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An Informativity Approach to the Data-Driven Algebraic Regulator Problem

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***Abstract*—In this article, the classical algebraic regulator problem is studied in a data-driven context. The endosystem is assumed to be an unknown system that is interconnected to a known exosystem that generates disturbances and reference signals. The problem is to design a regulator so that the output of the (unknown) endosystem tracks the reference signal, regardless of its initial state and the incoming disturbances. In order to do this, we assume that we have a set of input-state data on a finite time-interval. We introduce the notion of data informativity for regulator design, and establish necessary and sufficient conditions for a given set of data to be informative. Also, formulas for suitable regulators are given in terms of the data. Our results are illustrated by means of two extended examples.**

***Index Terms*—Data-driven control, informativity, linear control systems, tracking and regulation.**

I. Introduction

Recently the paradigm of data-driven control has gained a lot of attention in the analysis and controller design of linear systems [1], [3], [4], [7], [8], [11], [16]–[18], [20], [22]–[24]. Instead of using an explicit mathematical model, the data-driven approach uses only data obtained from the unknown system for verifying its system theoretic properties and for constructing controllers. Recently, it was argued in [23] that the data-driven approach can also be useful in cases where the given data do not give sufficient information to identify the "true" model for the system; for example, due to the fact that the data are not persistently exciting. Indeed, in [23], the notion of *informativity* of data was introduced to cover situations in which a given set of data gives rise to a whole *family of system models* that are compatible with the data. In other words, situations in which it is impossible to distinguish between models on the basis of the given data. A set of data is called informative for a given system property if the property holds for all systems compatible with the data. In [23], the notion of informativity was also developed in the context of controller design. In particular, conditions for the informativity of data for the following control problems were given as: state feedback stabilization, deadbeat control, linear quadratic optimal control, and stabilization by dynamic output feedback. Also, formulas (in terms of the data) were given to compute suitable controllers.

The aim of this article is to extend the framework of informativity to the classical algebraic regulator problem (see e.g., [9], [12], [13], and [15], and the textbooks [19] and [21]). This is the problem of finding a feedback controller (called a regulator) that makes the output of the controlled system track some *a priori* given reference signal, regardless of the disturbance input entering the system, and the initial state. In the context of the algebraic regulator problem, the relevant reference signals and disturbances (such as step functions, ramps, or sinusoids) are signals that are generated as solutions of suitable autonomous linear systems. Given such reference signal and class of disturbance signals, one first constructs a suitable generating autonomous system (called the exosystem). Next, this exosystem is interconnected to the control system (called the endosystem), and a new output is defined as the difference between the original system output and the reference signal. A regulator should, then, be designed to make the output of the interconnection converge to zero for all disturbances and initial states.

In this article, the "true" endosystem is assumed to be unknown, and therefore no mathematical model is available. Instead, we have collected data on the input, endosystem state, and exosystem state in the form of samples on a finite time-interval. The exosystem is assumed to be known, because this system models the reference signals and possible disturbance inputs. Also, the matrices in the output equations are assumed to be known, because these specify the design specification (namely the output that should converge to zero) on the controlled system. A given set of data will, then, be called informative for regulator design if the data contain sufficient information to design a single regulator for the entire family of systems that are compatible with this set of data. We will establish necessary and sufficient conditions for a given set of data to be informative for regulator design. In particular, it will be shown how to replace the characteristic regulator equations by their data-driven counterparts, and to compute suitable regulators.

We note that the data-driven regulator design was studied before in [10] and [6], albeit from a rather different perspective. We also mention alternative methods that deal with tracking objectives, such as iterative feedback tuning and virtual reference feedback tuning, as developed in [14] and [5], respectively. These methods do, however, not address the classical regulator problem, and are, thus, quite different from the work that will be presented in this article.

The main contributions of this article are the following.

1) We give a definition of the problem of data-driven tracking and regulation using the concept of informativity.

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2) We give necessary and sufficient conditions for data to be informative for regulator design, i.e., for the existence of a single regulator for all systems compatible with the given data.
3) We establish formulas for computing these regulators, entirely in terms of the data.

It should be noted that these regulators may be called *robust*, in the sense that a single regulator works for the whole set of systems that are compatible with the given data, see also [10].

The outline of this article is as follows. In Section II, we illustrate the data-driven problem of tracking and regulation using an extended example. Subsequently, we put the problem in a general framework, and define the concept of informativity for regulator design. In Section III, we review some classical basic material on the regulator problem. Then, in Section IV, we formulate our main result, giving necessary and sufficient conditions for the informativity for regulator design, and formulas to compute regulators. The main result is illustrated by means of two extended examples. Finally, Section V concludes this article.

II. Data-Driven Tracking and Regulation

We will first illustrate the problem to be considered in this article by means of an extended example.

Example 1: Consider the scalar linear time-invariant discrete-time system

$$\boldsymbol{x}(t+1) = a_s\boldsymbol{x}(t) + b_s\boldsymbol{u}(t) + \boldsymbol{d}(t) \tag{1}$$

where $\boldsymbol{x}$ is the state, $\boldsymbol{u}$ the control input, and $\boldsymbol{d}$ a disturbance input. The values of a_s and b_s in this system representation are unknown. We assume that the disturbance can be any constant signal of finite amplitude. Suppose that we want the state $\boldsymbol{x}(t)$ to track the given reference signal $\boldsymbol{r}(t) = \cos\frac{\pi}{2}t$, for any constant disturbance input, regardless of the initial state of the system. We want to design a control law for (1) that achieves this specification. We assume that $\boldsymbol{r}$, $\boldsymbol{x}$, and $\boldsymbol{d}$ are available for feedback and allow control laws of the form

$$\boldsymbol{u}(t) = k_1\boldsymbol{r}(t) + k_2\boldsymbol{r}(t+1) + k_3\boldsymbol{d}(t) + k_4\boldsymbol{x}(t). \tag{2}$$

Interconnecting (1) and (2) results in the controlled system

$$\begin{aligned}\boldsymbol{x}(t+1) &= (a_s + b_sk_4)\boldsymbol{x}(t) + (b_sk_3 + 1)\boldsymbol{d}(t)\\ &\quad + b_sk_1\boldsymbol{r}(t) + b_sk_2\boldsymbol{r}(t+1)\end{aligned}$$

where the gains k_i should be designed such that $\boldsymbol{x}(t) - \boldsymbol{r}(t) \to 0$ as $t \to \infty$ for any constant disturbance input $\boldsymbol{d}$ and initial state $\boldsymbol{x}(0)$. It is also required that the controlled system is internally stable, in the sense that $a_s + b_sk_4$ is stable.[1]

The values of a_s and b_s that represent the true system are unknown, but in the data-driven context, it is assumed that we do have access to certain data. In particular, it is assumed that we have finite sequences of samples of $\boldsymbol{x}(t)$, $\boldsymbol{u}(t)$, and $\boldsymbol{d}(t)$ on a given time interval $\{0, 1, \ldots, \tau\}$, given by

$$U_- := \begin{bmatrix} u(0) & u(1) & \cdots & u(\tau-1)\end{bmatrix} \tag{3a}$$

$$X := \begin{bmatrix} x(0) & x(1) & \cdots & x(\tau)\end{bmatrix} \tag{3b}$$

$$D_- := \begin{bmatrix} d(0) & d(1) & \cdots & d(\tau-1)\end{bmatrix} \tag{3c}$$

where, in this particular example, by assumption $d(t) = d(0)$ for $t = 1, 2, \ldots \tau - 1$. Define

$$X_+ := \begin{bmatrix} x(1) & x(2) & \cdots & x(\tau)\end{bmatrix}$$

$$X_- := \begin{bmatrix} x(0) & x(1) & \cdots & x(\tau-1)\end{bmatrix}.$$

It is assumed that these data are generated by the true system, so we must have $X_+ = a_sX_- + b_sU_- + D_-$. For this example, the problem of data-driven control design is now to use the data (3) to determine whether a suitable controller (2) exists, and to compute the associated gains k_1, k_2, k_3 and k_4 using only these data.

Note that, as we discussed before, both the reference signal and the disturbance signals are generated by the autonomous linear system

$$\begin{bmatrix}\boldsymbol{r}_1(t+1)\\ \boldsymbol{r}_2(t+1)\\ \boldsymbol{d}(t+1)\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\boldsymbol{r}_1(t)\\ \boldsymbol{r}_2(t)\\ \boldsymbol{d}(t)\end{bmatrix} \tag{4}$$

with initial state $\boldsymbol{r}_1(0) = 1$ and $\boldsymbol{r}_2(0) = 0$, and $\boldsymbol{d}(0)$ arbitrary. Indeed, it can be seen that the reference signal $\boldsymbol{r}(t) = \cos\frac{\pi}{2}t$ is equal to $\boldsymbol{r}_1(t)$. In addition, the solutions $\boldsymbol{d}(t)$ are all constant signals of finite amplitude. The autonomous system (4) is called the *exosystem*.

The interconnection of the (unknown) to be a controlled system (1) (called the *endosystem*) with the exosystem (4) is represented by

$$\begin{bmatrix}\boldsymbol{r}_1(t+1)\\ \boldsymbol{r}_2(t+1)\\ \boldsymbol{d}(t+1)\\ \boldsymbol{x}(t+1)\end{bmatrix} \begin{bmatrix}0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 1 & a_s\end{bmatrix}\begin{bmatrix}\boldsymbol{r}_1(t)\\ \boldsymbol{r}_2(t)\\ \boldsymbol{d}(t)\\ \boldsymbol{x}(t)\end{bmatrix} + \begin{bmatrix}0\\0\\0\\b_s\end{bmatrix}\boldsymbol{u}(t). \tag{5}$$

In this representation, the part corresponding to the exosystem is known, but the part corresponding to the endosystem (specifically: a_s and b_s) is unknown. We now also specify a (known) output equation

$$\boldsymbol{z}(t) = \begin{bmatrix}1 \; 0 \; 0 \; -1\end{bmatrix}\begin{bmatrix}\boldsymbol{r}_1(t)\\ \boldsymbol{r}_2(t)\\ \boldsymbol{d}(t)\\ \boldsymbol{x}(t)\end{bmatrix}.$$

Then, the problem of our example can be rephrased as: design a full *state feedback* control law

$$\boldsymbol{u}(t) = k_1\boldsymbol{r}_1(t) + k_2\boldsymbol{r}_2(t) + k_3\boldsymbol{d}(t) + k_4\boldsymbol{x}(t)$$

for the system (5) such that in the controlled system, we have $\boldsymbol{z}(t) \to 0$ as $t \to \infty$ for the initial states $\boldsymbol{r}_1(0) = 1$, $\boldsymbol{r}_2(0) = 0$, and $\boldsymbol{d}(0)$ arbitrary, while internal stability is achieved in the sense that $a_s + b_sk_4$ is a stable matrix. In order to allow tracking of signals from the richer class of all reference signals of the form $\boldsymbol{r}(t) = A\cos(\frac{1}{2}\pi t + \omega)$ (A and ω are determined by the initial states $\boldsymbol{r}_1(0) = 1$ and $\boldsymbol{r}_2(0)$), we may slightly relax the problem formulation and require $\boldsymbol{z}(t) \to 0$ as $t \to \infty$ for *all* initial states $\boldsymbol{r}_1(0), \boldsymbol{r}_2(0)$ and $\boldsymbol{d}(0)$.

[1] We say that a matrix is *stable* if all its eigenvalues are contained in the open unit disk.

After having introduced our problem set up by means of the above-mentioned example, we will now formulate it in a general framework.

Consider an endosystem represented by

$$\boldsymbol{x}_2(t+1) = A_{2\,s}\boldsymbol{x}(t) + B_{2\,s}\boldsymbol{u}(t) + A_3\boldsymbol{x}_1(t). \tag{6}$$

Here, $\boldsymbol{x}_2$ is the n_2-dimensional state, $\boldsymbol{u}$ the m-dimensional input, and $\boldsymbol{x}_1$ the n_1-dimensional state of the exosystem

$$\boldsymbol{x}_1(t+1) = A_1\boldsymbol{x}_1(t). \tag{7}$$

that generates all possible reference signals and disturbance inputs. The matrices A_{2s} and B_{2s} are unknown, but the matrix A_1 is known. Also A_3 is a known matrix that represents how the endosystem interconnects with the exosystem. The output to be regulated is specified by

$$\boldsymbol{z}(t) = D_1\boldsymbol{x}_1(t) + D_2\boldsymbol{x}_2(t) + E\boldsymbol{u}(t) \tag{8}$$

where the matrices D_1, D_2, and E are known. By interconnecting the endosystem with the state feedback controller

$$\boldsymbol{u}(t) = K_1\boldsymbol{x}_1(t) + K_2\boldsymbol{x}_2(t) \tag{9}$$

we obtain the controlled system

$$\begin{bmatrix} \boldsymbol{x}_1(t+1) \\ \boldsymbol{x}_2(t+1) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ A_3 + B_2K_1 & A_{2\,s} + B_{2\,s}K_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1(t) \\ \boldsymbol{x}_2(t) \end{bmatrix}$$

$$\boldsymbol{z}(t) = (D_1 + EK_1)\boldsymbol{x}_1(t) + (D_2 + EK_2)\boldsymbol{x}_2(t).$$

If $\boldsymbol{z}(t) \to 0$ as $t \to \infty$ for all initial states $\boldsymbol{x}_1(0)$ and $\boldsymbol{x}_2(0)$, we say that the controlled system is *output regulated*. If $A_{2\,s} + B_{2\,s}K_2$ is a stable matrix, we call the controlled system *endo-stable*. If the control law (9) makes the controlled system both output regulated and endo-stable, we call it a *regulator*.

As illustrated in the above-mentioned example, we assume that we do not know the true endosystem (6), and therefore, the design of a regulator can only be based on available data. In the general framework, these are finite sequences of samples of $\boldsymbol{x}_1(t), \boldsymbol{x}_2(t)$, and $\boldsymbol{u}(t)$ on a given time interval $\{0, 1, \ldots, \tau\}$ given by

$$U_- := \begin{bmatrix} u(0) & u(1) & \cdots & u(\tau-1) \end{bmatrix}$$
$$X_{1-} := \begin{bmatrix} x_1(0) & x_1(1) & \cdots & x_1(\tau-1) \end{bmatrix}$$
$$X_2 := \begin{bmatrix} x_2(0) & x_2(1) & \cdots & x_2(\tau) \end{bmatrix}.$$

An endosystem with (unknown) system matrices (A_2, B_2) is called *compatible* with these data if A_2 and B_2 satisfy the equation

$$X_{2+} = A_2X_{2-} + A_3X_{1-} + B_2U_- \tag{10}$$

where we denote

$$X_{2-} := \begin{bmatrix} x_2(0) & x_2(1) & \cdots & x_2(\tau-1) \end{bmatrix}$$
$$X_{2+} := \begin{bmatrix} x_2(1) & x_2(2) & \cdots & x_2(\tau) \end{bmatrix}.$$

The set of all (A_2, B_2) that are compatible with the data is denoted by $\Sigma_{\mathcal{D}}$, i.e.,

$$\Sigma_{\mathcal{D}} := \{(A_2, B_2) \mid (10) \text{ holds}\}. \tag{11}$$

We assume that the true endosystem $(A_{2\,s}, B_{2\,s})$ is in $\Sigma_{\mathcal{D}}$, i.e., the true system is compatible with the data. In general, the (10) does not specify the true system uniquely, and many endosystems (A_2, B_2) may be compatible with the same data.

Now we turn to controller design based on the data (U_-, X_{1-}, X_2). Note that, since on the basis of the given data, we cannot distinguish between the true endosystem and any other endosystem compatible with these data; a controller will be a regulator for the true system only if it is a regulator for any system with (A_2, B_2) in $\Sigma_{\mathcal{D}}$. If such regulator exists, we call the data *informative for regulator design*.

Definition 2: We say that the data (U_-, X_{1-}, X_2) are informative for regulator design if there exist K_1 and K_2 such that the control law $\boldsymbol{u}(t) = K_1\boldsymbol{x}_1(t) + K_2\boldsymbol{x}_2(t)$ is a regulator for any endosystem with (A_2, B_2) in $\Sigma_{\mathcal{D}}$.

The problem that will be considered in this article is to find necessary and sufficient conditions on the data (U_-, X_{1-}, X_2) to be informative for the regulator design. Also, in case that these conditions are satisfied, we will explain how to compute a regulator using only these data. Before addressing this problem, in the next section, we will review some basic material on the regulator problem.

III. Regulator Problem

In this section, we briefly review some basic material on the regulator problem. Following [21], we distinguish between analysis and design.

We first consider the analysis question under what conditions a controlled system is endo-stable and output regulated. Consider the autonomous linear system represented by

$$\begin{aligned} \boldsymbol{x}_1(t+1) &= A_1\boldsymbol{x}_1(t) \\ \boldsymbol{x}_2(t+1) &= A_2\boldsymbol{x}_2(t) + A_3\boldsymbol{x}_1(t) \\ \boldsymbol{z}(t) &= D_1\boldsymbol{x}_1(t) + D_2\boldsymbol{x}_2(t). \end{aligned} \tag{12}$$

In accordance with the terminology introduced in Section II, we call this system endo-stable if A_2 is a stable matrix. We call it output regulated if $\boldsymbol{z}(t) \to 0$ as $t \to \infty$ for all initial states $\boldsymbol{x}_1(0)$ and $\boldsymbol{x}_2(0)$. The following is the discrete-time version in [21, Lemma 9.1]

Proposition 3: Assume that A_1 is antistable[2]. Then, the system (12) is endo-stable and output regulated if and only if A_2 is stable and there exists a matrix T satisfying the equations

$$TA_1 - A_2\,T = A_3,\; D_1 + D_2\,T = 0. \tag{13}$$

In this case, T is unique.

Next, we consider the design problem and review conditions under which, for a given interconnection of an endosystem and exosystem, there exists a regulator, i.e., a controller that makes the controlled system endo-stable and output regulated. For the endosystem $\boldsymbol{x}_2(t+1) = A_2\boldsymbol{x}_2(t) + B_2\boldsymbol{u}(t) + A_3\boldsymbol{x}_1(t)$ together with the exosystem (7) and output (8), the following

[2]We say that a matrix is *anti-stable* if all its eigenvalues λ satisfy $|\lambda| \geq 1$.

is well known and can be proven easily by extending results from [21] to the discrete-time case.

Proposition 4: Assume that A_1 is antistable. There exists a regulator of the form (9) if and only if (A_2, B_2) is stabilizable and there exist matrices T and V satisfying the regulator equations

$$TA_1 - A_2\,T - B_2\,V = A_3,\ D_1 + D_2\,T + EV = 0. \quad (14)$$

In this case, a regulator is obtained as follows: choose any K_2 such that $A_2 + B_2K_2$ is stable, and define $K_1 := -K_2\,T + V$.

IV. Data-Driven Regulator Problem

Clearly, a necessary condition for the data (U_-, X_{1-}, X_2) to be informative for regulator design is that they are informative for endo-stabilization.

Definition 5: We call the data (U_-, X_{1-}, X_2) are *informative* for endo-stabilization if there exists K_2 such that $A_2 + B_2K_2$ is a stable matrix for all (A_2, B_2) in $\Sigma_{\mathcal{D}}$.

In order to obtain necessary and sufficient conditions for informativity for endo-stabilization, we formulate the following.

Proposition 6: Let τ be a positive integer. Let Z and X be real $n \times \tau$ matrices and let U be a real $m \times \tau$ matrix. Consider the set $\Sigma_{(Z,X,U)} := \{(A, B) \mid Z = AX + BU\}$. Then, the following hold.

1) There exists a matrix K such that $A + BK$ is stable for all $(A, B) \in \Sigma_{(Z,X,U)}$ if and only if X has full row rank, and there exists a right inverse $X^\dagger$ such that $ZX^\dagger$ is stable. In that case, by taking $K := UX^\dagger$, we have $A + BK$ is stable for all $(A, B) \in \Sigma_{(Z,X,U)}$.
2) For any K such that $A + BK$ is stable for all $(A, B) \in \Sigma_{(Z,X,U)}$, there exists a right inverse $X^\dagger$ such that $K = UX^\dagger$, and, moreover, $A + BK = ZX^\dagger$ for all $(A, B) \in \Sigma_{(Z,X,U)}$.

Proof: The proof can be given by slightly adapting the proof in[23, Th. 16]. ■

This immediately gives the following conditions for informativity for endo-stabilization.

Lemma 7: The data (U_-, X_{1-}, X_2) are informative for endo-stabilization if and only if X_{2-} has full row rank, and there exists a right inverse $X_{2-}^\dagger$ of X_{2-} such that $(X_{2+} - A_3X_{1-})X_{2-}^\dagger$ is stable. In that case, by taking $K_2 := U_-X_{2-}^\dagger$, we have $A_2 + B_2K_2$ stable for all $(A_2, B_2) \in \Sigma_{\mathcal{D}}$.

The following theorem is the main result of this article. It gives necessary and sufficient conditions on the data to be informative for regulator design, and explains how suitable regulators are computed using only these data.

Theorem 8: Assume that A_1 is antistable and suppose, for simplicity, that is diagonalizable. Then, the data (U_-, X_{1-}, X_2) are informative for regulator design if and only if at least one of the following two conditions hold.[3]

1) X_{2-} has full row rank, and there exists a right inverse $X_{2-}^\dagger$ of X_{2-} such that $(X_{2+} - A_3X_{1-})X_{2-}^\dagger$ is stable and $D_2 + EU_-X_{2-}^\dagger = 0$. Moreover, $\operatorname{im} D_1 \subseteq \operatorname{im} E$. In this case, a regulator is found as follows: choose K_1 such that $D_1 + EK_1 = 0$ and define $K_2 := U_-X_{2-}^\dagger$.
2) X_{2-} has full row rank and there exists a right inverse $X_{2-}^\dagger$ of X_{2-} such that $(X_{2+} - A_3X_{1-})X_{2-}^\dagger$ is stable. Moreover, there exists a solution W to the linear equations

$$X_{2-}WA_1 - (X_{2+} - A_3X_{1-})W = A_3 \quad (15a)$$

$$D_1 + (D_2X_{2-} + EU_-)W = 0. \quad (15b)$$

In this case, a regulator is found as follows: choose $K_1 := U_-(I - X_{2-}^\dagger X_{2-})W$ and $K_2 := U_-X_{2-}^\dagger$.

[3]We denote by $\operatorname{im} M$ the image of the matrix M.

Before turning to the proof, we will explain how to apply this theorem. What we know about the system are the system matrices A_1, A_3, D_1, D_2, and E and the data (U_-, X_{1-}, X_2). The aim is to use this knowledge to compute a *single* regulator (K_1, K_2) that works *for all* endosystems (A_2, B_2) in the set Σ_D defined by (11).

In order to check the existence of such regulator, we verify the two conditions 1) and 2) in Theorem 8. On the one hand, if neither of the two conditions holds, then the data are not informative. On the other hand, if condition 1) holds then a regulator (K_1, K_2) is computed as follows:

1) find a right inverse $X_{2-}^\dagger$ of X_{2-} such that the matrix $(X_{2+} - A_3X_{1-})X_{2-}^\dagger$ is stable and $D_2 + EU_-X_{2-}^\dagger = 0$,
2) compute K_1 as a solution of $D_1 + EK_1 = 0$,
3) define $K_2 := U_-X_{2-}^\dagger$.

If condition 2) holds, then a regulator is computed as follows:

1) find a right inverse $X_{2-}^\dagger$ of X_{2-} such that the matrix $(X_{2+} - A_3X_{1-})X_{2-}^\dagger$ is stable,
2) find a solution W of the data-driven regulator (15),
3) define $K_1 := U_-(I - X_{2-}^\dagger X_{2-})W$,
4) define $K_2 := U_-X_{2-}^\dagger$.

Proof: $(\Rightarrow)$ We first prove sufficiency. Assume that the condition 1) holds. Since $(X_{2+} - A_3X_{1-})X_{2-}^\dagger$ is stable, the data are informative for endo-stabilization and by taking $K_2 := U_-X_{2-}^\dagger$ we have $A_2 + B_2K_2$ is stable for all $(A_2, B_2) \in \Sigma_{\mathcal{D}}$. Since A_1 is assumed to be antistable, this implies that for all $(A_2, B_2) \in \Sigma_{\mathcal{D}}$ there exists a unique solution T to the Sylvester equation $TA_1 - (A_2 + B_2K_2)T = A_3 + B_2K_1$. By the fact that $D_1 + EK_1 = 0$ and $D_2 + EK_2 = 0$, this solution T also satisfies $D_1 + EK_1 + (D_2 + EK_2)T = 0$. Thus, for all $(A_2, B_2) \in \Sigma_{\mathcal{D}}$, there exists a matrix T that satisfies the equations (13). It follows from Proposition 3 that for all $(A_2, B_2) \in \Sigma_{\mathcal{D}}$ the controlled system is endo-stable and output regulated.

Next, assume that condition 2) holds. By Lemma 7, the data are informative for endo-stabilization and by taking $K_2 := U_1X_-^\dagger$, we have $A_2 + B_2K_2$ stable for all $(A_2, B_2) \in \Sigma_{\mathcal{D}}$. Let W satisfy the (15). Define $T := X_{2-}W$ and $V := U_-W$. Then, the pair (T, V) satisfies the regulator (14) for all $(A_2, B_2) \in \Sigma_{\mathcal{D}}$. Then, by Proposition 4, for each such (A_2, B_2), a regulator is given by the pair (K_1, K_2), with $K_1 = -K_2\,T + V = -K_2X_{2-}W + U_-W = U_-(I - X_{2-}^\dagger X_{2-})W$. This completes the proof of the sufficiency part.

We will now turn to the necessity part. Assume that the data are informative for regulator design. By Proposition 3, there exist K_1 and K_2 and for any $(A_2, B_2) \in \Sigma_{\mathcal{D}}$ a matrix $T_{(A_2,B_2)}$ such that $A_2 + B_2K_2$ is stable and

$$T_{(A_2,B_2)}A_1 - (A_2 + B_2K_2)T_{(A_2,B_2)} = A_3 + B_2K_1$$
$$D_1 + EK_1 + (D_2 + EK_2)T_{(A_2,B_2)} = 0.$$

We emphasize that $T_{(A_2,B_2)}$ may depend on the choice of $(A_2, B_2) \in \Sigma_{\mathcal{D}}$. However, since $A_2 + B_2K_2$ is stable for all $(A_2, B_2) \in \Sigma_{\mathcal{D}}$, by Proposition 6, there exists a right inverse $X_{2-}^\dagger$ of X_{2-} such that $A_2 + B_2K_2 = (X_{2+} - A_3X_{1-})X_{2-}^\dagger$ for all $(A_2, B_2) \in \Sigma_{\mathcal{D}}$. The latter matrix is independent of (A_2, B_2).

Call it M. Define

$$\Sigma_{\mathcal{D}}^0 := \{(A_0, B_0) \mid \begin{bmatrix} A_0 & B_0 \end{bmatrix} \begin{bmatrix} X_{2-} \\ U_- \end{bmatrix} = 0\}.$$

Note that $\Sigma_{\mathcal{D}}^0$ is the solution space of the homogeneous version of the defining equation (10) for $\Sigma_{\mathcal{D}}$ [see (11)]. We now distinguish two cases, namely (i) $B_0K_1 = 0$ for all $(A_0, B_0) \in \Sigma_{\mathcal{D}}^0$, and (ii) $B_0K_1 \neq 0$ for some $(A_0, B_0) \in \Sigma_{\mathcal{D}}^0$.

First consider case (i). Then, for all $(A_2, B_2), (\bar{A}_2, \bar{B}_2) \in \Sigma_{\mathcal{D}}$, we have $B_2K_1 = \bar{B}_2K_1$. Thus, there exists a *common* matrix T that solves the equations

$$\begin{aligned} TA_1 - MT &= A_3 + B_2K_1 \\ D_1 + EK_1 + (D_2 + EK_2)T &= 0 \end{aligned}$$

for all $(A_2, B_2) \in \Sigma_{\mathcal{D}}$. From this, we obtain

$$TA_1 - \begin{bmatrix} A_2 & B_2 \end{bmatrix} \begin{bmatrix} T \\ K_2\,T + K_1 \end{bmatrix} = A_3$$

for all $(A_2, B_2) \in \Sigma_{\mathcal{D}}$, and therefore

$$\begin{bmatrix} A_0 & B_0 \end{bmatrix} \begin{bmatrix} T \\ K_2\,T + K_1 \end{bmatrix} = 0$$

for all $(A_0, B_0) \in \Sigma_{\mathcal{D}}^0$. This implies

$$\operatorname{im} \begin{bmatrix} T \\ K_2\,T + K_1 \end{bmatrix} \subseteq \operatorname{im} \begin{bmatrix} X_{2-} \\ U_- \end{bmatrix}.$$

As a consequence, there exists a matrix W such that

$$\begin{bmatrix} T \\ K_2\,T + K_1 \end{bmatrix} = \begin{bmatrix} X_{2-} \\ U_- \end{bmatrix} W.$$

Clearly, W satisfies the (15), showing that condition 2) holds.

Next, consider case (ii). Let S be a real $(n_2 + m) \times r$ matrix such that[4]

$$\ker \begin{bmatrix} X_{2-} \\ U_- \end{bmatrix}^T = \operatorname{im} S.$$

Partition $S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$. Then, $(A_0, B_0) \in \Sigma_{\mathcal{D}}^0$ if and only if $A_0 = NS_1^T$ and $B_0 = NS_2^T$ for some $n_2 \times r$ matrix N. Note that, by hypothesis, $S_2^TK_1 \neq 0$.

Let $(A_2, B_2) \in \Sigma_{\mathcal{D}}$. Recall that for any such (A_2, B_2), there exists a unique $T_{(A_2,B_2)}$ such that

$$\begin{aligned} T_{(A_2,B_2)}A_1 - MT_{(A_2,B_2)} &= A_3 + B_2K_1 \\ D_1 + EK_1 + (D_2 + EK_2)T_{(A_2,B_2)} &= 0 \end{aligned} \tag{16}$$

Now let N be any real $n_2 \times r$ matrix. Then also $(A_2 + NS_1^T, B_2 + NS_2^T) \in \Sigma_{\mathcal{D}}$. Define $T_N := T_{(A_2,B_2)} - T_{(A_2+NS_1^T, B_2+NS_2^T)}$. Then, T_N is the unique solution to

$$T_NA_1 - MT_N = NS_2^TK_1 \tag{17}$$

which in addition satisfies $(D_2 + EK_2)T_N = 0$. Consider now a spectral decomposition $A_1 = Q^{-1}\Lambda Q$, where Λ is the diagonal matrix $\Lambda = \operatorname{diag}(\lambda_1, \ldots \lambda_{n_1})$ and

$$Q = \begin{bmatrix} q_1 \\ \vdots \\ q_{n_1} \end{bmatrix}, \; Q^{-1} = \begin{bmatrix} \hat{q}_1 & \ldots & \hat{q}_{n_1} \end{bmatrix}.$$

Then, for fixed N, the unique solution T_N to the Sylvester (17) can be expressed as

$$T_N = \sum_{i=1}^{n_1} (\lambda_i I - M)^{-1} NS_2^TK_1\hat{q}_iq_i$$

(see [2]), which implies that T_NQ^{-1} is equal to

$$\begin{bmatrix} (\lambda_1 I - M)^{-1}NS_2^TK_1\hat{q}_1 & \ldots & (\lambda_{n_1} I - M)^{-1}NS_2^TK_1\hat{q}_{n_1} \end{bmatrix}.$$

Note that the matrices $\lambda_i I - M$ are indeed invertible because M is stable and the eigenvalues λ_i of A_1 satisfy $|\lambda_i| \geq 1$. Since, in addition, $(D_2 + EK_2)T_N = 0$, we see that for all $i = 1, \ldots, n_1$, we have

$$(D_2 + EK_2)(\lambda_1 I - M)^{-1}NS_2^TK_1\hat{q}_i = 0.$$

Since $S_2^TK_1 \neq 0$, there must exist an index i such that $S_2^TK_1\hat{q}_i \neq 0$. For this i, let z be a real vector such that $z^TS_2^TK_1\hat{q}_i \neq 0$. Now choose $N := e_jz^T$, where e_j denotes the jth standard basis vector in $\mathbb{R}^{n_2}$. By the previous discussion, we obtain $(D_2 + EK_2)(\lambda_1 I - M)^{-1}e_j = 0$. Since this holds for any j, we actually find $(D_2 + EK_2)(\lambda_1 I - M)^{-1} = 0$, so $D_2 + EK_2 = 0$. Using (16), we must also conclude that $D_1 + EK_1 = 0$, which implies that $\operatorname{im} D_1 \subseteq \operatorname{im} E$. Since K_2 is stabilizing, it must be of the form $U_-X_{2-}^\dagger$ for some right inverse $X_{2-}^\dagger$. This implies that $(X_{2+} - A_3X_{1-})X_{2-}^\dagger$ is stable and $D_2 + EU_-X_{2-}^\dagger = 0$, that is, condition 1) holds. This completes the proof of Theorem 8. ∎

Remark 9: In order to avoid technicalities, in Theorem 8, we have assumed that the matrix A_1 is diagonalizable. The theorem, however, also holds if we drop this assumption. We omit the proof here.

Remark 10: According to Theorem 8, the data are informative for regulator design if and only if at least one of the conditions 1) or 2) holds. Condition 2) is in terms of solvability of the "data-driven regulator" (15a) and (15b). These equations hold for all (A_2, B_2) compatible with the data. In the end, a matrix T is defined as $T := X_{2-}W$, and together with $V := U_-W$, the classical regulator (14) is then satisfied for all (A_2, B_2) compatible with the data. This is, then, "the classical design," and the difference $x_2(t) - Tx_1(t)$ converges to 0 as t runs off to infinity ([21], p. 199)

If Condition 2) does not hold, but instead Condition 1) holds, then the only way to get output regulation is to make the entire output $z = (D_1 + EK_1)x_1 + (D_2 + EK_2)x_2$ equal to 0 *pointwise*. This is done by making $D_1 + EK_1 = 0$ (possible because $\operatorname{im} D_1 \subseteq \operatorname{im} E$) and $D_2 + EK_2 = 0$, where $K_2 = U_-X_{2-}^\dagger$ also makes the system endo-stable.

Note that Theorem 8 gives a characterization of all data that are informative for regulator design, and gives a method to design a suitable regulator. Nonetheless, the procedure to compute this regulator is not entirely satisfactory. Indeed, in

[4] We denote by $\ker M$ the kernel of the matrix M.

the case that condition 2) holds, it is not clear how to find a right inverse of X_{2-} such that $(X_{2+} - A_3X_{1-})X_{2-}^{\dagger}$ is stable. In the case of condition 1), the additional constraint $D_2 + EU_-X_{2-}^{\dagger} = 0$ needs to be satisfied. In general, X_{2-} has many right inverses, and $(X_{2+} - A_3X_{1-})X_{2-}^{\dagger}$ can be stable, with or without $D_2 + EU_-X_{2-}^{\dagger} = 0$, depending on the choice of the particular right inverse $X_{2-}^{\dagger}$. To deal with this problem and to solve the problem of regulator design, we formulate the problem of finding a suitable right inverse in terms of feasibility of linear matrix inequalities (LMI's).

Theorem 11: Let (U_-, X_{1-}, X_2) be given data. Then the following hold.

1) X_{2-} has full row rank and has a right inverse $X_{2-}^{\dagger}$ such that $(X_{2+} - A_3X_{1-})X_{2-}^{\dagger}$ is stable if and only if there exists a matrix $\Theta \in \mathbb{R}^{T\times n}$ such that

$$X_{2-}\Theta = (X_{2-}\Theta)^{\top} \tag{18}$$

and

$$\begin{bmatrix} X_{2-}\Theta & (X_{2+} - A_3X_{1-})\Theta \\ \Theta^{\top}(X_{2+} - A_3X_{1-})^{\top} & X_{2-}\Theta \end{bmatrix} > 0. \tag{19}$$

2) X_{2-} has full row rank and has a right inverse $X_{2-}^{\dagger}$ such that $(X_{2+} - A_3X_{1-})X_{2-}^{\dagger}$ is stable with, in addition, $D_2 + EU_-X_{2-}^{\dagger} = 0$ if and only if there exists a solution $\Theta \in \mathbb{R}^{T\times n}$ of (18) and (19) that satisfies the linear equation

$$(D_2X_{2-} + EU_-)\Theta = 0.$$

In both cases, a suitable right inverse is given by $X_{2-}^{\dagger} := \Theta(X_{2-}\Theta)^{-1}$.

Proof: The proof can be given by adapting the proof in [23, Th. 17]. ■

Example 12: We will now apply Theorem 8 to Example 1. Putting the example in our general framework, we have

$$\boldsymbol{x}_1 = \begin{bmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \boldsymbol{d} \end{bmatrix}, \quad \boldsymbol{x}_2 = \boldsymbol{x}, \quad A_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad D_2 = -1, \quad E = 0.$$

Assume $\tau = 3$, and the data on the disturbance input are $D_- = [\, d(0)\ d(1)\ d(2)\,] = [\, \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\,]$. Since the signal to be tracked is $\cos \frac{1}{2}\pi t$, we must have $\boldsymbol{r}_1(0) = 1$ and $\boldsymbol{r}_2(0) = 0$, so $\boldsymbol{r}_1(t) = \cos \frac{1}{2}\pi t$ and $\boldsymbol{r}_2(t) = \cos \frac{1}{2}\pi(t+1)$. This leads to

$$X_{1-} = \begin{bmatrix} r_1(0) & r_1(1) & r_1(2) \\ r_2(0) & r_2(1) & r_2(2) \\ d(0) & d(1) & d(2) \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Assume that $U_- = [\, u(0)\ u(1)\ u(2)\,] = [\,1\ 0\ 0\,]$ and $X_2 = [\, x_2(0)\ x_2(1)\ x_2(2)\ x_2(3)\,] = [\,0\ \frac{3}{2}\ 2\ \frac{5}{2}\,]$. It can be checked that condition 2) of Theorem 8 holds. Indeed, a solution W to the linear (15) is given by

$$W = \begin{bmatrix} -1 & 1 & -1 \\ \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Furthermore, $X_{2-}^{\dagger} = [\, -\frac{1}{2}\ \frac{2}{3}\ 0\,]^T$ is a right inverse of X_{2-} and $(X_{2+} - A_3X_{1-})X_{2-}^{\dagger} = \frac{1}{2}$ is stable. A regulator is then given by $K_1 = U_-(I - X_{2-}^{\dagger}X_{2-})W = [\, -\frac{1}{2}\ 1\ -1\,]$ and $K_2 := U_-X_{2-}^{\dagger} = -\frac{1}{2}$.

It can be checked that the above-mentioned data are compatible with the true endosystem $a_s = 1, b_s = 1$. In fact, in this particular example, the true system is uniquely determined by the data. Indeed, this follows from the fact that

$$X_{2+} = \begin{bmatrix} a_s & b_s \end{bmatrix} \begin{bmatrix} X_{2-} \\ U_- \end{bmatrix} + D_-$$

in which $\begin{bmatrix} X_{2-} \\ U_- \end{bmatrix}$ has full row rank. Thus, a regulator could also have been computed directly from the regulator equations (14) after first identifying the true endosystem $a_s = 1$, $b_s = 1$. It can indeed be verified that $T = [\,1\ 0\ 0\,]$ together with $V = [\,-1\ 1\ -1\,]$ satisfy the regulator equations (14) for the true endosystem. By choosing $K_2 = -\frac{1}{2}$, this would, then, lead to the same regulator as previously with $K_1 = -K_2\,T + V = [\,-\frac{1}{2}\ 1\ -1\,]$.

We note that, in general, the true endosystem may *not* be uniquely determined by the data. This is illustrated by the following example.

Example 13: Consider the two-dimensional (2-D) endosystem

$$\boldsymbol{x}_2(t+1) = A_{2\,s}\boldsymbol{x}_2(t) + B_{2\,s}\boldsymbol{u}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \boldsymbol{d}(t)$$

where $A_{2\,s}$ and $B_{2\,s}$ are unknown 2×2 and 2×1 matrices, respectively. Let $\boldsymbol{x}_2 = [\, \boldsymbol{x}_{21}\ \boldsymbol{x}_{22}\,]^T$. The disturbance input $\boldsymbol{d}$ is assumed to be a constant signal with finite amplitude, so is generated by $\boldsymbol{d}(t+1) = \boldsymbol{d}(t)$. We want to design a regulator so that $2\boldsymbol{x}_{21} + \frac{1}{2}\boldsymbol{x}_{22}$ tracks a given reference signal. In this example, the reference signals $\boldsymbol{r}$ are assumed to be generated by a given autonomous linear system with state-space dimension, say, n_1. Its representation will be irrelevant here. The total exosystem will then have state space dimension $n_1 + 1$, and our output equation is given by $\boldsymbol{z}(t) = D_1\boldsymbol{x}_1(t) + D_2\boldsymbol{x}_2(t) + E\boldsymbol{u}(t)$, with D_1 a $1\times(n_1+1)$ matrix such that $D_1\boldsymbol{x}_1 = -\boldsymbol{r}$ and $D_2 = \begin{bmatrix} 2 & \frac{1}{2} \end{bmatrix}$. We take $E = 2$. Also note that $A_3 = \begin{bmatrix} 0_{1\times n_1} & 0 \\ 0_{1\times n_1} & 1 \end{bmatrix}$. Here, $0_{1\times n_1}$ denotes $1\times n_1$ zero matrix. Suppose that $\tau = 2$ and assume we have the following data:

$$U_- = \begin{bmatrix} -1 & -1 \end{bmatrix}, \; D_- = \begin{bmatrix} 1 & 1 \end{bmatrix}, \; X_2 = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 2 & \frac{5}{2} \end{bmatrix}.$$

These data can be seen to be generated by the true endosystem $A_{2\,s} = \begin{bmatrix} 2 & \frac{1}{8} \\ 4 & \frac{5}{4} \end{bmatrix}$, $B_{2\,s} = \begin{bmatrix} \frac{3}{2} \\ 3 \end{bmatrix}$. We now check the Condition 1) of Theorem 8. First note that, indeed, $\operatorname{im} D_1 \subseteq \operatorname{im} E$. Also, X_{2-} is nonsingular and $(X_{2+} - A_3X_{1-})X_{2-}^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ 1 & \frac{1}{2} \end{bmatrix}$. This matrix has eigenvalues $\frac{1}{2} \pm \frac{1}{2}i$, so is stable. Finally, $D_2 + EU_-X_{2-}^{-1} = 0$. According to Theorem 8, a regulator for all endosystems compatible with the given data is given by

$$K_2 = U_1 X_{2-}^{-1} = \begin{bmatrix} -1 & -\frac{1}{4} \end{bmatrix}, \; K_1 = -\frac{1}{2} D_1. \tag{20}$$

It can be verified that the set of endosystems compatible with our data is equal to the affine set

$$\Sigma_{\mathcal{D}} = \left\{ \left(\begin{bmatrix} a & \frac{1}{4}a - \frac{3}{8} \\ b & \frac{1}{4}b + \frac{1}{4} \end{bmatrix}, \begin{bmatrix} a - \frac{1}{2} \\ b - 1 \end{bmatrix} \right) \mid a, b \in \mathbb{R} \right\}.$$

The controller given by (20) is a regulator for all these endosystems.

Remark 14: It is also possible to consider the situation that, in addition to A_2 and B_2, also the matrix A_3 (representing how the exosignal $\boldsymbol{x}_1$ enters the endosystem) is unknown. In that case, the set all endosystems compatible with the data (U_-, X_2, X_-) is defined as follows:

$$\Sigma_D = \{(A_2, B_2, A_3) \mid X_{2+} = A_2 X_{2-} + B_2 U_- + A_3 X_{1-}\}.$$

The data are then called informative for regulator design if there exists a single regulator $u = K_1 x_1 + K_2 x_2$ for all endosystems in Σ_D. The analogue of Theorem 8 for this situation is as follows. Both in Conditions 1) and 2), an additional condition $X_{1-} X_{2-}^{\dagger} = 0$ should be imposed on a suitable right inverse of X_{2-}. In addition, in Condition 2), the old data-driven regulator (15) should be replaced by

$$X_{2-} W A_1 - X_{2+} W = 0 \tag{21a}$$

$$X_{1-} W = I \tag{21b}$$

$$D_1 + (D_2 X_{2-} + E U_-) W = 0. \tag{21c}$$

Note that, as expected, A_3 no longer appears in the equations (it is unknown). In both cases, the formulas for K_1 and K_2 are the same as in Theorem 8. Due to space limitations, the proof is omitted.

V. Conclusion

We have introduced the notion of data informativity in the context of the classical algebraic regulator problem. Our main results are necessary and sufficient conditions for a given set of data to be informative for regulator design, and formulas to compute regulators using only this set of data. We have recasted the computation of suitable regulators in terms of feasibility of LMI's. Our results have been illustrated by means of two extended examples. In this article, only static state feedback regulators have been considered. As an open problem for future research, we mention the extension to dynamic output feedback regulators. Results obtained in [23] on the problem of stabilization by dynamic output feedback (both in terms of input-state-output data and input–output data) are expected to be relevant here. Another possible venue for future research is to consider the situation that, in addition to A_2, A_3, and B_2, the matrix A_3 is unknown. Finally, it would be interesting to include noise in the problem formulation, and to consider the situation in which, in addition to the modeled disturbances, bounded noise may enter the unknown endosystem (see also [22]).

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