b) *a finite-dimensional regular bilinear space over* k_0*, and* $W \subseteq \mathrm{O}(V,b)$ *a finite subgroup of the orthogonal group of* (V,b)*. Assume that the* W*-invariants* $\mathrm{S}(V)^W$ *of the symmetric algebra* $\mathrm{S}(V)$ *of* V *are a polynomial algebra over* k_0*.*

Then a cohomological invariant of degree n *of* W *with values in a cycle module* M_* *over* k_0

$$a\colon\ \mathrm{H}^1(\,-,W) \to \mathrm{M}_n(\,-\,)$$

is zero if and only if its restrictions to all abelian subgroups generated by reflections are zero.

Note that by the Chevalley–Shephard–Todd theorem a group W as in the theorem above is generated by orthogonal reflections in $\mathrm{O}(V,b)$, and is therefore a so-called orthogonal reflection group. On the other hand by the same result, if $W \subseteq \mathrm{O}(V,b)$ is generated by reflections and the order of W is prime to the characteristic of the base field k_0 then $\mathrm{S}(V)^W$ is a polynomial algebra over k_0.

The proof of the theorem makes use of the explicit description of a versal W-torsor over k_0, which we give in Section 4.4. This construction uses the fact that W is a subgroup of the orthogonal group of some regular symmetric bilinear space over k_0. Hence, although the finite group W is defined as a finite group scheme over an arbitrary field, for our proof of the splitting principle we have to assume that the group W has an orthogonal representation $W \hookrightarrow \mathrm{O}(V,b)$ over k_0, such that $\mathrm{S}(V)^W$ is a polynomial algebra over k_0.

Serre [32] has pointed out to us that thanks to a theorem of Demazure [8] essentially the same arguments show the splitting principle for Weyl groups as long as the characteristic of the base field is not one of the torsion primes, see Sections 4.7 and 4.8. More general, Serre has shown that for invariants of Weyl groups with values in abelian Galois cohomology with finite coefficients (these are cycle modules in the sense of Rost) the splitting principle holds as long as the characteristic of the base field is not two, see Remark 4.

Note that Ducoat in the unpublished preprint [10] claims the splitting principle for the special case of invariants of Coxeter groups with values in Galois cohomology with finite coefficients over (big enough) fields of characteristic zero.

This article as well as its sequel [16] are based on the 2010 Diploma thesis [15] of the second named author. This diploma thesis does not deal with cycle modules of Rost, but with Morel's [23] $\mathbb{A}^1$-invariants sheaves with $\mathrm{K}^M_*/2$-structure. However the proof of the splitting principle in [15] has some gaps and flaws.

Our original intention was to write this article in the same setting. We refrained from this for the following two reasons. On the one hand, it has turned out to be much easier and shorter (at least for us) to give the proof in the slightly more restrictive setting of cycle modules. And on the other hand, we believe that the most interesting invariants are anyway Galois cohomology-, or Milnor K-theory (modulo some integer) invariants, which are both cycle modules, or Witt invariants. Although Witt groups are not cycle modules our arguments work for these invariants as well, see Section 5. An advantage of this restriction is also that the article is readable for readers only interested in such invariants. They can assume throughout that the cycle module M_* in question is one of their favourite theories.

Acknowledgement. We would like to thank Fabien Morel for advice and fruitful discussions around this work. Furthermore we would like to thank Jean-Pierre Serre for advice and comments on this article and its sequel [16], in particular for pointing out to us unnecessary restrictions on the characteristic of the base field, and for explaining to us that thanks to a theorem of Demazure our arguments apply also to (most) Weyl groups.

This work has started (and slept long in between) more than 10 years ago, when one of us (S.G.) was Assistant and the other (C.H.) Diploma Student of Fabien Morel at the LMU Munich. The first named author would also like to thank Volodya Chernousov, now his colleague at the University of Alberta. Visiting Volodya in March 2008 has made a crucial impact on this work.

Finally we would like to thank the referees for their thorough reading, corrections, and very useful suggestions.

2. Preliminaries: Cycle modules, Galois cohomology and torsors

2.1. Notations

Given a field k_0 we denote by $(\mathrm{k}_0)_s$ its separable closure, and by Γ_{k_0} its absolute Galois group $\mathrm{Gal}((\mathrm{k}_0)_s/\mathrm{k}_0)$.

We denote by $\mathrm{Fields}_{\mathrm{k}_0}$ the category of all field extensions of k_0. More precisely, the objects of $\mathrm{Fields}_{\mathrm{k}_0}$ are pairs (L, j), where L is a field and $j : \mathrm{k}_0 \to L$ a homomorphism of fields. A morphism $(E, i) \to (L, j)$ is a morphism of fields $\varphi : E \to L$, such that

$$\begin{tikzcd} E \arrow[rr, "\varphi"] & & L \\ & \mathrm{k}_0 \arrow[ul, "i"] \arrow[ur, "j"'] & \end{tikzcd}$$

commutes. For ease of notation the structure morphism will not be mentioned, i.e., we only write L instead of (L, j).

If $(\ell, \iota) \in \mathrm{Fields}_{\mathrm{k}_0}$ then Fields_ℓ can be identified with a full subcategory of the category $\mathrm{Fields}_{\mathrm{k}_0}$ via the embedding $(L, j) \mapsto (L, j \circ \iota)$, which depends on the structure morphism $\iota : \mathrm{k}_0 \to \ell$.

The symbol $\mathfrak{F}_{\mathrm{k}_0}$ denotes the full subcategory of $\mathrm{Fields}_{\mathrm{k}_0}$ consisting of finitely generated field extensions of k_0, i.e., of pairs (L, j), where L is a field and j :

$\mathrm{k}_0 \to L$ a homomorphism of fields giving L the structure of a finitely generated field extension of k_0. Again we can identify $\mathfrak{F}_\ell$ with a full subcategory of $\mathfrak{F}_{\mathrm{k}_0}$ for all $\ell \in \mathfrak{F}_{\mathrm{k}_0}$.

2.2. Cycle modules

These have been invented by Rost [27] to facilitate Chow group computations. We refer to this article for details and more information, in particular for the long list of axioms, of which we recall here only those which play a role in this work.

The prototype of a cycle module is Milnor K-theory, which has been introduced by Milnor [21], and which we denote by

$$\mathrm{K}_*^M(F) := \bigoplus_{n\geq 0} \mathrm{K}_n^M(F)$$

for a field F. Recall that this is a graded ring and as abelian group generated by the pure symbols $\{x_1, \ldots, x_n\} \in \mathrm{K}_n^M(F)$, where $x_1, \ldots, x_n$ are non-zero elements of F.

A *cycle module over a field* k_0 is a covariant functor

$$\mathrm{M}_*\colon \mathfrak{F}_{\mathrm{k}_0} \to \mathfrak{grAb}, \quad F \mapsto \mathrm{M}_*(F) = \bigoplus_{n\in\mathbb{Z}} \mathrm{M}_n(F),$$

where $\mathfrak{grAb}$ denotes the category of graded abelian groups, which is subject to the (long list of) axioms in Rost [27, Sects. 1 and 2]. In particular, $\mathrm{M}_*(F)$ is a graded $\mathrm{K}_*^M(F)$-module for all $F \in \mathfrak{F}_{\mathrm{k}_0}$. Following the foundational paper of Rost on cycle modules and deviating from more usual customs we denote by φ_{M} the morphism $\mathrm{M}_*(F) \to \mathrm{M}_*(E)$ induced by a morphism $\varphi : F \to E$ in $\mathfrak{F}_{\mathrm{k}_0}$.

2.3. The second residue map

Let v be a discrete valuation of $F \in \mathfrak{F}_{\mathrm{k}_0}$ of geometric type which is trivial on k_0. By this we mean that there exists a normal integral k_0-scheme X of finite type, such that the function field $\mathrm{k}_0(X)$ is equal to F, and such that v corresponds to a regular codimension one point of X. Then there is a $\mathrm{K}_*^M(\mathrm{k}_0)$-linear homomorphism, the so-called (*second*) *residue map*:

$$\partial_v\colon \mathrm{M}_*(F) \to \mathrm{M}_{*-1}(F(v)),$$

where $F(v)$ is the residue field of v.

Associated with this homogenous homomorphism of degree -1 there is a homogenous homomorphism of degree 0, the so-called *specialization homomorphism*:

$$s_v^\pi\colon \mathrm{M}_*(F) \to \mathrm{M}_*(F(v)), \quad x \mapsto \partial_v\big(\{\pi\}\cdot x\big),$$

which depends on the choice of a uniformizer π for v.

We recall the following four axioms, which play some role here. Let $F, v, F(v)$ be as above and $\varphi : F \to E$ a finite field extension. Assume there is a geometric valuation w on E with residue field $E(w)$ and with $w|_F = v$. Let $e_{w|v}$ be the ramification index and $\bar{\varphi} : F(v) \to E(w)$ the induced homomorphism of the residue fields. Then the following holds (numbering as in Rost [27, p. 329]):

- **(R2a)** $\varphi_{\mathrm{M}}(x\cdot z) = \varphi_{\mathrm{K}^M}(x)\cdot\varphi_M(z)$ for all $x \in \mathrm{K}_*^M(F)$ and $z \in \mathrm{M}_*(F)$;
- **(R3a)** $\partial_w \circ \varphi_{\mathrm{M}} = e_{w|v}\cdot\bar{\varphi}_{\mathrm{M}} \circ \partial_v$;
- **(R3c)** if w is trivial on F, and so $F(v) = F$, then $\partial_w \circ \varphi_{\mathrm{M}} = 0$; and
- **(R3d)** if w is as in **(R3c)** and π is a uniformizer for w then $s_w^\pi \circ \varphi_{\mathrm{M}} = \bar{\varphi}_{\mathrm{M}}$.

2.4. Unramified cycle modules

Let X be an integral scheme, which is essentially of finite type over a field k_0. By the latter we mean that X is a finite type k_0-scheme or a localization of such a scheme. We denote by $X^{(1)}$ the set of points of codimension 1 in X. If a point x in $X^{(1)}$ is regular, then its local ring $\mathcal{O}_{X,x}$ is a discrete valuation ring and we get a valuation v_x on the function field $\mathrm{k}_0(X)$ of X.

Given a cycle module M_* over k_0 and a regular codimension one point x of X we have then a second residue map

$$\partial_x := \partial_{v_x} \colon \mathrm{M}_*(\mathrm{k}_0(X)) \to \mathrm{M}_{*-1}(\mathrm{k}_0(x)),$$

where $\mathrm{k}_0(x)$ denotes the residue field of x, as well as a specialization map

$$s_x^\pi := s_{v_x}^\pi : \mathrm{M}_*(\mathrm{k}_0(X)) \to \mathrm{M}_*(\mathrm{k}_0(x))$$

for every uniformizer $\pi \in \mathcal{O}_{X,x}$.

If X is regular in codimension one ∂_x exists for all $x \in X^{(1)}$ and so we can define

$$\mathrm{A}^0(X, \mathrm{M}_n) := \mathrm{Ker}\left(\mathrm{M}_n(\mathrm{k}_0(X)) \xrightarrow{(\partial_x)_{x\in X^{(1)}}} \bigoplus_{x\in X^{(1)}} \mathrm{M}_{n-1}(\mathrm{k}_0(x))\right),$$

the here so-called *unramified* M_n*-cohomology group* of X.

Remark 1. If K is a field one considers also the subgroup $\mathrm{M}_n(K)_{\mathrm{unr}}$ of unramified elements of $\mathrm{M}_n(K)$, which is defined as the intersection of the kernels of all residue maps associated with geometric discrete valuations of K. Note that if $K = \mathrm{k}_0(X)$ is as above we have $\mathrm{M}_n(\mathrm{k}_0(X))_{\mathrm{unr}} \subseteq \mathrm{A}^0(X, \mathrm{M}_n)$.

2.5. Non-abelian Galois cohomology

We recall now – mainly to fix notations – some definitions and properties of torsors and non-abelian Galois cohomology sets. We refer to Serre's well-known book [29] and also to [18, §28 and §29] for details and more information.

Let F be a field and G a linear algebraic group over F. We denote by $\mathrm{H}^1(F, G)$ the *first non-abelian Galois cohomology* set, i.e., $\mathrm{H}^1(F, G) = \mathrm{H}^1(\Gamma_F, G(F_s))$. If a continuous map $c : \Gamma_F \to G(F_s)$, $\sigma \mapsto c_\sigma$, is a 1-cocycle, we denote its class in $\mathrm{H}^1(F, G)$ by $[c]$.

If $\varphi : F \to E$ is a morphism of fields we denote the induced *restriction map* $\mathrm{H}^1(F, G) \to \mathrm{H}^1(E, G)$ by r_φ, or if φ is clear from the context by $r_{E/F}$.

If $\theta : H \to G$ is a morphism of linear algebraic groups over F we denote following [18] the induced homomorphism $\mathrm{H}^1(F, H) \to \mathrm{H}^1(F, G)$ by θ^1.

In the proof of Theorem 1 below we also consider the first non-abelian étale cohomology set $\mathrm{H}^1_{\mathrm{et}}(X, G)$, where X is a scheme over F and G a linear algebraic group over F. If $f : X \to Y$ is a morphism of such schemes we denote the pull-back map $\mathrm{H}^1_{\mathrm{et}}(Y, G) \to \mathrm{H}^1_{\mathrm{et}}(X, G)$ by r_f. We write then T_X instead of $r_f(T)$ for $T \in \mathrm{H}^1_{\mathrm{et}}(Y, G)$ if f is clear from the context.

Note that since G is smooth the set $\mathrm{H}^1_{\mathrm{et}}(X, G)$ can be identified with the isomorphism classes of G-torsors $\pi : \mathcal{T} \to X$ over X, see, e.g., [20, Chap. III.4]. We denote the class of a G-torsor $\pi : \mathcal{T} \to X$ over X by $[\mathcal{T} \to X]$.

We use in the following also affine notations, i.e., we write $\mathrm{H}^1_{\mathrm{et}}(R, G)$ instead of $\mathrm{H}^1_{\mathrm{et}}(X, G)$ if $X = \mathrm{Spec}\, R$ is affine. Note that if $X = \mathrm{Spec}\, K$ is the spectrum of a field then $\mathrm{H}^1_{\mathrm{et}}(X, G)$ is naturally isomorphic to $\mathrm{H}^1(K, G)$.

Example 1. Let G be a finite group with trivial Γ_F-action, where F is a field. Then the non-abelian Galois cohomology set $\mathrm{H}^1(F,G)$ can be identified with the isomorphism classes of G-Galois algebras, see, e.g., [18, §18B].

A particular example of such a G-Galois algebra is a finite Galois extension $E \supset F$ with group $\mathrm{Gal}(E/F) = G$. Then the continuous and surjective homomorphism of groups $c : \Gamma_F \to G$, $\sigma \mapsto \sigma|_E$, represents the class of the G-Galois algebra E in $\mathrm{H}^1(F,G)$.

In this situation, if $\theta : H \subset G$ is a subgroup with fixed field L then the class of the restriction $r_{L/F}([c])$ is represented by the continuous homomorphism $c|_{\Gamma_L} : \Gamma_L \subset \Gamma_F \xrightarrow{c} G$, whose image is in the subgroup H. It follows that $r_{L/F}([c]) = \theta^1([c'])$, where $[c'] \in \mathrm{H}^1(L,H)$ is represented by $c' : \Gamma_L \xrightarrow{c|_{\Gamma_L}} H$.

2.6. Versal torsors

Let $T \in \mathrm{H}^1_{\mathrm{et}}(X,G)$ be a G-torsor over the smooth integral F-scheme X with function field $K = F(X)$. Assume that for any infinite field $L \in \mathfrak{F}_F$ and every element $\widetilde{T} \in \mathrm{H}^1(L,G)$ there exists a dense set of L-points x of X, such that $T_{F(x)} = \widetilde{T}$ in $\mathrm{H}^1(F(x),G) = \mathrm{H}^1(L,G)$. Then $T_K \in \mathrm{H}^1(K,G)$ is called a *versal G-torsor*, see [31, Def. 5.1].

Such torsors exists for every linear algebraic group over a field, see [31, 5.3].

Example 2. Let G be a finite group which acts faithfully on a finite-dimensional k_0-vector space V, where k_0 is a field. Then G acts on the dual space $V^\vee := \mathrm{Hom}_{\mathrm{k}_0}(V,\mathrm{k}_0)$ via $(h.f)(v) := f(h^{-1}.v)$ for all $f \in V^\vee$, $v \in V$, and $h \in G$. This induces a G-action on $\mathbb{A}(V) := \mathrm{Spec}\,\mathrm{S}(V^\vee)$, where $\mathrm{S}(V^\vee)$ denotes the symmetric algebra of $V^\vee$. For $g \in G$ denote by $\mathcal{V}_g$ the closed subset of $\mathbb{A}(V)$ defined by the ideal generated by all $f \circ (g - \mathrm{id}_V) \in V^\vee$, $f \in V^\vee$. The group G acts freely on the open set

$$U := \mathbb{A}(V) \setminus \bigcup_{\mathrm{id}_V \neq g \in G} \mathcal{V}_g,$$

and so the quotient morphism $q : U \to U/G$ is a G-torsor. The generic fiber of this torsor is a versal G-torsor, see [31, 5.4 and 5.5] for a proof. Note that the function field $\mathrm{k}_0(U/G)$ is the fraction field of the invariant ring $\mathrm{S}(V^\vee)^G$. Hence the class of this versal G-torsor in $\mathrm{H}^1(\mathrm{k}_0(U/G),G)$ is the class of the G-Galois algebra $\mathrm{k}_0(U)$ over $\mathrm{k}_0(U/G)$. In other words, the class of the Galois extension $\mathrm{k}_0(U) \supseteq \mathrm{k}_0(U)^G = \mathrm{k}_0(U/G)$ is a versal G-torsor over k_0.

3. Invariants in cycle modules

Definition 1. Let G be a linear algebraic group and M_* a cycle module over the field k_0. A *cohomological invariant of degree n* of G with values in the cycle module M_* is a natural transformation of functors

$$a\colon\ \mathrm{H}^1(\,-,G) \to \mathrm{M}_n(\,-\,),$$

i.e., for all $\varphi : F \to E$ in $\mathfrak{F}_{\mathrm{k}_0}$ the following diagram commutes:

$$\begin{tikzcd} \mathrm{H}^1(E,G) \arrow[r, "a_E"] & \mathrm{M}_n(E) \\ \mathrm{H}^1(F,G) \arrow[r, "a_F"'] \arrow[u, "r_\varphi"] & \mathrm{M}_n(F) \arrow[u, "\varphi_{\mathrm{M}}"'] \end{tikzcd} .$$

This definition is due to Serre and a special case of the one given in his lectures [31], but it includes the main players of Serre's text, cohomological invariants of algebraic groups with values in $\mathrm{H}^n(\,-, C)$ for some finite discrete Γ_{k_0}-module C of order prime to $\operatorname{char} \mathrm{k}_0$. Note however that there is a subtle difference as we consider here only the category $\mathfrak{F}_{\mathrm{k}_0}$ of finitely generated field extensions of k_0 and not the category $\mathrm{Fields}_{\mathrm{k}_0}$ of all field extensions of k_0. This is forced by the fact that for technical reasons an "abstract" cycle module over a field k_0 is not defined for all field extensions of k_0 but only for the finitely generated ones, cf. Rost [27, p. 328]. If one is only interested in "concrete" cycle modules as for instance Milnor K-theory, or Galois cohomology with finite coefficients, there is no need for this restriction. (Note however that by the detection principle, which is proven in Serre's lecture [31] for Galois cohomology invariants with finite coefficients and for Milnor K-theory below, these invariants are determined by their values on the smaller category $\mathfrak{F}_{\mathrm{k}_0}$.)

Following Serre's lectures [31] we denote the set of cohomological invariants of degree n of the group G with values in M_* by $\mathrm{Inv}^n_{\mathrm{k}_0}(G, \mathrm{M}_*)$. The set $\mathrm{Inv}^n_{\mathrm{k}_0}(G, \mathrm{M}_*)$ has the structure of an abelian group as $\mathrm{M}_n(F)$ is one for all $F \in \mathfrak{F}_{\mathrm{k}_0}$.

We set

$$\mathrm{Inv}_{\mathrm{k}_0}(G, \mathrm{M}_*) := \bigoplus_{n\in\mathbb{Z}} \mathrm{Inv}^n_{\mathrm{k}_0}(G, \mathrm{M}_*),$$

and call elements of this direct sum *cohomological invariants* of G with values in M_*. Note that the K^M_*-structure of M_* induces a $\mathrm{K}^M_*(\mathrm{k}_0)$ operation on the graded abelian group $\mathrm{Inv}_{\mathrm{k}_0}(G, \mathrm{M}_*)$ making the set of cohomological invariants of G with values in M_* a graded $\mathrm{K}^M_*(\mathrm{k}_0)$-module.

Example 3. We have $\mathrm{Inv}_{\mathrm{k}_0}(G, \mathrm{M}_*) \neq 0$ if $\mathrm{M}_*(\mathrm{k}_0) \neq 0$. In fact, if $x \in \mathrm{M}_*(\mathrm{k}_0)$ then

$$c_L \colon \mathrm{H}^1(L, G) \to \mathrm{M}_*(L),\ t \mapsto (\iota_L)_{\mathrm{M}}(x),$$

where $L \in \mathfrak{F}_{\mathrm{k}_0}$ with structure map $\iota_L : \mathrm{k}_0 \to L$, defines an invariant. Such invariants are called *constant*, respectively, *constant of degree* n if $x \in \mathrm{M}_n(\mathrm{k}_0)$, and we write $c \equiv x \in \mathrm{M}_*(\mathrm{k}_0)$.

3.1. Restriction of invariants

Let $\theta : H \to G$ be a morphism of linear algebraic groups over k_0 and M_* a cycle module over k_0. Composing $a \in \mathrm{Inv}_{\mathrm{k}_0}(G, \mathrm{M}_*)$ with the map θ^1:

$$\mathrm{H}^1(\,-, H) \xrightarrow{\theta^1} \mathrm{H}^1(\,-, G) \xrightarrow{a} \mathrm{M}_*(\,-\,)$$

we get an invariant $\theta^*(a) \in \mathrm{Inv}_{\mathrm{k}_0}(H, \mathrm{M}_*)$. In case $\theta : H \subseteq G$ is the embedding of a closed subgroup we denote following Serre's lecture [31] the induced homomorphism $\theta^* \colon \mathrm{Inv}_{\mathrm{k}_0}(G, \mathrm{M}_*) \to \mathrm{Inv}_{\mathrm{k}_0}(H, \mathrm{M}_*)$ by Res^H_G and call it the *restriction.*

Example 4. Let H be a subgroup of a finite group G, M_* a cycle module over k_0, and $a \in \mathrm{Inv}_{\mathrm{k}_0}(G, \mathrm{M}_*)$. If g is an element of the normalizer $\mathrm{N}_G(H)$ we denote by ι_g the inner automorphism of G defined by g. The isomorphism ι_g acts also on H and so consequently via $\iota_g^* : a \mapsto a \circ \iota_g^1$ on $\mathrm{Inv}_{\mathrm{k}_0}(H, \mathrm{M}_*)$ for all $g \in \mathrm{N}_G(H)$ giving $\mathrm{Inv}_{\mathrm{k}_0}(H, \mathrm{M}_*)$ the structure of a $\mathrm{N}_G(H)$-module. We claim that Res_G^H maps $\mathrm{Inv}_{\mathrm{k}_0}(G, \mathrm{M}_*)$ into the subgroup $\mathrm{Inv}_{\mathrm{k}_0}(H, \mathrm{M}_*)^{\mathrm{N}_G(H)}$ of $\mathrm{N}_G(H)$-invariant elements in $\mathrm{Inv}_{\mathrm{k}_0}(H, \mathrm{M}_*)$. In fact, by [31, Prop. 13.1] the map $\iota_g^1 : \mathrm{H}^1(L, G) \to \mathrm{H}^1(L, G)$ is the identity for all $L \in \mathfrak{F}_{\mathrm{k}_0}$. Since we have $\mathrm{Res}_G^H \big(\iota_g^*(a)\big) = \iota_g^*\big(\mathrm{Res}_G^H(a)\big)$ for all $a \in \mathrm{Inv}_{\mathrm{k}_0}(G, \mathrm{M}_*)$ this implies the claim.

3.2. Specialization theorem and detection principle

The following specialization theorem is an analog of a result of Rost on Galois cohomology invariants. Its proof follows essentially the pattern of arguments in Serre's lecture [31, Thm. 11.1]. However it is more involved as we can not work with the henselization of the field K which is in general not in $\mathfrak{F}_{\mathrm{k}_0}$ anymore.

Theorem 1. *Let X be an integral scheme with function field $K = \mathrm{k}_0(X)$, which is essentially of finite type over the field k_0. Let further M_* be a cycle module over k_0, $a \in \mathrm{Inv}_{\mathrm{k}_0}(G, \mathrm{M}_*)$, where G is a linear algebraic group over k_0, and $T \in \mathrm{H}^1_{\mathrm{et}}(X, G)$. Let $x \in X$ be a regular codimension one point. Then we have:*

(i) $\partial_x\big(a_K(T_K)\big) = 0$*; and*
(ii) $s_x^\pi(a_K(T_K)) = a_{\mathrm{k}_0(x)}(T_{\mathrm{k}_0(x)})$ *for all local uniformizers $\pi \in \mathcal{O}_{X,x}$.*

In particular, if X is regular in codimension one, then $a_K(T_K) \in \mathrm{A}^0(X, \mathrm{M}_*)$.

Proof. Replacing X by $\mathrm{Spec}\,\mathcal{O}_{X,x}$ we can assume that $X = \mathrm{Spec}\,R$ for a discrete valuation ring R, which is essentially of finite type over k_0, and that x is the closed point of X. We denote $k = \mathrm{k}_0(x)$ the residue field of R, $q : R \to k$ the quotient map, $\eta : R \to K$ the embedding of R into its field of fractions K, and v the discrete valuation of K corresponding to R. Then $\partial_x = \partial_v : \mathrm{M}_*(K) \to \mathrm{M}_{*-1}(k)$, and for (i) we have to show

$$\partial_v\big(a_K(T_K)\big) \;=\; 0\,. \tag{1}$$

To prove this we 'construct' a discrete valuation ring S, which is a local étale extension of R with the same residue field k, and which contains a subfield ℓ with G-torsor $\widetilde{T}$, such that $T_S = S \times_R T \simeq S \times_\ell \widetilde{T}$. Having this ring S at our disposal (1) is then essentially a formal consequence of the axioms of a cycle module.

To get the desired local étale extension, let $\psi : R \to R^h$ be the henselization of R with fraction field K^h. This is also a discrete valuation ring with the same residue field k, and there exist local étale R-algebras $\varphi_i : R \to R_i$, $i \in I$, such that $R^h = \varinjlim_{i\in I} R_i$, see [26, Chap. VIII]. The rings R_i are also discrete valuation rings with k as residue field. We denote by v_i the induced valuation on the fraction field K_i of R_i.

We get a commutative diagram of homomorphisms of rings for all $i \in I$:

$$\begin{tikzcd} K \arrow[r, "\varphi'_i"] & K_i \arrow[r, "\psi'_i"] & K^h \\ R \arrow[u, "\eta"] \arrow[r, "\varphi_i"'] & R_i \arrow[u, "\eta_i"'] \arrow[r, "\psi_i"'] & R^h \arrow[u, "\eta^h"'] \end{tikzcd},$$

where the up arrows are the respective inclusions of the rings R, R_i, and R^h into their fraction fields. Note that $\psi = \psi_i \circ \varphi_i$ for all $i \in I$.

The homomorphism $\varphi_i : R \to R_i$ is unramified at the maximal ideal of R_i, and so by the cycle module Axiom **(R3a)**, see Section 2.3, the square on the right-hand side of the following diagram commutes

$$\begin{tikzcd} \mathrm{H}^1(K_i, G) \arrow[r, "a_{K_i}"] & \mathrm{M}_*(K_i) \arrow[r, "\partial_{v_i}"] & \mathrm{M}_{*-1}(k) \\ \mathrm{H}^1(K, G) \arrow[u, "r_{\varphi'_i}"] \arrow[r, "a_K"] & \mathrm{M}_*(K) \arrow[u, "(\varphi'_i)_{\mathrm{M}}"] \arrow[r, "\partial_v"] & \mathrm{M}_{*-1}(k) \arrow[u, "="'] \end{tikzcd} \tag{2}$$

for all $i \in I$. Since a is an invariant also the one on the left-hand side is commutative. Therefore to prove $\partial_v(a_K(T_K)) = 0$ it is enough to show that there exists $i \in I$, such that

$$\partial_{v_i}\big(a_{K_i}(T_{K_i})\big) = \partial_{v_i}\big(a_{K_i}(r_{\varphi'_i}(T_K))\big) \; = \; 0\,.$$

To find this $i \in I$ we use the fact that there exists a splitting $j : k \to R^h$ of the quotient morphism $q^h : R^h \to k$, i.e., $q^h \circ j = \mathrm{id}_k$. In fact, if $\widehat{R}$ is the completion of R we have a splitting $\widehat{j} : k \to \widehat{R}$ of the quotient map $\widehat{R} \to k$ by (a special case of) the Cohen structure theorem, see, e.g., [6, Chap. 8, §5, no. 2, Cor. 3 of Thm. 1]. This splitting factors via R^h since the henselian local domain R^h is excellent by [12, Cor. 18.7.6], and therefore satisfies the approximation property by [3, Sect. 3.6, Cor. 9], which implies in particular, that the splitting $\widehat{j}$ of $\widehat{R} \to k$ factors via R^h.

The composition of maps $\mathrm{H}^1(k, G) \xrightarrow{r_j} \mathrm{H}^1_{\mathrm{et}}(R^h, G) \xrightarrow{r_{q^h}} \mathrm{H}^1(k, G)$ is the identity, and by [9, Chap. XXIV, Prop. 8.1] the map $r_{q^h} : \mathrm{H}^1_{\mathrm{et}}(R^h, G) \to \mathrm{H}^1(k, G)$ is an isomorphism. Therefore $r_j : \mathrm{H}^1(k, G) \to \mathrm{H}^1_{\mathrm{et}}(R^h, G)$ is one as well, and moreover we have $r_j(T_k) = T_{R^h}$ since $r_{q^h}(T_{R^h}) = r_{q^h}(r_\psi(T)) = r_q(T) = T_k$.

Let k_i be the pre-image of $j(k)$ under the homomorphism $\psi_i : R_i \to R^h$. The set $k_i \setminus \{0\}$ is contained in the set of units of R_i and so k_i is a field. This implies also that v_i is trivial on k_i. The k_0-linear quotient homomorphism $q_i : R_i \to k$ maps k_i onto a subfield of k. It follows, see [5, Chap. 5, §14, no 7, Cor. 3], that k_i is also a finitely generated field extension of k_0 and so in $\mathfrak{F}_{\mathrm{k}_0}$ for all $i \in I$.

By the definition of the fields k_i we have a commutative diagram

$$\begin{tikzcd} k \arrow[r, "j"] & R^h \\ k_i \arrow[u, "\bar{\psi}_i"] \arrow[r, "j_i"'] & R_i \arrow[u, "\psi_i"'] \end{tikzcd} \tag{3}$$

for all $i \in I$, where j_i is the inclusion $k_i \subset R_i$ and $\bar{\psi}_i$ the homomorphism induced by ψ_i. Note that $\bar{\psi}_i = q_i \circ j_i$ as $q \circ j = \mathrm{id}_k$.

Diagram (3) gives in turn a commutative diagram of pointed non-abelian cohomology sets

$$\begin{tikzcd} \mathrm{H}^1(k,G) \arrow[r, "r_j"] & \mathrm{H}^1_{\mathrm{et}}(R^h,G) \\ \mathrm{H}^1(k_i,G) \arrow[u, "r_{\bar{\psi}_i}"] \arrow[r, "r_{j_i}"'] & \mathrm{H}^1_{\mathrm{et}}(R_i,G) \arrow[u, "r_{\psi_i}"'] \end{tikzcd} . \tag{4}$$

We have $k = \lim\limits_{i\in I} k_i$ and therefore by [2, Chap. VII, Thm. 5.7] (or by direct verification) that $\mathrm{H}^1(k,G) = \lim\limits_{i\in I} \mathrm{H}^1(k_i,G)$. Hence there exists $i_0 \in I$ and $T_{i_0} \in \mathrm{H}^1(k_{i_0},G)$, such that

$$r_{\bar{\psi}_{i_0}}(T_{i_0}) = T_k \in \mathrm{H}^1(k,G)\,. \tag{5}$$

By (4) we have

$$r_{\psi_{i_0}}\big(r_{j_{i_0}}(T_{i_0})\big) = r_j\big(r_{\bar{\psi}_{i_0}}(T_{i_0})\big) = r_j(T_k) = T_{R^h} = r_{\psi_{i_0}}(T_{R_{i_0}})\,.$$

Now $R^h = \lim\limits_{i\in I} R_i$ and so by [2, Chap. VII, Thm. 5.7] again we have

$$\lim_{i\in I} \mathrm{H}^1_{\mathrm{et}}(R_i,G) = \mathrm{H}^1_{\mathrm{et}}(R^h,G)\,.$$

Hence replacing i_0 by a 'larger' element of I if necessary we can assume that also

$$r_{j_{i_0}}(T_{i_0}) = T_{R_{i_0}}\,. \tag{6}$$

We claim that this index i_0 does the job, i.e., we have $\partial_{v_{i_0}}\big(a_{K_{i_0}}(T_{K_{i_0}})\big) = 0$. In fact, since a is a cohomological invariant we have a commutative diagram

$$\begin{tikzcd} \mathrm{H}^1(K_{i_0},G) \arrow[r, "a_{K_{i_0}}"] & \mathrm{M}_*(K_{i_0}) \\ \mathrm{H}^1(k_{i_0},G) \arrow[u, "r_{(\eta_{i_0}\circ j_{i_0})}"] \arrow[r, "a_{k_{i_0}}"'] & \mathrm{M}_*(k_{i_0}) \arrow[u, "(\eta_{i_0}\circ j_{i_0})_{\mathrm{M}}"'] \end{tikzcd} ,$$

and therefore taking (6) into account

$$a_{K_{i_0}}(T_{K_{i_0}}) = a_{K_{i_0}}\big(r_{(\eta_{i_0}\circ j_{i_0})}(T_{i_0})\big) = (\eta_{i_0}\circ j_{i_0})_{\mathrm{M}}\big(a_{k_{i_0}}(T_{i_0})\big)\,.$$

But $v_{i_0}|_{k_{i_0}} \equiv 0$ and so by the cycle module Axiom **(R3c)**, see Section 2.3, we have $\partial_{v_{i_0}}(z) = 0$ for all $z \in \mathrm{M}_*(K_{i_0})$, which are in the image of $(\eta_{i_0}\circ j_{i_0})_{\mathrm{M}}$. We have proven (i).

For the proof of (ii) we continue with the above notation, i.e., $R = \mathcal{O}_{X,x}$, $R^h = \lim_{i \in I} R_i$, and so on. We fix further a uniformizer π of R. Since the extensions $\varphi_i : R \to R_i$ are unramified the image of the uniformizer π in R_i is also one, which we denote by π as well. We have $s_x^\pi = s_v^\pi$, and so taking the right-hand side of the commutative diagram (2), the definition of the specialization map, as well as Axiom **(R2a)**, see Section 2.3, into account we have $s_{v_{i_0}}^\pi \circ (\varphi'_{i_0})_{\mathrm{M}} = s_v^\pi$.

We get then

$$\begin{aligned}
s_v^\pi(a_K(T_K)) &= s_{v_{i_0}}^\pi\big((\varphi'_{i_0})_{\mathrm{M}}(a_K(T_K))\big) \\
&= s_{v_{i_0}}^\pi\big(a_{K_{i_0}}(r_{\varphi'_{i_0}}(T_K))\big) && a \text{ is invariant} \\
&= s_{v_{i_0}}^\pi\big(a_{K_{i_0}}(r_{\eta_{i_0}}(T_{R_{i_0}}))\big) && \text{since } \eta_{i_0} \circ \varphi_{i_0} = \varphi'_{i_0} \circ \eta \\
&= s_{v_{i_0}}^\pi\big(a_{K_{i_0}}(r_{(\eta_{i_0} \circ j_{i_0})}(T_{i_0}))\big) && \text{by (6)} \\
&= s_{v_{i_0}}^\pi\big((\eta_{i_0} \circ j_{i_0})_{\mathrm{M}}(a_{k_{i_0}}(T_{i_0}))\big) && a \text{ is invariant} \\
&= (\bar{\psi}_{i_0})_{\mathrm{M}}\big(a_{k_{i_0}}(T_{i_0})\big) && \text{by } \textbf{(R3d)} \\
&= a_k\big(r_{\bar{\psi}_{i_0}}(T_{i_0})\big) && a \text{ is invariant} \\
&= a_k(T_k) && \text{by (5)}
\end{aligned}$$

as claimed. We are done. □

A consequence of this result is the following corollary, which is the analog of [31, 12.2] for cycle module invariants.

Corollary 2. *Let R be a regular local ring, which is essentially of finite type over the field k_0. Denote by K and k the fraction and residue field, respectively, of R. Let further G be a linear algebraic group over k_0, and M_* a cycle module over k_0. Then*

$$a_K(T_K) = 0 \quad \Longrightarrow \quad a_k(T_k) = 0$$

for all $T \in \mathrm{H}^1_{\mathrm{et}}(R, G)$ and all $a \in \mathrm{Inv}_{\mathrm{k}_0}(G, \mathrm{M}_*)$.

Proof. The proof is the same as the one of [31, 12.2]. We recall the arguments for the convenience of the reader.

If $\dim R = 1$ this follows from part (ii) of the theorem above, so let $d := \dim R \geq 2$, and $t \in R$ a regular parameter. Then R/Rt is also a regular local ring with the same residue field k, and which is essentially of finite type over k_0. The fraction field of R/Rt is the residue field K_t of the discrete valuation ring R_{Rt} (the localization at the codimension one prime ideal Rt). By the dimension one case we have $a_{K_t}(T_{K_t}) = 0$, and so by induction $a_k(T_k) = 0$. □

Finally we state and prove the following detection principle, which is the cycle module analog of [31, Thm. in 12.3]. Again the proof is the same as in Serre's lecture and only recalled for the convenience of our reader.

Theorem 3. *Let G be a linear algebraic group over the field k_0, and $T \in \mathrm{H}^1(K, G)$ a versal G-torsor. Then we have for a given cycle module M_* over k_0 and $a, b \in \mathrm{Inv}_{\mathrm{k}_0}(G, \mathrm{M}_*)$:*

$$a_K(T) = b_K(T) \quad \Longrightarrow \quad a = b\,.$$

Proof. Replacing a by $b-a$ it is enough to show that $a_K(T)=0$ implies $a\equiv 0$. We have to show $a_k(S)=0$ for all $k\in\mathfrak{F}_{\mathrm{k}_0}$ and all $S\in \mathrm{H}^1(k,G)$.

Replacing k by the rational function field $k(T)$ if necessary, we can assume that k is an infinite field. In fact, since a is an invariant the following diagram commutes:

$$\begin{tikzcd} \mathrm{H}^1(k(T),G) \arrow[r, "a_{k(T)}"] & \mathrm{M}_*(k(T)) \\ \mathrm{H}^1(k,G) \arrow[u, "r_\iota"] \arrow[r, "a_k"] & \mathrm{M}_*(k) \arrow[u, "\iota_{\mathrm{M}}"'] \end{tikzcd},$$

where $\iota : k\hookrightarrow k(T)$ is the natural embedding. By Rost [27, Prop. 2.2 **(H)**] the homomorphism $\iota_{\mathrm{M}}\colon \mathrm{M}_*(k)\to \mathrm{M}_*(k(T))$ is injective and so $a_{k(T)}(S_{k(T)})=0$ implies $a_k(S)=0$.

To prove the claim for an infinite field k we use that since $T\in \mathrm{H}^1(K,G)$ is a versal G-torsor there exists a G-torsor $\mathcal{T}\to X$ over a smooth integral scheme X with function field K, such that the generic fiber of $\mathcal{T}\to X$ is isomorphic to T, and such that there exists $x\in X(k)$ with $S=[\mathcal{T}\to X]_{\mathrm{k}_0(x)}$. We have $a_K(T_K)=0$ by assumption, and so we conclude that $a_{\mathrm{k}_0(x)}([\mathcal{T}\to X]_{\mathrm{k}_0(x)})=0$ by Corollary 2 above (note that $\mathcal{O}_{X,x}$ is regular since X is smooth). □

Remark 2.

(i) As one referee has pointed out to us Guillot [14, §6] also considers cohomological invariants of algebraic groups with values in Rost cycle modules. However he does not prove basic results as for instance the detection principle above.

(ii) A further generalization of cohomological invariants is due to Pirisi [25], who introduced cohomological invariants of algebraic stacks with values in cycle modules.

4. The splitting principle

4.1. Pseudo-reflections

We recall first some definitions and properties of reflection groups and their root systems, merely to fix our notations. We refer to the standard reference Bourbaki [7] for details and more information.

Let V be a finite-dimensional vector space over the field k_0. We denote by $\mathrm{GL}(V)$ the group of k_0-linear automorphisms of V. An element s of $\mathrm{GL}(V)$ is called a *pseudo-reflection* if $\mathrm{rank}(s-\mathrm{id}_V)=1$. A finite subgroup of $\mathrm{GL}(V)$ is called a *pseudo-reflection group* if it is generated by pseudo-reflections. A pseudo-reflection is called a *reflection* if it has exponent 2.

We have then the following well-known result. A proof can be found for instance in [17, Sects. 18.1 and 19.1]; see also [7, Chap. V, §5, no 3, Thm. 3 and Chap. V, §5, Ex. 7 and 8], or Serre [28].

Theorem 4 (Chevalley–Shephard–Todd–Bourbaki–Serre)**.** *Let V be a finite-dimensional vector space over* k_0 *and $W\subseteq \mathrm{GL}(V)$ a finite subgroup. Consider the following two assertions:*

(i) *The algebra of invariants* $\mathrm{S}(V)^W$ *is a polynomial algebra.*
(ii) *The subgroup* W *of* $\mathrm{GL}(V)$ *is generated by pseudo-reflections, i.e.,* W *is a pseudo-reflection group.*

Then (i) *implies* (ii), *and if* $\operatorname{char} \mathrm{k}_0$ *and* $|W|$ *are coprime* (i) *and* (ii) *are equivalent.*

The following lemma, which is due to Serre, is well known. For the sake of completeness we give a proof following Nakajima [24, Proof of Lem. 1.4]. Recall here that if a group W acts on a vector space V then it acts on the dual space $V^\vee$ by $(w.f)(x) := f(w^{-1}.x)$ for all $w \in W$, $f \in V^\vee$, and $x \in V$. We set then also

$$W_x := \{ w \in W \mid w.x = x \} \quad \text{and} \quad W_{\pm x} := \{ w \in W \mid w.x = \pm x \}$$

for $x \in V$ or $x \in V^\vee$.

Lemma 5. *Let* V *be a finite-dimensional* k_0*-vector space and* $W \subseteq \mathrm{GL}(V)$ *a finite group, such that* $\mathrm{S}(V)^W$ *is a polynomial algebra. Then* $\mathrm{S}(V)^{W_f}$ *is isomorphic to a polynomial algebra in* $n = \dim V$ *variables over* k_0, *and so* W_f *is generated by pseudo-reflections* (*in* $\mathrm{GL}(V)$) *for all* $f \in V^\vee$.

Proof. We can assume $f \neq 0$. Let $\mathrm{S}(f) : \mathrm{S}(V) \to \mathrm{k}_0$ be the morphism of k_0-algebras induced by f and $\mathfrak{m}_f$ its kernel.

We claim that

$$W_f = \{ w \in W \mid w(\mathfrak{m}_f) = \mathfrak{m}_f \}.$$

One inclusion is clear. For the other, let $w \in W \setminus W_f$. Then $w^{-1}.f \neq f$ and so there exists $x \in V$, such that $f(w.x) \neq f(x)$. It follows that $w.(x - f(x)) \notin \mathfrak{m}_f$, and hence the claim.

Therefore W_f is the decomposition group of $\mathfrak{m}_f$, and so by [4, Chap. 5, §2, no 2, Prop. 4] the extension of rings $\mathrm{S}(V)^{W_f}_{\mathfrak{m}_f \cap \mathrm{S}(V)^{W_f}} \supseteq \mathrm{S}(V)^{W}_{\mathfrak{m}_f \cap \mathrm{S}(V)^{W}}$ is unramified. Since by assumption $\mathrm{S}(V)^W$ is a polynomial ring the localization $\mathrm{S}(V)^{W}_{\mathfrak{m}_f \cap \mathrm{S}(V)^{W}}$ is a regular local ring. It follows that $\mathrm{S}(V)^{W_f}_{\mathfrak{m}_f \cap \mathrm{S}(V)^{W_f}} = \left(\mathrm{S}(V)_{\mathfrak{m}_f}\right)^{W_f}$ is a regular local ring as well.

We introduce a new grading on $\mathrm{S}(V)$ by assigning to $v - f(v)$, $0 \neq v \in V$, the degree 1. We denote $\mathrm{S}(V)$ with this grading by $S := \bigoplus_{i \geq 0} S_i$. Then $\mathfrak{m}_f = S_+ = \bigoplus_{i \geq 1} S_i$. It follows that $S^{W_f}_{(S^{W_f})_+} = \left(\mathrm{S}(V)_{\mathfrak{m}_f}\right)^{W_f}$ is a regular local ring, and so by [30, Chap. IV, App. III, Thm. 1] we get that $S^{W_f} = \mathrm{S}(V)^{W_f}$ is a polynomial ring in $n = \dim V$ variables over k_0.

The last assertion is a consequence of Theorem 4 above. $\square$

4.2. Orthogonal reflection groups

In the rest of this section we denote by k_0 a field of characteristic $\neq 2$.

Let (V, b) be a regular symmetric bilinear space of finite dimension over k_0, and $v \in V$ an anisotropic vector, i.e., $b(v, v) \neq 0$. Then

$$s_v \colon V \to V, \quad w \mapsto w - \frac{2b(v, w)}{b(v, v)} \cdot v,$$

is an element of the orthogonal group $\mathrm{O}(V,b)$, called the (*orthogonal*) *reflection* associated with v.

Note that a pseudo-reflection s in $\mathrm{O}(V,b)$ is automatically an orthogonal reflection. In fact, such an s has determinant ± 1. If $\det(s) = 1$ we have $s = \mathrm{id}_V$, see, e.g., [13, Prop. 5.7], and if $\det(s) = -1$ the number -1 is an eigenvalue and s is diagonalizable. Let w be an eigenvector for -1. Then we have $b(v,w) = b(s(v),s(w)) = -b(v,w)$ for all $v \in \ker(s - \mathrm{id}_V)$ and so $V = \ker(s - \mathrm{id}_V) \perp \mathrm{k}_0 \cdot w$. It follows that $s = s_w$.

Definition 2. Let (V,b) be a regular symmetric bilinear space of finite dimension over k_0. A finite subgroup of $\mathrm{O}(V,b)$ that is generated by orthogonal reflections is called a (*finite*) *orthogonal reflection group* over the field k_0.

4.3. The root system of an orthogonal reflection group

Given an orthogonal reflection group $W \subseteq \mathrm{O}(V,b)$ let R_W be the set of reflections in W. Recall now that

$$w \circ s_\alpha \circ w^{-1} = s_{w.\alpha}\,.$$

Hence the set R_W is the disjoint union of conjugacy classes $R_W = \bigcup_{i=1}^{m} R_i$. For every R_i we choose an anisotropic vector β_i with $s_{\beta_i} \in R_i$. Then we have $R_i = \{s_{w.\beta_i} \mid w \in W\}$ for all $1 \le i \le m$. Let

$$\Delta_i := \left\{ w.\beta_i \mid w \in W \right\}$$

for all $1 \le i \le m$, and set

$$\Delta := \bigcup_{i=1}^{m} \Delta_i\,.$$

Note that the sets Δ_i are W-invariant by definition. The set Δ is called a *root system* associated with W. It has the following properties:

(R1) if $\alpha \in \Delta$ then $\lambda \cdot \alpha \in \Delta$ for $\lambda \in \mathrm{k}_0$ if and only if $\lambda = \pm 1$, and
(R2) for all $\alpha, \beta \in \Delta$ we have $s_\alpha.\beta \in \Delta$.

(In fact, if $w.\alpha = \lambda \cdot \alpha$ then $b(\alpha,\alpha) = b(w.\alpha, w.\alpha) = b(\lambda\cdot\alpha, \lambda\cdot\alpha) = \lambda^2 b(\alpha,\alpha)$, and so $\lambda = \pm 1$, hence **(R1)**. Property **(R2)** is by construction.)

Moreover, also by construction the set $\{s_\alpha \mid \alpha \in \Delta\}$ is the set of all reflections in W, and so in particular W is generated by all s_α, $\alpha \in \Delta$.

Since b is a regular bilinear form the homomorphism $\widehat{b} : V \to V^\vee$, $v \mapsto b(v, -)$, is an isomorphism. We use this isomorphism to equip $V^\vee$ with a regular symmetric bilinear form:

$$b^\vee \colon V^\vee \times V^\vee \to \mathrm{k}_0, \quad (f,g) \mapsto b(\widehat{b}^{-1}(f), \widehat{b}^{-1}(g))\,.$$

The isomorphism $\widehat{b}$ is then a W-linear isometry $(V,b) \xrightarrow{\simeq} (V^\vee, b^\vee)$, which induces a W-linear isomorphism of k_0-algebras $\mathrm{S}(\widehat{b}) : \mathrm{S}(V) \xrightarrow{\simeq} \mathrm{S}(V^\vee)$. Hence $\mathrm{S}(V)^W$ is a polynomial algebra over k_0 if and only if $\mathrm{S}(V^\vee)^W$ is so.

For $\alpha \in \Delta$ we set

$$\alpha^\vee(v) := \frac{2b(\alpha,v)}{b(\alpha,\alpha)}$$

for all $v \in V$. With this notation we have $s_\alpha(x) = x - \alpha^\vee(x) \cdot \alpha$.

For later use we record the following:

$$\alpha^\vee(\beta) = 0 \ \Leftrightarrow \ \beta^\vee(\alpha) = 0 \ \Leftrightarrow \ s_\alpha \circ s_\beta = s_\beta \circ s_\alpha \tag{7}$$

for all $\alpha, \beta \in \Delta$.

Lemma 6. *Let* k_0 *be a field,* (V, b) *a finite-dimensional regular bilinear space over* k_0*, and* W *a finite subgroup of the orthogonal group* $\mathrm{O}(V, b)$*. Assume that* $\mathrm{S}(V)^W$ *is a polynomial ring over* k_0*, and so* W *is in particular an orthogonal reflection group by Theorem* 4*. Let* Δ *be a root system of* W*. Then we have:*

(i) $W_\alpha = W_{\alpha^\vee}$ *and* $W_{\pm\alpha} = W_{\pm\alpha^\vee}$ *for all* $\alpha \in \Delta$*.*

(ii) $W_{\pm\alpha} = \langle s_\alpha \rangle . W_\alpha \simeq \mathbb{Z}/2 \times W_\alpha$*, where* $\langle s_\alpha \rangle = \{\mathrm{id}_V, s_\alpha\}$ *is the subgroup generated by* s_α*, for all* $\alpha \in \Delta$*.*

(iii) $\mathrm{S}(V)^{W_\alpha}$ *and* $\mathrm{S}(V)^{W_{\pm\alpha}}$ *are polynomial algebras in* $n = \dim V$ *variables over* k_0*, and so the groups* W_α *and* $W_{\pm\alpha}$ *are generated by orthogonal reflections in* $\mathrm{O}(V, b)$*.*

(iv)
$$\bigcup_{\alpha \in \Delta} \mathrm{Ker}(\alpha^\vee) = \bigcup_{\mathrm{id}_V \neq w \in W} \mathrm{Ker}(w - \mathrm{id}_V)\,.$$

Proof. (i) This follows since b is a regular bilinear form.

(ii) If $w.\alpha = -\alpha$ then $s_\alpha \circ w \in W_\alpha$ and so $W_{\pm\alpha}$ is equal to the semidirect product $\langle s_\alpha \rangle \ltimes W_\alpha$.

To show that this is a direct product and so $W_{\pm\alpha} \simeq \mathbb{Z}/2 \times W_\alpha$, we have to show that s_α commutes with all elements of W_α. This can be seen as follows. By assumption, $\mathrm{S}(V)$ is a polynomial algebra over k_0, and so by Lemma 5 the subgroup $W_{\alpha^\vee}$, which coincides with W_α by (i), is generated by reflections. Hence it is enough to show $s_\beta \circ s_\alpha = s_\alpha \circ s_\beta$ for all $s_\beta \in W_\alpha$. But s_β is in W_α if and only if $\beta^\vee(\alpha) = 0$, and this implies that s_α and s_β commute with each other by (7).

(iii) Since $W_\alpha = W_{\alpha^\vee}$ and $\mathrm{S}(V)^W$ is a polynomial ring over k_0 by assumption, we get from Lemma 5 that the algebra $\mathrm{S}(V)^{W_\alpha}$ is a plynomial algebra over k_0.

Let now $H := \mathrm{Ker}\,\alpha^\vee$. We have then $V = \mathrm{k}_0 \cdot \alpha \oplus H$, and so

$$\mathrm{S}(V)^{W_\alpha} \simeq \mathrm{S}(\mathrm{k}_0 \cdot \alpha) \otimes_{\mathrm{k}_0} \mathrm{S}(H)^{W_\alpha|_H} \simeq \mathrm{k}_0[t] \otimes_{\mathrm{k}_0} \mathrm{S}(H)^{W_\alpha|_H}\,.$$

Using now the decomposition $W_{\pm\alpha} \simeq \mathbb{Z}/2 \times W_\alpha$ from (ii) we get that $\mathrm{S}(V)^{W_{\pm\alpha}}$ is isomorphic as k_0-algebra to

$$\mathrm{S}(\mathrm{k}_0 \cdot \alpha)^{\mathbb{Z}/2} \otimes_{\mathrm{k}_0} \mathrm{S}(H)^{W_\alpha|_H} \simeq \mathrm{k}_0[y^2] \otimes_{\mathrm{k}_0} \mathrm{S}(H)^{W_\alpha|_H} \simeq \mathrm{k}_0[t] \otimes_{\mathrm{k}_0} \mathrm{S}(H)^{W_\alpha|_H}\,.$$

Consequently, $\mathrm{S}(V)^{W_{\pm\alpha}}$ is a polynomial algebra over k_0 as well.

The last assertion of (iii) is a consequence of Theorem 4.

Finally we prove (iv). Since $\mathrm{Ker}\,\alpha^\vee = \mathrm{Ker}(s_\alpha - \mathrm{id}_V)$ the left-hand side is contained in the right-hand side. For the other inclusion let $x \in V \setminus \bigcup_{\alpha \in \Delta} \mathrm{Ker}(\alpha^\vee)$, and assume that $W_x \neq \{\mathrm{id}_V\}$. Since b is non-degenerate we have $W_x = W_{b(x,\,-\,)}$, and so by Lemma 5 there exists a reflection s_α in W_x. We get the contradiction $\alpha^\vee(x) = 0$. $\square$

We can state and prove now our main theorem.

Theorem 7. *Let (V,b) be a finite-dimensional regular symmetric bilinear space over the field k_0 and $W \subseteq \mathrm{O}(V,b)$ a finite subgroup, such that $\mathrm{S}(V)^W$ is a polynomial ring over k_0 (in particular, W is an orthogonal reflection group). Further, let M_* be a cycle module over k_0. Then a cohomological invariant*

$$a\colon\ \mathrm{H}^1(\,-,W) \to \mathrm{M}_n(\,-\,)$$

over k_0 is zero if and only if its restrictions to all abelian subgroups generated by reflections are zero.

For the proof we have to describe a versal W-torsor over k_0. For this, let Δ be a root system of W.

4.4. A versal torsor for *W*

Define $U \subset \mathbb{A}(V) = \operatorname{Spec}\mathrm{S}(V^\vee)$ as in Example 2, i.e., U is the open complement of the union of closet sets $\mathcal{V}_w$, $w \in W \setminus \{\mathrm{id}_V\}$, where $\mathcal{V}_w$ is the closed set defined by the ideal generated by all $f\circ(\mathrm{id}_V - w)$, $f \in V^\vee$. The group W acts freely on U, and the generic fiber of the quotient morphism $q : U \to U/W$ is a versal W-torsor over k_0, see Example 2.

By Lemma 6 (iv) we have for all field extensions $L \supseteq \mathrm{k}_0$ that

$$\bigcup_{\alpha\in\Delta} \operatorname{Ker}(\mathrm{id}_L \otimes \alpha^\vee) = \bigcup_{\mathrm{id}_V \neq w \in W} \operatorname{Ker}\big((\mathrm{id}_L \otimes w) - \mathrm{id}_{L\otimes_{\mathrm{k}_0} V}\big)\,. \tag{8}$$

We get $U = \operatorname{Spec}\big(\mathrm{S}(V^\vee)[g_\Delta^{-1}]\big)$, where we have set

$$g_\Delta := \prod_{\alpha\in\Delta} \alpha^\vee \ \in \mathrm{S}(V^\vee)\,.$$

Hence the quotient morphism $q : U \to U/W$ corresponds to the embedding of rings

$$\big(\mathrm{S}(V^\vee)\big)^W[g_\Delta^{-1}] \to \mathrm{S}(V^\vee)[g_\Delta^{-1}]\,.$$

The generic fiber of q is equal to the Galois extension $\operatorname{Spec}E \to \operatorname{Spec}E^W$ with Galois group W, where E denotes the fraction field of $\mathrm{S}(V^\vee)$. We set in the following $K := E^W$, and denote by $[E/K] \in \mathrm{H}^1(K,W)$ the class of the W-Galois algebra $E \supset K$, which is a versal W-torsor over k_0.

4.5. An unramified extension

Let Q be a prime ideal of height one in $\mathrm{S}(V^\vee)^W$ that is not in the open subscheme $U/W = \operatorname{Spec}\big(\mathrm{S}(V^\vee)^W[g_\Delta^{-1}]\big)$, i.e., $g_\Delta \in Q$. Since by our assumption $\mathrm{S}(V)^W \simeq \mathrm{S}(V^\vee)^W$ is a polynomial ring over k_0 the local ring $R := \big(\mathrm{S}(V^\vee)^W\big)_Q$ at Q is a discrete valuation ring. Let P be a prime ideal in $\mathrm{S}(V^\vee)$ above Q. Then there exists $\alpha \in \Delta$, such that $P = P_\alpha := \mathrm{S}(V^\vee)\cdot\alpha^\vee$. The Galois group W of $E \supset K$ acts transitively on the prime ideals above Q.

Since $w.\alpha = \pm\alpha$ is equivalent to $w.\alpha^\vee = \pm\alpha^\vee$ it is a consequence of **(R1)** that the decomposition group $W_{P_\alpha} = \{w \in W | w(P_\alpha) = P_\alpha\}$ is equal to

$$W_{\pm\alpha} = \big\{w \in W \,|\, w.\alpha = \pm\alpha\,\big\},$$

which in turn by Lemma 6 (ii) is equal to $\langle s_\alpha \rangle . W_\alpha \simeq \mathbb{Z}/2 \times W_\alpha$.

Denote by F_α the fixed field of $W_{\pm\alpha}$ in E, by ι_α the embedding $K \subseteq F_\alpha$, and by $\widetilde{S}$ the integral closure of R in F_α. We set $S_\alpha := \widetilde{S}_{\widetilde{S}\cap P_\alpha}$. This is a discrete valuation ring with maximal ideal $Q_\alpha := (\widetilde{S} \cap P_\alpha) \cdot \widetilde{S}_{\widetilde{S}\cap P_\alpha}$. By construction the extension of discrete valuation rings $S_\alpha \supseteq R$ is unramified, and the residue field $\mathrm{k}_0(Q_\alpha)$ of S_α is equal to the one of R, which we denote $\mathrm{k}_0(Q)$. Hence by the cycle module Axiom **(R3a)**, see Section 2.3, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{M}_*(F_\alpha) & \xrightarrow{\partial_{Q_\alpha}} & \mathrm{M}_{*-1}(\mathrm{k}_0(Q)) \\ {\scriptstyle (\iota_\alpha)_{\mathrm{M}}}\uparrow & & \uparrow{\scriptstyle =} \\ \mathrm{M}_*(K) & \xrightarrow{\partial_Q} & \mathrm{M}_{*-1}(\mathrm{k}_0(Q)) \end{array} \tag{9}$$

for all cycle modules M_* over k_0.

4.6. Proof of Theorem 7

The proof is by induction on $m = |W|$. If $m \leq 5$ there is nothing to prove, so let $m \geq 6$. Using the induction hypothesis we show first the following.

Claim. $a_K([E/K]) \in \mathrm{A}^0\big(\mathbb{A}(V)/W, \mathrm{M}_n\big)$.

To prove the claim we have to show

$$\partial_Q\big(a_K([E/K])\big) = 0 \tag{10}$$

for all prime ideals Q of height one in $\mathrm{S}(V^\vee)^W$. This is clear by Theorem 1 if Q is in the open subset $U/W \subset \mathbb{A}(V)/W = \mathrm{Spec}\big(\mathrm{S}(V^\vee)^W\big)$ since $[E/K]$ is by construction the generic fiber of $U \to U/W$.

So assume $Q \notin U/W$. Then Q contains g_Δ. Let Q_α and $\iota_\alpha : K \subseteq F_\alpha$ be as in Section 4.5. By Diagram (9) above it is enough to show

$$\partial_{Q_\alpha}\big((\iota_\alpha)_{\mathrm{M}}(a_K([E/K]))\big) = 0\,. \tag{11}$$

Since a is an invariant we have $(\iota_\alpha)_{\mathrm{M}}(a_K([E/K])) = a_{F_\alpha}(r_{\iota_\alpha}([E/K]))$, where r_{ι_α} denotes the pull-back $\mathrm{H}^1(K, W) \to \mathrm{H}^1(F_\alpha, W)$, see Section 2.5. Hence equation (11) is equivalent to

$$\partial_{Q_\alpha}\big(a_{F_\alpha}(r_{\iota_\alpha}([E/K]))\big) = 0\,. \tag{12}$$

Now we distinguish two cases:

(a) $F_\alpha = K$: Then $W = \langle s_\alpha \rangle . W_\alpha \simeq \mathbb{Z}/2 \times W_\alpha$ (in the notation of Section 4.5), and therefore we have

$$\mathrm{H}^1(-, W) \simeq \mathrm{H}^1(-, \mathbb{Z}/2) \times \mathrm{H}^1(-, W_\alpha)\,.$$

We claim that $a_\ell(x, y) = 0$ for all $(x, y) \in \mathrm{H}^1(\ell, \mathbb{Z}/2) \times \mathrm{H}^1(\ell, W_\alpha)$ and all $\ell \in \mathfrak{F}_{\mathrm{k}_0}$. This implies $a = 0$, and so $\partial_Q(a_K(T)) = 0$.

For this, let $\ell \in \mathfrak{F}_{\mathrm{k}_0}$, and $x \in \mathrm{H}^1(\ell, \mathbb{Z}/2)$, and consider $\mathfrak{F}_\ell$ as a full subcategory of $\mathfrak{F}_{\mathrm{k}_0}$, cf. Section 2.1. The maps

$$b^x_L \colon \mathrm{H}^1(L, W_\alpha) \to \mathrm{M}_n(L), \quad z \mapsto a_L(r_j(x), z),$$

where $j : \ell \to L$ is the structure map in $\mathfrak{F}_\ell$, define an invariant of degree n of W_α over ℓ with values in M_n, i.e., we have $b^x \in \mathrm{Inv}^n_\ell(W_\alpha, \mathrm{M}_*)$.

Let $H \subseteq W_\alpha$ be an abelian subgroup generated by reflections. Then the subgroup $H' := \langle s_\alpha \rangle.H$ of W is an abelian subgroup generated by reflections as well, and therefore by assumption the restriction $\mathrm{Res}^{H'}_W(a)$ is trivial. Now for $L \in \mathfrak{F}_\ell$ with structure map $j : \ell \to L$ and $z \in \mathrm{H}^1(L, H)$ we have

$$\mathrm{Res}^H_{W_\alpha}(b^x)_L(z) = \mathrm{Res}^{H'}_W(a)_L(r_j(x), z) = 0\,.$$

As $z \in \mathrm{H}^1(L, H)$ and $L \in \mathfrak{F}_\ell$ were arbitrary this implies $\mathrm{Res}^H_{W_\alpha}(b^x)$ is trivial. This holds for all abelian subgroups H of W_α generated by reflections and therefore since $\mathrm{S}(V)^{W_\alpha}$ is a polynomial algebra by Lemma 6(iii) above we can apply the induction assumption to conclude $b^x = 0$. In particular, we have $0 = b^x_\ell(y) = a_\ell(x, y)$ as claimed. We are done in the case $F_\alpha = K$.

(b) $F_\alpha \neq K$: Then $W \neq \langle s_\alpha \rangle.W_\alpha = W_{\pm\alpha}$. By Example 1 we have

$$[E/K]_{F_\alpha} = r_{\iota_\alpha}([E/K]) = \theta^1(T') \tag{13}$$

for some $T' \in \mathrm{H}^1(F_\alpha, W_{\pm\alpha})$, where $\theta : W_{\pm\alpha} \hookrightarrow W$ is the inclusion.

Let H be an abelian subgroup of $W_{\pm\alpha}$ generated by reflections. Then

$$\mathrm{Res}^H_{W_{\pm\alpha}}\big(\mathrm{Res}^{W_{\pm\alpha}}_W(a)\big) = \mathrm{Res}^H_W(a),$$

and since H is also an abelian subgroup of W generated by reflections we have $\mathrm{Res}^H_W(a) = 0$ by our assumption. It follows that

$$\mathrm{Res}^H_{W_{\pm\alpha}}\big(\mathrm{Res}^{W_{\pm\alpha}}_W(a)\big) = 0$$

for all abelian subgroups H of $W_{\pm\alpha}$ that are generated by reflections.

Since $\mathrm{S}(V)^{W_{\pm\alpha}}$ is a polynomial algebra over k_0, see Lemma 6 (iii), we conclude by induction that

$$\mathrm{Res}^{W_{\pm\alpha}}_W(a) \;=\; 0\,. \tag{14}$$

Using this we compute

$$\begin{aligned}
\partial_{Q_\alpha}\big(a_{F_\alpha}\big(r_{\iota_\alpha}([E/K])\big)\big) &= \partial_{Q_\alpha}\big(a_{F_\alpha}([E/K]_{F_\alpha})\big) & \\
&= \partial_{Q_\alpha}\big(a_{F_\alpha}(\theta^1(T'))\big) & \text{by (13)}\\
&= \partial_{Q_\alpha}\big(\mathrm{Res}^{W_{\pm\alpha}}_W(a)_{F_\alpha}(T')\big) & \text{by definition of } \mathrm{Res}^{W_{\pm\alpha}}_W\\
&= 0 & \text{by (14).}
\end{aligned}$$

Hence (12) holds also if $K \neq F_\alpha$ and we therefore have $\partial_Q(a_K([E/K])) = 0$ as claimed.

We have proven the claim, and can now finish the proof of the theorem. By assumption $\mathrm{S}(V)^W$ is a polynomial ring over k_0. We have $\mathrm{S}(V^\vee)^W \simeq \mathrm{S}(V)^W$, see Section 4.3, and so the k_0-scheme $\mathbb{A}(V)/W = \operatorname{Spec}\mathrm{S}(V^\vee)^W$ is an affine space over k_0. Therefore by homotopy invariance of the cohomology of cycle modules, see Rost [27, Prop. 8.6], we have $\mathrm{A}^0\left(\mathbb{A}(V)/W, \mathrm{M}_n\right) \simeq \mathrm{M}_n(\mathrm{k}_0)$. Hence the detection principle, Theorem 3, implies that the invariant a is constant. However the restriction of a to an abelian subgroup generated by reflections is zero, and so a has to be zero. □

In the corollary below we understand by a 'maximal abelian subgroup generated by reflections' a subgroup which is maximal with respect to the two properties (a) abelian and (b) generated by reflections.

Corollary 8. *Let* k_0 *and* W *be as in Theorem 7 and* M_* *a cycle module over* k_0*. Let further* $G_1, \ldots, G_r$ *be different maximal abelian subgroups generated by reflections, which represent all such subgroups up to conjugation, i.e., if* G *is a maximal abelian subgroup of* W *generated by reflections then* $G = wG_iw^{-1}$ *for some* $1 \leq i \leq r$ *and some* $w \in W$*.*

Then the product of restriction morphisms

$$\left(\operatorname{Res}_W^{G_i}\right)_{i=1}^r \colon \operatorname{Inv}_{\mathrm{k}_0}^n(W, \mathrm{M}_*) \to \bigoplus_{i=1}^r \operatorname{Inv}_{\mathrm{k}_0}^n(G_i, \mathrm{M}_*)^{N_W(G_i)}$$

is injective for all $n \in \mathbb{Z}$*. (Recall from Example 4 that the image of* $\operatorname{Res}_W^{G_i}$ *is contained in the subgroup* $\operatorname{Inv}_{\mathrm{k}_0}^n(G_i, \mathrm{M}_*)^{N_W(G_i)}$ *for all* $1 \leq i \leq r$*.)*

Proof. Let $a \in \operatorname{Inv}_{\mathrm{k}_0}^n(W, \mathrm{M}_*)$ be a non-trivial invariant. Then by Theorem 7 there exists an abelian subgroup H of W generated by reflections, such that $\operatorname{Res}_W^H(a) \neq 0$. Let G be a maximal abelian subgroup generated by reflections, which contains H. Then there exists $1 \leq i_0 \leq r$ and $w_0 \in W$, such that $w_0 G w_0^{-1} = G_{i_0}$. Let $H' \subseteq G_{i_0}$ be the image of H under the inner automorphism $\iota_{w_0} : g \mapsto w_0 \cdot g \cdot w_0^{-1}$ of W.

Since the morphism $\iota_{w_0}^* : \operatorname{Inv}_{\mathrm{k}_0}^n(W, \mathrm{M}_*) \to \operatorname{Inv}_{\mathrm{k}_0}^n(W, \mathrm{M}_*)$ is the identity by [31, Prop. 13.1] we get

$$\iota_{w_0}^*\left(\operatorname{Res}_W^{H'}(a)\right) = \operatorname{Res}_W^H\left(\iota_{w_0}^*(a)\right) = \operatorname{Res}_W^H(a) \neq 0,$$

and so $0 \neq \operatorname{Res}_W^{H'}(a) = \operatorname{Res}_{G_{i_0}}^{H'}\left(\operatorname{Res}_W^{G_{i_0}}(a)\right)$. It follows $\operatorname{Res}_W^{G_{i_0}}(a) \neq 0$. □

4.7. Weyl groups

The arguments above also apply to invariants of Weyl groups as long as the characteristic of the base field is not one of the torsion primes. We briefly indicate the details, recalling first the definition of Weyl groups.

A *reduced root system* in the sense of Bourbaki [7, Chap. VI], see also [9, Chap. XXI], consists of the following data: A free and finitely generated abelian group M, a finite subset $\Delta \subset M$, and a map $\rho : \Delta \to M^\vee := \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, $\alpha \mapsto \alpha^\vee$. The triple (M, ρ, Δ) is subject to the following axioms:

- **(R1)** $\lambda\alpha \in \Delta$ if and only if $\lambda = \pm 1$, and
- **(R2)** $s_\alpha(\beta) \in \Delta$ and $s_{\alpha^\vee}(\beta^\vee) \in \Delta^\vee := \rho(\Delta)$, where $s_\alpha(x) := x - \alpha^\vee(x) \cdot \alpha$ and $s_{\alpha^\vee}(f) = f - f(\alpha) \cdot \alpha^\vee$ for all $x \in M$, $f \in M^\vee$, and $\alpha \in \Delta$.

We assume in the following that the root system is *semisimple*, i.e., Δ generates the $\mathbb{Q}$-vector space $\mathbb{Q}\otimes_{\mathbb{Z}} M$.

The map ρ is injective and the triple $(M^\vee, \rho^{-1}, \Delta^\vee)$ is also a reduced root system, the *inverse root system* of (M,ρ,Δ). For ease of notation if M and ρ are clear from the context we call Δ a root system and $\Delta^\vee$ its inverse.

Let $\mathrm{GL}(M)$ be the group of all automorphisms of the abelian group M. The subgroup $W = W(\Delta)$ of $\mathrm{GL}(M)$ generated by all s_α, $\alpha\in\Delta$, is finite and called the *Weyl group* associated with the root system Δ.

Remark 3. A finite subset $\Delta\subset M$ satisfying only **(R2)** is called *(non-reduced) root system.* If Δ is irreducible it contains a reduced root system Δ_0, such that the associated Weyl group $W(\Delta_0)$ is equal the subgroup of $\mathrm{GL}(M)$ generated by all s_α, $\alpha\in\Delta$, see [7, Chap. VI, §1, no 4, Prop. 13], i.e., every Weyl group associated with a root system is equal to a Weyl group associated with a reduced root system.

The group W acts on the dual $M^\vee$ via $(w.f)(m) = f(w^{-1}.m)$. We have then $s_\alpha.f = s_{\alpha^\vee}(f)$ for all $\alpha\in\Delta$ and $f\in V^\vee$, and so $s_\alpha\mapsto s_{\alpha^\vee}$ induces an isomorphism of W onto the Weyl group $W^\vee := W(\Delta^\vee)\subset \mathrm{GL}(M^\vee)$ of the inverse root system.

Set $V := \mathrm{k}_0\otimes_{\mathbb{Z}} M$. If $\operatorname{char}\mathrm{k}_0 = 0$ then there is a regular symmetric bilinear form b on V, which is W-invariant, see, e.g., [7, Chap. VI, §1, no. 1, Prop. 3], and so $W\subseteq \mathrm{O}(V,b)$ is an orthogonal reflection group. Moreover by the Shephard–Todd–Bourbaki Theorem 4 we know that $\mathrm{S}(V)^W$ is a polynomial ring over k_0 and so the splitting principle Theorem 7 holds for W over fields of characteristic 0.

Using the following result of Demazure [8, Cor. of Thm. 2, Thm. 3, and Prop. 8] we can prove a more general splitting principle for Weyl groups.

Theorem 9 (Demazure)**.** *Let* k_0 *be a field* (*of characteristic* $\neq 2$) *and* (M,ρ,Δ) *a reduced semisimple root system with associated Weyl group* $W = W(\Delta)$*. Assume that* $\operatorname{char}\mathrm{k}_0\neq 3$ *if* Δ *has components of type* E_6, E_7, *or* F_4, *and* $\operatorname{char}\mathrm{k}_0\neq 3,5$ *if it has components of type* E_8*. Then the* k_0*-algebras* $\mathrm{S}(V)^W$ *and* $\mathrm{S}(V^\vee)^W$ *are polynomial* (*over* k_0)*.*

(For the claim that $\mathrm{S}(V^\vee)^W$ is a polynomial ring over k_0 note that $\mathrm{S}(V^\vee)^W = \mathrm{S}(V^\vee)^{W^\vee}$ by the isomorphism $W \xrightarrow{\simeq} W^\vee$, $s_\alpha\mapsto s_{\alpha^\vee}$.)

4.8. A splitting principle for Weyl groups

Let k_0, (M,ρ,Δ), W, and $V = \mathrm{k}_0\otimes_{\mathbb{Z}} M$ be as in the theorem of Demazure above. It follows then from Lemma 5 that $W_{\alpha^\vee}$ is generated by reflections and that $\mathrm{S}(V)^{W_{\alpha^\vee}}$ is a polynomial ring over k_0. We have $(w.\alpha)^\vee = w.\alpha^\vee$ (since $w\cdot s_\alpha\cdot w^{-1} = s_{w.\alpha}$), and so $W_\alpha = W_{\alpha^\vee}$ and also $W_{\pm\alpha} = W_{\pm\alpha^\vee}$. We conclude now as in the proof of Lemma 6(ii) and (iii) that $W_{\pm\alpha} = \langle s_\alpha\rangle.W_\alpha \simeq \mathbb{Z}/2\times W_\alpha$, and that $\mathrm{S}(V)^{\mathrm{W}_{\pm\alpha}}$ is a polynomial ring over k_0. Also the analog of Lemma 6(iv) holds, i.e., $\bigcup\limits_{\alpha\in\Delta}\operatorname{Ker}\alpha^\vee = \bigcup\limits_{\mathrm{id}_V\neq w\in W}\operatorname{Ker}(w-\mathrm{id}_V)$, with the same proof. One has only to observe that identifying $V\simeq V^{\vee\vee}$, it follows from Lemma 5 that W_x is generated by reflections for all $x\in V$ since $\mathrm{S}(V^\vee)^W$ is a polynomial ring over k_0 by Demazure's theorem above. Hence the analog of Lemma 6 holds in this situation.

Finally, since the s_α's are all reflections in W, see [7, Chap. VI, §1, no 1, Rem. 3)], the groups W_α and $W_{\pm\alpha}$, which are generated by reflections, correspond to Weyl groups of sub-root systems of Δ. We can copy now word by word the reasoning in Section 4.6 to get (taking Demazure's Theorem 9 above into account) the following result.

Theorem 10. *Let* k_0 *be a field* (*of characteristic not* 2), *and* W *a Weyl group associated with a reduced root system* Δ. *Assume that* $\operatorname{char}\mathrm{k}_0 \neq 3$ *if* Δ *has components of type* E_6, E_7, *or* F_4, *and that* $\operatorname{char}\mathrm{k}_0 \neq 3, 5$ *if it has components of type* E_8. *Further, let* M_* *be a cycle module over* k_0. *Then a cohomological invariant*

$$a\colon\ \mathrm{H}^1(\,-, W) \to \mathrm{M}_n(\,-\,)$$

over k_0 *is zero if and only if its restrictions to all abelian subgroups generated by reflections are zero.*

Remark 4. For W a symmetric group and $M_* = \mathrm{H}^*(\,-, C)$, where C is a finite Γ_{k_0}-module of order prime to the characteristic of the base field, the splitting priniciple holds in broader generality. Besides of being of characteristic $\neq 2$ the field k_0 can be arbitrary. This has been shown by Serre [31, Thm. 24.9] for the symmetric group, and in his letter [32] he informed us that he has a proof for the general case:

Let k_0 *be a field of characteristic* $\neq 2$ *and* W *a Weyl group. Further, let* C *be a finite* Γ_{k_0}*-module, whose order is prime to the characteristic of the base field* k_0. *Then a cohomological invariant*

$$a\colon\ \mathrm{H}^1(\,-, G) \to \mathrm{H}^*(\,-, C)$$

is zero if and only if its restrictions to all abelian subgroups generated by reflections are zero.

Note that the Galois cohomology groups $\mathrm{H}^*(\,-, C)$ with appropriate twisting by roots of unity (on C) are also a cycle module in the sense of Rost as long as the order of C is not divisible by the characteristic of the base field, see [27, Rem. 1.11].

5. On the splitting principle for Witt invariants

5.1. Witt groups and the fundamental ideal

Throughout this section we assume that all fields are of characteristic $\neq 2$.

Given a field F we denote by $\mathrm{W}(F)$ its Witt group (of bilinear forms), and by $\mathrm{I}^n(F)$, $n \in \mathbb{N}$, the nth power of the fundamental ideal $\mathrm{I}(F) \subset \mathrm{W}(F)$ of forms of even rank. We set $\mathrm{I}^n(F) := \mathrm{W}(F)$ for all integers $n \leq 0$, and $\mathrm{I}^*(F) := \bigoplus\limits_{n\geq 0} \mathrm{I}^n(F)$.

Given a valuation v on a field F with residue field $F(v)$, which is also assumed to be of characteristic $\neq 2$, and a uniformizer π there exists a (so-called) *second residue map*

$$\partial_{v,\pi}\colon\ \mathrm{W}(F) \to \mathrm{W}(F(v)),$$

which depends on the choice of the uniformizer π, see, e.g., [22, Chap. IV, §1]. Note that $\partial_{v,\pi}\big(\mathrm{I}^n(F)\big) \subseteq \mathrm{I}^{n-1}(F(v))$ for $n \geq 0$ by Arason [1, Satz 3.1].

The kernel of $\partial_{v,\pi}$ does not depend on π but only on the valuation v.

5.2. Unramified Witt groups

Let be X be a scheme over a field k_0, which is regular in codimension one. The *nth unramified Witt group* of X is then defined as

$$\mathrm{A}^0(X,\mathrm{I}^n) := \mathrm{Ker}\,\Big(\mathrm{I}^n(\mathrm{k}_0(X)) \xrightarrow{(\partial_{v_x,\pi_x})_{x\in X^{(1)}}} \bigoplus_{x\in X^{(1)}} \mathrm{I}^{n-1}(\mathrm{k}_0(x))\Big),$$

for all $n \geq 0$, where v_x denotes the valuation of the function field $\mathrm{k}_0(X)$ associated with the regular codimension one point x, $\mathrm{k}_0(x)$ denotes the residue field of x, and $\pi_x \in \mathcal{O}_{X,x}$ is a uniformizer.

Definition 3. Let G be a linear algebraic group over the base field k_0 and $n \geq 0$ an integer. A *Witt invariant of degree* n is a natural transformation

$$a\colon\ \mathrm{H}^1(\,-,G) \to \mathrm{I}^n(\,-\,),$$

where we consider $\mathrm{H}^1(\,-,G)$ and $\mathrm{I}^n(\,-\,)$ as functors on $\mathrm{Fields}_{\mathrm{k}_0}$, the category of all field extensions of k_0, with values in the category of pointed sets, respectively of abelian groups.

We denote by $\mathrm{Inv}^n_{\mathrm{k}_0}(G,\mathrm{I}^*)$ the set of all Witt invariants of G of degree n over the field k_0. The addition in the Witt ring induces a group structure on this set. We further set

$$\mathrm{Inv}_{\mathrm{k}_0}(G,\mathrm{I}^*) := \bigoplus_{n\geq 0} \mathrm{Inv}^n_{\mathrm{k}_0}(G,\mathrm{I}^*)\,.$$

If $H \subseteq G$ is a closed subgroup of the algebraic k_0-group G and $a \in \mathrm{Inv}^n_{\mathrm{k}_0}(G,\mathrm{I}^*)$ then the composition of natural transformations

$$\mathrm{H}^1(\,-,H) \to \mathrm{H}^1(\,-,G) \xrightarrow{a} \mathrm{I}^n(\,-\,)$$

is a Witt invariant of degree n of H, which we call the *restriction* of the invariant a to H, and denote it by $\mathrm{res}^H_G(a)$.

5.3. The splitting principle

For the proof of the splitting principle for Witt invariants of orthogonal reflection groups one needs the analogs of Theorem 1(i) and of the detection Theorem 3. These can be proven as for cycle modules in Section 3 above, or one can follow the arguments used in Serre's lectures, see [31, Sect. 27]. Part (ii), the detection principle, is also a special case of a general theorem [19, Thm. 3.3] of Merkurjev. Note however that Merkurjev's result does not apply to cycle modules as its proof uses completions.

Theorem 11. *Let* k_0 *be a field and* G *a linear algebraic group over* k_0*.*

(i) *Let* X *be an integral scheme, which is essentially of finite type over the base field* k_0*, with function field* $K = \mathrm{k}_0(X)$*, and* $T \in \mathrm{H}^1_{\mathrm{et}}(X, G)$*. Then for a regular codimension one point* x *of* X *with uniformizer* π_x *we have*

$$\partial_{v_x,\pi_x}\big(a_K(T_K)\big) = 0$$

for all $a \in \mathrm{Inv}_{\mathrm{k}_0}(G, \mathrm{I}^*)$*.*

(ii) *Let* K *be a finitely generated field extension of* k_0 *and* $T \in \mathrm{H}^1(K, G)$ *a versal* G*-torsor. Then* $a_K(T) = b_K(T)$ *implies that the invariants* a *and* b *are equal for all* $a, b \in \mathrm{Inv}_{\mathrm{k}_0}(G, \mathrm{I}^*)$*.*

With these results at our disposal we can now follow essentially word by word the arguments in Section 4.6 to prove the splitting principle for Witt invariants of orthogonal reflection groups. We have only to observe that the analog of the diagram (9) in Section 4.5 commutes since a uniformizer for R is also one for S_α as the extension $S_\alpha \supseteq R$ is unramified, and that by Fasel [11, Thm. 11.2.9] we have $\mathrm{A}^0(\mathbb{A}(V)/W, \mathrm{I}^n) \simeq \mathrm{I}^n(\mathrm{k}_0)$, since by assumption $\mathrm{S}(V^\vee)^W$ is a polynomial algebra over the base field k_0.

The same arguments apply also to the Witt invariants of Weyl groups as long as $\mathrm{char}\,\mathrm{k}_0$ does not divide the torsion primes.

Hence we have:

Theorem 12. *Let* k_0 *and* W *be as in Theorem* 7*, or as in Theorem* 10*. Then* $a \in \mathrm{Inv}_{\mathrm{k}_0}(W, \mathrm{I}^*)$ *is zero if and only if its restrictions to all abelian subgroups generated by reflections are zero.*

References

[1] J. Arason, *Kohomologische Invarianten quadratischer Formen*, J. Algebra **36** (1975), 448–491.

[2] M. Artin, A. Grothendieck, J. Verdier, *Théorie des Topos et Cohomologie Étale des Schémas.* Tome 2, Séminaire de Géométrie Algébrique du Bois Marie 1963/64 (SGA 4). Dirigé par M. Artin, A. Grothendieck, J. Verdier, Lecture Notes in Mathematics, Vol. 270, Springer-Verlag, Berlin, 1972.

[3] S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron Models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Vol. 21, Springer-Verlag, Berlin, 1990.

[4] N. Bourbaki, *Éléments de Mathématique. Fasc. XXX. Algèbre Commutative. Chapitre* 5*: Entiers. Chapitre* 6*: Valuations*, Actualités Scientifiques et Industrielles, No. 1308, Hermann, Paris, 1964.

[5] N. Bourbaki, *Éléments de Mathématique. Algèbre*, Chapitres 4 à 7, Masson, Paris, 1981.

[6] N. Bourbaki, *Éléments de Mathématique. Algèbre Commutative. Chapitre* 8*. Dimension. Chapitre* 9*. Anneaux locaux noethériens complets*, Masson, Paris, 1983.

[7] N. Bourbaki, *Éléments de Mathématique. Groupes et Algèbres de Lie*, Chapitres 4, 5 et 6, Masson, Paris, 1981.

[8] M. Demazure, *Invariants symétrique entiers des groupes de Weyl et torsion*, Invent. Math. **21** (1973), 287–301.

[9] M. Demazure, A. Grothendieck, *Schémas en Groupes.* III*: Structure des Schémas en Groupes éductifs*, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 153, Springer-Verlag, Berlin, 1970.

[10] J. Ducoat, *Cohomological invariants of finite Coxeter groups*, `arXiv:1112.6283` (2011).

[11] J. Fasel, *Groupes de Chow–Witt*, Mém. Soc. Math. Fr. (N.S.) **113** (2008).

[12] A. Grothendieck, J. Dieudonneé, *Éléments de Géométrie Algébrique IV. Étude Locale des Schémas et des Morphismes de Schémas* (*Quatrième Partie*), Publ. Math. Inst. Hautes Études Sci. **32** (1967), 5–343.

[13] L. Grove, *Classical Groups and Geometric Algebra*, Graduate Studies in Mathematics, Vol. 39, American Mathematical Society, Providence, RI, 2002.

[14] P. Guillot, *Geometric methods for cohomological invariants*, Doc. Math. **12** (2007), 521–545.

[15] C. Hirsch, *Cohomological Invariants of Reflection Groups*, Diplomarbeit, LMU Munich, 2010.

[16] C. Hirsch, *On the decomposability of cohomological invariants mod* 2 *of Weyl groups*, Comment. Math, Helv. (to appear).

[17] R. Kane, *Reflection Groups and Invariant Theory*, Springer-Verlag, New York, 2001.

[18] M. Knus, A. Merkurjev, M. Rost, J.-P. Tignol, *The Book of Involutions. With a preface in French by J. Tits*, American Mathematical Society Colloquium Publications, Vol. 44, American Mathematical Society, Providence, RI, 1998.

[19] A. Merkurjev, *Rost invariants of simply connected algebraic groups* (with a section by S. Garibaldi), in: *Cohomological Invariants in Galois Cohomology*, Univ. Lecture Ser., Vol. 28, Amer. Math. Soc., Providence, RI, 2003, pp. 101–158.

[20] J. Milne, *Étale Cohomology*, Princeton Mathematical Series, Vol. 33, Princeton University Press, Princeton, N.J., 1980.

[21] J. Milnor, *Algebraic K-theory and quadratic forms*, Invent. Math. **9** (1969/1970), 318–344.

[22] J. Milnor, D. Husemoller, *Symmetric Bilinear Forms*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 73, Springer-Verlag, New York, 1973.

[23] F. Morel, $\mathbb{A}^1$*-Algebraic Topology over a Field*, Lect. Notes Math., Vol. 2052, Springer, Heidelberg, 2012.

[24] H. Nakajima, *Invariants of finite groups generated by pseudo-reflections in positive characteristic*, Tsukuba J. Math. **3** (1979), 109–122.

[25] R. Pirisi, *Cohomological invariants of algebraic stacks*, Trans. Amer. Math. Soc. **370** (2018), 1885–1906.

[26] M. Raynaud, *Anneaux Locaux Hensélien*, Springer Lect. Notes Math., Vol. 169, Springer-Verlag, Berlin, 1970.

[27] M. Rost, *Chow groups with coefficients*, Doc. Math. **1** (1996), 319–393.

[28] J.-P. Serre, *Groupes finis d'automorphismes d'anneaux locaux réguliers*, Colloque d'Algèbre (Paris, 1967), Exp. 8, Secrétariat mathématique, Paris.

[29] J.-P. Serre, *Galois Cohomology*, Springer-Verlag, Berlin, 1997.

[30] J.-P. Serre, *Local Algebra*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2000.

[31] J.-P. Serre, *Cohomological invariants, Witt invariants, and trace forms*, in: *Cohomological Invariants in Galois Cohomology*, Univ. Lecture Ser., Vol. 28, Amer. Math. Soc., Providence, RI, 2003, pp. 1–100.

[32] J.-P. Serre, *Letter to second author*, dated August 15, 2020.