Non-collapsing on Hypersurfaces of Prescribed Curvature

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For my parents: Dumooa and Nasir

Declaration

The work in this thesis is my own except where otherwise stated.

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Abstract

The establishment and development of a non-collapsing technique have recently received a great deal of attention in geometric analysis. The area of study conducted in my thesis is motivated by this technique which is based on the application of the maximum principle to a two-point function that is defined on manifolds or relies on some global information of partial differential equations PDEs. This approach has been utilized in various settings to acquire some useful knowledge on the behaviour of geometric structures and properties of embedded hypersurfaces.

Our first purpose of study concerns the uniqueness of a class of embedded Weingarten hypersurfaces in the higher dimensional sphere $\mathbb{S}^{n+1}$. In particular, we consider a more general class of embedded Weingarten hypersurfaces with two distinct principal curvatures into $\mathbb{S}^{n+1}$. Also, we assume that these hypersurfaces satisfying a PDE of the principal curvatures with assumptions on the number of multiplicities of these curvatures. As a result, we deduce that these principal curvatures are both constant and consequently our hypersurfaces are congruent to a Clifford torus. One of the key ingredients of this work is based on the smart deployment of the maximum principle argument to a function of two variables that is defined on our hypersurfaces.

The second main object of study is the mean convex mean curvature flow in Minkowski space $\mathbb{L}^{n,1}$. Of particular interest is the study of mean convex embedded spacelike hypersurfaces evolving by the mean curvature flow into $\mathbb{L}^{n,1}$. In particular, we deduce a non-collapsing estimate by employing the parabolic version of Omori-Yau maximum principle to a bounded function that is defined on these spacelike hypersurfaces. More precisely, we compare the radius of the largest hyperbola which touches the spacelike hypersurface at a given point to the curvature at that point.

Contents

Chapter 1

Introduction

1.1 Previous work

The establishment and development of a non-collapsing technique have recently received a great deal of attention in geometric analysis. One of the main and challenging ingredients in such technique is the application of the maximum principle to a function which involves the geometry at several points. The most common version of this principle deals with a scalar function defined on a manifold, however, some remarkable versions have recently appeared in various settings. Of particular interest is the application of the maximum principle to a function that is defined at two points on manifolds or a function that relies on some global information of solutions of partial differential equations such as heat equations. This effective approach is used to acquire useful knowledge on the behaviour of geometric structures and properties of embedded surfaces in ambient spaces.

The study of hypersurfaces in a space form $\mathbb{Q}^{n+1}(c)$ of constant sectional curvature c is an extensive field of research emerging from the classical differential geometry of curves and surfaces. It has contributed to the development of differential geometry and partial differential equations. One of the main directions in this discipline is the study of geometric properties of minimal and constant mean curvature (CMC) hypersurfaces and other prescribed curvature problems in space forms: the Euclidean space $\mathbb{R}^{n+1}$ if $c = 0$, the sphere $\mathbb{S}^{n+1}$ if $c > 0$ and the hyperbolic space $\mathbb{H}^{n+1}$ if $c < 0$. More precisely, the classification of these immersed or embedded hypersurfaces into space forms has been an active field for mathematicians.

The application of non-collapsing machinery for hypersurfaces with prescribed curvatures has significantly contributed in the development of some geometric problems. Starting from the work of S. N. Kruzhkov [68] in 1967 where gradient estimates were deduced for a function defined as the difference between the solutions at two points for some parabolic partial differential equations in one-dimensional space by using the maximum principle and a suitable barrier argument. In 1998, G. Huisken [57] applied a refinement of these methods in his work on embedded solutions of the curve shortening flow proving the embeddedness is preserved and providing another proof for Grayson's theorem [49]. Precisely, he considered a one-parameter family of embedded curves and a two-point function which compares the arc length and the ambient distance between any pair of points, and deduced that these curves contract to a round point.

The theory of minimal surfaces, surfaces of least area or more generally surfaces with vanishing mean curvature, is one of the substantial topics across major disciplines of mathematics such as calculus of variations and geometric measure theory. The investigation of these surfaces in the 3-sphere $\mathbb{S}^3$ in particular has led to considerable interest, and offers some features which are not present for surfaces in Euclidean space. For example, there are no closed minimal surfaces in $\mathbb{R}^3$, whereas there exist closed minimal surfaces in $\mathbb{S}^3$ such as the equator and the Clifford torus. These are also important because the cones over such minimal surfaces in the sphere are minimal cones which are important in the understanding of singularities.

Questions regarding the uniqueness of minimal surfaces in $\mathbb{S}^3$ have attracted mathematicians over the centuries. In 1966, F. J. Almgren [9] showed that the only immersed minimal surface in the 3-sphere with genus $g = 0$ is the equator. In 1970, H. B. Lawson [71] showed that there is at least one compact embedded minimal surface in $\mathbb{S}^3$ with genus g, for any $g \in \mathbb{Z}^+$. Moreover, he provided the proof of the existence of at least two minimal surfaces in the case $g > 1$ for which g is not a prime number. In the same year, he also made an outstanding conjecture [72]. He conjectured that in the case $g = 1$, the only embedded minimal surface in $\mathbb{S}^3$ (up to ambient isometries) is the Clifford torus. In 2012, S. Brendle [23] was able to provide a positive and crucial answer for the conjecture of Lawson that is mentioned above. A powerful tool that was used in the proof of Lawsons's conjecture is the non-collapsing technique, in which the maximum principle argument was applied to an appropriate two-point function. Brendle's proof of Lawson's conjecture is provided in section 4.1.

Analogous questions in higher dimensions have also been paid considerable attention. Of particular interest are immersed or embedded hypersurfaces with prescribed curvatures in the $(n + 1)$-dimensional sphere $\mathbb{S}^{n+1}$. Note that the results of the two-dimensional case cannot be expected to hold with the same assumptions in higher dimensions, since there are examples of embedded CMC hypersurfaces arising from the theory of isoparametric hypersurfaces [78] which are not products of spheres. The special role of the two-dimensional setting in Brendle's proof arises in the argument that the largest principal curvature is non-vanishing, which uses the holomorphicity of a Hopf differential which is special to the two-dimensional case.

There are some crucial works on this area with respect to the norm of the second fundamental form. Precisely, assume that Σ is an n-dimensional minimal hypersurface immersed into the sphere $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ and $|A|^2$ is the squared norm of the second fundamental form of Σ. In the pioneering work of J. Simons [96], it was derived that if $|A|^2 \leq n$ on Σ, then either $|A|^2 = 0$, that is Σ is totally geodesic, or $|A|^2 = n$. Moreover, S. S. Chern, M. do Carmo and S. Kobayashi [33] and H. Lawson [70] determined all minimal hypersurfaces of $\mathbb{S}^{n+1}$ satisfying $|A|^2 = n$. In particular, they locally showed that $|A|^2 = n$ if and only if Σ is a Clifford torus, a direct product of spheres.

Other significant studies on embedded hypersurfaces with assumptions on the number of distinct principal curvatures and H_m in $\mathbb{S}^{n+1}$ where H_m refers to the m-th elementary symmetric function of the principal curvatures have tackled by mathematicians. T. Otsuki in his work ([83], [84]) proved that every compact embedded minimal hypersurface Σ^n of $\mathbb{S}^{n+1}$ of two distinct principal curvatures with multiplicities $m \geq 2$ and $k = n - m$ respectively at each point is locally congruent to the Clifford torus $\mathbb{S}^m\left(\sqrt{\frac{m}{n}}\right) \times \mathbb{S}^{n-m}\left(\sqrt{\frac{n-m}{n}}\right)$. Besides, he showed that there exist infinitely many immersed minimal hypersurfaces when the multiplicity of one of the two principal curvatures is one by solving an ordinary differential equation of second order. Following the work of Otsuki, H. Li and G. Wei [74] extended these hypersurfaces of two distinct principal curvatures to the case where the m-th mean curvature H_m vanishes. More precisely, they deduced that these minimal hypersurfaces are congruent to totally geodesic spheres and Clifford tori.

A recent beautiful work on the uniqueness for a class of Weingarten hypersurfaces in the $(n + 1)$-sphere was tackled using the non-collapsing technique. B. Andrews, Z. Huang and H. Li [17] considered a special class of embedded Weingarten hypersurfaces in $\mathbb{S}^{n+1}$ with some assumptions. They assumed that these hypersurfaces satisfying a linear relation between their distinct principal curvatures with imposing a condition on the number of these curvatures. As a result, they showed that these distinct principal curvatures are constant and thus these hypersurfaces should be congruent to a Clifford torus. Similar to Brendle's proof of Lawson's conjecture [23], the authors in [17] rely on the non-collapsing machinery to show this result, see section 4.3. In fact, this work provided a simple proof to the above results ([83], [84] and [74]). For more work on this direction of research (see [37], [21], [48], [95], [100], [106] and [85]).

The study of CMC surfaces in spaces $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$ is a classical subject in differential geometry. In 1853, J. H. Jellet [61] showed that CMC spheres that are star-shaped about some point in $\mathbb{R}^3$ are round. Also, H. Hopf raised a long-standing question on whether a compact CMC surface in $\mathbb{R}^3$ is necessarily a standard sphere or not. In the mid of twenty century, H. Hopf [54] proved that any CMC sphere in such ambient space must be totally umbilical by constructing the Hopf holomorphic differential for these surfaces. A. D. Alexandrov [6] showed that a CMC surface with the property of embeddedness in $\mathbb{R}^3$, a hemisphere $\mathbb{S}^3_+$ or $\mathbb{H}^3$ must be a round sphere. Also, the above result of Hopf was extended to the ambient spaces $\mathbb{S}^3$ and $\mathbb{H}^3$ by S. S. Chern [32]. In 1984, H. C. Wente [102] was the first to construct compact CMC surfaces of genus one in $\mathbb{R}^3$, thereby answering Hopf's question in the negative.

Note that the above technique provided by H. Hopf [54] was extended and applied in different contexts. For instance, E. Calabi applied this approach of H. Hopf for immersed minimal 2-dimensional spheres into the n-dimensional sphere [28] and S. .S. Chern [31] extended this to some more general ambient manifolds. Moreover, this method was used by U. Abresch and H. Rosenberg to study CMC spheres in homogeneous product spaces under some assumptions and derive that such surfaces are rotational (see [3] and [4]). The reader may refer to [62], [103], [99] and [60] for further information on the classification of these surfaces in $\mathbb{R}^3$ and to [7] and [32] for CMC surfaces in $\mathbb{S}^3$.

Shortly after the discovery of non-spherical CMC surfaces by Wente, many results were obtained on the structure and properties of Wente tori. U. Abresch [1] provided the classification of all CMC tori with planar curvature lines in $\mathbb{R}^3$ by solving an appropriate system of ordinary differential equations. Moreover, U. Pinkall and I. Sterling [88] classified generally all CMC tori in $\mathbb{R}^3$ by using a simple and implicit theorem. By considering this work, A. I. Bobenko [22] constructed CMC tori in $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$. Beside the paper of O. M. Perdomo [86], B. Andrews and H. Li [20] provided recently a complete classification of all embedded CMC tori in $\mathbb{S}^3$. In particular they showed that all embedded CMC tori in $\mathbb{S}^3$ are rotationally symmetric, confirming a conjecture of U. Pinkall and I. Sterling [88]. The proof of this beautiful result is based on an extension of

Brendle's argument for the proof of Lawson's conjecture for minimal surfaces [23] to the case of embedded CMC tori in $\mathbb{S}^3$, see section 4.2.

In addition to classifying embedded minimal and CMC hypersurfaces in $\mathbb{Q}^{n+p}(c)$, the study of hypersurfaces deforming by a normal velocity that is prescribed by their curvature has been a subject of increasing attention in differential equations and geometry. If the speed is monotone as a function of the principal curvatures, then these processes are described by fully nonlinear parabolic partial differential equations. These processes are also referred to as curvature flows. In ideal situations such flows converge to limiting states which are hypersurfaces of prescribed curvature, so these curvature flows are potentially useful both in studying prescribed curvature hypersurfaces and also in understanding the topology of hypersurfaces with given restrictions on curvature. Unfortunately, singularities can occur in some situations. A current active area of research is to understand the possible formation of singularity in various settings, see [55], [56], [59], [36], [41] and [94].

One of the most remarkable extrinsic geometric flows is the mean curvature flow (MCF). The global behaviour of smooth, compact, convex hypersurfaces evolving by the MCF in Euclidean spaces has been studied by many authors. In 1984, G. Huisken [55] showed that if the initial hypersurface is convex in $\mathbb{R}^{n+1}$ where $n \geq 2$, then there exists a unique solution to such flow. Also, he proved that the solution shrinks to a single point in finite time in such a way that, by using a proper rescaling about the final point, the resulting family of hypersurfaces converges smoothly to a round sphere. Furthermore, Gage and Hamilton [43] derived the analogous result for closed, embedded, curves in $\mathbb{R}^2$. In other words, they proved that every embedded, planar convex curve shrinks to a round point under the curve shortening flow (CSF), which is the one-dimensional case of the MCF.

The non-collapsing technique is an argument based on applying a maximum principle to a function depending on two points in a hypersurface, to prove a geometric non-collapsing property (precisely, comparing the radius of the largest ball which touches the hypersurface at a given point to the local geometry (curvature) at that point. This kind of estimate first arose in the context of mean curvature flow: The 2009 paper by Weimin Sheng and Xujia Wang [94] introduced a useful concept of non-collapsing for mean curvature flows, as the condition that every point of the evolving hypersurface is touched by a ball of radius bounded below by a multiple of the reciprocal of the mean curvature at that point. By a detailed analysis of the singularities and behaviour of the mean curvature flow they proved that any compact mean-convex initial hypersurface gives rise to a mean curvature flow which is non-collapsed in this sense. Similar estimates, thought not phrased in precisely this way, were also proved by White [104, 105]. In 2012, B. Andrews [12] showed that such a non-collapsing estimate can be proved directly using the two-point maximum principle (refer also to section 3.2 for more details).

Shortly after this work, S. Brendle adapted and refined the two-point maximum principle argument in the context of embedded minimal tori, and showed that it is possible to compare the size of the touching ball to the maximum principal curvature at each point. This was the basis of his proof of the Lawson conjecture mentioned above. Brendle's method was extended to the case of embedded constant mean curvature (CMC) tori in $\mathbb{S}^3$ by B. Andrews and H. Li [20], confirming the Pinkall-Sterling conjecture.

The non-collapsing argument has subsequently been extended and applied in various contexts such as fully nonlinear curvature flows (analogues of the mean curvature flow where the speed depends on a nonlinear function of the principal curvatures). The result in [12] was extended by B. Andrews, M. Langford and J. McCoy [19] to fully non-linear curvature flow where the speed function of the principal curvatures is homogeneous of degree one and convex or concave. In this study, the authors proved that the boundary curvature of the smallest exterior (largest interior) sphere which touches the hypersurface at each point is a supersolution (subsolution) of the linearized curvature flow if the speed function is convex (concave), see section 3.3. Further study was provided by B. Andrews, X. Han, H. Li and Y. Wei [16] showing that the previous result of the non-collapsing estimate can be extended to the ambient spaces $\mathbb{S}^{n+1}$ and $\mathbb{H}^{n+1}$, and by B. Andrews and M. Langford [18] to allow touching balls on both sides of the evolving hypersurface for some classes of fully nonlinear flows.

1.2 Results and outline of the thesis

The main results and the structure of the thesis are illustrated in this section.

The research work conducted throughout the thesis is motivated by the technique of non-collapsing that was used in the work of B. Andrews [12] for compact, embedded and mean convex hypersurfaces moving under the MCF in $\mathbb{R}^{n+1}$ where the maximum principle argument was applied to a two point function, implying a non-collapsing inequality. This technique in the previous work [12] was modified by S. Brendle in his work on the uniqueness conjecture of Lawson [23] to be applicable to the case of minimal surfaces.

Since then, various results have been proved based on developing this argument in different constructions such as the proof of the Pinkall-Sterling conjecture [20] on constant mean curvature hypersurfaces in the sphere. Also, the extension of the non-collapsing machinery to a special class of embedded hypersurfaces in $\mathbb{S}^{n+1}$ satisfying a linear relation between their distinct principal curvatures was tackled by B. Andrews, Z. Huang and H. Li [17] yielding the uniqueness result that such hypersurfaces are congruent to the Clifford torus. These results have motivated the research undertaken in this thesis.

The first main result of the thesis is an extension of work [17] on the uniqueness of a class of Weingarten hypersurfaces in S^{n+1}. In particular, we consider a larger class of embedded Weingarten hypersurfaces Σ^n with two distinct principal curvatures λ and μ at each point in S^{n+1}. We assume that this hypersurface satisfies the following form of PDE:

$$G(\lambda, \mu) = \lambda + \mu - \Phi(\lambda - \mu), \qquad \lambda > \mu$$

where $G(\lambda, \mu)$ is a symmetric function of principal curvatures. Also, further assumptions on $G(\lambda, \mu)$ are imposed as follows:

- $\frac{\partial G}{\partial \lambda}$, $\frac{\partial G}{\partial \mu}$, $\frac{\partial G}{\partial \lambda}\lambda$ and $\frac{\partial G}{\partial \mu}\mu$ are positive.
- The set $\{(\lambda, \mu) : G(\lambda, \mu) \geq 0\}$ is convex.
- The following inequality

  $$-\frac{\partial^2 G}{\partial \lambda^2}\left(\frac{\partial G}{\partial \mu}\right)^2 + 2\frac{\partial^2 G}{\partial \lambda \partial \mu}\frac{\partial G}{\partial \lambda}\frac{\partial G}{\partial \mu} - \frac{\partial^2 G}{\partial \mu^2}\left(\frac{\partial G}{\partial \lambda}\right)^2 \leq \frac{2}{\lambda - \mu}\frac{\partial G}{\partial \lambda}\frac{\partial G}{\partial \mu}\left(\frac{\partial G}{\partial \lambda} + \frac{\partial G}{\partial \mu}\right)$$

  holds, where $G(\lambda, \mu) = 0$

Precisely, we prove the following theorem:

Theorem 1.1. *Let $F : \Sigma \to \mathbb{S}^{n+1}$ be a compact embedded hypersurface with principal curvatures λ and μ, where $\lambda > \mu$. Assume that these principal curvatures λ and μ have multiplicities m and n-m respectively and satisfy the above linear relation $G(\lambda, \mu)$. Also, suppose that $\Phi(t)$ is a function satisfying the conditions: $0 \leq t\Phi'(t) < min\{\Phi(t), t\}$ and $0 \leq t\Phi''(t) < 1 - \Phi'(t)^2$, where $t = \lambda - \mu$. Then λ and μ are constant and Σ is congruent to the Clifford torus.*

The proof of such result involves the application of the maximum principle to a two-point function that is defined on our class of embedded Weingarten hypersurfaces of $\mathbb{S}^{n+1}$. More precisely, we define a quantity that characterises the curvature of the largest ball in the enclosed region which touches the hypersurface at a given point x and is denoted by $\bar{k}(x)$. Then, we show that this inscribed ball curvature $\bar{k}(x)$ satisfies a natural differential inequality in a viscosity sense by applying the maximum principle argument to this function.

In order to clarify the proof of this theorem, we begin by illustrating some geometric concepts of embedded Weingarten hypersurafces of $\mathbb{S}^{n+1}$ assuming a condition on the number of multiplicities of the principal curvatures. Also, under these assumptions we show that this class of hypersurafces is rotationally symmetric. In the second section, we derive the analogous identity of Simon for the maximum principal curvature λ of our class of hypersurfaces in the proposed setting. In the next section, we deduce that the inscribed ball curvature $\bar{k}(x)$ satisfies a differential inequality in the viscosity sense using the maximum principle argument. In the final section, we arrive to the equality $\lambda = \bar{k}$ everywhere on Σ by combining the previous two results. Hence, such equality enables us to deduce that the principal curvatures are constant and then yields to the result of our theorem. This is the structure of chapter 5.

The second main area of study concerns the flow of mean convex spacelike hypersurfaces evolving by their mean curvature in Minkowski space $\mathbb{L}^{n,1}$, particularly, the work of Andrews [12] (see section 3.2) is extended to mean convex embedded spacelike hypersurfaces evolving by the MCF of $\mathbb{L}^{n,1}$. More precisely, we derive a non-collapsing estimate by employing the Omori-Yau maximum principal to a quantity that relies on two points of these spacelike hypersurfaces. The proof of such result involves comparing the radius of the largest hyperbola which touches the spacelike hypersurface at a given point to the curvature at that point.

To clarify, we assume that $F : \Sigma^n \times [0, T) \to \mathbb{L}^{n,1}$ is an embedding of mean convex spacelike hypersurface into $\mathbb{L}^{n,1}$ and satisifes the MCF that is given by

$$\frac{\partial F(x,t)}{\partial t} = -H(x,t)\nu(x,t),$$

for $x \in \Sigma$, $t \in [0, T)$ and where H and ν refer to the mean curvature and future-directed unit normal vector field on spacelike hypersurfaces, respectively. The mean convexity condition indicates that the mean curvature of spacelike hypersurfaces is positive everywhere. By utilizing the machinery of non-collapsing for these embedded spacelike hypersurfaces, we deduce that the interior curvature of the touching hyperbola satisfies a differential inequality in the viscosity sense.

Hence, the second main theorem of the thesis is given by the following:

Theorem 1.2. *Let Σ^n be a mean convex spacelike hypersurface, and $F : \Sigma^n \times [0,T) \to \mathbb{L}^{n,1}$ a family of smooth embeddings deforming by the MCF in $\mathbb{L}^{n,1}$. Then, the interior curvature function $\overline{Z}$ is a viscosity subsolution of the equation*

$$\frac{\partial \overline{Z}}{\partial t} = \Delta \overline{Z} - |A|^2 \overline{Z}.$$

The fundamental content of our proof is based on a non-collapsing technique where the Omori-Yau maximum principle is applied in this situation.

In order to explain our second result, we first introduce some notions of Semi-Riemannian manifolds and the geometry of spacelike hypersurfaces in $\mathbb{L}^{n,1}$. Then, we deduce the evolution equations of some related geometric quantities such as the metric, the second fundamental form, the Weingarten map and the mean curvature. Also, we represent the generalized Omori-Yau maximum principle in the next section. In the final and main section of this result, we provide the proof of a non-collapsing estimate for our evolving spacelike hypersurfaces by applying the Omori-Yau maximum principle to the boundary curvature of the touching hyperbola defined on these spacelike hypersurfaces. This is the structure of chapter 6.

The content of this thesis is structured as follows: some preliminaries and conceptual principals are introduced in chapter 2. These include the geometry of hypersurfaces in the Euclidean space, curvature functions and the mean curvature flow MCF. Also, we represent one of the basic applications of the maximum principle argument to some parabolic equations in the one-dimensional space and on manifolds.

In chapter 3, we present some existing results for several extrinsic geometric flows in $\mathbb{R}^{n+1}$ employing the non-collapsing technique. We first illustrate the proof of some basic properties on the CSF involving the avoidance principle, the preservation of embeddedness and Huisken's estimate [55]. Then, we include the proof of a sharp estimate that was deduced by B. Andrews [12] for embedded mean convex hypersurfaces moving under the MCF in $\mathbb{R}^{n+1}$. In the final section of such chapter, we represent the extension of such work [12] to fully non linear curvature flow where the speed function of the principal curvatures is homogeneous of degree one, convex or concave by B. Andrews, M. Langford and J. McCoy [19].

In chapter 4, we illustrate some classification results on hypersurfaces of prescribed curvatures in the unit sphere. We first describe Brendle's proof [23] of Lawson's conjecture [72] that the only embedded minimal tori in $\mathbb{S}^3$ is the Clifford torus. We then represent the proof of B. Andrews and H. Li [20] on the classification of constant mean curvature tori in $\mathbb{S}^3$ in which they showed that theses surfaces should be a surface of rotation. We also involve an interesting application of the non-collapsing method for a special class of embedded Weingarten hypersurfaces with two distinct principal curvatures satisfying a linear partial differential equation in $\mathbb{S}^{n+1}$ in the last section of this chapter. This study was tackled by B. Andrews, Z. Huang and H. Li [17] and they derived that such hypersurfaces are congruent to a Clifford torus under some conditions on multiplicities of their principal curvatures. The authors in the previous results [23], [20] and [17] also extended the previous argument of non-collapsing technique that was used in [12] [23].

The chapters 5 and 6 contain the first and second main results of the thesis that are mentioned above, respectively.

Chapter 2

Preliminaries and background

In order to clarify the notation and background results used in the thesis, some preliminaries and conceptual principles are introduced in this chapter. In the first section, we recall some basic notation and formulas of the geometry of hypersurfaces in the Euclidean space. These include defining some terminology: the tensor bundles, first and second fundamental forms, connections and curvature tensors. Also, some important identities are introduced such as the Gauss and Codazzi relations and Simons' type identities. Since we study the behaviour of embedded hypersurfaces with prescribed curvatures, it is useful to introduce curvature functions. In particular, we consider embedded hypersurfaces Σ^n moving under a speed function $\mathcal{S}$ that is given by symmetric functions f in section 2.2. Also, some basic results of f and properties of both $\mathcal{S}$ and f are illustrated in this section.

Moreover, we consider an extrinsic geometric flow, the mean curvature flow (MCF), in the next section 2.3. More precisely, we introduce some basic concepts of this flow including evolution equations for geometric quantities under the MCF. Since the maximum principle is a significant tool that will be applied throughout the thesis, we illustrate its application to tensor inequalities in section 2.4.

In the final section, we present one of the main applications of the maximum principle argument to some parabolic equations, precisely heat equations, in one-dimensional space and on manifolds.

2.1 Geometric preliminaries

The theory of submanifolds has been extensively studied since the appearance of differential geometry of curves and surfaces, in the late 17^{th} century. The study of this discipline has significantly contributed to the development of differential geometry (see [81] and [38]). In order to have a good understanding of some results that are related to such area in this thesis, it will be useful to recall some general notions and concepts of the geometry of hypersurfaces, which are submanifolds with codimension 1 in the Euclidean space $\mathbb{R}^{n+1}$.

The main purpose of this section is to introduce the necessary background on immersed hypersurfaces in the Euclidean space $\mathbb{R}^{n+1}$. In particular, this section includes a discussion of geometric structures on manifolds such as tangent bundles, tensor bundles, metrics, connections and curvature tensors which are used in our later computations. Besides, the specific application of these concepts in the setting of hypersurfaces in $\mathbb{R}^{n+1}$ is described, as well as the additional structure such as normal vectors and second fundamental form which arise in the hypersurface setting. Moreover, some crucial identities are included such as the Gauss Identity, which relates the curvature tensor to the second fundamental form, and the Codazzi equations. We conclude this section with the proof of a useful generalisation of an identity of Simons [96] that is used frequently in computations throughout the thesis.

2.1.1 Manifolds

Our terminology and conventions for manifolds will be consistent with that in the books by Lee [73] and Do Carmo [38]. In particular the manifolds M^n we consider are Hausdorff, paracompact topological spaces equipped with an atlas of homeomorphic charts $\varphi_\alpha : U_\alpha \subset M \to V_\alpha \subset \mathbb{R}^n$ such that $\{U_\alpha\}$ is an open cover of M and for which the transition maps $\varphi_\beta \circ \varphi_\alpha^{-1}$ are smooth. Functions on M are smooth when their composition with the inverse of a chart is smooth, and maps between manifolds are smooth when their composition with charts is a smooth map between Euclidean spaces.

A tangent vector v to M at a point x are differential operators which act $\mathbb{R}$-linearly on germs of smooth functions defined near x:

$$v(af + bg) = av(f) + bv(g)$$

for all smooth f, g defined near x and $a, b \in \mathbb{R}$. This is required to be consistent with the Leibniz rule:

$$v(fg) = f(x)v(g) + g(x)v(f).$$

Such a differential operator is always realised by partial differentiation in some direction in a chart for M about x. The tangent space T_xM is the vector space of such tangents, with vector space structure given by addition and scalar multiplication of operators, and also consistent with the operations on vectors in $\mathbb{R}^n$ in any chart. In particular, given a chart φ near x a basis for T_xM is provided by the partial derivatives ∂_i which act according to

$$\partial_i f = \frac{\partial}{\partial x^i}\left(f \circ \varphi^{-1}\right)\big|_{\varphi(x)}.$$

Any smooth curve $\gamma : \ I \subset \mathbb{R} \to \Sigma$ has at each point a *tangent vector* $\gamma'(s) \in T_{\gamma(s)}\Sigma$, defined by its action as a differential operator on functions f according to

$$(\gamma'(s))f = \frac{d}{d\sigma} f(\gamma(\sigma))\Big|_{\sigma=s}. \tag{2.1}$$

The *tangent bundle* TM is the smooth manifold of dimension $2n$ consisting of all tangent vectors to points of M. Given an atlas of charts $\varphi_\alpha : \ U_\alpha \to V_\alpha$ for M, local charts for TM are defined by sending a tangent vector $v = \sum_{i=1}^n v^i \partial_i \in T_xM$ with $x \in U_\alpha$ to the point $(\varphi(x), (v^1, \cdots, v^n)) \in V_\alpha \times \mathbb{R}^n$.

A *vector field* on Σ is a smooth map X from Σ to $T\Sigma$ which associates to each $x \in \Sigma$ a tangent vector in $T_x\Sigma$. Equivalently, X is differential operator on smooth functions which is $\mathbb{R}$-linear and satisfies the Leibniz rule $X(fg) = fX(g)+gX(f)$. The space of vector fields is denoted by $\mathcal{X}(M)$, or $\Gamma(TM)$ (the space of *sections* of TM, see below). The *Lie bracket* $[X, Y]$ of two vector fields $X, Y \in \Gamma(T\Sigma)$ is the vector field which acts on a smooth function f as the commutator of the differential operators corresponding to the two vector fields, so that $[X.Y]f = X(Yf) - Y(Xf)$.

2.1.2 Bundles, metrics and connections

We start by briefly reviewing some basic terminology about tensors and tensor products, as well as vector bundles and their associated tensor bundles and tensor fields, as well as the metrics and connections which are induced by this construction. More details can be found in [67, 66] and [73].

Let Σ be a finite-dimensional smooth manifold. A vector bundle V of rank k over Σ is a smooth manifold of dimension $n+k$ equipped with a smooth submersion $\pi : \ V \to \Sigma$ and smooth local trivialisations, which are diffeomorphisms Ψ_α from $\pi^{-1}(U_\alpha)$ to $U_\alpha \times \mathbb{R}^k$ which agree with π on the first component, and such that $\Psi_\beta \circ \Psi_\alpha^{-1}$ restricts to a linear isomorphism of $\{x\} \times \mathbb{R}^k$ for each $x \in U_\alpha \cap U_\beta$. The local trivialisations provide a vector space structure on each *fibre* $V_x := \pi^{-1}(x)$. A smooth section of V is a smooth map ξ from Σ to V with $\pi \circ \xi$ equal to the identity map, so that $\xi(x) \in V_x$ for each $x \in \Sigma$. Any section ξ can be written locally over the base U_α as a C^∞-linear combination $\sum_{i=1}^k \xi(x)\psi_\alpha(x)$ of the basis of smooth local sections given by $\psi_\alpha(x) := \Psi_\alpha^{-1}(x, e_i)$, $i = 1, \cdots, k$. Conversely, any collection of sections $\{\psi_\alpha\}$ of V over a subset U of Σ which form a basis for the fibre V_x at each point $x \in U$ defines a local trivialisation of V over $\pi^{-1}(V)$ via the map $\xi = \sum_\alpha \xi^\alpha \psi_\alpha \mapsto (\pi(\xi), \xi^1, \cdots, \xi^k)$.

We note some important constructions of bundles:

If V is a vector bundle over Σ, the *dual bundle* V^* is the bundle $\{(x, \omega) : \ x \in \Sigma, \ \omega \in (V_x)^*\}$, equipped with local trivialisations $\{\psi_\alpha^*\}$ defined by the pointwise dual basis of the basis of local trivialisations $\{\psi_\alpha\}$ for V: That is, $\psi_\alpha^*(x)$ is the linear functional on V_x defined by $(\psi_\alpha^*(x))(\psi_\beta(x)) = \delta_{\alpha\beta}$.

Given any bundles V of rank k and W of rank ℓ over Σ, the *tensor product* bundle is the bundle of rank $k\ell$ over Σ with fibre at x given by $V_x \otimes W_x$ (the reader may find it convenient to identify this with the space of real bilinear forms acting on $V^* \times W^*$), with a smooth basis of local sections provided by $\{\psi_\alpha \otimes \eta_\beta : \ 1 \leq \alpha \leq k, \ 1 \leq \beta \leq \ell\}$ where $\{\psi_\alpha : \ 1 \leq \alpha \leq k\}$ and $\{\eta_\beta : \ 1 \leq \beta \leq \ell\}$ are bases of smooth local sections for V and W respectively.

If $\psi : \ \Sigma \to M$ is a smooth map between manifolds, and E is a vector bundle of rank k over M, then the *pullback bundle* of E by F, denoted F^*E, is the vector bundle of rank k over Σ defined by $F^*E = \{(x, v) : \ x \in \Sigma, \ v \in E_{F(x)}\}$, and given a local trivialisation $(y, v) \in E_y \mapsto (y, \Psi(y, v)$ of E near $F(x)$, the map $(x, v) \mapsto (x, \Psi(F(x), v))$ provides a local trivialisation of F^*E near x. In particular if $\{\psi_\alpha\}_{\alpha=1}^k$ is a basis of smooth local section of E near $F(x)$, then $\{\psi_\alpha \circ F\}_{\alpha=1}^k\}$ is a basis of smooth local sections for F^*E.

Assume that V and V^* are a vector bundle and its dual bundle respectively over Σ. The tensor product of these bundles, denoted by $V^* \otimes V$, is naturally identified with the bundle of linear maps of V, and its fibers at a point $x \in \Sigma$ are given by $V_x \otimes V_x^*$. This product bundle is isomorphic to its dual $V \otimes V^*$, the space of linear endomorphisms of V^*, under the correspondence given by associating to each linear map $L \in V^* \otimes V$ the adjoint map $L^T \in V \otimes V^*$ defined by $(L^T(\omega))(v) := \omega(L(v))$ for each $v \in V$ and $\omega \in V^*$.

A metric g on a vector bundle V is defined by assigning to each fibre V_x an inner product g_x, which varies smoothly in the sense that $g(\xi, \eta)$ is smooth whenever $\xi, \eta \in \Gamma(V)$. The metric g is itself is a section of a bundle associated to V, specifically the vector bundle $V^* \otimes V^*$, of bilinear forms on V. Such a metric on V induces an isomorphism $\iota_g : V \to V^*$ that is given by

$$(\iota_g(u))(v) = g(u, v),$$

for all u and v in V_x. Also, this metric can be extended to other tensor bundles. For example, there exists a unique metric that is also denoted by g on V^* such that ι_g is an isometry:

$$g(\iota_g(u), \iota_g(v)) = g(u, v),$$

for all $u, v \in V_x$.

If V and W are tensor bundles equipped with metrics g and h, there is a unique induced metric $g \otimes h$ on $V \otimes W$ such that $g \otimes h(u \otimes v, w \otimes z) = g(u, w)h(v, z)$.

One of the basic and crucial tools of differentiating over vector bundles is a connection. A connection on V over Σ is a bilinear map $\nabla : \Gamma(T\Sigma) \times \Gamma(V) \to \Gamma(V)$ satisfying

$$\nabla_X(fu) = f\nabla_X u + (Xf)u,$$

for $f \in C^\infty(\Sigma)$, $X \in \Gamma(T\Sigma)$ and $u \in \Gamma(V)$. The notations $\nabla_X u$ and Xf refer to the covariant derivative of u and the derivative of f respectively in the X-direction. Given a metric g on V, a connection ∇ on V is said to be metric-compatible (with g) if for any $u, v \in \Gamma(V)$ and $X \in \Gamma(T\Sigma)$ we have

$$Xg(u, v) = g(\nabla_X u, v) + g(u, \nabla_X v).$$

If ∇ is a connection on V, then there is an induced connection on V^* (which we also denote by ∇) defined by

$$(\nabla_X \omega)(\xi) = X(\omega(\xi)) - \omega(\nabla_X \xi).$$

If ∇ is compatible with ametric g on V, then the induced connection on V^* is also compatible with the induced metric on V^*.

If V and W are vector bundles equipped with connections, then there is a unique induced connection on $V \otimes W$ defined by

$$\nabla_X(\xi \otimes \eta) = (\nabla_X \xi) \otimes \eta + \xi \otimes (\nabla_X \eta).$$

An important special case of this arises when differentiating a tensor field T acting on sections of vector bundles: In this case

$$(\nabla_X T)(\xi_1, \ldots, \xi_k) = X(T(\xi_1, \ldots, \xi_k)) - T(\nabla_X \xi_1, \ldots, \xi_k) - \ldots - T(\xi_1, \ldots, \nabla_X \xi_k). \tag{2.2}$$

If V and W are equipped with metrics, and the connections are compatible, then the induced metric on $V \otimes W$ is also compatible with the induced metric.

If F^*E is the pullback bundle over Σ of a vector bundle E over M by a smooth map $F: \Sigma \to M$, and ∇ is a connection on E, then there is a unique connection ${}^F\nabla$ on F^*E defined by the requirement that ${}^F\nabla_u(\xi \circ F) = \nabla_{F_*u}\xi$ for any section ξ of E (so $\xi \circ F$ is a section of F^*E).

Observe that the connection ∇ can be used to define a tensor $R : \Gamma(T\Sigma) \times \Gamma(T\Sigma) \times \Gamma(V) \to \Gamma(V)$ called the curvature tensor of ∇ and defined by

$$R(u, v, w) = \nabla_v \nabla_u \xi + \nabla_u \nabla_v \xi + \nabla_{[u,v]}\xi.$$

The curvature of bundles constructed via duality and tensor products can be expressed simply in terms of the curvature on the original bundles: For example, for a tensor T acting on k vector bundles with scalar values,

$$(R(X, Y)T)(\xi_1, \cdots, \xi_k) = -T(R(X, Y)\xi_1, \ldots, \xi_k) - \ldots - T(\xi_1, \ldots, R(X, Y)\xi_k).$$

In the case $k = 1$ this provides an expression for the curvature of the dual bundle. The curvature tensor can be seen as measuring the local obstruction to the existence of parallel sections of the bundle V. It will be most important to us in the context of the Riemannian manifolds of the next section.

2.1.3 Riemannian manifolds

A Riemannian manifold is a smooth manifold Σ equipped with a Riemannian metric g (a metric on the tangent bundle $T\Sigma$). The metric g allows the definition of the length of any smooth curve $\gamma : I \to \Sigma$:

$$L[\gamma] = \int_I \sqrt{g_{\gamma(s)}(\gamma'(s), \gamma'(s))}\, ds, \tag{2.3}$$

where $\gamma'(s)$ is the tangent vector defined by (2.1). This in turn allows the definition of the *Riemannian distance* on each connected component of Σ by

$$d(x, y) = \inf\{L[\gamma] : \gamma : [0, 1] \to \Sigma,\ \gamma(0) = x,\ \gamma(1) = y\} \tag{2.4}$$

A Riemannian manifold also comes equipped with a preferred choice of connection on the tangent bundle, referred to as the *Riemannian connection* or *Levi-Civita* connection. This is the unique connection on $T\Sigma$ which is compatible with the Riemannian metric g and is *symmetric*, in the sense that $\nabla_X Y - \nabla_Y X = [X, Y]$ for all $X, Y \in \Gamma(T\Sigma)$.

If $\gamma : I \to \Sigma$ is a smooth curve, then the tangent vector γ' defines a section of the pull-back bundle $\gamma^*(T\Sigma)$ over the interval I. The curve γ is called a *geodesic* if ${}^{\gamma}\nabla_s\gamma' = 0$ at every point of I. Given any point $(x, v) \in T\Sigma$ there exists (for sufficiently small parameters s) a unique geodesic γ with $\gamma(0) = x$ and $\gamma'(0) = v$, denoted by $\gamma(s) = \exp_x(sv)$. For fixed $x \in \Sigma$ the map $v \in T_x\Sigma$ defines a local diffeomorphism from a neighbourhood of $0 \in T_x\Sigma$ to a neighbourhood of x in Σ. The Gauss Lemma implies that a geodesic γ achieves the distance between its endpoints provided that $\exp_x$ is a diffeomorphism on the ball of radius $L[\gamma]$ about the origin in $T_{\gamma(0)}\Sigma$.

The curvature of the Riemannian connection is called the *Riemann curvature tensor*, and satisfies the symmetries

$$R(X, Y, W, Z) = -R(Y, X, W, Z) = R(Y, X, Z, W)$$

and

$$R(X, Y, W, Z) + (R(Y, W, X, Z) + R(W, X, Y, Z) = 0$$

for each $X, Y, W, Z \in T_x\Sigma$, where $R(X, Y, W, Z) := g(R(X, Y, W), Z)$. It follows also that the symmetry $R(X, Y, W, Z) = R(W, Z, X, Y)$ holds.

Given a pair of orthogonal unit vectors u and v in $T_x\Sigma$, the *sectional curvature* of the plane $u \wedge v$ generated by u and v in $T_x\Sigma$ is defined by $\sigma(u]wedgev) := R(u, v, u, v)$. This is independent of the choice of orthonormal basis.

For a fixed vector $v \in T_x\Sigma$, the *flag curvature* in direction v is the bilinear form on $T_x\Sigma$ defined by $R_v(X, Y) = R(u, X, u, Y)$. The bilinear form $R_v(X, X)$ gives zero for $X \in \mathbb{R}v$, and returns $|X|^2$ times the sectional curvature of the plane spanned by v and X if $X \perp v$. The trace of R_v (equivalently, the sum of principal curvatures of $v \wedge e_i$ over an orthonormal basis $e_1, \cdots, e_{n-1}$ for $v^\perp \subset T_x\Sigma$) is called the *Ricci curvature* in direction v. The polarisation of this in v defines a bilinear form Rc on $T_x\Sigma$ defined by $\mathrm{Rc}(X, Y) = g^{kl}R_{X,\partial_k,Y,\partial_l}$. The trace of the Ricci tensor is called the *scalar curvature* $R = g^{ij}g^{kl}R_{ikjl} = g^{ij}R_{ij}$.

2.1.4 Hypersurfaces

In order to introduce some geometric structures of smooth hypersurfaces in $\mathbb{R}^{n+1}$, let F be an immersion of a smooth n-dimensional manifold Σ into $\mathbb{R}^{n+1}$. The pullback bundle $F^*T\mathbb{R}^{n+1}$ is equipped with the metric and compatible connection induced by the standard inner product $\langle \cdot, \cdot \rangle$ and derivative D on $\mathbb{R}^{n+1}$. The derivative F_* is a linear map from $T_x\Sigma$ to $T_{F(x)}\mathbb{R}^{n+1} = (F^*T\mathbb{R}^{n+1})_x$, and so defines a section of $T^*\Sigma \otimes F^*T\mathbb{R}^{n+1}$. The fibre $F^*T\mathbb{R}^{n+1}$ decomposes into the orthogonal direct product of the image of F_* (which is isomorphic to $T\Sigma$ via F_* since F_* is injective) and the orthogonal complement $N\Sigma$, which we refer to as the *normal bundle* of Σ. In the case of hypersurfaces $N\Sigma$ is a vector bundle of rank 1, and if Σ is embedded then $N\Sigma$ is trivial, so that there is a globally defined unit normal vector ν.

The restriction of the induced metric on $F^*T\mathbb{R}^{n+1}$ induces a Riemannian metric g (also called the *first fundamental form* of Σ) by restriction to $F_*(T\Sigma)$: Explicitly,

$$g_x(u, v) = \langle F_*(u), F_*(v) \rangle .$$

The induced connection ${}^F D$ on $F^*T\mathbb{R}^{n+1}$ defines a connection ∇ (called the *submanifold connection* on $T\Sigma$, and a bilinear form h called the *second fundamental form* of Σ, by orthogonal projection onto the tangential and normal sub-spaces:

$${}^F D_X(F_*Y) = F_*(\nabla_X Y) - h(X, Y)\nu. \tag{2.5}$$

The submanifold connection is symmetric and compatible with the induced metric g, and so coincides with the Levi-Civita connection of g. Similarly, D induces an operator $\mathcal{W} \in \Gamma(T^*\Sigma \otimes T\Sigma)$ by projection onto the tangential part:

$${}^F D_X \nu = F_*(\mathcal{W}(X)). \tag{2.6}$$

The Weingarten map $\mathcal{W}$ and the second fundamental form h describe the *extrinsic curvature* of the hypersurface. The relation between these is given by by *Weingarten relation*:

$$h(u, v) = \langle \mathcal{W}(u), v \rangle,$$

for all $u, v \in T_x\Sigma$.

The eigenvalues of $\mathcal{W}_x$, which are also the eigenvalues of h with respect to g, are denoted by λ_i where $i = 1, ..., n$ and called the principal curvatures. The eigenvectors of $\mathcal{W}_x$ are called the principal directions of Σ at x. If the principal curvatures are equal at a point $x \in \Sigma$, we say that x is an umbilic point of Σ. If all points of Σ are umbilic, then we say that Σ is a totally umbilical hypersurface. For example, the hyperplanes and hyperspheres are totally umbilical hypersurfaces in $\mathbb{R}^{n+1}$.

One of the fundamental examples of an embedded hypersurface is the n-sphere $\mathbb{S}^n(r)$ with a radius $r > 0$, defined by

$$\mathbb{S}^n(r) = \{q \in \mathbb{R}^{n+1} : g_{\mathbb{R}^{n+1}}(q, q) = r\}.$$

The hypersurface $\mathbb{S}^n$ is embedded in the Euclidean space $\mathbb{R}^{n+1}$ by the inclusion map ι, and the metric induced as above on $T\Sigma$ is called the *round metric* on S^n. In this example, the unit normal ν can be taken to be $\nu(x) = \frac{x}{r}$, and so the Weingarten map is $\frac{1}{r}$ times the identity map on TS^n. By the Weingarten relation, the second fundamental form h is $\frac{1}{r}$ times the metric g, and the principal curvatures are all equal to $\frac{1}{r}$.

The second fundamental form h and the curvature tensor R of the Riemannian metric g induced on $T\Sigma$ satisfy the following identities:

$$R(u, v, w, z) = h(v, w)h(u, z) - h(u, w)h(v, z), \tag{2.7}$$

and

$$(\nabla h)(u, v, w) = (\nabla h)(v, w, u) = (\nabla h)(w, u, v), \tag{2.8}$$

where ∇h is defined according to Equation (2.2):

$$(\nabla h)(u, v, w) = u\,(h(v, w)) - h(\nabla_u v, w) - h(v, \nabla_u w).$$

The two crucial identities (2.7) and (2.8) are known as the Gauss and Codazzi equations for Σ respectively and can be locally expressed as follows

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk},$$

and

$$\nabla_i h_{jk} = \nabla_j h_{ik},$$

respectively. Therefore, for the interchange of two covariant derivatives applied to a vector $u = u^i \partial_i$ we have

$$\begin{aligned} \nabla_i \nabla_j u^m &= \nabla_j \nabla_i u^m + R^m_{ijk} u^k \\ &= \nabla_j \nabla_i u^m + (h_{ik}h_{jl} - h_{lk}h_{ij})g^{lm}u^k. \end{aligned}$$

Using the last relation and the Codazzi equation (2.8) yields the following well-known Simons' identity [96, 55]:

$$\Delta h_{ij} = \nabla_i \nabla_j H + H h_{ik} g^{kl} h_{lj} - |A|^2 h_{ij}. \tag{2.9}$$

In local coordinates the structural equation (2.5) becomes

$$\partial_i \partial_j F = \Gamma_{ij}{}^k \partial_k F - h_{ij}\nu, \tag{2.10}$$

where the Christoffel symbols ${\Gamma_{ij}}^k$ are defined by $\nabla_i \partial_j = {\Gamma_{ij}}^k \partial_k$; and (2.6) becomes

$$\partial_i \nu = h_{ij} g^{jk} \partial_k F. \tag{2.11}$$

Finally, considering the previous identities (2.10) and (2.11) enables us to derive the following remarkable commutation identity which generalises the Simons' Identity (2.9):

Proposition 2.1. *Let u, v, w, z be tangent vectors in $T_p\Sigma$ at any $p \in \Sigma$ and ν be a unit normal vector. Then*

$$\begin{aligned}\nabla_u \nabla_v h(w, z) = \nabla_w \nabla_z h(u, v) &+ h(u, v) h^2(w, z) - h(w, v) h^2(u, z)\\ &+ h(u, z) h^2(v, w) - h(w, z) h^2(u, v),\end{aligned}$$

where $h^2(u, v) = h(u, \mathcal{W}(v))$ (or in coordinates $h^2_{ij} = h_{ip} g^{pq} h_{qj}$).

Proof. From the equations (2.10) and (2.11) and the symmetry of h, we have

$$\begin{aligned}\nabla_u \nabla_v h(w, z) &= \nabla_u \nabla_w h(v, z)\\ &= \nabla_w \nabla_u h(v, z) + (R(w, u)h)(v, z)\\ &= \nabla_w \nabla_u h(z, v) - h(R(w, u)v, z) - h(v, R(w, u)z)\\ &= \nabla_w \nabla_z h(u, v) - h(h(w, v)\mathcal{W}(u) - h(u, v)\mathcal{W}(w), z)\\ &- h(h(w, z)\mathcal{W}(u) - h(u, z)\mathcal{W}(w))\\ &= \nabla_w \nabla_z h(u, v) - h(w, v) h^2(u, z) + h(u, v) H^2(w, z)\\ &- h(w, z) h^2(u, v) + h(u, z) h^2(v, w).\end{aligned}$$ □

2.2 Functions of eigenvalues

One of the main purposes of this thesis is to study immersed hypersurfaces Σ^n with principal curvatures λ_i $(i = 1, ..., n)$ satisfying at each point an equation involving a function of the principal curvatures of the form

$$\mathcal{S} = f(\lambda_1, ..., \lambda_n),$$

where the function f is assumed to be symmetric and defined on the positive cone

$$\Gamma_+ = \{\lambda = (\lambda_1, ..., \lambda_n) : \lambda_i > 0, i = 1, ..., n\}.$$

Therefore, it is useful to briefly mention some preliminary results on symmetric functions f in this section. Moreover, we discuss some important properties of $\mathcal{S}$ as a function of the components of the second fundamental form.

Let $f : \Gamma_+ \subset \mathbb{R}^n \to \mathbb{R}$ be a smooth function defined on the positive cone Γ_+ which is symmetric, in the sense that $f(\lambda_1, ..., \lambda_n) = f(\lambda_{\sigma(1)}, ..., \lambda_{\sigma(n)})$ for any permutation σ of the set $\{1, ..., n\}$. For more details on symmetric functions see [11] and [69]. Denote by $\frac{\partial f}{\partial \lambda_j}$ the derivative of f with respect to λ_j. The following observation will be useful in our later computations:

Lemma 2.2. *If f is convex (concave) on a symmetric convex cone $\Gamma \subset \mathbb{R}^n$, then*

$$\left(\frac{\partial f}{\partial \lambda_i}\Big|_\lambda - \frac{\partial f}{\partial \lambda_j}\Big|_\lambda \right) (\lambda_i - \lambda_j) \geq 0 \quad (\leq 0)$$

for any $i \neq j$ and any $\lambda = (\lambda_1, \cdots, \lambda_n) \in \Gamma$.

Proof. Fix $i \neq j$ and $\lambda \in \Gamma$, and let $\xi = e_i - e_j$, where $\{e_i\}$ is the standard basis for $\mathbb{R}^n$. Then $D_\xi f = \frac{\partial f}{\partial \lambda_i} - \frac{\partial f}{\partial \lambda_j}$. Let γ be the line segment defined by $\gamma(s) = \lambda + \frac{1-s}{2}(\lambda_j - \lambda_i)\xi$ where $s \in [-1, 1]$, so that $\gamma_i(s) = \frac{1+s}{2}\lambda_i + \frac{1-s}{2}\lambda_j$, $\gamma_j(s) = \frac{1-s}{2}\lambda_i + \frac{1+s}{2}\lambda_j$, and $\gamma_k(s) = \lambda_k$ for $k \neq i, j$. We note that $\gamma(1) = \lambda$, and that $\gamma(-s)$ is obtained from $\gamma(s)$ by applying the permutation which switches the ith and jth components. The convexity and symmetry of Γ implies that $\gamma(s) \in \Gamma$ for each $s \in [-1, 1]$. The symmetry of f then implies that $f(\gamma(s))$ is an even function of s which is convex (concave), so that the derivative is non-negative (non-positive) for $s \geq 0$, and in particular this is true for $s = 1$. This gives

$$\frac{(\lambda_i - \lambda_j)}{2} \left(\frac{\partial f}{\partial \lambda_i}\Big|_\lambda - \frac{\partial f}{\partial \lambda_j}\Big|_\lambda \right) = \frac{\lambda_i - \lambda_j}{2} D_\xi f(\lambda) = \frac{d}{ds}(f(\gamma(s)))\big|_{s=1} \geq 0 \quad (\leq 0),$$

and the claimed inequality follows. $\square$

An equivalent statement of Lemma 2.2 is as follows: If the eigenvalues λ_i are arranged in increasing order, so that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, then the derivatives $\frac{\partial f}{\partial \lambda_i}$ are in increasing order if f is convex:

$$\frac{\partial f}{\partial \lambda_1}\Big|_\lambda \leq \frac{\partial f}{\partial \lambda_2}\Big|_\lambda \leq \cdots \leq \frac{\partial f}{\partial \lambda_n}\Big|_\lambda.$$

The reverse inequalities hold in the case where f is concave.

We now consider the properties of a function of the eigenvalues when considered as a function of the components of the matrix. Denote the space of positive definite symmetric matrices by $\mathfrak{S}_+ := \{A \in \mathrm{Sym}(n) : \lambda(A) \in \Gamma_+\}$. Given a symmetric function f defined on Γ_+ as above, let $\mathcal{S} : \mathfrak{S}_+ \to \mathbb{R}$ be the real-valued function that is defined by the following equation:

$$\mathcal{S}(A) = f(\lambda(A)), \tag{2.12}$$

where $\lambda(A)$ is the n-tuple of eigenvalues of A. Note that $\lambda(A)$ is well-defined if we make the requirement that the eigenvalues are in increasing order. However the symmetry of f also means that changing the order of the eigenvalues does not change the value of $\mathcal{S}$. If $\mathbb{O}$ is an orthogonal transformation, then $\lambda(\mathbb{O}^T A \mathbb{O}) = \lambda(A)$, and therefore $\mathcal{S}(\mathbb{O}^T A \mathbb{O}) = \mathcal{S}(A)$. That is, $\mathcal{S}$ is an $O(n)$-invariant function defined on $\mathfrak{S}_+$. Conversely, an $O(n)$-invariant function $\mathcal{S}$ on $\mathfrak{S}_+$ arises in this way from the symmetric function f defined by $f(\lambda_1, \cdots, \lambda_n) = \mathcal{S}(\operatorname{diag}(\lambda_1, \cdots, \lambda_n))$, where $\operatorname{diag}(\lambda_1, \cdots, \lambda_n)$ is the diagonal matrix with entries $\lambda_1, \cdots, \lambda_n$ on the diagonal.

Lemma 2.3. *For a symmetric function f and an $O(n)$-invariant function $\mathcal{S}$ related by $\mathcal{S}(A) = f(\lambda(A))$, f is C^∞ if and only if $\mathcal{S}$ is C^∞.*

Proof. If f is symmetric and smooth on Γ_+, then for any compact symmetric $K \subset \Gamma_+$ there exists a smooth extension $\hat{f}$ of $f|_K$ to $\mathbb{R}^n$. By averaging over the orbit of the group of permutations we can ensure that $\hat{f}$ is symmetric. A theorem of Glaeser [47] implies that $\hat{f} = g \circ \rho$, where g is smooth and $\rho = (\sigma_1, \ldots, \sigma_n)$ is the vector of elementary symmetric functions $\sigma_k = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$. Since the elementary symmetric functions $E_k(A)$ of the eigenvalues of A are a polynomial function of the components of A, it follows that $\mathcal{S}$ is smooth in the components of A on $\lambda^{-1}(K) \subset \mathfrak{S}_+$. Since K is arbitrary, $\mathcal{S}$ is smooth on $\mathfrak{S}_+$.

Conversely, If $\mathcal{S}$ is smooth and $O(n)$-invariant on $\mathfrak{S}_+$, then for any compact $K \subset \mathfrak{S}_+$ there is a smooth $O(n)$-invariant extension $\hat{\mathcal{S}}$ to $\operatorname{Sym}(n)$. A theorem of Schwarz [93] implies that $\hat{\mathcal{S}} = G(E_1, \cdots, E_n)$ for some smooth G on $\mathbb{R}^n$, since $\{E_1, \cdots, E_n\}$ generate the algebra of $O(n)$-invariant polynomials on $\operatorname{Sym}(n)$. Therefore on $\lambda(K)$ we have $f(\lambda) = G(\sigma_1(\lambda), \cdots, \sigma_n(\lambda))$ is smooth. Since K is arbitrary, f is smooth on Γ_+. $\square$

In the situation of bundles over manifolds, it is necessary to consider a slightly more general situation: For example for hypersurfaces, the principal curvatures are determined not only by the second fundamental form A, but also by the metric g, so we have a more general form $\mathcal{S}(A, g) = f(\lambda(A, g))$. If f is symmetric under permutation of the eigenvalues, then $\mathcal{S}$ invariant under the action of $GL(n)$ on $\operatorname{Sym(n)} \times \mathfrak{S}_+$ defined by $(L, A, g) \mapsto (L^T A L, L^T g L)$ for any $A \in \operatorname{Sym}(n)$, $g \in \mathfrak{S}_+$

and $L \in GL(n)$, since these transformations do not change the eigenvalues of A with respect to g. The smoothness of S as a function of the components of A and g can be deduced as follows: Given $g \in \mathfrak{S}_+$, the Gram-Schmidt procedure produces from the standard basis $e_1, \cdots, e_n$ for $\mathbb{R}^n$ an orthonormal basis $E_1, \cdots, E_n$ with respect to g, and these clearly depend smoothly on the components of g. Let L be the matrix with columns $E_1, \cdots, E_n$ (equivalently, the linear map in $GL(n)$ which takes each e_i to E_i). Then by construction $L^T g L = I$, and we have $\mathcal{S}(A, g) = S(L^T AL, I)$. Therefore $\mathcal{S}$ is smooth in the components of A and g if and only if $\tilde{\mathcal{S}}(M) = \mathcal{S}(M, I)$ is smooth in the components of M. Since $\tilde{\mathcal{S}}$ is $O(n)$-invariant, $\tilde{\mathcal{S}}(M)$ is smooth in the components of M if and only if $f(\lambda) := \tilde{\mathcal{S}}(\mathrm{diam}(\lambda))$ is smooth, by Lemma 2.3.

The above observation enables the construction from a smooth symmetric function f on $\Gamma_+ \subset \mathbb{R}^k$ of a smooth $O(E)$-invariant function $\mathcal{S}$ on the bundle of positive definite symmetric 2-tensors on a metric vector bundle E of rank k over M: If $E_1, \cdots, E_k$ are a basis of smooth local sections for E, then for $A \in \mathrm{Sym}_+(E)$ we can define $A_{ij} = A(E_i, E_j)$ and $g_{ij} = g(E_i, E_j)$, and then set $\mathcal{S}(A) := \hat{\mathcal{S}}((A_{ij}), (g_{ij}))$. The $GL(k)$-invariance of $\hat{\mathcal{S}}$ implies that $\mathcal{S}$ is well-defined (independent of the choice of basis), and the smoothness follows from the argument above.

For $\mathcal{S}$ smooth and any $A \in \mathfrak{S}_+$, the first and second partial derivatives $\dot{\mathcal{S}}_+$ and $\ddot{\mathcal{S}}_+$ of $\mathcal{S}$ at A are defined by

$$\left.\frac{\partial}{\partial \eta}\right|_{\eta=0} \mathcal{S}(A + \eta B) = \dot{\mathcal{S}}^{kl}(A) B_{kl},$$

and

$$\left.\frac{\partial^2}{\partial \eta^2}\right|_{\eta=0} \mathcal{S}(A + \eta B) = \ddot{\mathcal{S}}^{kl,rs}(A) B_{kl} B_{rs},$$

respectively. Moreover, for abbreviation we consider the notation

$$\frac{\partial f}{\partial \lambda_i}(\lambda) = \dot{f}^i(\lambda),$$

and

$$\frac{\partial^2 f}{\partial \lambda_i \partial \lambda_j}(\lambda) = \ddot{f}^{ij}(\lambda).$$

Some well-known properties of f and $\mathcal{S}$ are given by the following lemma:

Lemma 2.4. *Let f and $\mathcal{S}$ be defined as above.*

- *If f is smooth, then $\mathcal{S}$ is smooth.*

- *If f is strictly increasing $\dot{f}^i > 0$ for each $i = 1, ..., n$ at every point in Γ, then $\dot{\mathcal{S}}(B)$ is positive definite.*

- *If f is concave or convex, then $\mathcal{S}$ is also concave or convex, that is*

$$\ddot{\mathcal{S}}^{kl,rs}(A)B_{kl}B_{rs} \leq 0 \quad or \quad \ddot{\mathcal{S}}^{kl,rs}(A)B_{kl}B_{rs} \geq 0.$$

For the proof of this lemma (see [26] and [10]).

We now need to recall a crucial result of the differentiability properties of $\mathcal{S}$ that is related to those of f. More precisely, a formula which involves the derivatives of functions of symmetric matrices that are defined with respect to their eigenvalues is given by the next theorem

Theorem 2.5. *Let f and $\mathcal{S}$ be defined as above such that $\mathcal{S}(A) = f(\lambda(A))$. Let A be a diagonal matrix in $\mathfrak{S}_+$ with distinct eigenvalues. Then the second derivative of $\mathcal{S}$ at $A = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$ takes the form*

$$\ddot{\mathcal{S}}^{pq,rs}(A)B_{pq}B_{rs} = \sum_{p,q} \ddot{f}^{pq} B_{pp}B_{qq} + 2\sum_{p>q} \frac{\dot{f}^p(\lambda) - \dot{f}^q(\lambda)}{\lambda_p - \lambda_q}(B_{pq})^2. \tag{2.13}$$

The proof of such theorem is straightforward using fundamental facts but lengthy (refer to [11] and [44] for more details).

2.3 Mean curvature flow

The study of relations between local invariants (such as extrinsic and intrinsic curvature) and global invariants (such as topology, diameter, volume, or eigenvalues of the Laplacian) of Riemannian manifolds has been an active field of research in differential geometry. One of the recent active areas of investigation has involved applying processes in which submanifolds move with a normal velocity prescribed by their curvature. If the speed is monotone as a function of the principal curvatures, then these processes are described by fully nonlinear parabolic partial differential equations. These processes are also referred to as curvature flows. In ideal situations such flows converge to limiting states which are hypersurfaces of prescribed curvature, so these curvature flows are potentially useful both in studying prescribed curvature hypersurfaces and also in understanding the topology of hypersurfaces with given restrictions on curvature. Unfortunately, singularities can occur in some situations. A current active area of research is to understand the possible formation of singularity in various settings, see [55], [56], [59], [36], [41] and [94].

In this section, we consider the mean curvature flow MCF which is one of the most significant curvature flows. In particular, we introduce some basic facts and remarks on the evolution of hypersurfaces in Euclidean space, including some applications and examples of this flow. Also, other methods for the MCF are illustrated, such as defining the evolving submanifolds locally as graphs, and interpreting MCF as a gradient flow. The final part of this section includes the evolution equations for some geometric quantities of these hypersurfaces.

The study of the MCF dates back to the work of J. Von Neumann on metal surfaces in 1952 [98] where soap foams were discussed with interfaces having constant mean curvature. Soon after this study Mullins [77] illustrated coarsing in metals in 1956, however the mean curvature of the interface is not constant in general. Therefore, the explicit formulation of the mean curvature equation might first be considered by Mullins.

The applications of the MCF has been extensively considered in various disciplines of science such as geometry and physics. This flow and other related geometric flows have been beautifully used to model phenomena in physics such as phase boundaries, bubbles and grain growth and have also been applied in engineering and image processing.

We now define the evolution of hypersurfaces in Euclidean spaces. Let Σ be a smooth compact manifold of dimension n, and let $F(\cdot, t) : \Sigma \times [0, T) \to \mathbb{R}^{n+1}$ be a time-dependent and smooth embedding where $t \in [0, T)$. We say that the hypersurfaces $F(\Sigma, t) = \Sigma_t$ evolve by mean curvature if

$$\frac{\partial}{\partial t} F(x, t) = \vec{H}(x, t) = -\nu(x, t) H(x, t), \tag{2.14}$$

for all $x \in \Sigma$ and $t \in [0, T)$, where $\vec{H}(x, t)$ is the mean curvature vector, H is the mean curvature, and $\nu(x, t)$ is the unit outward normal vector of Σ_t at the point (x, t) (in the situation where Σ_t bounds a region).

Some explicit solutions of (2.14) are minimal hypersurfaces where $\vec{H}(x, t) = 0$, see [82], and the homothetically shrinking round spheres $\mathbb{S}^n_r$ of radius $r_t > 0$ that is represented by $F(x, t) = \sqrt{r_0^2 - 2ntx}$ where $(x, t) \in \mathbb{S}^n_1 \times [0, r_0^2/2n)$ [55]. In this thesis, we usually consider the mean curvature flow of hypersurfaces with positive mean curvature, namely the mean convex flow, see chapter 3 and 6.

The mean curvature flow evolution equation is a geometric analogue of the heat equation. It is called the curve shortening flow in the case $n = 1$ of curves in the plane. Points on hypersurfaces moving in the inward (outward)-pointing normal direction are called mean convex (mean concave) points, respectively. Note that hypersurfaces are evolved in time in order to decrease its volume. The velocity function is represented by curvature and the hypersurface flows faster when its curvature becomes larger and distinct at some finite time.

After a suitable time-dependent reparametrisation, the hypersufaces Σ_t can be locally expressed as graphs over a region $\mathcal{U} \in \mathbb{R}^n$ such that $F(x,t) = (x, u(x,t))$. The evolution equation (2.14) takes the following form

$$\frac{\partial}{\partial t} u(x,t) = \left(\delta_{ij} - \frac{D_i u D_j u}{1 + |Du|^2} \right) D_i D_j u.$$

From this we see that the mean curvature flow is equivalent to a quasilinear scalar parabolic evolution equation for the graph function u.

Note that the mean curvature flow is equivalent to the gradient flow for the area functional: This can be simply seen by differentiating the n-dimensional surface area with respect to the time in order to obtain the first variation formula as follows

$$\frac{d}{dt} Vol(\Sigma_t) = \int_{\Sigma_t} \langle \partial_t p, \nu H \rangle,$$

where $p = F(x,t)$ is the position vector of Σ_t. From this formula, the gradient is given by

$$\nabla Vol(\Sigma_t) = H\nu.$$

Therefore the gradient flow (or steepest descent flow) for the volume functional is given by

$$\partial_t p = -\nabla Vol(\Sigma_t) = -H\nu.$$

which is the mean curvature flow. Thus, the evolution equation for the area functional is given by

$$\frac{d}{dt} Vol(\Sigma_t) = -\int \langle \nu H, \nu H \rangle = -\int_{\Sigma_t} H^2.$$

We now want to represent the formulas of how geometric quantities of Σ_t change since it is evolved by the MCF in the normal direction. In other words, we will compute evolution equations for the metric g_{ij}, the normal vector ν, the second fundamental form h_{ij} and the mean curvature H, for more information see [55], [50] and [58].

Lemma 2.6. *The metric g_{ij} of Σ_t satisfies the evolution equation*

$$\partial_t g_{ij} = -2Hh_{ij}.$$

Proof. By using the fact that $\partial_i F$ is tangential to Σ_t and $h_{ij} = \langle \partial_i \nu, \partial_j F \rangle$, we obtain

$$\begin{aligned} \partial_t g_{ij} &= \partial_t\big(\partial_i F \cdot \partial_j F\big) \\ &= \partial_i\big(-H\nu\big) \cdot \partial_j F + \partial_i F \cdot \partial_j\Big(-H\nu\Big) \\ &= -H\big(\partial_i \nu \cdot \partial_j F\big) - H\big(\partial F_i \cdot \partial_j \nu\big) \\ &= -2Hh_{ij}. \end{aligned}$$

□

We also deduce the evolution equation of the unit normal vector ν to Σ_t as follows:

Lemma 2.7. *The evolution of the unit normal ν is given by*

$$\partial_t \nu = \nabla H.$$

Proof. By straightforward computations we have

$$\begin{aligned} \partial_t \nu &= \big(\partial_t \nu \cdot \partial_i F\big) g^{ij} \partial_j F \\ &= -\big(\nu \cdot \partial_t \partial_i F\big) g^{ij} \partial_j F \\ &= \Big(\nu \cdot \partial_i\big(H\nu\big)\Big) g^{ij} \partial_j F \\ &= \partial_i H g^{ij} \partial_j F \\ &= \nabla H. \end{aligned}$$

□

Now we show that the evolution equation for h_{ij} is given by the following:

Theorem 2.8. *The evolution equation for the second fundamental form is given by*

$$\partial_t h_{ij} = \Delta_{\Sigma_t} h_{ij} - 2Hh_{im} g^{lm} h_{jl} + |A|^2 h_{ij}. \tag{2.15}$$

Proof. Using the definition of the second fundamental form and the Gauss-Weingarten identities (2.7) and (2.8) we can compute

$$\begin{aligned} \partial_t h_{ij} &= -\partial_t\big(\partial_i \partial_j F \cdot \nu\big) \\ &= \big(\partial_i \partial_j (H\nu) \cdot \nu\big) - \big(\partial_i \partial_j F \cdot \partial_k H g^{kl} \partial_l F\big) \\ &= \partial_i \partial_j H + H\big(\partial_i (h_{jl} g^{lm} \partial_m F) \cdot \nu\big) + \big((h_{ij}\nu - \Gamma_{ij}{}^q \partial_q F) \cdot \partial_k H \partial_l F g^{kl}\big) \\ &= (\partial_i \partial_j H - \Gamma_{ij}{}^q \partial_q H) - Hh_{im} g^{ml} h_{lj} \\ &= \nabla_i \nabla_j H - Hh_{im} g^{ml} h_{lj}. \end{aligned}$$

Thus, we obtain (2.15) by substituting the identity (2.9). □

As a consequence of (2.15), we derive the evolution equation for the mean curvature as follows:

$$\frac{\partial H}{\partial t} = \Delta H + |A|^2 H. \tag{2.16}$$

Note that these evolution equations are reconsidered in section 6.3 for embedded spacelike hypersurfaces in the Minkowski space.

2.4 A maximum principle for tensors

There have been some substantial developments on the geometric analysis of partial differential equations depending on the application of the maximum principle argument. This principle allows us to obtain useful information about the properties of solutions of partial differential equations [25] and [13]. Also, it emerges in various settings with different versions, the most common version of this principle deals with a scalar function defined on a manifold. Recently, other sophisticated versions have raised. In other words, the maximum principle can be used to functions with several variables.

In this section we focus on the application of the maximum principle to two-point functions. More precisely, we illustrate such argument for a tensor by considering the following theorem:

Theorem 2.9. *Let $F : \Sigma^n \times [0,T) \to \mathbb{R}^{n+1}$ be a one-parameter smooth family of immersions of a compact hypersurface Σ^n into the Euclidean space. Assume that f is a real-valued function that is locally bounded and $\mathcal{S}$ is a symmetric tensor satisfying*

$$\nabla_t \mathcal{S}\big|_{(x,t)}(u,u) \geq \Delta \mathcal{S}\big|_{(x,t)}(u,u) + f(x,t)\mathcal{S}\big|_{(x,t)}(u,u),$$

where $(x,t,u) \in T\Sigma^n \times [0,T)$. If $\mathcal{S}$ is initially greater than or equal 0, then the same remains for all $(x,t) \in \Sigma^n \times [0,T)$.

Proof. The aim here is to show that $\mathcal{S}\big|_{(x,t)} \geq 0$ for all $(x,t) \in \Sigma^n \times [0,\tau]$, where $\tau \in (0,T)$ is small with respect to a constant $C_\tau = \max_{\Sigma^n \times [0,\tau)} |f|$. For every $\epsilon > 0$, we assume that

$$\mathcal{S}^{\epsilon,\tau} = \mathcal{S} + \epsilon e^{(C_\tau+1)t} g > 0, \quad \forall (x,t) \in \Sigma^n \times [0,\tau].$$

Suppose that this does not hold, that is, there exist $t_0 \in (0,\tau]$, $x_0 \in \Sigma^n$ and a non-vanishing vector $U_0 \in T_{x_0}\Sigma^n$ such that $\mathcal{S}^{\epsilon,\tau} > 0$ for all $(x,t) \in \Sigma^n \times [0,t_0)$.

However, we have $\mathcal{S}^{\epsilon,\tau}(U_0, U_0) = 0$ at (x_0, t_0). We can locally extend the vector U_0 to a vector field around x_0 in space by solving the following along radial geodesics $\mathcal{C}$ with respect to g_{t_0} passing through x_0

$$\nabla_{\mathcal{C}'} U = 0.$$

Also, we extend such resulting vector by differentiating it with respect to time as follows

$$\nabla_t U = 0.$$

These imply to both $\nabla U|_{(x_0,t_0)} = 0$ and $\nabla_t U|_{(x_0,t_0)} = 0$. We also assume that $\Delta U|_{(x_0,t_0)} = 0$. In order to explain this we consider an orthonormal frame $\{e_i\}_{i=1}^n$ at x_0 that is transported parallel along the geodesic $\mathcal{C}$ such that $\mathcal{C}_i' = e_i$ along $\mathcal{C}_i$ and $\mathcal{C}_i'(0) = e_i$. Hence,

$$\Delta U\Big|_{(x_0,t_0)} = \sum_{i=1}^n \Big(\nabla_{e_i}(\nabla_{e_i} U) - \nabla_{\nabla_{e_i} e_i} U \Big)\Big|_{(x_0,t_0)} = 0.$$

We now assume that

$$\mathcal{S}^{\epsilon,\tau}(U,U)\Big|_{(x,t)} = s_{\epsilon,\tau}(x,t),$$

for (x,t) in a neighbourhood of (x_0, t_0). Therefore, we obtain that $s_{\epsilon,\tau}(x,t) \geq 0$ for (x,t) contained in $B_r(x_0,t_0) \times (t_0 - r^2, t_0]$ around (x_0,t_0) where $B_r(x_0,t_0)$ is a ball centered at (x_0,t_0) with radius r. Also, we have $s_{\epsilon,\tau}|_{(x_0,t_0)} = 0$ at (x_0,t_0). These yield to the following inequality:

$$\begin{aligned}
0 &\geq (\partial_t - \Delta) s_{\epsilon,\tau}|_{(x_0,t_0)} \\
&= (\nabla_t - \Delta)\mathcal{S}^{\epsilon,\tau}|_{(x_0,t_0)}(U_0,U_0) \\
&\geq f(x_0,t_0)\mathcal{S}_{(x_0,t_0)}(U_0,U_0) + \epsilon(C_\tau + 1)e^{(C_\tau+1)t_0} g_{(x_0,t_0)}(U_0,U_0) \\
&= -\epsilon f(x_0,t_0)e^{(C_\tau+1)t_0} g_{(x_0,t_0)}(U_0,U_0) + \epsilon(C_\tau + 1)e^{(C_\tau+1)t_0} g_{(x_0,t_0)}(U_0,U_0) \\
&\geq \epsilon e^{(C_\tau+1)t_0} g_{(x_0,t_0)}(U_0,U_0) > 0,
\end{aligned}$$

which contradicts our assumption. This implies that $\mathcal{S}^{\epsilon,\tau} > 0$ in the interval $[0,\tau]$. The assertion follows for the entire interval by repeating such result. $\square$

2.5 Modulus of continuity of heat equations on manifolds

The application of the maximum principle to a function of two variables is an extremely useful tool which has been used in various settings of geometric analysis.

In this section, we represent straightforward applications of this argument to the classical heat equation and a more general type of parabolic equations. In particular, we illustrate how this technique can preserve the modulus of continuity for solutions of some parabolic partial differential equations in one-dimensional Euclidean space and on Riemannian manifolds. These works provide us with useful information regarding sharp eigenvalue estimates, however, we are merely interested of understanding the non-collapsing machinery in terms of such simple versions in this section, see [13] and [15].

We begin by introducing the concept of the modulus of continuity for a function satisfying the one-dimensional heat equation. The modulus of continuity was defined for a real-valued function in one dimensional space in 1910 by H. Lebesgue in order to measure the continuity of this function at a point. Let $w(s)$ be a function of a positive variable s, we say that $w(s)$ is the modulus of continuity for a smooth function $f(x)$ if and only if the inequality $|f(y) - f(x)| \leq w(s)$ holds for all x and y defined in the domain of f and $s = |y - x|$.

To simplify the computation of this section, we make a small change of the above definition. Let $w(s,t)$ be the modulus of continuity, we say that the function $f(x,t)$ admits $w(s,t)$ for each time t if it satisfies the following inequality

$$|f(y,t) - f(x,t)| \leq w\left(\frac{|y-x|}{2}, t\right),$$

for all x and y on the domain of f and for all t. Hence, the smallest modulus of continuity is given by

$$w_f(s,t) = \sup\left\{\frac{|f(y,t) - f(x,t)|}{2} : s = \frac{|y-x|}{2}\right\}. \tag{2.17}$$

Besides the above definition of the modulus of continuity, we show that the modulus of continuity $w(s,t)$ for solutions of the heat equation on the Euclidean space of dimension one is preserved. We explain this by considering the following straightforward and crucial lemma

Lemma 2.10. *Let $f(x)$ be an odd, non-decreasing and concave function on the positive real line $\mathbb{R}_+$, then we obtain*

$$w_f(s) = f(s),$$

for all $s > 0$.

Proof. We consider that $|f(y) - f(x)| = f(y) - f(x)$ and $y > x$ since f is non-decreasing . We assume that $\eta(x)$ is an even function of the variable x that is defined as $\eta(x) = f(x+s) - f(x-s)$ where f is odd. We want to show that $\eta(x)$ has a maximum point along all points x and y satisfying $s = \frac{|y-x|}{2}$ at $x = -s$ and $y = s$. Also, suppose that $x \geq s > 0$, then we can rewrite the points s and $x - s$ as follows

$$s = \frac{x}{x+s}(0) + \frac{s}{x+s}(x+s),$$

and

$$x - s = \frac{x-s}{x+s}(x+s) + \frac{2s}{x+s}(0),$$

respectively. Thus the function f with respect to these points satisfies the following inequalities:

$$f(s) \geq \frac{s}{x+s} f(x+s),$$

$$f(x-s) \geq \frac{x-s}{x+s} f(x+s),$$

where f vanishes at $x = 0$ and it is concave on the positive interval $[0, x+s]$.

Combining these inequalities yields to

$$f(x-s) + 2f(s) \geq f(x+s).$$

From the definition of η, we obtain

$$\eta(x) - \eta(0) \leq 0.$$

We also have the same inequality in the case $0 < x < s$ where f is concave on the positive interval $[x-s, x+s]$. Hence $|f(y) - f(x)|$ approaches its supremum for all x and y at $x = -s$ and $y = s$ since the absolute maximum of η vanishes. □

Remark that if the above function $f(s)$ is evolved by the classical heat equation

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial s^2}, \tag{2.18}$$

then all the properties of f are preserved for all $t > 0$. Thus, the equality $w(s,t) = f(s,t)$ still holds for all points s and t. In other words, the modulus of continuity $w(s,t)$ satisfies (2.18) in one spatial variable.

The behaviour of $w(s,t)$ for solutions of one-dimensional heat equations for later times can be illustrated by applying the argument of zero-counting. Unfortunately, such argument could not be effectively extended to control the modulus of continuity in higher dimensions. Therefore, the application of the maximum principle can be considered to such situation in the higher dimensional setting with some assumptions on the boundary like the Neumann boundary condition or on the solution such as the periodicity condition [14].

We now want to extend the above argument of the classical heat equation in the Euclidean setting to other equations in compact Riemannian manifolds. In particular, we consider a more general type of parabolic equations which is a quasi-linear equation where the coefficients are based on the gradient. This equation may be expressed as isotropic flow where diffusion coefficients are preserved under any rigid motions of space and vertical translation and the gradient vector is fixed. This equation is formulated as follows

$$\frac{\partial u}{\partial t} = \mathcal{L}[u] := \left[a_1(|Du|)\frac{u_i u_j}{|Du|^2} + a_2(|Du|)\left(\delta^{ij} - \frac{u_i u_j}{|Du|^2}\right)\right]. \tag{2.19}$$

The positive functions a_1 and a_2 refer to the diffusion rate in the gradient and orthogonal directions, respectively. Such quasilinear flow is characterized as the following: If $a_1 = a_2 = 1$ then we have the classical heat equation and if $a_1 = |Du|^{q-2}$ and $a_2 = (q-2)|Du|^{q-2}$ then another class of heat equations is obtained which is called the q-Laplacian heat equation. Also, the equation (2.19) is called the flow of the graphical mean curvature when $a_1 = \frac{1}{1+|Du|^2}$ and $a_2 = 1$.

Let $(\mathcal{M}, g)$ be a compact Riemannian manifold of constant sectional curvature κ and u be a smooth function on $\mathcal{M}$. Assume that the Ricci curvature of $\mathcal{M}$ is non-negative. Then, we have a similar form of the modulus of continuity (2.17) for u that is given by

$$w_u(s) = \sup\left\{\frac{u(y)-u(x)}{2} : s = \frac{d(x,y)}{2}\right\}, \quad \forall x, y \in \Sigma \tag{2.20}$$

where $d(x,y) = \inf[L(\mathcal{C})]$ is the Riemannian distance function, where the infimum is taken over all $\mathcal{C} : [0,1] \to \mathcal{M}$ such that $x = \mathcal{C}(0)$ and $y = \mathcal{C}(1)$.

To clarify the argument of the next proposition, it will be useful to derive a one-dimensional heat equation. More precisely, we illustrate the formulation of the appropriate equation for which the modulus of continuity is satisfied in the viscosity sense. In other words, we want to deduce solutions of (2.19) which is defined on one-dimensional spaces in warped product spaces $\bar{\mathcal{M}}_\kappa$ with constant sectional curvature κ.

Let $F : \Sigma \times [a,b] \to \bar{\mathcal{M}}$ be an embedding of a totally geodesic hypersurface Σ times the interval $[a,b]$ into the warped product space given by the exponential map $F(z,s) = exp(sn(z))$, where $(z,s) \in \Sigma \times [a,b]$ and n is a unit normal vector to Σ. The induced Riemannian metric of $\bar{\mathcal{M}}$ is defined by $\bar{g} = ds^2 + \zeta(s)^2 g_\Sigma$, where g_Σ is the Riemannian metric on Σ. We assume that the warping function $\zeta(s)$ satisfies the differential equation $\zeta'' + \kappa\zeta = 0$ with conditions $\zeta(0) = 1$ and $\zeta'(0) = 0$, amounting to the requirement that sectional curvatures of planes containing the ∂_s direction are equal to κ and that the surface $s = 0$ is totally geodesic.

Let u be a solution of (2.19) that depends only on the warping variable s and on time. We can choose $\{e_i\}_{i=1}^n$ to be an orthonormal basis for $T_{x,0}\bar{M}$, extended by parallel transport along the s direction, and with $e_n = \partial_s$. Then the second covariant derivative of u is given by $u_{ij} = u' s_{ij} + u'' s_i s_j$. Denoting $\Sigma_s = F(\Sigma, s)$, so that $\{e_i\}_{i=1}^{n-1}$ are tangent to Σ_s and e_n is equal to ∂_s, we obtain $u_{ij} = u' s_{ij}$, $u_{in} = 0$ and $u_{nn} = u''$. Note that the second covariant derivative s_{ij} along Σ_s is the same as the second fundamental form of Σ_s which is equivalent to the derivative of the metric with respect to s divided by two. Thus, $s_{ij} = \frac{\zeta'}{\zeta} g_{ij}$ and we have

$$\begin{aligned} u_t &= \left[a_1(|Du|) \frac{u_i u_j}{|Du|^2} + a_2(|Du|) \left(\delta^{ij} - \frac{u_i u_j}{|Du|^2} \right) \right] u_{ij} \\ &= a_1(u') u^{''} + (n-1) a_2(u') \frac{\zeta'}{\zeta} u'. \end{aligned}$$

We now describe the application of the maximum principle to (2.20) for arbitrary solutions of (2.19) [13]:

Proposition 2.11. *Let $u : \Sigma \times [0, T) \to \mathbb{R}$ be a solution of (2.19) on Σ with diameter D and with the possibility that the boundary is non-empty and locally convex with the Neumann boundary condition. Then the modulus of continuity $w : [0, D/2] \times [0, T) \to \mathbb{R}$ is a viscosity solution of the differential inequality*

$$w_t \leq a_1(w') w^{''} + (n-1) a_2(w') \frac{\zeta'}{\zeta} w'. \tag{2.21}$$

Proof. We want to show that the modulus of continuity $w(s, t)$ satisfies the given differential inequality in the viscosity sense by using the maximum principle argument. Since the Riemannian distance function $d(x, y)$ is generally Lipschitz (that is, not necessarily smooth), we expect that $w(s, t)$ is not smooth in general. Therefore, we use instead the viscosity notion of solution, which means that whenever we have a smooth function ϕ which lies above w near (s, t) and touches w at the point (s_0, t_0) then the inequality holds for ϕ at (s_0, t_0). Let $\mathcal{C}$ be a smooth path immersed into the Riemannian manifold Σ such that $\mathcal{C}(0) = x$ and $\mathcal{C}(1) = y$, then we have the following inequality

$$u(\mathcal{C}(1), t) - u(\mathcal{C}(0), t) - 2\phi \left(\frac{L[\mathcal{C}]}{2}, t \right) \leq 0, \tag{2.22}$$

where $t \leq t_0$ and $\frac{L[\mathcal{C}]}{2}$ close to (s_0, t_0). Since $\frac{d(x,y)}{2} = s$ and the last inequality reaches its equality at (s_0, t_0), then there exist points x_0 and y_0 in Σ such that $\frac{d(x_0,y_0)}{2} = s_0$ and the inequality can be rewritten as

$$u(y_0, t_0) - u(x_0, t_0) - 2\phi\bigg(\frac{d(x_0, y_0)}{2}, t_0\bigg) = 0.$$

When $\mathcal{C}_0$ is a length-minimizing geodesic between x_0 and y_0, then we get

$$u(y_0, t_0) - u(x_0, t_0) - 2\phi\bigg(\frac{L[\mathcal{C}_0]}{2}, t_0\bigg) = 0.$$

Let $\mathcal{C} : (r, s) \to \mathcal{C}_r(s)$ be a smooth family of paths where the parameter r is the variation through such family and these curves pass through $\mathcal{C}_0$. Therefore, we can derive some inequalities by computing the first and second derivatives along such curves $\mathcal{C}_r(s)$. In particular, we differentiate the length $L[\mathcal{C}_r]$ with respect to this smooth variation at $\mathcal{C}_0$ as follows

$$\frac{\partial}{\partial r} L[\mathcal{C}_r]\bigg|_{r=0} = \int_0^1 \frac{\partial}{\partial r}|\mathcal{C}_r| ds = \int_0^1 g(T, \nabla_r \mathcal{C}_s) ds = g(T, \mathcal{C}_r)\bigg|_0^1, \tag{2.23}$$

where the unit vector T is tangential to $\mathcal{C}_0$. Hence, for an arbitrary variation of curves of the inequality (2.22) at $r = 0$ we obtain

$$\langle \nabla u(y_0, t_0) - \phi' \mathcal{C}_0'(1), \mathcal{C}_r(1)\rangle - \langle \nabla u(x_0, t_0) - \phi' \mathcal{C}_0'(0), \mathcal{C}_r(0)\rangle = 0, \tag{2.24}$$

which implies to $\nabla u(y_0, t_0) = \phi' \mathcal{C}_0'(1)$ and $\nabla u(x_0, t_0) = \phi' \mathcal{C}_0'(0)$.

Also, we need to compute the second derivative along $\mathcal{C}_r$ as follows

$$\frac{\partial^2}{\partial r^2} L[\mathcal{C}_r]\bigg|_{r=0} = \frac{1}{L}\int_0^1 \Big(\big|(\nabla_s \mathcal{C}_r)^{\perp}\big|^2 - R(\mathcal{C}_s, \mathcal{C}_r, \mathcal{C}_s, \mathcal{C}_r)\Big) ds + g(T, \nabla_r \mathcal{C}_r)\bigg|_0^1. \tag{2.25}$$

We now want to derive inequalities from this second variation identity by choosing suitable variations. The deduced inequalities must have the condition that equality holds in one-dimensional situation illustrated above. Hence, the inequality (2.22) becomes equal if the path $\mathcal{C}(s)$ is given by the immersion $F\bigg(z, \frac{L(2s-1)}{2}\bigg)$ where z and L are fixed. Therefore, we have to assume that the smooth families of paths $\mathcal{C}(r, s)$ take the same form of F and to obtain the right paths we consider the suitable coordinates that work along the length-minimizing geodesic $\mathcal{C}_0$. Suppose that $\{e_i\}$ is chosen to be an orthonormal basis to the tangent space at x_0 with respect to the metric g, with $e_n = T(0)$ of $\mathcal{C}_0$ at x_0. Then, using parallel transport along the geodesic $\mathcal{C}_0$ allows to have an orthonormal basis $e_i(s)$ for each tangent space $T_{\mathcal{C}_0(s)}\Sigma$ with $e_n(s) = T(s)$ for each s.

We consider two directions of variation $\mathcal{C}(r,s)$ along the geodesic $\mathcal{C}_0$ in order to obtain the appropriate inequalities from the second derivative. One direction is to assume that $\mathcal{C}(r,s)=\mathcal{C}_0\Big(\frac{r(2s-1)}{L}+s\Big)$ which means that the length is varying (increasing) while z is not changed, then we have $\mathcal{C}_r(s)=\mathcal{C}_0'\frac{2s-1}{L}$. It is straightforward to get the first and second derivatives of the length by substituting $\partial_r L=2$ and $\partial_r^2 L=0$ into (2.25). Thus we obtain the following inequality of second derivative where $\mathcal{C}_r(1)=T(1)=e_n(1)$ and $\mathcal{C}_r(0)=-T(0)=-e_n(0)$

$$u_{nn}(y_0,t_0)-u_{nn}(x_0,t_0)-2\phi^{''}\le 0. \tag{2.26}$$

The second direction is perpendicular to the previous one such that keeping now the length fixed and changing z which requires the variation $\mathcal{C}_r(s)$ to correspond to a Jacobi field of the warped product spaces. Hence, it takes the form $\mathcal{C}_r(s)=\frac{\zeta(L(2s-1)/2)}{\zeta(L/2)}e_i$ where $i=1,...,n-1$. Therefore, the first derivative is given by $\partial_r L=0$ and the second derivative becomes

$$\partial_r^2 L=2\frac{\zeta'(L/2)}{\zeta(L/2)}-\int_{-L/2}^{L/2}\frac{\zeta(x)^2}{\zeta(L/2)^2}\Big(R(e_i,T,e_i,T)-\kappa\Big)dx,$$

where $\frac{L(2s-1)}{2}$ is replaced by the variable x and the integration by parts is used. This direction of variation yields to the following second derivative inequality

$$u_{ii}\Big|_{(y_0,t_0)}-u_{ii}\Big|_{(x_0,t_0)}-\phi^{'}\left(2\frac{\zeta'(L/2)}{\zeta(L/2)}-\int_{-L/2}^{L/2}\frac{\zeta(x)^2}{\zeta(L/2)^2}\Big(R(e_i,T,e_i,T)-\kappa\Big)dx\right)\le 0,$$

such inequality can be rewritten, where $i=1,...,n-1$, as follows

$$\sum_i^{n-1}\Big(u_{ii}\Big|_{(y_0,t_0)}-u_{ii}\Big|_{(x_0,t_0)}\Big)-2(n-1)\frac{\zeta'(L/2)}{\zeta(L/2)}\phi^{'}$$
$$+\phi^{'}\int_{-L/2}^{L/2}\frac{\zeta(x)^2}{\zeta(L/2)^2}\Big(Ric(T,T)-(n-1)\kappa\Big)dx\le 0.$$

Since we assume that the Ricci curvature is non-negative, the last term of this inequality is removed, that is

$$\sum_i^{n-1}\Big(u_{ii}\Big|_{(y_0,t_0)}-u_{ii}\Big|_{(x_0,t_0)}\Big)-2(n-1)\frac{\zeta'(L/2)}{\zeta(L/2)}\phi^{'}\le 0. \tag{2.27}$$

By multiplying the inequalities (2.26) and (2.27) by $a_2(\phi')$ and $a_1(\phi')$ respectively and combining them, we arrive to

$$\mathcal{L}[u]\big|_{(y_0,t_0)}-\mathcal{L}[u]\big|_{(x_0,t_0)}-2a_1(\phi')\phi^{''}-2(n-1)a_2(\phi')\frac{\zeta'(L/2)}{\zeta(L/2)}\phi^{'}\le 0. \tag{2.28}$$

However, differentiating with respect to time implies

$$\mathcal{L}[u]\big|_{(y_0,t_0)} - \mathcal{L}[u]\big|_{(x_0,t_0)} - 2\phi_t \geq 0. \tag{2.29}$$

The required inequality is deduced by adding the last two inequalities as follows

$$\phi_t \leq a_1(\phi')\phi^{''} + (n-1)a_2(\phi')\frac{\zeta'}{\zeta}\phi'.$$

This confirms that the modulus of continuity $w(s,t)$ satisfies (2.21) in the viscosity sense. $\square$

Chapter 3

Noncollapsing for Extrinsic Geometric Flows in $\mathbb{R}^{n+1}$

The non-collapsing technique is a powerful tool that has been employed in various contexts with useful results in partial differential equations and differential geometry. The application of this approach can be extended to embedded hypersurfaces evolving by some curvature flows such as the CSF, the MCF and fully nonlinear curvature flows. In this chapter, we represent the proofs of several results which are based on applying the maximum principle to a function depending on two points in a hypersurface in order to prove a geometric non-collapsing property.

The first section illustrates the proofs of some basic properties on the CSF involving the avoidance principle, the preservation of embeddedness and Huisken's estimate [57] of the ratio between the intrinsic and extrinsic distance along this flow in the plane.

The second section contains the work of B. Andrews [12] of a sharp estimate for embedded and mean convex hypersurfaces evolving by the MCF (2.14) in $\mathbb{R}^{n+1}$. More precisely, he compared the radius of the largest ball which touches the hypersurface at a given point to the local geometry (curvature) at that point to provide a self-contained proof of the non-collapsing estimate. In other words, the author showed that every point of the evolving hypersurface is touched by a ball of radius bounded below by a multiple of the reciprocal of the mean curvature at that point.

The non-collapsing argument has subsequently been extended and applied to fully nonlinear curvature flows (analogues of the mean curvature flow where the speed depends on a nonlinear function of the principal curvatures). In section

3, we represent the result of B. Andrews, M. Langford and J. McCoy [19] with respect to this curvature flow where the speed function of the principal curvatures is homogeneous of degree one and convex or concave. In particular, the authors proved that the boundary curvature of the smallest exterior (largest interior) sphere which touches the hypersurface at each point is a supersolution (subsolution) of the linearized curvature flow if the speed function is convex (concave). Finally we just mention the extension of these results of [12] and [19] into the sphere and the hyperbolic spaces by [16].

3.1 Huisken's distance comparison estimate

The CSF is an extrinsic geometric flow which is identified as a geometric heat equation acting on a curve over time. In particular, such flow moves each point of an immersed curve inwards with a speed proportional to its curvature. This flow has various applications in mathematics and physics. Also, several important techniques in this context are involved such as the maximum principle, monotonicity formulas and the Harnack inequality.

The main objective of this section is to provide the proof of Huisken's estimate [55] that is qualitatively equivalent to the preservation of embeddedness feature for solutions of the CSF. More precisely, if the curve $\mathcal{C}_0$ deforming by the CSF is embedded at $t = 0$, then the same holds for $\mathcal{C}_t$ for each $t \geq 0$.

We start here by representing a basic and nice geometric principle which states that any two disjoint solutions of the curve shortening flow at $t = 0$ in the plane will remain disjoint for each $t > 0$ which is referred to as the avoidance principle and given by the following:

Theorem 3.1. *(The avoidance principle) Let $F_i : \Sigma_i^1 \times [0, T) \to \mathbb{R}^2$ be a family of embedded curves in the plane deforming by the CSF and satisfying*

$$F_1(\Sigma_1^1, 0) \bigcap F_2(\Sigma_2^1, 0) = \phi.$$

Then the same holds as the flow progresses in time.

Proof. Consider a two-point function $d(x, y, t) : \Sigma_1 \times \Sigma_2 \times [0, T) \to \mathbb{R}$ that is defined on a product of two curves by

$$d(x, y, t) = |F_2(y, t) - F_1(x, t)|. \tag{3.1}$$

Note that such distance function is positive at the initial time $t = 0$ since the curves are compact, that is

$$d_0 \doteqdot \inf\{d(x, y, 0) : (x, y) \in \Sigma_1 \times \Sigma_2\} > 0.$$

The main aim of the proof is to deduce that $de^{\eta(1+t)} > d_0$ for each $\eta > 0$ by assuming this is not true and applying the maximum principle argument to d as defined by (3.1).

Therefore, since the function $\inf\{d(x, y, t)e^{\eta(1+t)} : (x, y) \in \Sigma_1 \times \Sigma_2\}$ is positive at $t = 0$ and continuous in the time t, then we can find a positive first time $t_0 \in (0, T)$ such that

$$d_0 = \inf\{d(x, y, t_0)e^{\eta(1+t_0)} : (x, y) \in \Sigma_1 \times \Sigma_2\}.$$

Also there exists a point $(x_0, y_0) \in \Sigma_1 \times \Sigma_2$ such that $d_0 = d(x_0, y_0, t_0)$ since the product of curves is compact. However, computing the first and second derivatives of d at such point (x_0, y_0, t_0) implies to a contradiction. Note that the inequality $\frac{\partial}{\partial t}\left(de^{\eta(1+t)}\right) \leq 0$ holds at (x_0, y_0, t_0). Also, we have that the first derivatives of d in the x and y directions on $\Sigma_1 \times \Sigma_2$ are zero and the matrix of second derivatives is positive definite at this point.

In order to simplify these computation, it is useful to choose the arc length to be a parameter of the curves Σ_i by the immersion $F_i(\cdot, t_0)$. Also we consider that $w = \frac{F_2 - F_1}{d}$ which is the unit vector and d is defined above as the distance function between $F_2(y, t)$ and $F_1(x, t)$. Also, we assume that T_1 and T_2 are the unit vectors that are tangential to the parametrized curves $F_1(\cdot, t_0)$ and $F_2(\cdot, t_0)$ at x_0 and y_0, respectively.

Sine the first spatial derivatives vanish at (x_0, y_0, t_0), then we obtain

$$0 = \frac{\partial d}{\partial x}\bigg|_{(x_0,y_0,t_0)} = -\langle w, T_1\rangle,$$

and

$$0 = \frac{\partial d}{\partial y}\bigg|_{(x_0,y_0,t_0)} = \langle w, T_2\rangle.$$

These identities tell us that T_1 and T_2 are orthogonal to the unit vector w. This unit vector can be considered as the normal unit vector by changing the orientation of the parametrization of curves if necessary. Hence, we have $T_1(x_0, t_0) = T_2(y_0, t_0)$.

In order to compute the second derivatives, recall that the derivatives of the unit tangent vectors are given by $\frac{\partial T_1}{\partial x} = \kappa_1 N_1$ and $\frac{\partial T_2}{\partial x} = \kappa_2 N_2$ where κ_1 and κ_2

are curvatures of the plane curves $F_1(x_0, t_0)$ and $F_2(y_0, t_0)$, respectively. Thus, the second derivatives of d at any point (x, y, t) are given by

$$\frac{\partial^2 d}{\partial x^2} = \frac{\langle T_1 - \langle w, T_1\rangle w, T_1\rangle}{|F_1 - F_2|} - \langle w, \kappa_1 N_1\rangle,$$

$$\frac{\partial^2 d}{\partial y^2} = \frac{\langle T_2 - \langle w, T_2\rangle w, T_2\rangle}{|F_1 - F_2|} - \langle w, \kappa_2 N_2\rangle,$$

and

$$\frac{\partial^2 d}{\partial x \partial y} = -\frac{\langle T_1 - \langle w, T_1\rangle w, T_2\rangle}{|F_1 - F_2|}.$$

Since we have $w = N_1 = N_2$ and $T_1 = T_2$ at (x_0, y_0, t_0), then the last equations are rewritten as

$$\frac{\partial^2 d}{\partial x^2} = \frac{1}{d} - \kappa_1,$$
$$\frac{\partial^2 d}{\partial y^2} = \frac{1}{d} - \kappa_2,$$

and

$$\frac{\partial^2 d}{\partial x \partial y} = \frac{1}{d}.$$

Since the matrix of the second derivatives is non-negative definite at such point, we arrive at

$$\frac{\partial^2 d}{\partial x^2} + 2\frac{\partial^2 d}{\partial x \partial y} + \frac{\partial^2 d}{\partial y^2} = \kappa_2 - \kappa_1 \geq 0.$$

Differentiating $de^{\eta(1+t)}$ with respect to the time at (x_0, y_0, t_0) implies that

$$\begin{aligned} 0 &\geq \frac{\partial}{\partial t}\left(de^{\eta(1+t)}\right)\Big|_{(x_0,y_0,t_0)} \\ &= e^{\eta(1+t)}\big(d\eta + \langle w, \kappa_2 N_2 - \kappa_1 N_1\rangle\big) \\ &> \langle w, \kappa_2 N_2 - \kappa_1 N_1\rangle \\ &= \kappa_2 - \kappa_1. \end{aligned}$$

However, from the inequality of the second derivatives we also have $\kappa_2 - \kappa_1 \geq 0$. We then arrive to a contradiction to our assertion, and thus we deduce that the inequality $de^{\eta(1+t)} > d_0$ holds. In other words, the distance between d and d_0 is not decreasing in time and then there is no intersection between the curves $\Sigma_1^1(x, t)$ and $\Sigma_2^1(y, t)$ moving by the curve shortening flow for each $t \in [0, T)$. $\square$

The above principle deals with two curves that are initially disjoint and they continue to be disjoint under the evolution, while the next theorem is with respect to the preservation of embeddedness under the CSF. Precisely, we consider a curve that is initially embedded, without self-intersection, and derive that such property still holds for each $t \in [0, t_0]$ as the following:

Theorem 3.2. *Assume that $F : \mathcal{S}^1 \times [0, t_0) \to \mathbb{R}^2$ is a family of smooth curves satisfying the CSF. If $F(\cdot, 0)$ is initially embedded, then the embedding property is preserved under such flow.*

Proof. Let $d(x, y, t)$ and $\ell(x, y, t)$ be the extrinsic and intrinsic distance functions respectively between the points x and y in the curve $\mathcal{S}^1$. Assume that the function $d(x, y, t)$ is defined by

$$d = |F(y, t) - F(x, t)|.$$

We want here to apply the maximum principle argument to the function d in order to deduce that $F(\cdot, t)$ is injective for every $t \in [0, t_0]$ since the curve is compact. However, the obstacle is that the extrinsic distance vanishes when $x = y$, that means when the points belong to the diagonal subset, and thus d will not be uniformly positive on the remaining region.

To rule out this issue, we assume that the curvature $\kappa(x)$ is bounded and then this estimate enables us to apply the maximum principle to d on the complement of a suitable neighbourhood of the diagonal subset, by first using the curvature bound to control d on the boundary. The following lemma accomplishes what we need by providing a lower bound for d at points which are sufficiently close to each other, assuming a curvature bound:

Lemma 3.3. *Let $F : \mathcal{S}^1 \to \mathbb{R}^2$ be an immersion, and suppose the curvature $\kappa(x)$ is bounded in magnitude by a constant c for all $x \in \mathcal{S}^1$. Then, we obtain the following estimate:*

$$|F(y) - F(x)| \geq \frac{2}{c} \sin\left(\frac{c \cdot \ell(x, y)}{2}\right),$$

where $x, y \in \mathcal{S}^1$ and $\ell(x, y, t) \leq \frac{\pi}{c}$.

Proof. Assume that x and y lie on the curve $\mathcal{S}^1$ and satisfy $\ell(x, y) \leq \frac{\pi}{c}$. Let $\mathcal{S}^1$ be parametrized by an arc length s such that $s(x) = -\frac{\ell}{2}$ and $s(x) = \frac{\ell}{2}$. If s is bounded by $-\frac{\ell}{2} \leq s \leq \frac{\ell}{2}$, then we have

$$|\alpha(s) - \alpha(0)| \leq \int_0^s |\kappa| ds \leq c \cdot s \leq c\frac{\ell}{2} \leq \frac{\pi}{2}.$$

.

We now calculate the following:

$$\begin{aligned}
|F(y)-F(x)| &\geq \langle F(y)-F(x), T(0)\rangle \\
&= \int_{s(x)}^{s(y)} \langle T(s), T(0) ds\rangle \\
&= \int_{s(x)}^{s(y)} \langle \cos(|\alpha(s)-\alpha(0)|) ds\rangle \\
&\geq \int_{s(x)}^{s(y)} \langle \cos(c|s|) ds\rangle \\
&= \frac{2}{c}\sin\left(\frac{c\cdot \ell(x,y)}{2}\right).
\end{aligned}$$

where we used the fact that $\cos x$ is decreasing in x for $0 \leq x \leq \frac{\pi}{2}$. This completes the proof of the lemma. □

In order to apply the lemma we require a bound on the magnitude of curvature. But this is guaranteed for any smooth solution on $\mathbb{S}^1 \times [0, t_0$ by compactness.

Now define the region $\mathcal{O} = \{(x,y,t) \in \mathbb{S}^1 \times \mathbb{S}^1 \times [0,t_0] : \ell(x,y,t) \geq \frac{\pi}{c}\}$. By the previous lemma, the function d is bounded below by $\frac{2}{c}$ on the boundary of this domain where $\ell(x,y,t) = \frac{\pi}{c}$. Therefore the maximum principle can be applied to d on the set $\mathcal{O}$ to show that a new minimum of d with value less than $\frac{2}{c}$ cannot arise, since such a minimum must occur at a point in the spatial interior, and then we have

$$\begin{aligned}
\frac{\partial d}{\partial t} &= \left(\frac{\partial^2}{\partial s_x^2} + \frac{\partial^2}{\partial s_y^2} + 2T_x \cdot T_y \frac{\partial^2}{\partial s_x \partial s_y}\right) d \\
&\quad - \frac{1}{d}\left(\left(\frac{\partial d}{\partial s_x}\right)^2 + \left(\frac{\partial d}{\partial s_y}\right)^2 - 2T_x \cdot T_y \frac{\partial d}{\partial s_y}\frac{\partial d}{\partial s_x}\right).
\end{aligned}$$

By applying the maximum principle, we obtain

$$d(x,y,t) = \min\left\{\inf\{d(x,y,0) : \ell(x,y,0) \geq \frac{\pi}{c}\}, \frac{2}{c}\right\}.$$

We obtain that $\inf\{d(x,y,0) : \ell(x,y,0) \geq \frac{\pi}{c}\} > 0$ since $F(.,0)$ is injective, and this implies that $F(.,t)$ remains injective, and hence an embedding, as long as the solution remains smooth. □

The above study of the preservation of embeddedness in the plane was beautifully refined by G. Huisken [57] by comparing the extrinsic and intrinsic distance along the curve shortening flow. More specifically, G. Huisken developed the previous arguments by considering that d is bounded from below by the intrinsic distance ℓ and the total length $L(t)$ instead of a bound on the curvature as explained above. We will explain the argument below.

Let $F : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ be solutions of the curve shortening flow. Assume that the total length of the curve is denoted by $L(t)$ for each $t \in [0, T)$ and the arc length between $F(x, t)$ and $F(y, t)$ is denoted by $\ell(x, y, t)$ that is referred to as the intrinsic curvature where $x, y \in \mathbb{S}^1$. Also, the extrinsic curvature is denoted by $d(x, y, t)$ and defined as above. G. Huisken considered the following quantity:

$$\mathcal{Q}(t) := \sup_{x \neq y} \frac{L(t)}{\pi d(x, y, t)} \sin\left(\frac{\pi \ell(x, y, t)}{L(t)}\right),$$

where x, y in $\mathbb{S}^1$. Note that if the curve is the round circle, then $\mathcal{Q}(t) = 1$ for every x, y, while otherwise $\mathbb{Q} \geq 1$. Our aim now is to show the following estimate that was derived by Huisken:

Theorem 3.4. *(Huisken [57], 1984) Let $F : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ be a family of embedded curves in the plane evolving by CSF, then $\mathcal{Q}(t)$ is monotone decreasing in time.*

The proof of this theorem is based on the argument of the strict maximum principle applied to a function of two variables (see [57] and [25]).

Proof. Assume that the statement is invalid. Then $\mathcal{Q}(t)$ is not monotone decreasing in time, so there exist t_1 and t_2 where $t_1 < t_2$ and a real number q satisfying

$$\mathcal{Q}(t_1) < q,$$

and

$$\mathcal{Q}(t_2) > q,$$

where $q > \pi$.

Defining the function $Z(x, y, t)$ on $S^1 \times S^1 \times [0, T)$ by the following:

$$Z_q(x, y, t) = qd(x, y, t) - L(t) \sin \frac{\pi \ell_t(x, y)}{L(t)}.$$

We can find a time $\bar{t} \in (t_1, t_2)$ and also there exist two distinct points $\bar{x}, \bar{y} \in \mathbb{S}^1$ for which the function Z_q vanishes, and for all $x, y \in \mathbb{S}^1$ and $t \in (t_1, \bar{t}]$ we have $Z(x, y, t) \geq 0$.

Let x and y be chosen as local parameters around the points $\bar{x}$ and $\bar{y}$, respectively, which we can take to be arc length parameters for the embedding at time $\bar{t}$. Also, we choose the orientations in such a way that $\partial_x \ell = -1$ and $\partial_y \ell = 1$ at $(\bar{x}, \bar{y}, \bar{t})$. Note that $\frac{\partial F}{\partial x}(\bar{x}, \bar{t})$ and $\frac{\partial F}{\partial y}(\bar{y}, \bar{t})$ are the unit tangent vectors.

We now begin by computing the first spatial derivatives at $(\bar{x}, \bar{y}, \bar{t})$: We have

$$0 = \frac{\partial Z_q}{\partial x}\bigg|_{(\bar{x},\bar{y},\bar{t})} = q\frac{\langle F(\bar{x}, \bar{t}) - F(\bar{y}, \bar{t}), \frac{\partial F}{\partial x}(\bar{x}, \bar{t})\rangle}{d} + \pi \cos \frac{\pi \ell_{\bar{t}}(\bar{x}, \bar{y})}{L(t)},$$

and

$$0 = \frac{\partial Z_q}{\partial y}\bigg|_{(\bar{x},\bar{y},\bar{t})} = -q\frac{\langle F(\bar{x}, \bar{t}) - F(\bar{y}, \bar{t}), \frac{\partial F}{\partial y}(\bar{y}, \bar{t})\rangle}{d} - \pi \cos \frac{\pi \ell_{\bar{t}}(\bar{x}, \bar{y})}{L(t)}.$$

Since the last two identities vanish at $(\bar{x}, \bar{y}, \bar{t})$, then by adding them we obtain

$$\langle F(\bar{x}, \bar{t}) - F(\bar{y}, \bar{t}), \frac{\partial F}{\partial x}(\bar{x}, \bar{t})\rangle = \langle F(\bar{x}, \bar{t}) - F(\bar{y}, \bar{t}), \frac{\partial F}{\partial y}(\bar{y}, \bar{t})\rangle.$$

This identity tells us that the tangent vector $\frac{\partial F}{\partial y}(\bar{y}, \bar{t})$ can be expressed in terms of $\frac{\partial F}{\partial x}(\bar{x}, \bar{t})$ by using the reflection principle across the line that is orthogonal to $F(\bar{x}, \bar{t}) - F(\bar{y}, \bar{t})$.

We then compute the second derivatives with respect to the x-direction at $(\bar{x}, \bar{y}, \bar{t})$ as follows

$$\frac{\partial^2 Z_q}{\partial x^2}\bigg|_{(\bar{x},\bar{y},\bar{t})} = \frac{q}{d}\left(1 - \langle w, \frac{\partial F}{\partial x}(\bar{x}, \bar{t})\rangle^2\right) - q\kappa(\bar{x})\langle w, N(\bar{x})\rangle + \frac{\pi}{L(\bar{t})} \sin \frac{\pi \ell_{\bar{t}}(\bar{x}, \bar{y})}{L(\bar{t})},$$

while in the y-direction, we get

$$\frac{\partial^2 Z_q}{\partial y^2}\bigg|_{(\bar{x},\bar{y},\bar{t})} = \frac{q}{d}\left(1 - \langle w, \frac{\partial F}{\partial y}(\bar{y}, \bar{t})\rangle^2\right) - q\kappa(\bar{y})\langle w, N(\bar{y})\rangle + \frac{\pi}{L(\bar{t})} \sin \frac{\pi \ell_{\bar{t}}(\bar{x}, \bar{y})}{L(\bar{t})},$$

and also we obtain

$$\frac{\partial^2 Z_q}{\partial x \partial y}\bigg|_{(\bar{x},\bar{y},\bar{t})} = -\frac{q}{d}\langle \frac{\partial F}{\partial x}(\bar{x}, \bar{t}), \frac{\partial F}{\partial y}(\bar{y}, \bar{t})\rangle - \langle \frac{\partial F}{\partial x}(\bar{x}, \bar{t}), w\rangle\langle w, \frac{\partial F}{\partial y}(\bar{y}, \bar{t})\rangle - \frac{\pi}{L(\bar{t})} \sin \frac{\pi \ell_{\bar{t}}(\bar{x}, \bar{y})}{L(\bar{t})}.$$

Note that $\frac{\partial^2 F}{\partial x^2} = N(\bar{x}, \bar{t})\kappa(\bar{x}, \bar{t})$ and $\frac{\partial^2 F}{\partial y^2} = N(\bar{y}, \bar{t})\kappa(\bar{y}, \bar{t})$. Combining the last three equations implies

$$\begin{aligned}\left(\frac{\partial^2 Z_q}{\partial x^2} + \frac{\partial^2 Z_q}{\partial y^2} - 2\frac{\partial^2 Z_q}{\partial x \partial y}\right)\bigg|_{(\bar{x},\bar{y},\bar{t})} &= -q\kappa(\bar{x}, \bar{t})\langle w, N(\bar{x}, \bar{t})\rangle \\ &\quad + q\kappa(\bar{y}, \bar{t})\langle w, N(\bar{y}, \bar{t})\rangle \\ &\quad + \frac{4\pi}{L(\bar{t})} \sin \frac{\pi \ell_{\bar{t}}(\bar{x}, \bar{y})}{L(\bar{t})}.\end{aligned}$$

We finally need to calculate the time derivative of Z_q as follows:

$$\begin{aligned}\frac{\partial Z_q}{\partial t}\bigg|_{(\bar{x},\bar{y},\bar{t})} &= -q\langle w, \kappa(\bar{x},\bar{t})N(\bar{x},\bar{t}) - \kappa(\bar{y},\bar{t})N(\bar{y},\bar{t})\rangle \\ &+ \left(\frac{1}{\pi}\sin\frac{\pi\ell_{\bar{t}}(\bar{x},\bar{y})}{L(\bar{t})} - \frac{\ell_{\bar{t}}(\bar{x},\bar{y})}{L(\bar{t})}\cos\frac{\pi\ell_{\bar{t}}(\bar{x},\bar{y})}{L(\bar{t})}\right)\int_{\mathbb{S}^1}\kappa^2 \\ &+ \cos\frac{\pi\ell_{\bar{t}}(\bar{x},\bar{y})}{L(\bar{t})}\int_{\bar{x}}^{\bar{y}}\kappa^2.\end{aligned}$$

Recall that the curvature of the curve $F(\mathbb{S}^1,\bar{t})$ is not constant as the function Z vanishes at $(\bar{x},\bar{y},\bar{t})$ and $q > \pi$. Hence, the integrals can be estimated by applying the Cauchy-Schwarz inequality and the Gauss-Bonnet theorem since their coefficients are positive. Thus, we obtain

$$\int_{\mathbb{S}^1}\kappa^2 > \frac{1}{L(\bar{t})}\left(\int_{\mathbb{S}^1}\kappa\right)^2 = \frac{4\pi^2}{L(\bar{t})}, \tag{3.2}$$

and

$$\int_{\bar{x}}^{\bar{y}}\kappa^2 \geq \frac{1}{\ell_{\bar{t}}(\bar{x},\bar{y})}\left(\int_{\bar{x}}^{\bar{y}}\kappa\right)^2 = \frac{4\xi^2}{\ell_{\bar{t}}(\bar{x},\bar{y})}, \tag{3.3}$$

where ξ satisfies

$$\cos\xi = \langle w, \frac{\partial F}{\partial x}(\bar{x},\bar{t})\rangle = \frac{\pi}{q}\cos\frac{\pi\ell_{\bar{t}}(\bar{x},\bar{y})}{L(\bar{t})}.$$

Therefore, by substituting (3.2) and (3.3) into the time derivative of Z_q above and then combining them we conclude that

$$\begin{aligned}0 &\geq \left(\frac{\partial Z_q}{\partial t} - \frac{\partial^2 Z_q}{\partial x^2} - \frac{\partial^2 Z_q}{\partial y^2} + 2\frac{\partial^2 Z_q}{\partial x\partial y}\right)\bigg|_{(\bar{x},\bar{y},\bar{t})} \\ &> \frac{4\pi}{\ell_{\bar{t}}(\bar{x},\bar{y})}\left(\xi^2 - \frac{\pi^2\ell_{\bar{t}}^2(\bar{x},\bar{y})}{L^2(\bar{t})}\right)\cos\frac{\pi\ell_{\bar{t}}(\bar{x},\bar{y})}{L(\bar{t})} > 0.\end{aligned}$$

Since we have $q > \pi$, then $\cos\xi \leq \cos\frac{\pi\ell_{\bar{t}}(\bar{x},\bar{y})}{L(\bar{t})}$. Hence, we arrive at a contradiction since $\xi \geq \frac{\pi\ell_{\bar{t}}(\bar{x},\bar{y})}{L(\bar{t})}$. □

We conclude this section by pointing out that the previous argument of Huisken plays an important role to prove the most famous result regarding the CSF, Grayson's theorem [49]. This theorem states that if the initial curve is embedded, then the evolving curve shrinks to a point, becoming circular in the process. In particular, the solution continues to exist as long as the length remains positive as follows:

Theorem 3.5 (G. Huisken [55], 1984; Gage-Hamilton [43], 1986). *Let $F : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ be an embedded solution of the curve shortening flow which is defined on a maximal time interval. Then, as t approaches the singular time T, the curves $F(\cdot, t)$ shrink to a point, and converge to a circle after suitable rescaling.*

In fact, Huisken's estimate is not enough by itself to deduce that the solution continues to exist while the length remains positive, but this can be deduced from Huisken's estimate by combining it with a blow-up argument and a classification of type I and type II blowup limits: Huisken's estimate rules out the latter, while the former are known to be shrinking circles by the monotonicity formula and a classification of self-similar solutions by Abresch and Langer [2].

3.2 Non-collapsing of mean convex MCF

This section contains the direct proof of B. Andrews's result [12] on a non-collapsing estimate for compact, embedded and mean convex hypersurfaces evolving by the MCF (2.14) in $\mathbb{R}^{n+1}$ using the argument of the two-point maximum principle. This estimate is a qualitative manifestation of the fact that embeddedness is preserved under the MCF.

This estimate was also provided in the work of Weimin Sheng and Xujia Wang [94] on singularities which arise at the first time of the mean convex flow in the Euclidean space by using a detailed analysis of the singularities and behaviour of the MCF. In particular, they provided a description of the suitable limits of the rescaled flows over a singularity for which the compactness theorem and contradiction argument were applied.

Similar estimates, though not phrased in precisely this way, were also deduced by B. White [104, 105] in his work on the structure of singularities. One of the significant consequences of this study is that the estimate on the inscribed radius can be derived. The author used some techniques from the perspective of geometric measure theory involving a detailed analysis of some properties for solutions of the mean convex flow.

On the other hand, B. Andrews [12] provided a self-contained technique to prove this theorem which is based on the application of the maximum principle argument to a two-point function. In this part, we concentrate on presenting this approach.

The main theorem of this section is stated as follows:

Theorem 3.6. *(B. Andrews [12] and W. Sheng, X-J. Wang [94]) Assume that* Σ^n *is a connected and compact hypersurface and let* $F : \Sigma^n \times [0,T) \to \mathbb{R}^{n+1}$ *be a one-parameter family of smooth, embedded and mean convex hypersurfaces moving under the MCF (2.9). If the inscribed radius of the initial hypersurface is bounded from below by* $\frac{\epsilon}{H}$ *where* ϵ *is a positive constant, then this remains valid for all positive times in* $[0,T)$.

Before beginning the proof, we need to define an appropriate function of two variables defined on the hypersurface for which the maximum principle argument can be applied. Assume that $F : \Sigma \times [0,T) \to \mathbb{R}^{n+1}$ is a compact, embedded and mean convex hypersurface that satisfies the mean curvature flow in the Euclidean space. Let U be the enclosed region such that the unit normal vector ν is chosen to be outward pointing of U. For a point $F(x,t)$ where $x \in \Sigma$, there is a touched ball $B_r(c)$ where $r > 0$ and $c = F(x,t) - r\nu(x,t)$ are the radius and the center of the ball, respectively. Assume that $B_r(c)$ is inside the region U which is equivalent to the following statement:

$$\begin{aligned}
B_r(c) \subset \Sigma \quad &\iff \quad |F(y,t) - c|^2 \geq r^2, \quad \forall y \in \varSigma; \\
&\iff \quad |F(y,t) - F(x,t) + r\nu(x,t)|^2 \geq r^2, \quad \forall y \in \Sigma; \\
&\iff \quad |F(y,t) - F(x,t)|^2 + 2r\langle F(y,t) - F(x,t), \nu(x,t)\rangle \geq 0, \quad \forall y \in \Sigma; \\
&\iff \quad \frac{2\langle F(x,t) - F(y,t), \nu(x,t)\rangle}{|F(y,t) - F(x,t)|^2} \leq \frac{1}{r}, \quad \forall y \in \Sigma.
\end{aligned}$$

Assume that

$$k(x,y,t) = \frac{2\langle F(x,t) - F(y,t), \nu(x,t)\rangle}{|F(y,t) - F(x,t)|^2}, \quad \forall y \in \Sigma$$

and the boundary curvature of the largest ball enclosed in U at $F(x,t)$ is given by

$$\bar{k}(x,t) = \sup\{k(x,y,t) : y \in \Sigma \backslash \{x\}\}. \tag{3.4}$$

where $\bar{k}(x,t)$ refers also to the inscribed ball curvature.

The main idea of the proof is the application of the maximum principle to the function $\bar{k}(x,t)$. More precisely, we show that $\bar{k}(x,t)$ satisfies the following natural differential inequality in a viscosity sense

$$\frac{\partial \bar{k}}{\partial t} \leq \Delta \bar{k} + |A|^2 \bar{k}. \tag{3.5}$$

Such equation is called the linearized mean curvature flow.

Proof. In order to prove that $\bar{k}$ is a subsolution of the linearized mean curvature flow in the viscosity sense, we assume that $\rho(x,t)$ is a smooth function that is defined on a neighbourhood of an arbitrary point $(x_0,t_0) \in \Sigma \times [0,t_0]$ and lies above $\bar{k}(x,t)$ such that $\rho = \bar{k}$ at (x_0,t_0) where $t \le t_0$. From the definition of k, we also have $k(x,y,t) \le \rho(x,t)$ for all $(x,y,t) \in \Sigma \times \Sigma \times [0,t_0]$.

There are two directions to be considered. The first simple direction just includes the evolution equation of the second fundamental form h. In this case we assume that the supremum is not reached, then there should be a sequence of points y approaching to x_0 where the inscribed ball curvature $\bar{k}$ and the maximum principal curvature $\lambda_{\max}$ are equal at (x_0,t_0). Moreover, for any smooth unit vector field defined around (x_0,t_0), we have $\rho(x,t) \ge \bar{k} \ge h_{(x,t)}(e,e)$ such that the equality holds at (x_0,t_0). These implying to the following inequality:

$$\left(\frac{\partial}{\partial t} - \Delta\right)\rho|_{(x_0,t_0)} \le \left(\frac{\partial}{\partial t} - \Delta\right)h(e,e)|_{(x_0,t_0)}. \tag{3.6}$$

where e is the direction of the largest principal curvature at (x_0,t_0). By using the parallel transport from the point (x_0,t_0) for the vector e and the mean curvature flow, we arrive to

$$\left(\frac{\partial}{\partial t} - \Delta\right)h(e,e)|_{(x_0,t_0)} = -|A|^2\lambda_{\max} = -|A|^2\rho. \tag{3.7}$$

Hence, in this case $\bar{k}$ satisfies the linearized mean curvature flow in the viscosity sense.

The second situation is the assumption that the supremum can be reached at some point, meaning that $\bar{k}(x_0,t_0) = k(x_0,y_0,t_0)$ where $x_0 \ne y_0$, then we have $\rho(x,t) \ge \bar{k}(x,t) \ge k(x,y,t)$ with equality at (x_0,y_0,t_0). Define the two-point function $Z(x,y,t)$ on $\Sigma \times \Sigma \times [0,T)$ by

$$Z(x,y,t) = \frac{2\langle F(x,t) - F(y,t), \nu(x,t)\rangle}{|F(y,t) - F(x,t)|^2} - \rho(x,t) \le 0, \tag{3.8}$$

for all $x \ne y$ and $t \le t_0$.

In order to obtain the first and second derivatives of Z, we choose x^i and y^i to be local normal coordinates for Σ near x_0 and y_0, respectively. Let ∂_i^x and ∂_i^y be orthonormal coordinates for $T_{x_0}\Sigma$ and $T_{y_0}\Sigma$, respectively. For simplicity, some abbreviations are considered such that $d = |F(y,t) - F(x,t)|$, the unit vector $w = \frac{F(y,t)-F(x,t)}{d}$, $\nu_x = \nu(x,t)$, $\frac{\partial\rho}{\partial x^i} = \rho_i$, $\partial_i^x = \frac{\partial F}{\partial x_i}$ and similarly $\partial_i^y = \frac{\partial F}{\partial y_i}$. We now start by computing the first spatial derivatives

$$(\partial_i^x + \partial_i^y)Z = \frac{2}{d^2}\big((\partial_i^x - \partial_i^y)\cdot(\nu_x + (Z+\rho)dw) - dw\cdot(h_x)_i^p\partial_p^x\big) - \partial_i\rho. \tag{3.9}$$

Since the first derivatives vanish at (x_0, y_0, t_0), then we have

$$\rho_i = \frac{2}{d^2}\big(\rho\partial_i^x \cdot dw) - (h_x)_i^p \partial_p^x \cdot dw\big) = \frac{2}{d}\big(\rho\delta_i^p - (h_x)_i^p\big)\partial_p^x \cdot w. \tag{3.10}$$

The geometric interpretation for this situation is since the inscribed ball $B_r(c)$ meets the hypersurface Σ at both $F(x_0, t_0)$ and $F(y_0, t_0)$ where these points lie on the boundary of $B_r(c)$, then from the symmetry of the sphere we obtain that the tangent spaces $T_{x_0}\Sigma_{t_0}$ and $T_{y_0}\Sigma_{t_0}$ are equivalent to the tangent spaces of the boundary of $B_r(c)$ and related by the reflection function R_w in the hyperplane orthogonal to w, that means $R_w(v) = v - 2(v \cdot w)w$. Hence, the normal coordinates are chosen such that $R_w(\partial_i^x) = \partial_i^y$ for each i.

The second derivatives are given as the following

$$\begin{aligned}\sum(\partial_i^x + \partial_i^y)^2 Z = -\frac{2}{d^2}\Bigg(&(\nu_x + \rho dw)\cdot(H_x\nu_x - H_y\nu_y) + \nabla H_x \cdot dw \\ &+ 2\sum(\partial_i^y - \partial_i^x)\big((h_x)_i^q \partial_q^x + \rho_i dw\big) \\ &+ \sum|\partial_i^y - \partial_i^x|^2\rho + |A|^2 dw\cdot\nu_x\Bigg) - g^{ij}\frac{\partial^2\rho}{\partial x^i \partial x^j}.\end{aligned}$$

The computation of the time derivative gives the inequality

$$Z_t = \frac{2}{d^2}\big((H_y\nu_y - H_x\nu_x)\cdot(\nu_x + \rho dw) - dw\cdot\nabla H_x\big) - \rho_t \geq 0 \tag{3.11}$$

where the mean curvature flow $\frac{\partial F}{\partial t} = -H\nu$ and $\partial_t\nu = \nabla H$ are considered. Finally, by using the symmetry of $B_r(c)$ and the definition of Z at (x_0, y_0, t_0), we have $\partial_i^y = \partial_i^x - 2(\partial_i^x \cdot w)w$ and $\nu_x + \rho dw = \nu_y$ such that $2d^2\rho = -dw\cdot\nu_x$ and $\nu_x\cdot\nu_y = 1 - \frac{d^2\rho^2}{2}$. Also, by combining the second spatial and time derivatives that are computed above, it follows that

$$\begin{aligned}\rho_t &\leq \Delta\rho + |A|^2\rho + \frac{8}{d^2}\Bigg(\big(\rho\delta_i^q - (h_x)_i^q\big)\partial_i^x \cdot w\partial_q^x \cdot w - 4\rho_i dw\cdot\partial_i^x\Bigg) \\ &= \Delta\rho + |A|^2\rho - 2\big((\rho I - h_x)^{-1}\big)^{ij}\rho_i\rho_j,\end{aligned}$$

where the identity of the first derivative (3.10) is used in the last negative term. We arrive to the required inequality(3.6), proving that $\bar{k}$ satisfies the following inequality in the viscosity sense:

$$\partial_t\bar{k} \leq \Delta\bar{k} + |A|^2\bar{k}.$$

$\square$

According to [12], such result is analogous to curvature pinching $Hg \geq \epsilon h$ at $F(x,t)$. Moreover, if the mean convexity condition of the mean curvature is cancelled, that means in the case of hypersurface initially star-shaped, then the result remains valid. Also, it still works for the case of curve shortening flow where a curve is convex. Furthermore, the non-collapsing method can also be applied to the exterior case where the ball is touched the hypersurface at the point $F(x,t)$ from outside such that the radius is $\frac{|\epsilon|}{H}$. As a consequence, this gives the lower curvature pinching.

The above argument of non-collapsing has subsequently been extended with promising results in various contexts. In the next section, we describe the non-collapsing technique which was extended by B. Andrews, M. Langford and J. McCoy [12] to embedded hypersurfaces evolving by fully nonlinear curvature flows (analogues of the mean curvature flow where the speed depends on a nonlinear function of the principal curvatures).

3.3 Fully non-linear curvature flows

In this section we describe an analogous result to the previous work (section 3.2) of the non-collapsing estimate for compact, embedded and mean convex hypersurfaces deforming by the mean curvature flow in $\mathbb{R}^{n+1}$. In particular, we consider the study of B. Andrews, M. Langford and J. McCoy [19] on embedded hypersurfaces evolving by a more general class of fully nonlinear parabolic equations in $\mathbb{R}^{n+1}$. This class of curvature flows is formulated by

$$\frac{\partial X}{\partial t} = -\mathcal{S} \cdot \nu, \tag{3.12}$$

where $\mathcal{S}$ is the speed function of the principal curvatures that is homogeneous of degree one and ν is the unit normal vector, see section 2.2 for more information of this type of flow.

The authors [19] showed that a function which represents the curvature of the largest sphere inscribed in the region enclosed by the hypersurface and meeting at each point on this hypersurface satisfies a natural differential inequality in the viscosity sense. Precisely, they derived that if the curvature function $\mathcal{S}$ is convex (concave), then the curvature of the exscribed (inscribed) sphere is a supersolution (subsolution) to the equation of the linearized curvature flow.

The proof of such non-collapsing estimate for this class of curvature flows is also based on employing the maximum principle argument to the two-point function defined on the evolving hypersurface. As it has been shown in the previous section that the geometric structure for the inscribed sphere curvature touching the hypersurface at a point (x,t) is equivalent to the non-collapsing quantity

$$Z(x,y,t) = \frac{2\langle X(x,t) - X(y,t), \nu(x,t)\rangle}{\|X(x,t) - X(y,t)\|^2} \leq f(x,t),$$

where $f(x,t)$ is the reciprocal of the radius of the inscribed sphere at (x,t). Also, the curvature of inscribed (exscribed) sphere is defined as the supremum (infimum) of the function Z and denoted by $\overline{Z}$ ($\underline{Z}$), respectively such that

$$\overline{Z} = \sup\{Z(x,y,t) : y \in \Sigma, y \neq x\},$$

and

$$\underline{Z} = \inf\{Z(x,y,t) : y \in \Sigma, y \neq x\}.$$

More precisely, they deduced the following theorem:

Theorem 3.7. *(B. Andrews, and M. Langford, and J. McCoy [19]). Let $F : \Sigma \times [0,T) \to \mathbb{R}^{n+1}$ be a family of smooth embedded hypersurfaces evolving by nonlinear curvature flows (3.27). Assume that $\mathcal{S}$ is a concave function of the principal curvatures, then $\overline{Z}$ is a subsolution of the linearized equation*

$$\frac{\partial \overline{Z}}{\partial t} \leq \dot{\mathcal{S}}^{kl}\nabla_k\nabla_l\overline{Z} + \dot{\mathcal{S}}^{kl}h_k^p h_{pl}\overline{Z} \tag{3.13}$$

in the viscosity sense. An analogous statement holds for $\underline{Z}$ in the case $\mathcal{S}$ is convex.

Now we illustrate the proof of this theorem according to [19]

Proof. Assume that we have a one-parameter family of smooth and embedded hypersurfaces Σ_t moving under the nonlinear curvature flow (3.12). The idea is to deduce that the curvature of the inscribed sphere $\overline{Z}$ is a viscosity subsolution of the linearized flow (3.13).

In order to apply the maximum principle to $\overline{Z}(x,t)$, we should have an appropriate compact region in which Z is defined. Hence, Z can be extended to a manifold with boundary $\widetilde{\Sigma}$ by using the tubular neighbourhood theorem on the compact diagonal set $D = \{(x,x) : x \in \Sigma\}$ and then blowing up along D such that $\widetilde{\Sigma} = \{(\Sigma \times \Sigma)\backslash D\} \cup S\Sigma\}$ where $S\Sigma = \{(x,u) \in T\Sigma : \|u\| = 1\}$.

Therefore, since Z is extended to $\widetilde{\Sigma} \times [0,T)$ for which $(\Sigma \times \Sigma)\backslash D$ is compact, then we define

$$Z(x,y,t) = \frac{2\langle X(x,t) - X(y,t), \nu(x,t)\rangle}{\|X(x,t) - X(y,t)\|^2},$$

for $(x, y) \in (\Sigma \times \Sigma)\backslash D$ and $t \in [0, T)$. Also, for $(x, u) \in S\Sigma$ we have

$$Z(x, u, t) = h_{(x,t)}(u, u),$$

where $h_{(x,t)}$ refers to the second fundamental form of Σ at (x, t). Note that since Z is continuous on $\widetilde{\Sigma}$, then the function Z attains its supremum $\overline{Z}$ and infimum $\underline{Z}$ over $\widetilde{\Sigma}$.

To show that $\overline{Z}$ is a viscosity subsolution of (3.13) where $\mathcal{S}$ is concave, we define a smooth function $\rho(x, t)$ that lies above $\overline{Z}(x, t)$ on a region over a point $(x_0, t_0) \in \Sigma \times [0, T)$ such that $\rho = \overline{Z}$ at (x_0, t_0) where $t \leq t_0$. Thus, we have $\rho(x, t) \geq \overline{Z}(x, t) \geq Z(x, y, t)$ for all (x, t) close to (x_0, t_0) where $t \leq t_0$ with $\rho(x_0, t_0) = \overline{Z}(x_0, t_0)$ and $\rho(x, t) \geq \overline{Z}(x, t) \geq Z(x, u, t)$ for all $u \in S_x\Sigma$ with also equality holds for the first inequality at (x_0, t_0).

We have two possibilities: the first case is if the supremum is not attained at the point (x_0, t_0), then there is a sequence of points close to x_0 and a smooth unit normal vector v defined over (x_0, t_0) such that $v(x_0, t_0) = v_0$. This vector can be extended in this region for x close to x_0 and for $t \leq t_0$ close to t_0 by using the parallel transport from the point (x_0, t_0) along the geodesic and by the evolution equation $\frac{\partial v}{\partial t} = \mathcal{S}\mathcal{W}(v)$ such that $\mathcal{W}$ is the Weingarten map. Such extension yields that the first and second spatial derivatives of $v(x_0, t_0)$ are equal to zero. In this case, the evolution equation of the second fundamental form h at (x_0, t_0) implies that

$$\frac{\partial}{\partial t} h(v, v) = \dot{\mathcal{S}}^{kl} \nabla_k \nabla_l (h(v, v)) + \ddot{\mathcal{S}}^{kl,pq} \nabla_v h_{kl} \nabla_v h_{pq} + h(v, v) \dot{\mathcal{S}}^{kl} h_k^p h_{pl}. \tag{3.14}$$

Note that $\rho \geq h(v, v)$ for points x near x_0 and $t \leq t_0$ with equality at (x_0, t_0). Hence, the inequalities $\nabla^2 \rho \geq \nabla^2(h(v, v))$ and $\frac{\partial \rho}{\partial t} \leq \frac{\partial}{\partial t}(h(v, v))$ hold at such point (x_0, t_0). Also, since $\mathcal{S}$ is assumed to be concave, then its second derivative is non-positive. These imply the required differential inequality for $\rho(x_0, t_0)$, that means

$$\frac{\partial \rho}{\partial t} \leq \dot{\mathcal{S}}^{kl} \nabla_k \nabla_l \rho + \rho \dot{\mathcal{S}}^{kl} h_k^p h_{pl}. \tag{3.15}$$

The second possibility is assuming that the supremum is attained at (x_0, y_0) where $x_0 \neq y_0$ such that $\overline{Z}(x_0, t_0) = Z(x_0, y_0, t_0) = \rho(x_0, t_0)$ and $\rho(x, t) \geq \overline{Z}(x, t) \geq Z(x, y, t)$ for all (x, t) close to (x_0, t_0) where $t \leq t_0$ and $(x, y) \in \Sigma \times \Sigma \backslash D$. Therefore, the derivative in the time direction gives the inequality $\frac{\partial}{\partial t}(\rho - Z) \leq 0$ at (x_0, y_0, t_0). Also, the first derivatives of $\rho - Z$ in the direction of x and y vanish at (x_0, y_0, t_0) while the second derivatives of $\rho - Z$ are non-negative.

In order to compute the first and second spatial derivatives of $\rho - Z$, let x^i and y^i be local normal coordinates near x_0 and y_0 for Σ respectively. Also, we assume that ∂_i^x and ∂_i^y are orthonormal coordinates for $T_{x_0}\Sigma$ and $T_{y_0}\Sigma$, respectively. For simplicity, we consider $d = |X(y,t) - X(x,t)|$, $w = \frac{X(y,t)-X(x,t)}{d}$ and write $\nu_x = \nu(x,t)$, $\frac{\partial \rho}{\partial x^i} = \rho_i$, $\partial_i^x = \frac{\partial X}{\partial x_i}$ and similarly $\partial_i^y = \frac{\partial X}{\partial y_i}$.

We now start by computing the first spatial derivatives in the $y-$direction

$$\frac{\partial}{\partial y^i}(\rho - Z) = \frac{2}{d^2}\langle \partial_i^y, \nu_x - Zdw\rangle, \tag{3.16}$$

and differentiating in the $x-$direction gives

$$\frac{\partial}{\partial x^i}(\rho - Z) = \frac{2}{d}(Z\langle \partial_i^x, w\rangle - (h_x)_i^p\langle \partial_p^x, w\rangle) + \frac{\partial \rho}{\partial x^i}, \tag{3.17}$$

recall that the last two equations vanish at (x_0, y_0, t_0).

The second derivatives are computed as the following:

$$\begin{aligned}\frac{\partial^2}{\partial y^i \partial y^j}(\rho - Z) &= \frac{2}{d^2}\Big(Z\langle \partial_i^y, \partial_j^y\rangle + \langle Zdw - \nu_x, (h_y)_{ij}\nu_y\rangle \Big) \\ &= \frac{2}{d^2}(Z\delta_{ij} - (h_y)_{ij}),\end{aligned}$$

where we use here the fact that the first derivatives of Z in the $y-$direction vanish. Also, we obtain

$$\begin{aligned}\frac{\partial^2}{\partial x^i \partial x^j}(\rho - Z) = \frac{2}{d^2}(Z\delta_{ij} - (h_y)_{ij}) + Z(h_x)_{jp}\delta^{pq}(h_x)_{qi} - \frac{2}{d}\nabla_p (h_x)_{ij}\delta^{pq}\langle w, \partial_q^x\rangle \\ - Z^2(h_x)_{ij} + \frac{2}{d}\frac{\partial \rho}{\partial x^j}\langle w, \partial_i^x\rangle + \frac{2}{d}\frac{\partial \rho}{\partial x^i}\langle w, \partial_j^x\rangle + \frac{\partial^2 \rho}{\partial x^i \partial x^j},\end{aligned}$$

and

$$\frac{\partial^2}{\partial x^j \partial y^i}(\rho - Z) = \frac{2}{d^2}\big((h_x)_j^p - Z\delta_j^p\langle \partial_i^y, \partial_p^x\rangle\big) - \frac{2}{d}\frac{\partial \rho}{\partial x^j}\langle w, \partial_i^x\rangle.$$

The computation of the time derivative implies to the inequality

$$\begin{aligned}\frac{\partial}{\partial t}(\rho - Z) &= \frac{\partial \rho}{\partial t} + \frac{2\mathcal{S}_x}{d^2} - \frac{2\mathcal{S}_y}{d^2}\langle \nu_y, \nu_x - Zdw\rangle - \frac{2}{d}\langle w, \nabla X_x\rangle - Z^2 X_x \\ &= \frac{\partial \rho}{\partial t} + \frac{2\mathcal{S}}{d^2} - \frac{2\mathcal{S}_y}{d^2} - \frac{2}{d}\langle w, \nabla \mathcal{S}_x\rangle - Z^2 X_x.\end{aligned}$$

By combining the previous equations and inequalities, we arrive to the following inequality

$$
\begin{aligned}
\frac{\partial}{\partial t}(\rho - Z) \leq \dot{\mathcal{S}}_x^{ij}\left(\frac{\partial^2}{\partial x^i \partial x^j} + 2\frac{\partial^2}{\partial x^i \partial y^j} + \frac{\partial^2}{\partial y^i \partial y^j}\right)(\rho - Z) \\
= -\frac{\partial \rho}{\partial t} + \dot{S}_x^{ij}\nabla_i\nabla_j\rho + \rho\dot{S}_x^{ij}(h_x)_{ip}\delta^{pq}(h_x)_{qj} \\
- \frac{4\mathcal{S}_x}{d^2} + \frac{4}{d^2}\dot{S}_x^{ij}(h_x)_{iq}\delta^{qp}\langle\partial_j^y, \partial_p^x\rangle \\
+ \frac{2\mathcal{S}_y}{d^2} - \frac{2}{d^2}\dot{S}_x^{ij}(h_y)_{ij} + \frac{4Z}{d^2}\dot{S}_x^{ij}\delta_{ij} \\
- \frac{4Z}{d^2}\dot{S}_x^{ij}\langle\partial_j^y, \partial_i^x\rangle + \frac{4}{d}\dot{S}_x^{ij}\frac{\partial \rho}{\partial x^i}\langle w, \partial_j^x - \partial_j^y\rangle.
\end{aligned}
\tag{3.18}
$$

We can rewrite the two terms before the last term of such inequality (3.19) as the following

$$
\frac{4Z}{d^2}\dot{S}_x^{ij}\delta_{ij} - \frac{4Z}{d^2}\dot{S}_x^{ij}\langle\partial_j^y, \partial_i^x\rangle = \frac{4Z}{d^2}\dot{S}_x^{ij}\big(\delta_{ij} - \langle\partial_j^y, \partial_i^x\rangle\big).
$$

Since $\mathcal{S}$ is homogeneous, $\mathcal{S}_x = \dot{\mathcal{S}}_x^{ij}(h_x)_{ij}$, then we have

$$
-\frac{4\mathcal{S}_x}{d^2} + \frac{4}{d^2}\dot{S}_x^{ij}(h_x)_{iq}\delta^{qp}\langle\partial_j^y, \partial_p^x\rangle = -\frac{4}{d^2}\dot{S}_x^{ij}(h_x)_{iq}\delta^{qp}(\delta_{jp} - \langle\partial_j^y, \partial_p^x\rangle.
$$

Moreover, since $\mathcal{S}$ is concave, then we obtain that

$$
\mathcal{S}_y \leq \dot{S}_x^{ij}(h_y)_{ij}. \tag{3.19}
$$

Therefore, by substituting these identities, the last inequality (3.20) and the vanishing of the identity (3.18) we arrive at

$$
\begin{aligned}
0 \leq -\frac{\partial \rho}{\partial t} + \dot{S}_x^{ij}\nabla_i\nabla_j\rho + \rho\dot{S}_x^{ij}(h_x)_{ip}\delta^{pq}(h_x)_{qj} \\
+ \frac{4}{d^2}\dot{S}_x^{ij}(Z\delta_{ip} - (h_x)_{qj})\delta^{pq}\big(\delta_{qj} - \langle\partial_j^y, \partial_q^x\rangle + 2\langle w, \partial_q^x\rangle\langle w, \partial_j^y - \partial_j^x\rangle\big).
\end{aligned}
\tag{3.20}
$$

We need now to check the sign of the term in the second line of (3.21). We notice that the factors of $\dot{S}_x^{ij}(Z\delta_{ip} - (h_x)_{qj})\delta^{pq}$ are each non-negative definite and commute. Hence, such matrix is positive definite and symmetric. Besides, the next lemma shows that the term $\delta_{qj} - \langle\partial_j^y, \partial_q^x\rangle + 2\langle w, \partial_q^x\rangle\langle w, \partial_j^y - \partial_j^x\rangle$ is non-positive as follows

Lemma 3.8. *We have*

$$
\delta_{qj} - \langle\partial_j^y, \partial_q^x\rangle + 2\langle w, \partial_q^x\rangle\langle w, \partial_j^y - \partial_j^x\rangle \leq 0
$$

Proof. We consider the computation at the minimum point (x_0, y_0, t_0). Let $\{x^i\}$ and $\{y^i\}$ be the local coordinates, we choose them so that $\partial_i^x = \partial_i^y$ for $i = 1, ..., n-1$. Hence ∂_n^x and ∂_n^x are coplanar with w. As a result, if $p = q = n$, then we find that $\delta_{qj} - \langle \partial_j^y, \partial_q^x \rangle + 2\langle w, \partial_q^x \rangle \langle w, \partial_j^y - \partial_j^x \rangle$ does not vanish.

Therefore, the proof will be divided into two parts: $\langle w, n_x \rangle \leq 0$ and $\langle w, n_x \rangle \geq 0$. If $\langle w, n_x \rangle \leq 0$, then we may assume that $\langle w, n_x \rangle = -\sin\theta$, where $\theta \in [0, \frac{\pi}{2})$. We can adjust the direction of ∂_n^x such that $\langle w, \partial_n^x \rangle = \cos\theta$. Assume that we have the conditions $\langle \partial_n^x, \partial_n^y \rangle = -\cos 2\alpha$ and $\langle \partial_n^y, \nu_x \rangle = \sin 2\alpha$ where $\alpha \in [0, \frac{\pi}{2})$ and the orientation of ∂_n^y are satisfied. By using the first derivative of $(\phi - Z)$ in particular the equation (3.17) with respect to the direction y^n and the definition of Z, we obtain

$$\begin{aligned}
\frac{\partial}{\partial y^n}(\phi - Z) &= \frac{2}{d^2}\langle \partial_n^y, \nu_x - Zdw \rangle \\
&= 2\langle \partial_n^y, \nu_x \rangle - 2\langle \partial_n^y, Zdw \rangle \\
&= \langle \partial_n^y, \nu_x \rangle - 2\langle \partial_n^y, w \rangle \langle w, \nu_x \rangle \\
&= \sin 2\alpha \cos 2\theta - \sin 2\theta \cos 2\alpha = 0 \\
&= \sin(2\alpha - 2\theta) = 0,
\end{aligned}$$

thus $\theta = \alpha$ and $\langle \partial_n^y, w \rangle = -\cos\alpha$. We substitute into the following

$$\begin{aligned}
\delta_{qj} - \langle \partial_j^y, \partial_q^x \rangle + 2\langle w, \partial_q^x \rangle \langle w, \partial_j^y - \partial_j^x \rangle &= 1 + \cos(2\alpha) - 2\cos\alpha(\cos\alpha + \cos\alpha) \\
&= 2\cos^2\alpha - 4\cos^2\alpha = -2\cos^2\alpha \leq 0.
\end{aligned}$$

In the case of $\langle w, n_x \rangle \geq 0$, the proof is similar to the previous case. We define $\theta \in [0, \frac{\pi}{2})$ such that $\langle w, n_x \rangle = \sin\theta$. We have $\langle w, \partial_n^x \rangle = -\cos\theta$ by directing ∂_n^x . Also, we can choose the the orientation of ∂_n^y and $\alpha \in [0, \frac{\pi}{2})$ to meet the conditions $\langle \partial_n^x, \partial_n^y \rangle = -\cos 2\alpha$ and $\langle \partial_n^y, \nu_x \rangle = \sin 2\alpha$. We use again the equation (3.17) and the definition of Z, we then have

$$\begin{aligned}
\langle \partial_n^y, \nu_x \rangle &= 2\langle \partial_n^y, w \rangle \langle w, \nu_x \rangle \\
\sin(2\alpha - 2\theta) &= 0,
\end{aligned}$$

this implies that $\theta = \alpha$ and $\langle \partial_n^y, w \rangle = \cos\alpha$. Hence, the computation is given by the following

$$\begin{aligned}\delta_{qj} - \langle \partial_j^y, \partial_q^x \rangle + 2\langle w, \partial_q^x \rangle \langle w, \partial_j^y - \partial_j^x \rangle &= 1 + \cos(2\alpha) - 2\cos\alpha(-\cos\alpha - \cos\alpha) \\ &= 2\cos^2\alpha - 4\cos^2\alpha = -2\cos^2\alpha \leq 0.\end{aligned}$$

□

Therefore, we arrive at

$$\frac{\partial \rho}{\partial t} \leq \dot{\mathcal{S}}_x^{ij} \nabla_i \nabla_j \rho + \rho \dot{\mathcal{S}}_x^{ij} (h_x)_{ip} g_x^{pq} (h_x)_{qj}. \tag{3.21}$$

This implies that ρ satisfies the above natural differential inequality in the viscosity sense.

Similarly, in the case where the function $\mathcal{S}$ is convex and all the inequalities are reversed we deduce the following:

Corollary 3.9. *Let $X : \Sigma \times [0, T) \to \mathbb{R}^{n+1}$ be a family of smooth embedded hypersurfaces evolving by nonlinear curvature flows* (3.12). *Assume that $\mathcal{S}$ is convex, then $\underline{Z}$ is a viscosity supersolution of the linearized equation*

$$\frac{\partial \underline{Z}}{\partial t} = \dot{\mathcal{S}}^{kl} \nabla_k \nabla_l \underline{Z} + \dot{\mathcal{S}}^{kl} h_k^p h_{pl} \underline{Z}$$

This completes the proof of Theorem 3.6. □

The non-collapsing estimates, that were derived in the previous results of B. Andrews [12] and B. Andrews, M. Langford and J. McCoy [19] on hypersurfaces evolving by mean convex and fully non-linear curvature flows in the Euclidean space respectively, were generalized to the ambient spaces: the sphere $\mathbb{S}^{n+1}$ and the hyperbolic space $\mathbb{H}^{n+1}$. In particular, B. Andrews and X. Han and H. Li and Y. Wei [16] deduced the non-collapsing results for embedded hypersurfaces evolving by curvature flows in $\mathbb{S}^{n+1}$ and $\mathbb{H}^{n+1}$. In the sphere case, they deduced the following statement

Theorem 3.10. *(B. Andrews and X. Han and H. Li and Y. Wei [16]) Let $X : \Sigma^n \times [0, T) \to \mathbb{S}^{n+1}$ be a family of smooth embedded hypersurfaces moving under the flow (3.27) where the function of the principal curvatures $\mathcal{S}$ is non-negative. Also,*

- *If $\mathcal{S}$ is convex, then*
$$\frac{\underline{Z}}{\mathcal{S}} \geq \frac{1}{n} + \alpha_1 e^{-2nt},$$
where $\alpha_1 = \inf\{\frac{\underline{Z}}{\mathcal{S}} - \frac{1}{n}\} \leq 0$ at $(x, 0)$ for every $x \in \Sigma$.

- *If $\mathcal{S}$ is concave, then*
$$\frac{\overline{Z}}{\mathcal{S}} \leq \frac{1}{n} + \alpha_2 e^{-2nt},$$
where $\alpha_2 = \sup\{\frac{\overline{Z}}{\mathcal{S}} - \frac{1}{n}\} \geq 0$ at $(x, 0)$ for every $x \in \Sigma$.

While in the ambient space $\mathbb{H}^{n+1}$, they derived a non-collapsing estimate only for the MCF as follows:

Theorem 3.11. *(B. Andrews and X. Han and H. Li and Y. Wei. [16]) Let $X(\Sigma^n)$ be an embedded solution of the MCF in $\mathbb{H}^{n+1}$.*

- *If the initial hypersurface is mean-convex that is $H > 0$, then for $(x, t) \in \Sigma \times [0, T)$ we have*
$$\frac{\overline{Z}}{H} \leq \frac{1}{n} + \alpha_3 e^{2nt}, \quad \frac{\underline{Z}}{H} \geq \frac{1}{n} + \alpha_4 e^{2nt}$$
where $\alpha_3 = \sup\{\frac{\overline{Z}}{H} - \frac{1}{n}\}$ and $\alpha_4 = \inf\{\frac{\underline{Z}}{\mathcal{S}} - \frac{1}{n}\}$ at $(x, 0)$.

- *If the initial hypersurface satisfies $H > n$, then*
$$\frac{\overline{Z}}{H} \leq \frac{1}{n} + \alpha_5 e^{2nt},$$
where $\alpha_5 = \sup\{\frac{\overline{Z}}{H-n} - \frac{1}{n}\}$ at $(x, 0)$.

Chapter 4

Classification of Hypersurfaces with Prescribed Curvatures in the Sphere

The study of hypersurfaces of prescribed curvature in Riemannian manifolds such as the unit sphere has been a field of extensive research. This has included work on minimal and constant mean curvature hypersurfaces and also on hypersurfaces with other curvature functions constant such as the elementary symmetric functions of principal curvatures (see for example [83], [101], [7] and [45]). In this chapter, we present a short survey of results relevant to the new work carried out in the next chapter.

We begin the first section by describing a crucial result on the uniqueness of surfaces of vanishing mean curvature with genus 1 in the three-dimensional sphere: Brendle's proof [23] of Lawson's conjecture [72] that the only embedded minimal torus in $\mathbb{S}^3$ is the Clifford torus (up to rigid motion). A key component of the proof of this result is the application of the non-collapsing technique that was developed by B. Andrews in his work on mean convex mean curvature flow [12], which utilized a maximum principle argument to compare the boundary curvature of the largest ball touching the hypersurface to the positive mean curvature at each point, see section 3.2. In the case of minimal surfaces the mean curvature vanishes, but Brendle succeeded in using the largest principal curvature instead of the mean curvature in this setting.

In the second section, we present the proof of the Pinkall-Sterling conjecture [88] about constant mean curvature tori in $\mathbb{S}^3$ that was affirmatively answered by B. Andrews and H. Li [20]. Precisely, they classified embedded tori of constant mean curvature in $\mathbb{S}^3$, proving that these surfaces are surfaces of rotation. The authors of the work [20] extended the non-collapsing technique used in [12] and [23] to embedded tori of constant mean curvature in $\mathbb{S}^3$ by finding a suitable replacement for the mean curvature or maximum principal curvature.

We consider in the last section of this chapter the work of B. Andrews, Z. Huang and H. Li [17] regarding the uniqueness of a class of embedded Weingarten hypersurfaces satisfying a linear partial differential equation equation into in the $(n+1)$-dimensional sphere. They assumed some conditions on multiplicities of the principal curvatures. The non-collapsing method was applied to these hypersurfaces to deduce that they must be congruent to a Clifford torus.

4.1 Uniqueness of minimal tori in the three-sphere and Lawson's conjecture

Complete minimal surfaces are nicely constructed in the Euclidean space $\mathbb{R}^3$ by selecting an appropriate holomorphic 1-form and a meromorphic function for the representation formula of Weierstrass. Unfortunately, this useful method is no longer valid when the ambient space is the sphere $\mathbb{S}^3$. Therefore, the construction and uniqueness of embedded minimal surfaces in the sphere have been challenging questions. The only known examples of this kind of surface in $\mathbb{S}^3$ of genus 0 and 1 were the equator and the Clifford torus, respectively. However, other constructions were given in the work of Lawson in 1970 [71] where he showed that there exist infinitely many compact embedded minimal surfaces in $\mathbb{S}^3$, including examples of every genus g. In the same year, he also made an outstanding conjecture [72] that the only embedded minimal surface with $g = 1$ in the 3-sphere is the Clifford torus. This conjecture was successfully confirmed by Brendle in 2012 [23]. The main goal of this section is to illustrate Brendle's proof of Lawson's conjecture.

Before we explain Brendle's work on the uniqueness conjecture of Lawson, we will briefly mention some previous efforts towards the construction of compact embedded minimal surfaces in $\mathbb{S}^3$. H. B. Lawson [71] showed that there is at least one compact embedded minimal surface in $\mathbb{S}^3$ for each genus $g \in \mathbb{Z}^+$ and at least two such surfaces if $g > 1$ and g is not a prime number. Additional and different examples of these types of surfaces were obtained in 1988 by H. Karcher, U. Pinkall and I. Sterling [64] with genus 3, 5, 6, 7, 11, 19, 73 and 601. Moreover, Kapouleas and Yang [63] constructed another family of these surfaces of genus

$m^2 + 1$ for m sufficiently large. In 2015, J. Choe and M. Soret [34] showed the existence of an infinite family of compact embedded minimal surfaces with genus $1 + 4m(m-1)$ in $\mathbb{S}^3$.

The techniques that were applied in the previous results [71] and [64] for developing embedded minimal surfaces in $\mathbb{S}^3$ have some similarities. More precisely, Lawson's construction [71] is based on finding a minimal disk D in a simple closed (Jordan) curve Γ consisting of geodesic segments forming some of the sides of a tetrahedron, and then extending the surface by π-rotation about each of the geodesic sements of Γ in order to obtain a compact embedded minimal surface. The method that is used in [64] extends this idea by finding a minimal disk D embedded in the fundamental cell T of a tesselation of the 3-sphere, in such a way that D can be extended beyond the boundary of T by reflections to produce new compact embedded minimal surfaces in $\mathbb{S}^3$. The arguments in [34] are somewhat related, using a tesselation of the sphere by a family of intersecting Clifford tori into pentahedral cells. The technique in [63] is different in nature and technique: This uses uses a more technical doubling construction which involves first constructing an approximate minimal surface by joining the two boundary surfaces of a fattened Clifford torus using many small catenoidal bridges, and then solving the delicate perturbation problem required to find a minimal surface close to the approximate one.

Questions regarding the uniqueness of immersed or embedded minimal surfaces of different genus in the sphere have been tackled by a number of mathematicians over the last several decades. In 1966, F. J. Almgren [9] was the first to prove that the totally geodesic 2-dimensional spheres are the only immersed minimal surfaces in $\mathbb{S}^3$ with genus $g = 0$.

Theorem 4.1. *(F. J. Almgren [9]) Let Σ be an immersed minimal surface of $\mathbb{S}^3$ and the genus of Σ is zero, then Σ is congruent to the equator.*

In order to prove this assertion, the author applied the holomorphic differential method that was established by H. Hopf [54].

Proof. Assume that $F : \mathbb{S}^2 \to \mathbb{S}^3$ is a conformal, minimal and harmonic immersion and let $w = h(\frac{\partial}{\partial z}, \frac{\partial}{\partial z})dz^2$ be the Hopf differential, where h is the second fundamental form of Σ. Since the mean curvature vanishes, then we obtain that w is holomorphic from the Codazzi equations. Moreover, from [92] Section VI, we have $w = 0$ since $\mathbb{S}^2$ does not carry a non-zero holomorphic quadratic form. This implies that the second fundamental form vanishes and hence Σ is congruent to the equator. $\square$

Another crucial uniqueness result for minimal surfaces in the three-dimensional sphere states that the only embedded minimal surface with $g = 1$ in $\mathbb{S}^3$ is the Clifford torus. This was conjectured by H. B. Lawson in 1970 [72] and positively answered by S. Brendle in 2012 [23].

Theorem 4.2. *(S. Brendle [23]). Let Σ be an embedded minimal surface with $g = 1$ of $\mathbb{S}^3$, then Σ is congruent to the Clifford torus.*

The proof of Lawson's conjecture had been approached previously over several decades resulting in a variety of incomplete answers. In 1990, Urbano [97] studied these surfaces by considering their index. More precisely, he proved that if Morse index of the minimal torus is less than or equal 5, then such surface must be congruent to the Clifford torus. Furthermore, this conjecture of Lawson was proved by Ros [90] with further assumptions that the torus is symmetric through four pairwise orthogonal hyperplanes. More recently, Marques and Neves [76] verified that the minimal surface of genus greater than or equal one and with the smallest area $2\pi^2$ amongst other minimal surfaces in $\mathbb{S}^3$ is congruent to the Clifford torus. The main ingredient of the proof of such work is the application of the min-max theory for minimal surfaces, refer to the work of Pitts [89] and also Colding and De Lellis [35] for more details on this approach.

The conjecture of Lawson was proved by S. Brendle [23]. The essential assumption on the minimal surface in this conjecture is the embeddedness, otherwise there exist infinitely many immersed minimal surfaces that are constructed by applying methods of integrable systems. Another crucial condition is that Σ has genus 1 which enables Brendle to apply the following result of Lawson:

Proposition 4.3. *Let Σ be an immersed minimal surface in $\mathbb{S}^3$, If the genus of Σ is 1, then the surface has no umbilical points. Equivalently the second fundamental form does not vanish everywhere on Σ.*

The proof of this proposition is straightforward by utilizing a simplified version of the holomorphic Hopf differential.

Proof. Assume that Σ is an immersed minimal torus in $\mathbb{S}^3$, and let h be the second fundamental form of Σ. By the uniformisation theorem there is a conformal diffeomorphism from Σ to a flat torus $\mathbb{C}/\Gamma$ for some lattice Γ, and this induces a globally defined complex vector field $\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$ on Σ. This enables the definition of a complex function defined by $f = h(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}) = (h_{xx} - h_{yy}) - 2ih_{xy}$. The minimal surface equation, together with the conformality of the parametrisation and the Codazzi equations, imply that f is holomorphic, and therefore constant. Thus, either the second fundamental form vanishes identically, or it is nowhere zero. But the vanishing of the second fundamental form would imply that Σ is totally geodesic, and hence an equatorial sphere, contradicting the assumption that Σ has genus 1. Therefore, h is non-zero everywhere on Σ. □

Besides the previous result of Lawson, another useful remark regarding the geometric structure of the Clifford torus is relevant to argument which follows: This torus has the property that the boundary curvature of largest ball coincides with the maximum principal curvature. In particular, for each point x in the Clifford torus, we can touch x by a ball with boundary curvature equal to the largest principal curvature $\lambda_{max}(x)$. That is, we have $\bar{k}(x) = \lambda_{\max}(x)$, where $\bar{k}(x)$ is the boundary curvature of the largest touching ball, defined analogously to the function $\bar{k}$ defined by Equation (3.4) in the non-collapsing argument for mean curvature flow in section 3.2.

The above observations enabled Brendle to apply a refined version of the non-collapsing argument to prove the Lawson conjecture. More precisely, Brendle compared the size of balls touching minimal surfaces at every point to the maximum principal curvature, instead of the mean curvature as used in Section 3.2, since the latter vanishes in the case of minimal surfaces. It is useful to note that the intersection of $\mathbb{S}^3$ with balls belonging to the ambient space $\mathbb{R}^4$ produce geodesic balls in $\mathbb{S}^3$, and the geodesic curvature of the boundary of such a ball in $\mathbb{S}^3$ is the same as its curvature in $\mathbb{R}^4$. It follows that the formula (3.4) gives precisely the same interpretation of $\bar{k}(x)$ as the geodesic curvature of the largest sphere disjoint from Σ which touches at x, where we now interpret the position vector F and the normal vector ν as vectors in the ambient space $\mathbb{R}^4$. As a consequence of this modification, the machinery of the non-collapsing estimate can be extended directly to the context of minimal tori in the sphere $\mathbb{S}^3$.

Now we are ready to present the proof of the main theorem 4.2, which is based on the application of the maximum principle argument to the function $k(x, y, t)$ to deduce that $\bar{k}(x) = \lambda_{max}(x)$ for each point x of Σ.

Proof. The first aim here is to extend Andrews' work on the non-collapsing for mean curvature flow in the Euclidean space [12] into the 3-dimensional sphere. In other words, we want to derive a natural differential inequality of $\bar{k}(x)$ in the viscosity sense in $\mathbb{S}^3$.

Proposition 4.4. *Let $F : \Sigma \times [0, T) \to \mathbb{S}^3$ be a family of embedded surfaces mov-*

ing under the mean curvature flow in the sphere. Then $\bar{k}$ satisfies the inequality

$$\partial_t \bar{k} \leq \Delta \bar{k} + |A|^2 \bar{k} + 2H_x - 2\bar{k}. \tag{4.1}$$

in the viscosity sense.

Proof. Let $F : \Sigma \times [0, T) \to \mathbb{S}^3$ be a family of embedded surfaces which satisfies the mean curvature flow in the sphere. We assume that $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^4$. Also, we define the boundary curvature of the largest ball $\bar{k}(x, y, t)$ enclosed in the region U and touching the surface $F(\Sigma)$ at $F(x, t)$ similarly as in section (3.2) on mean convex mean curvature flow by the following:

$$\bar{k}(x, t) = \sup_{y \in \Sigma \setminus \{x\}} k(x, y, t) = \sup_{y \in \Sigma \setminus \{x\}} \left\{ \frac{2\langle F(x, t) - F(y, t), \nu(x, t) \rangle}{\|F(x, t) - F(y, t)\|^2} \right\}.$$

Such quantity refers to the interior ball curvature in the $-\nu(x, t)$ direction. Similarly, the exterior ball curvature $\underline{k}(x, t)$ of F is given by

$$\underline{k}(x, t) = \inf_{y \in \Sigma \setminus \{x\}} k(x, y, t),$$

where $\underline{k}(x, y, t)$ is the boundary curvature of the smallest ball $\bar{k}(x, y, t)$ that contains U and touches $F(\Sigma)$ at $F(x, t)$ in the opposite direction of $\nu(x, t)$.

In order to deduce that $\bar{k}(x, t)$ is a subsolution of a natural differential equation in the viscosity sense, we assume that $\rho(x, t)$ is a smooth function that is defined on a neighbourhood of an arbitrary point $(x_0, t_0) \in \Sigma \times [0, t_0]$ and lies above $\bar{k}(x, t)$ such that $\rho(x_0, t_0) = \bar{k}(x_0, t_0)$ where $t \leq t_0$. This implies to $k(x, y, t) \leq \rho(x, t)$ for all $(x, y, t) \in \Sigma \times \Sigma \times [0, t_0]$. We choose x^i and y^i for $i = 1, 2$ to be local normal coordinates for Σ near x_0 and y_0, respectively.

We now mostly follow the computation in the previous chapter, highlighting the differences caused by the fact that the ambient space is the sphere instead of the Euclidean space (see section 3.2, [12] and [16]) . Computing the first spatial derivatives of k gives

$$\left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i} \right) k = \frac{2}{d^2} \left(\left\langle \frac{\partial F}{\partial x^i} - \frac{\partial F}{\partial y^i}, \nu_x + k d\vec{w} \right\rangle - \left\langle d\vec{w}, (h_x)_i^p \frac{\partial F}{\partial x^p} \right\rangle \right).$$

Recall that $d = \|F(y, t) - F(x, t)\|$ is the distance function, $\vec{w} = \frac{F(y,t) - F(x,t)}{d}$ is the unit vector and $\nu_x = \nu(x, t)$ is the unit normal vector field. Also, remember that h and λ_i refer to the second fundamental form and the principal curvature of Σ for $i = 1, 2$. Since the last identity vanishes at (x_0, y_0, t_0), then we have

$$\frac{\partial \rho}{\partial x^i} = \frac{\partial k}{\partial x^i} = -\frac{2}{d}\big(k-\lambda_i\big)\left\langle \frac{\partial F}{\partial x^i}, w\right\rangle, \tag{4.2}$$

and

$$\frac{\partial k}{\partial y^i} = -\frac{2}{d^2}\left\langle \frac{\partial F}{\partial y^i}, \nu_x + kd\vec{w}\right\rangle = 0.$$

The calculation of the second derivatives of k at x_0 in the sphere is as follows:

$$\begin{aligned}
Lk &= \sum_i \left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i}\right)^2 k \\
&= \frac{2}{d^2}\Bigg(\langle \nu_x + kd\vec{w}, H_y\nu_y - H_x\nu_x + 2d\vec{w}\rangle - d\langle \vec{w}, \nabla H_x\rangle \\
&+ 2\sum_i \left\langle \frac{\partial F}{\partial x^i} - \frac{\partial F}{\partial y^i}, (h_x)_i^q \frac{\partial F}{\partial x^q} + \frac{\partial k}{\partial x^i} d\vec{w}\right\rangle \\
&- \sum_i \left\| \frac{\partial F}{\partial y^i} - \frac{\partial F}{\partial x^i}\right\|^2 k + d\langle \vec{w}, |A|^2\nu_x + H_x F(x)\rangle\Bigg), \\
&\geq \frac{2}{d^2}\big(\langle \nu_x + kd\vec{w}, H_y\nu_y - H_x\nu_x\rangle\big) - d\langle \vec{w}, \nabla H_x\rangle \\
&+ \frac{8}{d^2}(k-\lambda_i)\left\langle \vec{w}, \frac{\partial F}{\partial x^i}\right\rangle^2 - |A|^2 k - H_x + 2k.
\end{aligned}$$

Note that the equation (4.2) and the definition of $k(x,y)$ are used in the last inequality. The computation of the time derivative implies

$$\begin{aligned}
\frac{\partial k}{\partial t} &= -\frac{2}{d^2}\big(\langle \nabla H_x + H_x F(x), d\vec{w}\rangle + \langle \nu_x + kd\vec{w}, H_x\nu_x - H_y\nu_y\rangle\big), \\
&= -\frac{2}{d^2}\big(\langle \nabla H_x, d\vec{w}\rangle + \langle \nu_x + kd\vec{w}, H_x\nu_x - H_y\nu_y\rangle\big) + H_x.
\end{aligned}$$

Whereas the evolution of the unit normal $\nu(x)$ in the sphere is given by

$$\frac{\partial \nu}{\partial t} = \nabla H_x + H_x F(x).$$

It follows that

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - L\right)k &\leq |A|^2 k - \frac{8}{d^2}(k-\lambda_i)\left\langle \vec{w}, \frac{\partial F}{\partial x^i}\right\rangle^2 + 2H_x - 2k \\
&\leq |A|^2\rho + 2H_x - 2\rho.
\end{aligned}$$

This is precisely the statement we wanted to prove, that $\bar{k}$ satisfies the inequality (4.1) in the viscosity sense. □

To simplify the computation, it is possible to add the term $\frac{-|\nabla \bar{k}|^2}{\bar{k}}$ to the last inequality. Also, we have $H = 0$ since Σ is a minimal torus in $\mathbb{S}^3$. Therefore,

$$0 \leq \Delta \bar{k} + |A|^2 \bar{k} - 2\bar{k} - \frac{|\nabla \bar{k}|^2}{\bar{k}}. \tag{4.3}$$

In addition to such inequality, we need also to derive the following result which is analogous to the Simons-type identity [96] (see also [79]):

Proposition 4.5. *Let $F : \Sigma \to \mathbb{S}^3$ be an embedded minimal torus in the three-dimensional sphere. Then the maximum principle function λ_1 is smooth and strictly positive everywhere on Σ. Moreover, λ_1 satisfies the following partial differential equation:*

$$\Delta \lambda_1 + (|A|^2 - 2)\lambda_1 - \frac{|\nabla \lambda_1|^2}{\lambda_1} = 0 \tag{4.4}$$

Proof. From the definition of the mean curvature H, we have $\lambda_1 = \frac{|A|}{\sqrt{2}}$. Also, it was proven by Hopf in his paper [54] that a minimal torus in $\mathbb{S}^3$ is impossible to have umbilical points. Consequently, λ_1 is smooth and strictly positive at each point on the torus. We now directly compute to find $\Delta(|A|^2)$

$$\Delta(h_{ij}h_{ij}) = 2h_{ij}\Delta h_{ij} + 2\nabla_k h_{ij} \nabla_k h_{ij},$$

thus, we get

$$\Delta(|A|^2) = 2\langle |A|, \Delta |A| \rangle + 2|\nabla A|^2.$$

By commuting $\nabla_i \nabla_j h_{kl} - \nabla_k \nabla_l h_{ij}$ and then taking its trace over the components k and l, we obtain the following simon's identity [96] (see also [79]):

$$\Delta h_{ij} + (|A|^2 - 2)h_{ij} = 0.$$

It follows from the last two equations that

$$\Delta(|A|^2) + 2(|A|^2 - 2)|A|^2 - 2|\nabla A|^2 = 0,$$

Since $\Delta(|A|^2) = 2|A|(\Delta|A|) + |\nabla A|^2$ and $|\nabla A|^2 = 2|\nabla |A||^2$ from the Codazzi equations, then the last equation becomes

$$\Delta(|A|) + (|A|^2 - 2)|A| - \frac{|\nabla |A||^2}{|A|} = 0.$$

The assertion follows since $\lambda_1 = \frac{|A|}{\sqrt{2}}$. $\square$

Now by combining (4.3) and (4.4), we have the following inequality in the viscosity sense

$$0 \leq \Delta \log \left(\frac{\bar{k}}{\lambda_1} \right). \tag{4.5}$$

This implies that $\bar{k} = C\lambda_1$ by the strong maximum principle. However, we have $\bar{k} = \lambda_1$ at every point on the surface that means $C = 1$. More precisely, we show that $\bar{k} = \lambda_1$ and we can similarly derive that $\underline{k} = \lambda_2$ for each point in the embedded minimal torus, where λ_2 denote the minimum principal curvatures.

Finally, to complete the proof we need to show that $\nabla h = 0$. Let $\sigma : \mathbb{R} \to \Sigma$ be a geodesic such that $\sigma(0) = x_0$ and $\sigma'(0) = e_1$. From the definition of $\bar{k}$ and since $\bar{k} = \lambda_1$, we define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(s) = Z(\sigma(0), \sigma(s)) = \lambda_1(1 - \langle x_0, \sigma(s)\rangle) + \langle \nu(x_0), \sigma(s)\rangle,$$

where $f(s)$ is non-negative for all s. Therefore, by direct calculation we obtain the following:

$$\begin{aligned}
f'(s) &= -\langle \lambda_1(x_0)x_0 - \nu(x_0), \sigma'(s)\rangle, \\
f''(s) &= \langle \lambda_1(x_0)x_0 - \nu(x_0), \sigma(s)\rangle + h(\sigma'(s), \sigma'(s))\langle \lambda_1(x_0)x_0 - \nu(x_0), \nu(\sigma(s))\rangle, \\
f'''(s) &= \langle \lambda_1(x_0)x_0 - \nu(x_0), \sigma'(s)\rangle + h(\sigma'(s), \sigma'(s))\langle \lambda_1(x_0)x_0 - \nu(x_0), D_{\sigma'(s)}\nu\rangle \\
&\quad + D^{\Sigma}_{\sigma'(s)}h)(\sigma'(s), \sigma'(s))\langle \lambda_1(x_0)x_0 - \nu(x_0), \nu(\sigma(s))\rangle.
\end{aligned}$$

If $t = 0$, we have $f(0) = f'(0) = f''(0) = 0$ and also $f'''(0) = 0$ since $f(s) \geq 0$. This gives $(D_{e_1}h)(e_1, e_1) = 0$ everywhere on Σ since x_0 is arbitrary. Similarly for the exterior region, we obtain $(D_{e_2}h)(e_2, e_2) = 0$ everywhere. Hence, we deduce from the Codazzi equations that the second fundamental form is parallel that is $Dh = 0$. Moreover, from the Gauss equation, the Ricci curvature of the surface is constant and thus the induced metric is flat. It follows that $F(\Sigma)$ is congruent to the Clifford torus since Lawson showed that the only minimal torus in the three-sphere with parallel Ricci curvature is the Clifford torus. $\square$

We have illustrated Brendle's proof of the Lawson conjecture for the minimal torus. However, what is the situation if the mean curvature does not vanish. The explanation of constant mean curvature tori in $\mathbb{S}^3$ will be in the next section.

4.2 The Pinkall-Sterling conjecture for CMC tori in $\mathbb{S}^3$

In this section, the work of B. Andrews and H. Li [20] on embedded constant mean curvature tori in the three-sphere is represented. In particular, they showed that any embedded constant mean curvature torus in $\mathbb{S}^3$ must be rotationally symmetric. The proof of such result mainly depends on improving the similar argument of S. Brendle [23] on embedded minimal tori in $\mathbb{S}^3$ that is illustrated in the previous section 4.1. Consequently, they provided a complete classification of such rotationally symmetric surfaces in $\mathbb{S}^3$ together with Perdomo's work [86]. It was recognized later that the proof of B. Andrews and H. Li [20] precisely confirms the conjecture of U. Pinkall and I. Sterling [88]. The main ingredient of this part is to describe the proof provided in [20] that any embedded constant mean curvature torus in $\mathbb{S}^3$ is a surface of rotation.

The theory of constant mean curvature surfaces in space forms of constant sectional curvatures $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$ is a crucial direction of study in differential geometry. The basic examples of such surfaces in $\mathbb{S}^3$ are the totally umbilical two dimensional spheres and the Clifford torus $\mathbb{S}^1(r) \times \mathbb{S}^1(\sqrt{1-r^2})$, where $0 < r < 1$. In 1981, S. S. Chern [32] generalized the result of H. Hopf [53] that any immersed constant mean curvature surfaces with genus 0 in $\mathbb{R}^3$ is totally umbilical to the ambient spaces $\mathbb{S}^3$ and $\mathbb{H}^3$.

Many results with regard to the construction of such surfaces have been significantly investigated and contributed in the development of this field. In the lecture notes of H. Hopf on differential geometry [54], the following fundamental question was introduced: Is a compact surface with constant mean curvature immersed into the Euclidean space must be the standard embedded sphere?. This question was interestingly developed by H. C. Wente in [102] where he was the first to produce comapct immersed constant mean curvature surfaces with genus $g = 1$ in $\mathbb{R}^3$. Such Hopf's conjecture was partially confirmed in the spaces $\mathbb{R}^3$, $\mathbb{S}^3_+$ and $\mathbb{H}^3$ with imposing the condition of embeddedness by A. D. Alexandrov [6]. Moreover, the existence of immersed constant mean curvature surfaces into $\mathbb{R}^3$ with arbitrary genus $g > 3$ was shown by N. Kapouleas [62] in 1987.

Besides Wente's tori, the problem of constructing all constant mean curvature tori in $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$ is remarkably considered. Bobenko [22] provided other examples of such surfaces in $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$ by integrating the Gauss-Weingarten equations and deducing an explicit formula with the crucial property of periodicity. Moreover, U. Pinkall and I. Sterling [88] generally classified all constant mean curvature tori in $\mathbb{R}^3$ using the system of ordinary differential equations to study solutions of constant mean curvature tori. A conjecture that these surfaces are rotationally symmetric in $\mathbb{S}^3$ was explicitly presented in the later work [88] and positively confirmed by B. Andrews and H. Li [20].

Hence, such result along with Perdomo's contribution [86] complete the classification of constant mean curvature tori in $\mathbb{S}^3$ which is illustrated in the following theorem:

Theorem 4.6 (B. Andrews, H. Li [20], O. Perdomo [86]). *Let Σ be an embedded constant mean curvature torus in $\mathbb{S}^3$, then the following statements hold:*

(a) Every Σ must be rotationally symmetric.

(b) If H values at 0 or $\pm\frac{1}{\sqrt{3}}$, then every Σ with such H is congruent to the Clifford torus.

(c) Every Σ with H has different values from 0 and $\pm\frac{1}{\sqrt{3}}$ is non-isoparametric. In other words, Σ admits the group $O(2) \times Z_m$ in its group of isometries where $m \geq 2$ is a maximal integer.

(d) There exists at most one such non-isoparametric surface for any integer $m \geq 2$.

(e) If $m \geq 2$ and $\cot(\frac{\pi}{m}) < H < \frac{m^2-2}{2\sqrt{m^2-1}}$, then there exists a compact, embedded and non isoparametric surface Σ which admits the group $O(2) \times Z_m$.

The results (b), (c) and (e) are deduced by O. Perdomo [86] where the proof in the case of vanishing mean curvature is provided by S. Brendle [23] and explained in the previous section. Moreover, the components (a) and (d) are contributed by B. Andrews and H. Li [20] implying a complete classification of these tori in $\mathbb{S}^3$. Recall that an embeddedness is a necessary requirement in this work since A. I. Bobenko [22] constructed infinitely many immeresed constant mean curvature tori in the three sphere which are non-rotationally symmetric.

The main focus in this section is to illustrate the proof of the component (a) which is deduced in [20] by extending Brendle's argument of non-collapsing method to the constant mean curvature case. The argument has been explained in the previous chapter with respect to the mean convex mean curvature flow which was tackled by B. Andrews [16]. Also, it is mentioned above in S. Brendle's paper [23] after some modification of such approach to be applicable to the case of minimal surfaces.

It seems apparently improbable that such technique could be applied in this case, since there exist some embedded constant mean curvature tori in $\mathbb{S}^3$ which are not only product of circles. These constructions are classified as rotationally symmetric surfaces. For instance, O. Perdomo [86] showed the existence of embedded constant mean curvature tori with $H \notin \{0, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\}$ which are non-isoparametric. Fortunately, B. Andrews and H. Li [20] were able to extend the non-collapsing argument to the case of embedded constant mean curvature tori in $\mathbb{S}^3$.

The structure of the proof of item (a) in Theorem 4.6 will be divided into two parts. The first main part is to show that the interior ball curvature $\bar{\kappa}$ is equal to $\lambda_{\max}$ for every $x \in \Sigma$ in $\mathbb{S}^3$. In the second part, we obtain that some components of the derivative of the second fundamental form vanish. In other words, we deduce that one of the principal curvatures of the surface are constant which is sufficient to conclude that the surface is rotatioanlly symmetric. In other words, Σ is invariant under the group of rotations fixing $\Pi^2 \in \mathbb{R}^4$.

Proof. We begin by recalling some basic expressions and notions about the interior balls touching Σ at a point x (see also section 3.2, [16] and [19] for more details). Let $F(\Sigma)$ be an embedded constant mean curvature torus in $\mathbb{S}^3$. Assume that the region $U \subset \mathbb{S}^3$ is chosen to be enclosed by the embedding F such that the unit normal vector ν points in the outward direction of U. For every point $x \in \Sigma$, there is a touching ball B that is contained in U with the boundary curvature κ. The corresponding interpretation of such geometric idea is that other points of F are not contained in B which is precisely stated by the following inequality:

$$Z(\kappa, x, y) = \kappa(1 - F(x) \cdot F(y)) + \langle F(y), \nu(x)\rangle \geq 0, \tag{4.6}$$

where $x, y \in \Sigma$. Given x, the inequality $Z \geq 0$ will be satisfied for all y for κ sufficiently large, since we can always touch by a small enough ball at x. The interior ball curvature $\bar{\kappa}(x)$ is defined to be the infimum of all $\kappa > 0$ for which the inequality $Z(\kappa, x, y) \geq 0$ holds for all y.

Note that the function $Z(\bar{\kappa}(x), x, y)$ is non-negative everywhere on the surface. Hence, there exist two distinct points x_0 and y_0 for which Z vanishes. In other words, Z has a minimum value and thus its first derivative vanishes as well and the second derivatives of Z are non-negative at such point (x_0, y_0).

This construction of Z implies that the tangent spaces and the outward unit normal vectors of Σ at x_0 and y_0 coincide with those tangent spaces and unit normal vectors of the boundary of the touching ball B respectively. Moreover, since the ambient space is $\mathbb{S}^3$, then the reflection $R_{\vec{w}}$ maps the tangent space of Σ at x_0 to that at y_0 and is given as follows:

$$R_{\vec{w}}(\nu(x_0) = (\nu(y_0) = \nu(x_0) - 2\vec{w} \cdot \nu(x_0)\vec{w}, \tag{4.7}$$

and

$$R_{\vec{w}}\left(\frac{\partial F}{\partial x^i}(x_0)\right) = \frac{\partial F}{\partial y^i}(y_0). \tag{4.8}$$

Assume that the boundary curvature of the largest touching ball at every point of Σ is denoted by $\bar{\kappa}$ and called the interior ball curvature. Note that the inequality $\bar{\kappa} \geq \lambda_{\max}$ always holds for every $x \in \Sigma$ since it is impossible to have $\kappa(x) < \lambda_{\max}(x)$.

We now illustrate the first main part of the proof which is proving that the interior ball curvature $\bar{\kappa}$ is equal to $\lambda_{\max}$ for every $x \in \Sigma$ in $\mathbb{S}^3$, similar to the previous section.

Proposition 4.7. *Let $F(\Sigma)$ be an embedded constant mean curvature torus in the three-dimensional sphere. Then we have*

$$\bar{\kappa} = \lambda_{\max},$$

at every point x of Σ.

In order to prove this proposition we consider the opposite and arrive to a contradiction by applying the maximum principle argument.

Proof. We use similar notations to the proof of Theorem 4.2 in the previous section. Let (x_1, x_2) and (y_1, y_2) be orthonormal coordinates around the points x_0 and y_0, respectively. Since the second fundamental form h is diagonal such that $\sum_{i=1}^{2} h_{ij} = \lambda_i \delta_i^j$, then we have $h_{11} = \lambda_1 = \lambda_{max}$, $h_{22} = \lambda_2 = \lambda_{min}$ and $h_{12} = 0$ at x_0 where $\lambda_{\max}$ and λ_{min} refer to the maximum and minimum principal curvatures of Σ, respectively. By selecting a suitable direction of ν, we assume that H is non-negative and defined by

$$H = \sum_{i=1}^{2} \lambda_i - \sum_{i=1}^{2} \mu_i,$$

and also we have the following useful notation

$$|trA(x_0)|^2 = |A(x_0)|^2 - 2H^2.$$

If the mean curvature H vanishes, the result follows from Brendle's proof of Lawson's conjecture [23]. Hence, the focus here is on the case $H > 0$. Assume that $\bar{\kappa}$ is greater than $\lambda_{\max}$ at the point (x_0, y_0) and then differentiate the equation (4.6) in the direction $\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i}$. We have

$$\left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i}\right) Z = \frac{d^2}{2} \nabla_i \kappa + \left\langle \frac{\partial F}{\partial y^i} - \frac{\partial F}{\partial x^i}, \nu_x + kd\vec{w} \right\rangle + \left\langle d\vec{w}, (h_x)_i^p \frac{\partial F}{\partial x^p} \right\rangle,$$

where $\vec{w} = \frac{F(y)-F(x)}{d}$ is the unit vector and $d = \|F(y) - F(x)\|$ is the distance function. Since the first derivative of Z with respect to the x^i-direction vanishes at (x_0, y_0), we obtain the following identity:

$$\frac{\partial Z}{\partial x^i} = \frac{2}{d} (\kappa - \lambda_i) \left\langle \frac{\partial F}{\partial x^i}, \vec{w} \right\rangle, \tag{4.9}$$

and also we have $\frac{\partial \kappa}{\partial y^i} = 0$.

The calculation of the second derivatives of Z at (x_0, y_0) implies

$$\begin{aligned}
\sum_i \left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i}\right)^2 Z = \frac{d^2}{2} \Delta \kappa + \sum_i \left\| \frac{\partial F}{\partial y^i} - \frac{\partial F}{\partial x^i} \right\|^2 \kappa \\
- 2 \sum_i \left\langle \frac{\partial F}{\partial x^i} - \frac{\partial F}{\partial y^i}, h_i^q \frac{\partial F}{\partial x^q} + \nabla_i \kappa d\vec{w} \right\rangle \\
+ 2 \langle \nu_x + \kappa d\vec{w}, H_y \nu_y + F(y) - H_x \nu_x - F(x) \rangle \\
+ d \langle \vec{w}, 2\nabla H_x - |A|^2 \nu_x - H_x F(x) \rangle.
\end{aligned}$$

Note that the term $2\nabla H_x$ in the last line appears form the Codazzi equation such that $2\nabla^q H_x = \nabla^q h_{ii} = \nabla_i h_i^q$. Also, since the mean curvature is constant, then ∇H vanishes. Moreover, it follows from the reflection property of the tangent spaces (4.7)-(4.8) and the first derivative (4.9) of Z with respect to x^i that

$$\frac{\partial F}{\partial x^i} - \frac{\partial F}{\partial y^i} = 2\vec{w} \cdot \frac{\partial F}{\partial x^i} \vec{w} = \frac{\nabla_i \kappa}{\kappa - \lambda_i} d\vec{w}.$$

Using the identities (4.7), (4.8), $d\vec{w} \cdot \nu(x) = -\frac{d^2}{2}\kappa$ and also $d\vec{w} \cdot F(x) = -\frac{d^2}{2}$, we obtain

$$\sum_i \left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i}\right)^2 Z = \frac{d^2}{2} \left(\Delta \kappa - 2 \frac{|\nabla^2 \kappa|^2}{\kappa - \lambda_i} + (|A|^2 - 2 - 2H\kappa)\kappa + 2H \right).$$

Consequently, by considering the following inequality

$$\kappa - \lambda_2 = \kappa - (2H - \lambda_1) = \kappa - 2H + \lambda_1 \leq 2(\kappa - H),$$

where the definition of the mean curvature is used and then substituting this into the last equation, we arrive to

$$\sum_i \left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i}\right)^2 Z \leq \frac{d^2}{2}\left(\Delta\kappa - \frac{|\nabla^2\kappa|^2}{\kappa - H}\right) + \big((|A|^2 - 2 - 2H\kappa)\kappa + 2H\big). \quad (4.10)$$

Now we want to show that the maximum principal curvature $\lambda_{max} = \lambda_1$ is equivalent to the interior ball curvature $\bar{\kappa}$. Let α be any positive number and the curvature of the touching ball κ is defined as

$$\kappa = \alpha\mu + H.$$

We emphasize that $F(\Sigma)$ is embedded which implies that the interior ball curvature $\bar{\kappa}$ has an upper bound. Also, since the embedded surface is compact and its principal curvature μ is strictly positive, then μ is bounded from below by a positive value. Hence, we have

$$\alpha\mu + H > \bar{\kappa},$$

where α is sufficiently large and Z is positive.

Besides the inequality (4.10), we need to consider the following result which is analogous to the Simons-type identity which is deduced in Proposition 4.3 of the previous section (see also [96] and [79]):

Proposition 4.8. *Let $F : \Sigma \to \mathbb{S}^3$ be an embedded minimal torus in the three-sphere. Then the function μ is smooth and strictly positive everywhere on Σ and satisfies*

$$\Delta\mu - \frac{|\nabla\mu|^2}{\mu} + 2(\mu^2 + H^2 - 1) = 0. \quad (4.11)$$

Now we want to check the possible values of α. If $\gamma(t)$ denotes a geodesic passes through a point x in Σ, then we have

$$\frac{2}{d^2}\langle d\vec{w}, \nu(x)\rangle = -h_X(\gamma', \gamma') + O(t).$$

Thus, we obtain form the definition of Z

$$Z(\kappa, x, \gamma(t)) = \frac{1}{2}\big(\kappa - h_X(\gamma', \gamma')\big)t^2 + O(t^3).$$

The values of Z are negative when $\alpha < 1$. More precisely, let $\gamma'(0) = e_1$ where e_1 is the direction of λ_{max} that gives $h_X(\gamma', \gamma') = \lambda = H + \mu$. Thus, $Z(\kappa, x, \gamma(t)) < 0$ which contradicts the fact that Z is non-negative.

On the other hand if $\alpha > 1$, we obtain that $Z(\kappa, x, \gamma(t)) \geq \frac{1}{2}(\kappa - 1)\mu t^2 + O(t^3)$ from any direction of the point x. This implies that Z takes positive values only in a neighbourhood of the diagonal set $x = y$ in $\Sigma \times \Sigma$.

Now we consider $\bar{\alpha} = \inf\{\alpha > 0 : Z(\kappa, x, y) \geq 0\}$ for all $x, y \in \Sigma$. Therefore we can infer that $\bar{\alpha}$ takes the values $1 \leq \bar{\alpha} < \infty$. Finally, we have two cases whether $\bar{\alpha} > 1$ or $\bar{\alpha} = 1$. In the former case, we also arrive to a contradiction. More specifically, the function $Z(\kappa, x, y)$ in such case has positive values in a region of the diagonal set $\{(x, x) : x \in \Sigma\}$. Thus, there exists a minimum and distinct point (x_0, y_0) in $\Sigma \times \Sigma$ such that $Z(x_0, y_0) = 0$ and therefore the second derivatives of Z at such point are non-negative. Recall that if $\kappa > \lambda_i$ at (X_0, y_0), we deduce here the inequality (4.10). Considering such inequality and using Proposition 4.8 gives the following:

$$0 \geq \sum_i \left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i} \right)^2 Z \leq -(\alpha^2 - 1) d^2 \mu^2 H < 0,$$

which shows a contradiction.

The proof is completed by considering the later case $\bar{\alpha} = 1$. It confirms straightforwardly the assertion that $\bar{\kappa} = \lambda$ by just substituting in $\bar{\kappa} \leq \bar{\alpha}\mu + H = \lambda$. □

The second main part of the proof is the argument that Σ is rotationally symmetric:

Theorem 4.9. *Assume that $F : \Sigma \to \mathbb{S}^3$ is an embedded constant mean curvature torus for which $\lambda_{max} = \bar{\kappa}$ at every point of Σ. Then Σ is rotationally symmetric*

In order to ensure that the surface is rotationally symmetric, it is adequate to deduce that the derivatives of the components of the second fundamental form vanish and also there exists a plane $\Pi^2 \in \mathbb{R}^4$ such that Σ is invariant under the group of rotations fixing Π^2.

Proof. As explained in the previous section, we first derive that $(D_{e_1} h)(e_1, e_1) = 0$ and thus $(D_{e_1} h)(e_2, e_2) = 0$ everywhere on Σ where e_1 and e_2 are smooth eigenvector fields. Let $\sigma(s)$ be a geodesic that passes through $x_0 \in \Sigma$ such that $\sigma(0) = x_0$ and $\sigma'(0) = e_1$. Since we have $\bar{\kappa} = \lambda$ at each point of Σ (Proposition 4.7), then the real function $f : \mathbb{R} \to \mathbb{R}$ is given by

$$f(s) = Z(\lambda, \sigma(0), \sigma(s)) = \lambda(1 - \langle x_0, \sigma(s) \rangle) + \langle \nu(x_0), \sigma(s) \rangle,$$

where $f(s)$ is non-negative for all s and $f(0) = 0$. Differentiating $f(s)$ implies

$$f'(s) = -\langle \lambda(x_0)x_0 - \nu(x_0), \sigma'(s)\rangle,$$

note that f' vanishes at $s = 0$. Calculating the second derivative of $f(s)$ gives

$$f''(s) = \langle \lambda(x_0)x_0 - \nu(x_0), \sigma + h(\sigma', \sigma')\nu(\sigma)\rangle,$$

and also computing $f'''(s)$ as the following

$$f'''(s) = \langle \lambda(x_0)x_0 - \nu(x_0), \sigma' + h(\sigma', \sigma')D_{\sigma'}\nu + (D_{\sigma'}h)(\sigma', \sigma')\nu\rangle.$$

We have $f'' = f''' = 0$ at $s = 0$ since $f(s) \geq 0$. This gives $(D_{e_1}h)(e_1, e_1) = 0$ everywhere on Σ since x_0 is arbitrary. In particular, $e_1\lambda = 0$ and thus $e_1\mu = 0$ where $H = \lambda - \mu$.

Let U be a neighbourhood of any point x on Σ, then we can choose a local orthonormal frame $\{e_0, e_1, e_1\}$ where e_0 is the unit normal vector of Σ. Assume that its dual coframe is denoted by $\{\omega_1, \omega_2\}$. The Levi Civita connection forms are given by

$$de_1 = \omega_{12}e_2, \qquad de_2 = \omega_{21}e_1,$$

where $\omega_{12} + \omega_{21} = 0$.

The first covariant derivative of the second fundamental form h_{ij} with respect to e_k where $1 \leq i, j, k \leq 2$ is defined by

$$h_{ijk}\omega_k = dh_{ij} + h_{kj}\omega_{ki} + h_{ik}\omega_{kj}.$$

We have that $h_{ij,k}$ are symmetric from the Codazzi equations. Since the components of h_{ij} are diagonal, then $h_{11} = \lambda_1 = \lambda = H + \mu$, $h_{22} = \lambda_2 = \mu = H - \mu$ and $h_{12} = 0$. Considering that $i = 1$ and $j = 2$ in the last equation, we obtain

$$h_{122} = (h_{11} - h_{22})\omega_{12}(e_2) = 2\mu\omega_{12}(e_2), \tag{4.12}$$

and

$$h_{121} = 2\mu\omega_{12}(e_1). \tag{4.13}$$

From the equations of Codazzi, (4.12) and (4.13) and since we have $(\nabla_{e_1}h)(e_1, e_1) = 0$ and H is constant, then

$$\omega_{12}(e_2) = 0 \qquad 2\mu\omega_{12}(e_1) = e_2(\lambda) = e_2(\mu).$$

Hence, we deduce the following:

$$\nabla_{e_1}e_1 = \omega_{12}(e_1)e_2 = \frac{e_2(\mu)}{2\mu}e_2, \qquad \nabla_{e_2}e_1 = \omega_{12}(e_2) = 0, \tag{4.14}$$

and

$$\nabla_{e_2} e_2 = \omega_{21}(e_1)e_1 = 0, \qquad \nabla_{e_1} e_2 = \omega_{21}(e_1)e_1 = -\frac{e_2(\mu)}{2\mu}e_1. \tag{4.15}$$

The last equation (4.15) implies that the lines in direction e_2 are geodesics in Σ.

By using the Gauss equation

$$R_{ijkl} = (h_{ik}h_{jl} - h_{il}h_{jk}) + \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk},$$

and equations (4.14) and (4.15), we obtain

$$\begin{aligned} R_{1212} &= \lambda_1\lambda_2 + 1 \\ &= \langle \nabla_{e_1}\nabla_{e_2}e_2 - \nabla_{e_2}\nabla_{e_1}e_2 - \nabla_{[e_1,e_2]}e_2 \rangle \\ &= e_2\big(\frac{1}{2\mu}e_2(\mu)\big) - \big(\frac{1}{2\mu}e_2(\mu)\big)^2. \end{aligned}$$

Hence, we deduce the following equation

$$\frac{e_2(e_2(\psi))}{\psi} - \frac{1}{\psi^4} + H^2 + 1 = 0, \tag{4.16}$$

where $\psi = \frac{1}{\sqrt{\mu}}$. Multiplying (4.16) by the term $2\psi e_2(\psi)$ gives

$$(e_2(\psi))^2 + \psi^{-2} + (H^2+1)\psi^2 = C_0, \tag{4.17}$$

where C_0 is a constant.

Assume that Σ is embedded into the Euclidean space $\mathbb{R}^4$ and x is any arbitrary position vector, then we have $\langle \nu(x), \nu(x)\rangle = 1$, $\langle x, \nu(x)\rangle = 0$ and $\bar{\nabla}_v x = v$ where $\bar{\nabla}$ refers to the connection on $\mathbb{R}^4$. These provide us with the following:

$$\begin{aligned} \bar{\nabla}_{e_1}e_1 &= \frac{e_2(\mu)}{2\mu}e_2 - (x + \lambda_1\nu), \\ \bar{\nabla}_{e_2}e_2 &= -(x + \lambda_2\nu), \\ \bar{\nabla}_{e_2}x &= e_2, \\ \bar{\nabla}_{e_2}\nu &= \lambda_2 e_2. \end{aligned}$$

Moreover, let $\gamma(u)$ be a geodesic that satisfies $\gamma(0) = x_0 \in \Sigma$ and $\gamma'(0) = e_2(x_0)$. The above equation (4.17) is now rewritten with respect to $g(u)$ by taking $g(u) = \psi(\gamma(u))$ as

$$(g')^2 + g^{-2} + (1+H^2)g^2 + 2H = C_1 \tag{4.18}$$

where $C_1 = C_0 + 2H$ and $C_1 > 2(H + \sqrt{1 + H^2}$.

In order to solve Equation (4.18), the following non-negative polynomial is considered

$$P(s) = (C_1 - 2H)s^2 - (1 + H^2)s^4 - 1,$$

which is defined on an interval (s_0, s_1) for $s_1 > s_0 > 0$ and the value of $P(s)$ at s_0 and s_1 vanishes. The zeros of $P(s)$ are given by

$$s_0 = \sqrt{\frac{C_1 - 2H - A}{B}}, \qquad s_1 = \sqrt{\frac{C_1 - 2H + A}{B}}.$$

where $A = \sqrt{C_1^2 - 4(HC_1 + 1)}$ and $B = 2\sqrt{1 + H^2}$. Thus the solution of the periodic function $g(u)$ is given by

$$g(u) = \sqrt{\frac{C_1 - 2H + A\sin(2\sqrt{1 + H^2}u)}{B}}$$

with a period $T = \frac{\pi}{\sqrt{1+H^2}}$.

It is straightforward to derive that the vectors $\bar{\nabla}_{e_1}e_1$ and e_1 are the basis of the plane $\Pi^\perp$ and therefore it is constant on Σ. By differentiating the basis in the e_1 and e_2 directions and using the fact that the vectors $\frac{e_1}{\sqrt{\mu}}$ and e_2 are commute that is $[\frac{e_1}{\sqrt{\mu}}, e_2] = 0$, then we obtain that $\Pi^\perp$ is constant.

Assume that Σ is parametrized by s and u such that $x_0 = (0, 0)$. Since we have $[\frac{e_1}{\sqrt{\mu}}, e_2] = 0$, then E_1 and E_2 are also commute and hence we can consider that $\frac{\partial x}{\partial u} = e_2 = E_2$ and $\frac{\partial x}{\partial s} = \frac{e_1}{\sqrt{\mu}} = E_1$. If we suppose that

$$r(u) = \frac{g(u)}{\sqrt{C_1}} = \frac{1}{\sqrt{\mu C_1}},$$

then we obtain the following equations

$$\frac{r''}{r} + \lambda_1\lambda_2 = 0, \qquad (r')^2 + r^2(1 + \lambda_1^2) = 1, \tag{4.19}$$

where $e_1(\mu) = 0$ and $\frac{r'}{r} = -\frac{\mu'}{2\mu}$.

Since $\Pi^\perp$ that is generated by $\bar{\nabla}_{e_1}e_1$ and e_1 is constant, then the plane Π which is spanned by the basis p and q is also constant on Σ where $p = \frac{\lambda_1 x - \nu}{\sqrt{1+\lambda_1^2}}$ and $q = \frac{r(1+\lambda_1^2)e_2 - r'(x+\lambda\nu)}{\sqrt{1+\lambda_1^2}}$. Consequently, we can write the orthonormal basis for the space $\mathbb{R}^4$ as the following:

$$
\begin{aligned}
v_1 &= e_1,\\
v_2 &= \frac{\bar{\nabla}_{e_1}e_1}{|\bar{\nabla}_{e_1}e_1|} = -(r'e_2 + \lambda r\nu + rx),\\
v_3 &= p,\\
v_4 &= q,
\end{aligned}
$$

where $\{v_1, v_2\}$ and $\{v_3, v_4\}$ are the orthonormal basis for $\Pi^{\perp}$ and Π respectively.

The derivatives of these basis of $\mathbb{R}^4$ with respect to E_1 and E_2 give

$$E_1 v_1 = \frac{\bar{\nabla}_{e_1}e_1}{\sqrt{\mu}} = \frac{v_2}{r\sqrt{\mu}} = \sqrt{C_1}v_2. \tag{4.20}$$

Since v_1 and v_2 are perpendicular and $\Pi^{\perp}$ is preserved, then

$$E_1 v_2 = -\sqrt{C_1}v_2. \tag{4.21}$$

Moreover, we obtain

$$
\begin{aligned}
E_1 v_3 &= E_1 p\\
&= \frac{e_1(\lambda_1 x - \nu)}{\sqrt{\mu(1+\lambda_1^2)}}\\
&= 0,
\end{aligned}
$$

where $e_1\lambda_1 = 0$. We also have $E_1 v_4 = E_1 q = 0$ since Π is constant. The computation in the E_2 direction gives

$$E_2 v_1 = \bar{\nabla}_{e_2}e_1 = 0 \qquad E_2 v_2 = 0. \tag{4.22}$$

Furthermore, we obtain that $E_2 v_3 = \sigma v_4$ and $E_2 v_4 = -\sigma v_3$ where $\sigma = \frac{2\mu}{r(1+\lambda_1^2)}$

Lemma 4.10. *Assume that $A(u) = \int_0^u \sigma(\zeta)d\zeta$ and also $v_1 = \mathbf{a}$, $v_2 = \mathbf{b}$,$v_3 = \mathbf{c}$ and $v_4 = \mathbf{d}$ at $(0,0)$. Then,*

$$
\begin{aligned}
v_1(s,u) &= a\cos(\sqrt{C_1}S) + b\sin(\sqrt{C_1}S);\\
v_2(s,u) &= -a\sin(\sqrt{C_1}S) + b\cos(\sqrt{C_1}S);\\
v_3(s,u) &= c\cos(A(u)) + d\sin(A(u));\\
v_4(s,u) &= -c\sin(A(u)) + d\cos(A(u)).
\end{aligned}
$$

Proof. Since $\frac{\partial v_1}{\partial u} = \frac{\partial v_2}{\partial u} = 0$ and by using the identities (4.20) and (4.21), we have

$$\frac{\partial v_1}{\partial s} = \sqrt{C_1}v_2, \qquad \frac{\partial v_2}{\partial s} = -\sqrt{C_1}v_1.$$

The similar approach follows for v_3 and v_3. □

Finally, the equations of $v_1, ..., v_4$ imply to

$$x(s,u) = -rv_2 + \frac{\lambda}{\sqrt{1+\lambda_1^2}}v_3 - \frac{r'}{\sqrt{1+\lambda_1^2}}v_4.$$

Assume that $B(u)$ is given by

$$\cos B(u) = \frac{\lambda}{\sqrt{1+\lambda_1^2}\sqrt{1-r^2}}, \qquad \sin B(u) = \frac{r'}{\sqrt{1+\lambda_1^2}\sqrt{1-r^2}}$$

Then, we have from Lemma 4.10

$$\begin{aligned} x(s,u) &= r(u)\sin(\sqrt{C_1}s)\mathbf{a} - r(u)\cos(\sqrt{C_1}s)\mathbf{b} \\ &\quad + \sqrt{1-r^2(u)}\cos(A(u)-B(u))\mathbf{c} + \sqrt{1-r^2(u)}\sin(A(u)-B(u))\mathbf{d}. \end{aligned}$$

Assuming that

$$\begin{aligned} \mathbf{a} &= v_1(0,0) = (0,1,0,0) \\ \mathbf{b} &= v_2(0,0) = (-1,0,0,0) \\ \mathbf{c} &= v_3(0,0) = (0,0,\cos B(0), \sin B(0)) \\ \mathbf{d} &= v_4(0,0) = (0,0,-\sin B(0), \cos B(0)), \end{aligned}$$

then we obtain

$$x(s,u) = \Big(r(u)\cos(\sqrt{C_1}s), r(u)\sin(\sqrt{C_1}s), \sqrt{1-r^2(u)}\cos(\eta(u)), \sqrt{1-r^2(u)}\sin(\eta(u))\Big).$$

Note that $\eta(u) = A(u) - B(u) + B(0)$, $B(u) = \arctan\frac{r'}{\lambda}$ and $\eta(0) = 0$, thus

$$\begin{aligned} \eta'(u) &= A'(u) - B'(u) \\ &= \frac{2\mu}{(1+\lambda^2)r} - \frac{1}{1+(\frac{r'}{\lambda})^2}\left(\frac{r''}{\lambda} - \frac{r'\lambda'}{\lambda^2}\right) \\ &= \frac{\lambda r}{1-r^2}. \end{aligned}$$

Therefore, we have $\eta(u) = \int_0^u \frac{\lambda(\zeta) r(\zeta)}{1-r^2(\zeta)} d\zeta$. Writing $v = \sqrt{C_1} s$ for abbreviation and expressing Σ as the following:

$$F(u,v) = \left(r(u)\cos v, r(u)\sin v, \sqrt{1-r^2(u)}\cos\eta(u), \sqrt{1-r^2(u)}\sin\eta(u) \right),$$

where $v \in [0, 2\pi)$ and $0 \le u < \frac{\pi m}{\sqrt{1+H^2}}$ such that m is a positive number.

□

Combining Proposition 4.7 and Theorem 4.9, the proof of the item (a) in Theorem 4.6 is now completed. □

Before beginning the next section, we make some remarks on the work of the previous two sections 4.1 and 4.2: The result of [20] in this section is somewhat similar to the one described in section 4.1 on Brendle's proof [23] for the vanishing mean curvature case. In [23], the balls are touched from above and below at each point of the surface, implying that all derivatives of the second fundamental form are zero, so the Theorem follows from Lawson's rigidity result. However, in [20] the non-collapsing argument can only be applied for balls touching on one side of the surface, so that only some components of the derivative of second fundamental form vanish. However, these are sufficient to deduce that the surface is rotationally symmetric.

4.3 A Special class of Weingarten hypersurfaces in $\mathbb{S}^{n+1}$

The study of minimal hypersurfaces in higher dimensional space forms such as $\mathbb{S}^{n+1}$ turns to be a more interesting field of research in differential geometry. The non-collapsing technique, which is used in various results such as the proof of the Lawson conjecture [23] and also in the proof of the Pinkall and Sterling conjecture [20] (see also the above sections 4.1 and 4.2), extended to be applicable to the higher dimensional case. One of remarkable contributions in this area is the work of B. Andrews, Z. Huang and H. Li [17] on a class of embedded hypersurfaces in $\mathbb{S}^{n+1}$ satisfying a linear equation of their distinct principal curvatures. A further condition on the number of these curvatures is also imposed, implying that such hypersurfaces are congruent to the Clifford torus. The main aim of this section is to describe the key steps of the proof of this considerable result.

It was shown by T. Otsuki in [83] and [84] that an embedded minimal hypersurface with different principal curvatures λ and μ is congruent to a Clifford torus as the following:

Theorem 4.11. *(T. Otsuki [83], [84]): Let $F : \Sigma \to \mathbb{S}^{n+1}$ be a compact embedded minimal hypersurface with principal curvatures $\lambda \neq \mu$, then Σ is congruent to*

$$S^m\left(\sqrt{\frac{m}{n}}\right) \times S^{n-m}\left(\sqrt{\frac{n-m}{n}}\right), \qquad 1 \leq m \leq n-1.$$

Such result was extended by H. Li and G. Wei [74] to the case where the m-th mean curvature H_m of Σ vanishes. In particular, they proved that any compact embedded rotational hypersurfaces with $H_m = 0$ is congruent to round geodesic spheres and the product

$$S^{n-1}\left(\sqrt{\frac{n-m}{n}}\right) \times S^1\left(\sqrt{\frac{m}{n}}\right), \qquad 1 \leq m \leq n = 1.$$

In 2013, B. Andrews, Z. Huang and H. Li [17] considered a class of embedded Weingarten hypersurfaces satisfying a linear PDE equation into the $(n+1)$-dimensional sphere. Moreover, They assumed some conditions on multiplicities of their principal curvatures and deduced that such hypersurfaces is congruent to a Clifford torus. More precisely, they showed the following:

Theorem 4.12. *(B. Andrews, Z. Huang and H. Li [17]): Let $F : \Sigma \to \mathbb{S}^{n+1}$ be a compact embedded hypersurface with distinct principal curvatures λ and μ such that $\lambda = -c\mu$ for some $c > 0$. Assume also that λ and μ have multiplicities m and $n-m$ respectively where $m = 1, ..., n-1$. Therefore, λ and μ are constant and Σ is congruent to a Clifford torus*

$$S^m\left(\sqrt{\frac{1}{c+1}}\right) \times S^{n-m}\left(\sqrt{\frac{c}{c+1}}\right).$$

This result is somewhat motivated by the above work of Otsuki ([83], [84]) and H. Li and G. Wei [74]. The proof of such result [17] is based on using the maximum principle argument to a two-point function which is successfully extended to this context. As a consequence, the authors in [17] provided simple proofs of Otsuki's result for minimal hypersurfaces [83, 84] and of the work of H. Li and G. Wei [74] on hypersurfaces with vanishing m-th mean curvature.

Now we begin to describe the proof of the last theorem, more details are provided in [17].

Proof. We first consider the case that the multiplicity of λ is greater than 1 that is $m \geq 2$. From lemma 2.2 it is deduced that λ is constant and consequently μ is also constant by using the assumption $\lambda + c\mu = 0$. Since the covariant derivatives of λ and μ vanish, then Σ is an isoparametric hypersurface. Due to Cartan in his work on the classification of isoparametric hypersurfaces with two distinct principal curvatures [29], it implies that Σ is congruent to the Riemannian product as the following:

Corollary 4.13. *Let m and $n-m$ be the multiplicities of λ and μ respectively such that $m \geq 2$ and $n-m \geq 2$, then we have*

$$e_i(\lambda) = e_i(\mu) = 0, \qquad i = 1, ..., n.$$

and consequently Σ is congruent to the Clifford torus

$$S^m\left(\sqrt{\frac{1}{1+c}}\right) \times S^{n-m}\left(\sqrt{\frac{c}{1+c}}\right), \qquad 2 \leq m \leq n-2.$$

The second part of the proof concentrates on the case $m = 1$, in other words, when one of the principal curvatures is simple. In particular, we want to prove the following:

Theorem 4.14. *Let Σ be a compact embedded hypersurface into $\mathbb{S}^{n+1}$ where $n > 2$ with principal curvatures $\lambda \neq \mu$ with multiplicities 1 and $n-1$ respectively such that they are related by the equation*

$$\lambda + c\mu = 0,$$

where $c > 0$. Hence, the functions λ and μ are constant and Σ is congruent to the Clifford product

$$\mathbb{S}^1\left(\sqrt{\frac{1}{1+c}}\right) \times \mathbb{S}^{n_1}\left(\sqrt{\frac{c}{1+c}}\right).$$

Proof. We assume that Σ is a compact embedded hypersurface in $\mathbb{S}^{n+1}$ with distinct principal curvatures λ and μ with multiplicities 1 and $n-1$ respectively such that

$$\lambda_1 = \lambda, \qquad \lambda_2 = \lambda_3 = ... = \lambda_n = \mu$$

where these curvatures are related by the equation $\lambda + c\mu = 0$. Let $\{e_i : i = 1, ..., n\}$ be a local orthonormal frame of Σ where the vector e_1 is smooth and the principal direction of λ. Assume that h and ∇ are the second fundamental form and the Levi-Civita connection of Σ, respectively. The frame $\{e_i\}$ can be chosen such that h_{ij} is diagonal that is $h_{ij} = \lambda_i \delta_{ij}$. We also have that $h_{ij,k}$ are totally symmetric in i, j, k from the Codazzi equations.

In order to obtain some crucial information on the computation of the covariant derivatives of h and also of the second derivatives of one of the principal curvature, we consider the following lemmas:

Lemma 4.15. *consider the above assumptions, if i, j, k are distinct, then $h_{ij,k} = 0$ and we also obtain*

$$\begin{aligned} h_{11,1} &= e_1(\lambda), \\ h_{ii,1} &= e_1(\mu), \qquad i \geq 2, \\ h_{ii,j} &= 0 \qquad i \geq 1, j \neq 1. \end{aligned} \tag{4.23}$$

Proof. From the definition of the covariant derivatives of h_{ij}, we have

$$h_{ij,k}\theta_k = dh_{ij} + h_{ij}\theta_k + h_{ij}\theta_k = dh_{ij} + (\lambda_i - \lambda_j)\theta_{ij}.$$

If we assume that the components i, j, k are not equal, then $h_{ij,k}$ vanish. In other words, if $i, j = 2, ..., n$ and $i \neq j$, then $\lambda_i = \lambda_j = \mu$ and thus

$$h_{ij,k}\theta_k = dh_{ij} = 0.$$

where $k \geq 1$.

We need now to compute $h_{ii,j}$ and $h_{ii,i}$. In the case $i = j$, we have from the above definition

$$h_{ii,k} = e_k(\lambda_i),$$

for any k that is different from i, j. This implies to

$$\begin{aligned} h_{11,1} &= e_1(\lambda), \qquad & h_{11,1} &= e_k(\lambda) = 0 \\ h_{ii,1} &= e_1(\mu), \qquad & h_{ii,1} &= e_k(\mu) = 0. \end{aligned}$$

where i, k are greater than 1. □

We now want to look at the second derivatives of the principal curvature μ:

Lemma 4.16. *Let Σ be a complete hypersurface in $\mathbb{S}^{n+1}$ with two distinct principal curvatures λ and μ with multiplicities 1 and $n-1$ respectively, then*

$$\nabla^2\mu(e_1, e_1) = \frac{e_1\mu(2e_1\mu - e_1\lambda)}{\mu - \lambda} + (1 + \lambda\mu)(\mu - \lambda), \tag{4.24}$$

$$\nabla^2\mu(e_i, e_i) = \frac{(e_1\mu)^2}{\mu - \lambda}, \qquad i \geq 2. \tag{4.25}$$

Proof. By using the equations of Codazzi and Gauss we obtain

$$\begin{aligned}\nabla_k\nabla_l h_{ij} &= \nabla_k\nabla_i h_{lj} \\ &= \nabla_i\nabla_k h_{lj} + R^p_{kil}h_{pj} + R^p_{kij}h_{lp} \\ &= \nabla_i\nabla_j h_{lk} + h_{kl}h^2_{ij} - h_{il}h^2_{kj} + +g_{kl}h_{ij} - g_{il}h_{kj} \\ &+ h_{kj}h^2_{il} - h_{ij}h^2_{kl} + g_{kj}h_{il} + g_{ij}h_{kl}.\end{aligned} \tag{4.26}$$

Assuming $i = j = 1$ and $k = l = 2$ in (4.26) implies

$$\nabla_2\nabla_2 h_{11} = \nabla_1\nabla_1 h_{22} + (1+\lambda\mu)(\mu-\lambda). \tag{4.27}$$

We need now to compute the terms $\nabla_1\nabla_1 h_{22}$ and $\nabla_2\nabla_2 h_{11}$ in the last equation as follows:

$$\begin{aligned}\nabla_1\nabla_1 h_{22} &= e_1(\nabla_1 h_{22}) - \nabla_{\nabla_1 e_1}h_{22} - 2\nabla_1 h(e_2, \nabla_1 e_2) \\ &= e_1 e_1 \mu \\ &= \nabla^2\mu(e_1, e_1),\end{aligned} \tag{4.28}$$

and

$$\begin{aligned}\nabla_2\nabla_2 h_{11} &= e_2(\nabla_2 h_{11}) - \nabla_{\nabla_2 e_2}h_{11} - 2\nabla_2 h(e_1, \nabla_2 e_1) \\ &= -(\nabla_2 e_2 \cdot e_1)\nabla_1\lambda - 2(\nabla_2 e_1 \cdot e_2)\nabla_1\mu \\ &= \frac{e_1\lambda e_1\mu}{\lambda-\mu} - \frac{2(e_1\mu)^2}{\lambda-\mu},\end{aligned} \tag{4.29}$$

note that $\nabla_2 h_{11}$ vanishes from the previous lemma.

Substituting (4.28) and (4.29) into (4.27) gives the equality (4.24). In the case $i \geq 2$, we obtain the second equation (4.25):

$$\nabla^2\mu(e_i, e_i) = e_i(\nabla_i\mu) - \nabla_{\nabla_i e_i}\mu = \frac{(e_1\mu)^2}{\mu-\lambda}$$

where $e_i\mu = 0$. The proof is completed. $\square$

Let $\mathcal{L}$ be an elliptic differential operator of the form:

$$\mathcal{L} = \sum_{i,j=1}^{n} a_{ij}(\nabla_{e_i}\nabla_{e_j} - \nabla_{e_i}e_j),$$

where the coefficients a_{ij} are diagonal and defined as

$$a = e_1 \otimes e_1 + \frac{c}{n-1}\sum_{i=1}^{2} e_i \otimes e_i,$$

such that $a_{11} = 1$ and $a_{ii} = \frac{c}{n-1}$ for $i \geq 2$. It was shown in [17] using the last lemma that any connected complete hypersurface Σ in $\mathbb{S}^{n+1}$ with distinct principal curvatures λ and μ of multiplicity 1 and $n-1$ respectively is rotationally symmetric.

One of the main parts of this proof is to derive an analogue of Simon's identity [96] for our hyprsurface which is given by the following:

Proposition 4.17. *Let Σ be any connected complete hypersurface in $\mathbb{S}^{n+1}$ with two principal curvatures $\lambda \neq \mu$ of multiplicity 1 and $n-1$, respectively. If λ and μ satisfy $\lambda = -c\mu$ for some $c > 0$, then we have*

$$\mathcal{L}\lambda = \frac{2}{1+c}\frac{|\nabla\lambda|^2}{\lambda} + \frac{1+c}{c}\lambda(c - \lambda^2). \tag{4.30}$$

Proof. From the definition of the elliptic operator, we have

$$\mathcal{L}\mu = \nabla^2\mu(e_1, e_1) + \sum_{i=1}^{2} \frac{c}{n-1}\nabla^2\mu(e_i, e_i).$$

Substituting the equations from lemma 4.7 in the last equation and using the relation $\lambda = -c\mu$ and its derivative $e_1\lambda = -ce_1\mu$ imply to the required identity. $\square$

Another crucial part in the proof of the main theorem in this section is the application of the non-collapsing method which is recently employed in [23], [12] and [20]. In particular, the maximum principal argument is applied to a two-point function. Let x be any point in σ and consider the arguments in [20] and [13] that are explained in the previous sections, we obtain that the boundary curvature of a ball $B(C, r) \subset U$ centred at C with radius r, touching Σ at a point x is given by the following inequality:

$$k(x, y) = 2\frac{\langle x - y, \nu(x)\rangle}{|x - y|^2} \leq \frac{1}{r}, \qquad \forall y \in \Sigma. \tag{4.31}$$

Let y_0 be a point in Σ such that $x \neq y_0$ and assume that the supremum of $k(x, y)$ is attained at y_0, then we have

$$\bar{k}(x, y) = \sup\{k(x, y_0) : y_0 \neq x\}.$$

Such quantity refers to the boundary curvature of the largest ball touching Σ at points x and y_0.

Now we want to show that the boundary curvature $\bar{k}$ satisfies a differential inequality in a viscosity sense.

Proposition 4.18. *Let Σ be a compact embedded hypersurface in $\mathbb{S}^{n+1}$ with distinct principal curvatures λ and μ of multiplicity 1 and $n-1$, respectively. Assume that λ and μ are related by the equation $\lambda + c\mu = 0$ where c is any positive number. Then the function $\bar{k}$ satisfies*

$$\mathcal{L}\bar{k} \geq \frac{2(e_1\bar{k})^2}{\bar{k}-\lambda} + \frac{1+c}{c}(c-\lambda^2)\bar{k}, \tag{4.32}$$

in the viscosity sense on the set $\hat{\Sigma} = \{x : \bar{k}(x) > \lambda(x)\}$.

Proof. We begin by illustrating some useful information about the geometry of Σ. We know that Σ is invariant under the isometric action $O(\Pi^{\perp})$ and a subspace $\Pi \in \mathbb{R}^2$ is fixed. Thus, Σ can be given by the rotations $O(\Pi^{\perp})$ on a curve $\mathcal{C}$ in $\mathbb{S}^2$ that arises from the intersection of Σ and a three-dimensional subspace $\hat{\Pi}$ where $\Pi \subset \hat{\Pi}$. Note that for any point x in $\mathcal{C}$, $\hat{\Pi}$ is spanned by the tangent vector $e_1(x)$, x and $\nu(x)$.

Let x_0 be a fixed point in $\hat{\Sigma}$ and assume that $k(x,y)$ achieves its maximum at (x_0,y_0) such that $x_0 \neq y_0$. We obtain that x_0 is in the subspace $\hat{\Pi}$ since λ and k are invariant under the group of rotations $O(\Pi^{\perp})$ on Σ. As a result, $e_1(x_0)$ and $\nu(x_0)$ are also belong to $\hat{\Pi}$. With respect to the point y_0, we note from the touching ball $B(C,r)$ of Σ at both x_0 and y_0 that $C = x_0 - r(x_0)\nu(x_0) = y_0 - r(y_0)\nu(y_0)$. Also, we know that $\bar{k}(x_0) \geq \lambda(y_0) > \mu(y_0)$ and any point in Π is written as $q = \mu(y_0)y_0 - \nu(y_0)$. These imply that y_0 is in $\hat{\Pi}$ and hence in the curve $\mathcal{C}$.

Assume that $\vec{w} = \frac{y_0-x_0}{|y_0-x_0|}$ and $d = |y_0 - x_0|$, then $k = \frac{2}{d^2}\langle d\vec{w}, \nu\rangle$. Let $\mathcal{R}_{\vec{w}}$ denote the reflection map of the tangent vectors at x_0 across the hyperplane perpendicular to $\vec{w}$, so that $\mathcal{R}_{\vec{w}}(u) = u - 2\langle u, \vec{w}\rangle\vec{w}$. Since the center of the touching ball B is given by $C = x_0 - r\nu(x_0) = y_0 - r\nu(y_0) \in \Pi$, then we have

$$\nu(y_0) - \nu(x_0) = \bar{k}(x_0)(y_0 - x_0),$$

and consequently $\mathcal{R}_{\vec{w}}$ is defined as

$$\mathcal{R}_{\vec{w}}(\nu(x_0)) = \nu(x_0) - 2\langle\nu(x_0), \vec{w}\rangle\vec{w} = \nu(x_0) + \bar{k}(x_0)d\vec{w} = \nu(y_0).$$

In other words, $\mathcal{R}_{\vec{w}}$ maps the tangent vectors at x_0 isometrically to those at y_0.

The main aim of the proof of such proposition is to show that for any smooth function ϕ lies above $\bar{k}$ and is defined on an open region Ω around x_0 in $\hat{\Sigma}$ satisfies

$$\mathcal{L}\phi|_{x_0} \geq \left(\frac{2(e_1\phi)^2}{\phi-\lambda} + \frac{1+c}{c}(c-\lambda^2)\phi\right)\Big|_{x_0}.$$

Choose $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ to be geodesic normal coordinates in the neighbourhood of x_0 and y_0 respectively such that $\partial_i^x = \frac{\partial}{\partial x_i}$ and $\partial_i^y = \frac{\partial}{\partial y_i}$ refer to the coordinates of tangent vectors at x_0 and y_0, respectively. From the assumption $\phi(x) \geq \bar{k}(x)$ for all $x \in \Omega$, we obtain $\phi(x) \geq k(x,y)$ for all y. Note that the equality holds at (x_0, y_0). The derivatives of the smooth functions ϕ and k in the x- direction are the same, that is $\frac{\partial \phi}{\partial x^i} = \frac{\partial k}{\partial x^i}$, and k in the y- direction vanishes $\frac{\partial k}{\partial y^i} = 0$ at (x_0, y_0). We also have for any $v \in T_{x_0}\Sigma$ and $w \in T_{y_0}\Sigma$

$$D^2\phi|_{x_0}(v,v) \geq D^2 k|_{(x_0,y_0)}((v,w),(v,w)).$$

This implies that

$$\mathcal{L}\phi|_{x_0} \geq \sum_{i,j=1}^{n} a^{ij}(\partial_i^x + \partial_i^y)(\partial_j^x + \partial_j^y)|_{(x_0,y_0)}. \tag{4.33}$$

Now we compute the following at (x_0, y_0):

$$\begin{aligned} a^{ij}(\partial_j^x + \partial_j^y)k &= \frac{2}{d^2} a^{ij}\big((\partial_j^x - \partial_j^y)\cdot \nu_x - d\vec{w}\cdot h_j^p \partial_p^x + kd\vec{w}\cdot(\partial_j^x - \partial_j^y)\big) \\ &= \frac{2}{d^2}\bigg((\partial_j^x - \partial_j^y)\cdot\big(\nu_x + kd\vec{w}\big) - dh_j^p\partial_p^x\cdot\vec{w}\bigg). \end{aligned}$$

The second derivatives are given by

$$\begin{aligned} \sum_{i,j=1}^n a^{ij}(\partial_i^x + \partial_i^y)(\partial_j^x + \partial_j^y)k = \frac{2}{d^2}\sum_{i,j=1}^n a^{ij}\bigg(&-\nabla^p h_{ij}^x d\vec{w}\cdot\partial_p^x - 2(h^x)_j^p\partial_p^x\cdot(\partial_i^y - \partial_i^x) \\ &+ (h^x)_j^p d\vec{w}\cdot(h_{pi}^x\nu_x + g_{pi}x) \\ &+ \big((h^y)_{pj}\nu_y + g_{ij}y - (h^x)_{ij}\nu_x - g_{ij}x\big)\cdot(\nu_x + kd\vec{w}) \\ &- k(\partial_i^x - \partial_i^y)\cdot(\partial_j^x - \partial_j^y) + 2\partial_j k(\partial_j^x - \partial_j^y)\cdot d\vec{w}\bigg). \end{aligned}$$

Since the first derivatives of ϕ and k with respect to x are equal, then we have $e_1\phi = e_1 k = \frac{2(k-\lambda)\partial_1^x\cdot\vec{w}}{d}$ and $e_i k = 0$ for $i \geq 2$. We note that the first term of the last equation disappears as $a^{ij}h_{ij}^x = a^{ij}h_{ij}^y = 0$, ∇a^{ij} is off-diagonal and h_{ij} is diagonal in i and j. Recall that $h_{11}^x = \lambda(x)$, $h_{11}^y = \lambda(y)$, $h_{ii}^x = \mu(x)$ and $h_{ii}^y = \mu(y)$ for $i \geq 2$. Some identities are observed like $-\nu_x\cdot\vec{w} = \frac{kd}{2}$, $\vec{w} = \frac{-d}{2}$ and $\nu_x = \nu_x + kd\vec{w}$. We also have $a^{ij}g_{ij} = 1 + c$, $a^{ij}h_{jk}^x g^{kl}h_{li}^x = \lambda^2 + c\mu^2 = \frac{1+c}{c}\lambda^2$. Moreover, from the reflection map that defined above we obtain $\partial_j^y = \partial_j^x - 2\vec{w}\cdot\partial_j^x\vec{w}$ and consequently $\partial_1^y = \partial_1^x - \frac{e_1 k}{k-\lambda}d\vec{w}$ and $\partial_j^y = \partial_j^x$ for $i \geq 2$.

By substituting these identities into the last equation, we arrives to the following required inequality

$$\sum_{i,j=1}^{n} a^{ij}(\partial_i^x + \partial_i^y)(\partial_j^x + \partial_j^y)k = -\frac{1+c}{c}k\lambda^2 + (1+c)k + 2\frac{(e_1 k)^2}{k-\lambda}$$
$$= \frac{1+c}{c}(c-\lambda^2)\phi + +2\frac{(e_1\phi)^2}{\phi-\lambda}.$$

□

Assuming that $\psi = \frac{\bar{k}}{\lambda}$, it follows from (4.30) and (4.32) that ψ satisfies the following natural differential inequality in the viscosity sense:

Lemma 4.19. *If $\psi = \frac{\bar{k}}{\lambda}$, then ψ satisfies*

$$\mathcal{L}\psi \geq \frac{2(e_1\psi)^2}{\psi-1} + \frac{2(\psi+1)}{\lambda(\psi-1)}e_1\lambda e_1\psi + \frac{2\psi(c\psi+1)}{\lambda^2(1+c)(\psi-1)}(e_1\lambda)^2$$

in the viscosity sense on $\Omega = \{x : \bar{k} > \lambda\}$.

Proof. Assume that ξ is a smooth function lies above ψ with equality at x_0, thus $\phi \geq \bar{k}$ and the equality holds at x_0 where $\phi = \lambda\xi$. Hence, we have from the inequality (4.32)

$$\mathcal{L}\phi|_{x_0} \geq \frac{2(e_1\phi)^2}{\phi-\lambda} - \frac{c+1}{c}(\lambda^2 - c)\phi.$$

Substituting $\lambda\xi$ instead of ϕ in the last inequality and then differentiating the function ξ and using the identity (4.30) we obtain

$$\mathcal{L}\xi|_{x_0} \geq \frac{2(e_1\xi)^2}{\xi-1} + \frac{2(\xi+1)}{\lambda(\xi-1)}e_1\lambda e_1\xi + \frac{2\xi(c\xi+1)}{\lambda^2(1+c)(\xi-1)}(e_1\lambda)^2.$$

This implies that $\psi = \frac{\bar{k}}{\lambda}$ is a viscosity solution of the last equation. □

It is shown next that $\psi = 1$ and thus $\bar{k} = \lambda$ everywhere on Σ by using the contradiction and the maximum principle arguments.

Corollary 4.20. *$\bar{k}$ and λ are equal at every point of Σ.*

Proof. We have $\bar{k} \geq \lambda$ everywhere on Σ. Assume the case $\bar{k} \geq \lambda$ holds at some point in Ω, then $\psi > 1$ and equal 1 on the non-empty boundary of Ω. This implies that ψ has a maximum point on such region. By using the last lemma and applying the maximum principle argument we obtain $\psi > 1$ and constant, on the other hand $\psi = 1$ at the point of the largest maximum principle curvature on Σ. Therefore, it is a contradiction and as a result $\bar{k} = \lambda$ at every point of Σ. □

Finally we complete the proof by showing that $e_1\lambda = 0$ everywhere on Σ by using Similar arguments as in [23]. Let η be a curve in Σ such that $\eta(0) = x_0$. Since $\bar{k} = \lambda$ for any $x \in \eta$, then $k(x, \eta(t)) \leq \lambda(x)$. Therefore,

$$f(t) = \lambda(x)|x - \eta(t)|^2 - 2\nu(x) \cdot (x - \eta(t)),$$

where $f(t) \geq 0$. The first derivative of $f(t)$ gives

$$f'(t) = 2(\nu(x) - \lambda(x)(x - \eta(t)) \cdot \eta'(t).$$

The second derivative implies

$$f''(t) = -2(\nu(x) - \lambda(x)(x - \eta(t)) \cdot (\lambda_{\eta(t)}\nu_{\eta(t)} + \eta(t)) + 2\lambda(x)|\eta'(t)|^2.$$

Note that $f(t) = f'(t) = f''(t) = 0$ at $t = 0$, thus $f'''(t)$ also vanishes at $t = 0$ and consequently we have

$$f'''(0) = -2e_1\lambda(x) = 0.$$

Since Σ is a rotational hypersurface, then $\lambda(x)$ is constant everywhere on Σ. By the relation $\lambda + c\mu = 0$, we obtain that μ is also constant and hence Σ is congruent to the Clifford torus. This completes the proof. □

We now arrive at the proof of Theorem 4.12 by combining the results of Theorem 4.14 and Corollary 4.13. □

The main tool in the previous results of this chapter is mainly based on utilizing the non-collapsing technique. Such technique can be extended and applied in various contexts. One of the interesting applications of the non-collapsing method is a more recent work of S. Brendle [24] on embedded Weingarten tori in $\mathbb{S}^3$. By modifying such approach to a specific class of Weingarten surfaces in $\mathbb{S}^3$ satisfying a linear equation of its principal curvatures under certain structure conditions, Brendle showed that such surface is a surface of rotation. In particular, he proved the following:

Theorem 4.21. *S. Brendle [24]] Let $F\colon \Sigma \to \mathbb{S}^3$ be an embedded surface of genus 1 and has two principal curvatures λ and μ such that $\lambda + \mu = \Phi(\lambda - \mu)$ where $\lambda \geq \mu$. Assume that $\Phi(t)$ is an even function satisfying $0 \leq t\Phi'(t) < min\{\Phi(t), t\}$ and $0 \leq t\Phi''(t) < 1 - \Phi'(t)^2$, where $t = \lambda - \mu$. Then $F(\Sigma)$ must be a surface of rotation.*

Moreover, the proof of the Lawson's conjecture can be modified and applied in various contexts. In the next chapter, this method will be extended to a more general class of Weingarten hypersurfaces in higher dimensions.

Chapter 5

Uniqueness of a Class of Weingarten Hypersurfaces in Spheres

In Section 4.3, we described the work of B. Andrews, Z. Huang and H. Li [17] on the uniqueness of solutions of a particular family of Weingarten equations for hypersurfaces of S^{n+1}. In this chapter, we present the first main result of the thesis in which uniqueness results are proved for a much larger class of embedded Weingarten hypersurfaces. We assume that this hypersurface satisfies a partial differential equation (PDE) of distinct principal curvatures λ and μ with multiplicities m and $n-m$, respectively. Moreover, we impose some additional structure conditions on this PDE. As a result, we deduce that λ and μ are both constant and consequently our hypersurface is congruent to a Clifford torus.

The proof of this result involves the application of the maximum principle to a two-point function that is defined on our class of embedded Weingarten hypersurfaces of $\mathbb{S}^{n+1}$. More precisely, we define a quantity that characterises the curvature of the largest ball in the enclosed region which touches the hypersurface at a given point. Then, we show that this inscribed ball curvature of these hypersurfaces satisfies a natural differential inequality in a viscosity sense, allowing the application of maximum principles.

The method of non-collapsing is a powerful tool that is applied in various contexts such as in [23], [20], [12] and [17], see also the previous chapters. In particular, such approach is the key ingredient of both Brendle's argument [23] proving the Lawson conjecture on embedded minimal tori in $\mathbb{S}^3$, and of the work of B. Andrews and H. Li [20] confirming the conjecture of U. Pinkall and I. Sterling

[88] on embedded constant mean curvature (CMC) tori in $\mathbb{S}^3$. Another interesting application of the non-collapsing method is the more recent work of S. Brendle [24] on a specific class of embedded Weingarten tori in $\mathbb{S}^3$ that satisfies a linear equation of its principal curvatures under certain structure conditions, see Theorem 4.21.

Our purpose in this chapter is to apply the non-collapsing technique to a larger class of embedded Weingarten hypersurfaces Σ^n with two distinct principal curvatures λ and μ at each point of Σ^n in S^{n+1}. In particular, we assume that this hypersurface satisfying the following form of PDE:

$$G(\lambda, \mu) = \lambda + \mu - \Phi(\lambda - \mu), \qquad \lambda > \mu \tag{5.1}$$

where $G(\lambda, \mu)$ is a symmetric function of the principal curvatures. Also, further assumptions on $G(\lambda, \mu)$ are imposed as follows:

- $\frac{\partial G}{\partial \lambda}$, $\frac{\partial G}{\partial \mu}$, $\frac{\partial G}{\partial \lambda}\lambda$ and $\frac{\partial G}{\partial \mu}\mu$ are positive when $G(\lambda, \mu) = 0$.
- The set $\{(\lambda, \mu) : G(\lambda, \mu) \geq 0\}$ is convex.
- The following inequality
$$-\frac{\partial^2 G}{\partial \lambda^2}\left(\frac{\partial G}{\partial \mu}\right)^2 + 2\frac{\partial^2 G}{\partial \lambda \partial \mu}\frac{\partial G}{\partial \lambda}\frac{\partial G}{\partial \mu} - \frac{\partial^2 G}{\partial \mu^2}\left(\frac{\partial G}{\partial \lambda}\right)^2 \leq \frac{2}{\lambda - \mu}\frac{\partial G}{\partial \lambda}\frac{\partial G}{\partial \mu}\left(\frac{\partial G}{\partial \lambda} + \frac{\partial G}{\partial \mu}\right)$$
holds, when $G(\lambda, \mu) = 0$.

Precisely, our aim is to prove the below theorem:

Theorem 5.1. *Let $F : \Sigma \to \mathbb{S}^{n+1}$ be a compact embedded hypersurface with principal curvatures λ and μ, where $\lambda > \mu$. Assume that these principal curvatures λ and μ have multiplicities m and $n - m$ respectively and satisfy the relation* (5.1)*. Also, suppose that $\Phi(t)$ is a function satisfying the conditions: $0 \leq t\Phi'(t) < min\{\Phi(t), t\}$ and $0 \leq t\Phi''(t) < 1 - \Phi'(t)^2$, where $t = \lambda - \mu$. Then λ and μ are constant and Σ is congruent to the Clifford torus.*

In order to clarify the proof of Theorem 5.1, we begin in Section 5.1 by illustrating some geometric concepts of embedded Weingarten hypersurfaces of $\mathbb{S}^{n+1}$ assuming a condition on the multiplicity of the principal curvatures. Also, under these assumptions we show that this class of hypersurfaces is rotationally symmetric. In Section 5.2, we derive an identity analogous to Simons' identity for the maximum principal curvature for our class of hypersurfaces in the proposed setting. In Section 5.3, we deduce that the boundary curvature of the largest touching ball inscribed in the hypersurface satisfies a differential inequality in the viscosity sense using the maximum principle argument. In the final section, we deduce the equality of the maximum principal curvature and the inscribed ball curvature everywhere on Σ by combining the previous two results. This equality enables us to deduce that the principal curvatures are constant and then yields the result of our theorem.

5.1 Weingarten hypersurfaces in $\mathbb{S}^{n+1}$

In this section, we describe some basic concepts of the geometry of embedded hypersurfaces satisfying an equation that relates their two distinct principal curvatures in $\mathbb{S}^{n+1}$. Let $F : \Sigma^n \to \mathbb{S}^{n+1}$ be an isometric immersion of n-dimensional hypersurfaces Σ into $\mathbb{S}^{n+1}$. We assume that Σ is a compact hypersurface and h is its second fundamental form. Also, we suppose that Σ has two distinct principal curvatures λ and μ with multiplicities m and $n-m$ respectively such that

$$\lambda_1 = ... = \lambda_m = \lambda, \qquad \lambda_{m+1} = ... = \lambda_n = \mu.$$

Moreover, we consider an arbitrary point x_0 of Σ such that $x_0 \in U \subset \Sigma$ where U is a neighbourhood of x_0. Let the tangent space of the hypersurface at x_0 be denoted by $T_{x_0}(\Sigma)$ which is a subspace of the tangent space $T_{x_0}\mathbb{S}^{n+1}$ of the ambient space $\mathbb{S}^{n+1}$ at the same point. Note that the vectors in $T_{x_0}\mathbb{S}^{n+1}$ that are orthogonal to $T_{x_0}\Sigma$ with respect to the Riemannian metric g are defined as the normal space $N_{x_0}\Sigma$.

The following convention on the range of indices is used throughout this chapter

$$1 \leq a, b, c, ... \leq n+1, \qquad 1 \leq i, j, k, ... \leq n.$$

Let $\{e_a\}$ be a local orthonormal frame field around any point x_0 in U such that $\{e_i\} \in T_{x_0}(U)$ and $e_{n+1} = \nu \in N_{x_0}(\Sigma)$, where ν is a positively oriented unit normal vector. Assume that $\{\omega_a\}$ is the corresponding dual coframe of $\{e_a\}$ on $\mathbb{S}^{n+1}$. Associated with such frame, there are 1-forms $\{\omega_{ab}\}$ representing the connection on $\mathbb{S}^{n+1}$.

The structure equations of $\mathbb{S}^{n+1}$ are determined as follows:

$$d\omega_a = \omega_{ab} \wedge \omega_b, \qquad \omega_{ab} = -\omega_{ba},$$

and

$$d\omega_{ab} = \omega_{ac} \wedge \omega_{cb} + \frac{1}{2} Q_{abcd}\omega_c \wedge \omega_d,$$

where Q_{abcd} refers to the components of the curvature tensor of $\mathbb{S}^{n+1}$. If $\{\theta_a\}$ and $\{\theta_{ab}\}$ are the restriction of $\{\omega_a\}$ and $\{\omega_{ab}\}$ to Σ respectively, then we have $\theta_{n+1} = 0$ and its exterior derivative is given by

$$d\theta_{n+1} = -\theta_{n+1i} \wedge \theta_i = 0.$$

From Cartan's lemma, we obtain the following:

$$\theta_{n+1i} = h_{ij}\theta_j,$$

where the second fundamental form h_{ij} is a symmetric bilinear form with components i and j. Since it is symmetric, then we can write $h_{ij} = h_{ji}$. We also assume that h_{ij} is diagonal by choosing the orthonormal frame $\{e_i\}$ such that $h_{ij} = \lambda_i\delta_{ij}$. The covariant derivative of h_{ij} in the direction of e_k is defined as

$$h_{ij,k}\theta_k = dh_{ij} + h_{kj}\theta_{ki} + h_{ik}\theta_{kj}. \tag{5.2}$$

Also, the Codazzi equation is given by

$$h_{ij,k} = h_{ik,j}.$$

Therefore, $h_{ij,k}$ is totally symmetric in i, j and k.

We now deduce the structure equations of Σ from the previous structure equations of $\mathbb{S}^{n+1}$ as follows:

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0,$$

and

$$d\omega_{ij} = \sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj} + \frac{1}{2}\sum_{k,l=1}^{n} R_{ijkl}\omega_k \wedge \omega_l,$$

where R_{ijkl} refers to the components of curvature tensor of Σ. Thus, the Gauss equation is defined by

$$R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{5.3}$$

that is also given by

$$R_{iqjq} = (\lambda_i\lambda_q + 1)(\delta_{ij} - \delta_{iq}\delta_{jq}).$$

In this chapter, we define an elliptic operator $\mathcal{L}$ by

$$\mathcal{L} = a^{ij}(\partial_i^x + \partial_i^y)(\partial_j^x + \partial_j^y),$$

where

$$a^{ij} = \begin{cases} \beta_1 = \frac{\partial G}{\partial \lambda}, & i = j = 1 \\ 0, & i \neq j \\ \beta_2 = \frac{\partial G}{\partial \mu}, & i = j > 1. \end{cases}$$

Note that ∂_i^x and ∂_j^x are the partial derivatives in the direction of e_i and e_j at the point x respectively. Similarly, ∂_i^y and ∂_j^y are the partial derivatives in the direction of e_i and e_j respectively but at the point y.

In order to prove Theorem 5.1 we first consider the case $m \geq 2$. In this case we show that the derivative of λ in the direction of e_i vanishes for all $i = 1, ..., m$ as follows:

Lemma 5.2. *Let $\{e_i\}$ be a local orthonormal frame and assume that the multiplicity of λ is greater than one, then we have*

$$e_i(\lambda) = 0, \qquad \forall i = 1, ..., m.$$

Proof. From the definition (5.2) of the covariant derivative of h_{ij}, and the Codazzi equation, we get

$$h_{ij,k}\theta_k = dh_{ij} - (\lambda_j - \lambda_i)\theta_{ij}.$$

If we consider that $i \neq j$ and $\lambda_i = \lambda_j$, then $h_{ij} = 0$ (since h_{ij} is diagonal) and hence $h_{ij,k} = 0$. However, if $i = j$, we get $h_{ii,k}\theta_k = dh_{ii}$ which implies that $h_{ii,k} = e_k(\lambda_i)$. In the last case, we take $m = 2, ..., n$ and $i, j = 1, ..., m$ such that $i \neq j$, then we have

$$\lambda_i = \lambda_j = \lambda, \quad e_i(\lambda_j) = h_{jj,i} = h_{ij,j} = 0.$$

This implies $e_i(\lambda) = 0$ for $i = 1, ..., m$ as required. □

It follows from the previous lemma and the relation (5.1) that $e_i\mu = 0$ for all $i = 1, ..., n$ where the multiplicity of μ is greater than 1. Note that $\lambda = -c\mu$ is a special class of the relation (5.1). Hence, since Σ is an isoparametric hypersurface with λ and μ are constant, then Σ is congruent to the Clifford torus as follows:

Corollary 5.3. *Let Σ be a compact embedded hypersurface into $\mathbb{S}^{n+1}$ with two distinct principal curvatures λ and μ of multiplicities m and $n - m$ greater than two respectively. Consider that λ and μ satisfy the equation* (5.1)*. Therefore*

$$e_i\lambda = e_i\mu = 0, \quad i = 1, ..., m,$$

and Σ is congruent to the Clifford torus.

From now we will focus on the case where the multiplicity of one of principal curvatures is one. We assume that Σ is an n-dimensional hypersurface that is embedded in $\mathbb{S}^{n+1}$ with $\lambda \neq \mu$ of multiplicity 1 and $n-1$ respectively satisfying the relation (5.1). In order to find the derivatives of λ and μ in the e_1-direction, it is useful to consider the below lemma about the covariant derivatives of the second fundamental form h.

Lemma 5.4. *Let $\{e_i\}$ be a local orthonormal frame where h_{ij} are diagonal, then h_{ijk} are symmetric in i, j, k and*

$$h_{11,1} = \lambda_{,1}, \quad h_{ii,1} = \mu_{,1}, i > 1.$$

Also $h_{ij,k} = 0$, for i, j, k are distinct and $h_{ii,j} = 0$, for $i \geq 1, j \neq 1$.

Proof. The proof of this lemma relies on the covariant derivative of the second fundamental form (5.2) and the Codazzi equation. In particular, we know from the Codazzi equation that $h_{ij,k}$ is totally symmetric, that is $h_{ij,k} = h_{ik,j}$. Using the identity (5.2), we obtain

$$h_{ij,k}\theta_k = dh_{ij} + (\lambda_i - \lambda_j)\theta_{ij}.$$

Assume that i, j, k are distinct, then $\lambda_i = \lambda_j = \mu$ and the last equality becomes $h_{ij,k} = dh_{ij}$. Since h_{ij} is diagonal, then $h_{ij} = 0$ and therefore $h_{ij,k} = 0$ for any k. In order to find $h_{ii,k}$, take $i = j$, then we have

$$h_{ii,k} = e_k(\lambda_i).$$

Hence, we get

$$h_{11,j} = e_j(\lambda) = 0 \qquad h_{ii,j} = e_j(\mu) = 0$$
$$h_{11,1} = e_1(\lambda) \qquad h_{ii,1} = e_1(\mu),$$

where i, j are greater than 1. □

We want now to show that the hypersurface Σ we are considering in this chapter is rotationally symmetric.

Proposition 5.5 (B. Andrews, Z. Huang and H. Li [17]). *Let $F : \Sigma^n \to \mathbb{S}^{n+1}$ be a connected complete hypersurface with two distinct principal curvatures λ and μ of multiplicity 1 and $n-1$ respectively. Hence, Σ is a rotation hypersurface. In other words, Σ is invariant under the group of rotations fixing a two-dimensional subspace of $\mathbb{R}^{n+2}$.*

Proof. We first need to obtain the first and second derivatives of μ with respect to e_i. We start by using the covariant derivative of the second fundamental form (5.2):

$$e_k(h(e_i, e_j)) = \nabla_k h_{ij} + h(\nabla_k e_i, e_j) + h(e_i, \nabla_k e_j).$$

As we assume that $h_{ij} = \lambda_i \delta_{ij}$ and use the identity $g(\nabla_k e_i, e_j) = -g(e_i, \nabla_k e_j)$, then we have for $i \neq j$

$$0 = \nabla_k h_{ij} + \lambda_j \nabla_k e_i \cdot e_j + \lambda_i e_i \cdot \nabla_k e_j,$$

which is equivalent to

$$\nabla_k h_{ij} = \nabla_k e_i \cdot e_j (\lambda_i - \lambda_j).$$

Taking $i = 1$ and $j, k \geq 2$ implies

$$\frac{\nabla_1 h_{jk}}{(\lambda_i - \lambda_j)} = \nabla_k e_1 \cdot e_j.$$

Such identity becomes

$$\frac{\nabla_1 \mu}{(\lambda - \mu)} = \nabla_k e_1 \cdot e_j,$$

where $j = k$. However, if $j \neq k$, then $\nabla_1 h_{jk} = 0$.

We want now to deduce an equation for the second derivative of μ, analogous to Simons' identity for minimal hypersurfaces [96]. It follows from the Gauss and Codazzi equations that

$$\begin{aligned}
\nabla_i \nabla_j h_{kl} &= \nabla_i \nabla_k h_{jl} \\
&= \nabla_k \nabla_i h_{jl} + R^q_{ikj} h_{ql} + R^q_{ikl} h_{jq} \\
&= \nabla_k \nabla_l h_{ij} + h_{ij} h^2_{kl} - h_{jk} h^2_{il} + h_{il} h^2_{jk} - h_{kl} h^2_{ij} \\
&\quad + g_{ij} h_{kl} - g_{jk} h_{il} + g_{il} h_{jk} - g_{kl} h_{ij}.
\end{aligned}$$

Assuming that $i = j = 2$ and $k = l = 1$, then we obtain the following commutator identity

$$\nabla_2 \nabla_2 h_{11} = \nabla_1 \nabla_1 h_{22} + (\lambda\mu + 1)(\lambda - \mu).$$

By computing $\nabla_1 \nabla_1 h_{22}$, we have

$$\begin{aligned}
\nabla_1 \nabla_1 h_{22} &= \nabla_1(\nabla_1 h_{22}) - \nabla_{\nabla_1 e_1 h_{22}} - 2\nabla_1 h(e_2, \nabla_1 e_2) \\
&= \nabla^2 \mu(e_1, e_1),
\end{aligned}$$

and $\nabla_2\nabla_2 h_{11}$, we get

$$\begin{aligned}
\nabla_2\nabla_2 h_{11} &= \nabla_2(\nabla_2 h_{11}) - \nabla_{\nabla_2 e_2 h_{11}} - 2\nabla_2 h(e_1, \nabla_2 e_1)\\
&= -e_1\lambda(\nabla_2 e_2.e_1) - 2e_1\mu(\nabla_2 e_1.e_2)\\
&= \frac{e_1\lambda e_1\mu}{\lambda-\mu} - 2\frac{(e_1\mu)^2}{\lambda-\mu}.
\end{aligned}$$

Substituting the last two equations into the above commutator identity gives the second derivative of μ with respect to e_1

$$\nabla^2\mu(e_1, e_1) = \frac{e_1\mu(e_1\lambda - 2e_1\mu)}{\lambda-\mu} - (\lambda-\mu)(1+\lambda\mu). \tag{5.4}$$

Moreover, the second derivative of μ with respect to e_i is given by

$$\nabla^2\mu(e_i, e_i) = e_i(\nabla_i\mu) - \nabla_{\nabla_i e_i}\mu = -\frac{(e_1\mu)^2}{\lambda-\mu}, \tag{5.5}$$

where $e_i\mu = 0$ $(i \geq 2)$ from lemma (5.2).

We next consider a plane Π at each point of Σ which is defined as the span of vectors p and q where

$$p = \mu x - \nu \qquad q = (e_1\mu)x - (\lambda-\mu)e_1.$$

Computing the first derivatives of both p and q in the direction of e_1 implying to

$$e_1 p = q,$$

and

$$\begin{aligned}
e_1 q &= \lambda(\mu-\lambda)p + \frac{e_1\lambda - 2e_1\mu}{\lambda-\mu}q\\
&\quad + \Big(\nabla^2\mu(e_1, e_1) - \frac{e_1\mu(e_1\lambda - 2e_1\mu)}{\lambda-\mu} + (\lambda-\mu)(1+\lambda\mu)\Big)x.
\end{aligned}$$

While, in the e_i-direction $(i = 2, ..., n)$ we get

$$e_i p = e_i q = 0.$$

Note that the last term of the above derivative $e_1 q$ vanishes from the deduced identity (5.4), this implies that the derivatives of p and q belong to Π. We also have that Π is constant since Σ is connected.

Finally, let the tangent space to $\mathbb{S}^{n-1}$ is expressed by the group of rotations $O(\Pi^{\perp})$ at each point x of Σ. Since $O(\Pi^{\perp})$ is orthogonal to x, p and q, then this group of action is also orthogonal to x, ν and e_1 and implying to the coincidence of the tangent space with the span of $e_1, ..., e_{n-1}$. Hence, the orbit of action $O(\Pi^{\perp})$ is tangent to Σ at each point x and Σ is invariant under such group. Therefore, Σ is a rotation hypersurface. □

5.2 Simons' identity for the maximum principal curvature

In this section, a Simons-type identity [96] is deduced for our defined class of hypersurfaces. More precisely, our object here is to show the following result:

Proposition 5.6. *Let Σ be a compact hypersurface that is embedded in $\mathbb{S}^{n+1}$ with two distinct principal curvatures λ and μ of multiplicities 1 and $n-1$ respectively and satisfies the relation (5.1). Then we have*

$$\mathcal{L}\lambda = -\left(2\frac{(\beta_1-\beta_2)\beta_1^2}{(2\lambda-\Phi)\beta_2^2} - 4\frac{\Phi''}{\beta_2^2}\right)(e_1\lambda)^2 - \beta_2\Big(\big(\Phi-\lambda\big)\big(\lambda(\Phi-2\lambda)+1\big)-\lambda\Big),$$

where $\beta_1 = 1-\Phi'$, $\beta_2 = 1+\Phi'$ and the function Φ is restricted by the above assumptions.

Proof. Let Σ be a compact embedded hypersurface in $\mathbb{S}^{n+1}$ with second fundamental form h and two distinct principal curvatures λ and μ. Choose the set $\{e_i : i = 1, ..., n\}$ to be an orthonormal frame for TU, where U is a neighbourhood of any point x on Σ.

Differentiating the equation (5.1) gives

$$\sum_{i=1}^{n} \nabla_i\mu = -\frac{\beta_1}{\beta_2}\sum_{i=1}^{n}\nabla_i\lambda, \tag{5.6}$$

where $\beta_1 = 1-\Phi'$ and $\beta_2 = 1+\Phi'$. The second derivative of the identity (5.1) is given by

$$\beta_2\nabla_i\nabla_i\mu + \beta_1\nabla_i\nabla_i\lambda - \Phi''|_{\lambda-\mu}(\nabla_i\lambda - \nabla_i\mu)^2 = 0. \tag{5.7}$$

Now, we use the following equation that relates the second covariant derivatives of h to the Hessian of the function λ_k which is

$$\sum_{i,k=1}^{n} \nabla_i\nabla_i\lambda_k = \nabla_i\nabla_i h_{kk} - (-1)^k\frac{2}{\lambda_1-\lambda_2}(\nabla_i h_{12})^2. \tag{5.8}$$

In the case $k = 1$ where $\lambda_1 = \lambda$, we obtain

$$\nabla_i \nabla_i \lambda = \nabla_i \nabla_i h_{11} + \frac{2}{\lambda_1 - \lambda_2} (\nabla_i h_{12})^2,$$

while if $k > 1$ such that $\lambda_k = \mu$, we have

$$\nabla_i \nabla_i \mu = \nabla_i \nabla_i h_{kk} - (-1)^k \frac{2}{\lambda_1 - \lambda_2} (\nabla_i h_{12})^2.$$

Substituting the last two equations into (5.7) yields

$$\beta_2 \nabla_i \nabla_i h_{kk} + \beta_1 \nabla_i \nabla_i h_{11} + 2\frac{\beta_1 - \beta_2}{(\lambda - \mu)} (\nabla_i h_{12})^2 - \Phi''|_{\lambda-\mu} \sum_{i=1}^{n} (\nabla_i \lambda - \nabla_i \mu)^2 = 0. \quad (5.9)$$

The next step is to use the following commutator identity

$$\sum_{i,k=1}^{n} (\nabla_i \nabla_i h_{kk} - \nabla_k \nabla_k h_{ii}) = (\lambda_k^2 - 1)\lambda_i - \lambda_k(\lambda_i^2 - 1),$$

Taking $k = 1$, we have

$$\nabla_i \nabla_i h_{11} - \nabla_1 \nabla_1 h_{ii} = (\lambda_1^2 - 1)\lambda_i - \lambda_1(\lambda_i^2 - 1),$$

and $k \geq 2$ implies

$$\nabla_i \nabla_i h_{kk} - \nabla_k \nabla_k h_{ii} = (\lambda_k^2 - 1)\lambda_i - \lambda_k(\lambda_i^2 - 1).$$

Also, substituting the last two equations into (5.9) gives

$$\begin{aligned}
&\beta_2 \nabla_k \nabla_k h_{ii} + \beta_1 \nabla_1 \nabla_1 h_{ii} \\
&+ \beta_2(\mu^2 - 1)\lambda_i - \beta_2 \mu(\lambda_i^2 - 1) + \beta_1(\lambda^2 - 1)\lambda_i - \beta_1 \lambda(\lambda_i^2 - 1) \\
&+ 2\frac{\beta_1 - \beta_2}{\lambda - \mu} (\nabla_i h_{12})^2 - \Phi''|_{\lambda-\mu} (\nabla_i \lambda - \nabla_i \mu)^2 = 0.
\end{aligned}$$

Now we use again the relation (5.8) and obtain the following

$$\begin{aligned}
&\beta_2 \nabla_k \nabla_k \lambda_i + \beta_1 \nabla_1 \nabla_1 \lambda_i \\
&+ \beta_2(\mu^2 - 1)\lambda_i - \beta_2 \mu(\lambda_i^2 - 1) + \beta_1(\lambda^2 - 1)\lambda_i - \beta_1 \lambda(\lambda_i^2 - 1) \\
&+ 2\frac{(-1)^i}{\lambda - \mu} \big(\beta_1 (\nabla_1 h_{12})^2 + \beta_2 (\nabla_k h_{12})^2\big) \\
&+ 2\frac{\beta_1 - \beta_2}{\lambda - \mu} (\nabla_i h_{12})^2 - \Phi''|_{\lambda-\mu} (\nabla_i \lambda - \nabla_i \mu)^2 = 0.
\end{aligned}$$

We know from Lemma 5.4 that $\nabla_1 h_{12} = 0$ and $\nabla_k h_{12} = \nabla_1 \mu$ where $k \geq 2$. Moreover, we replace the terms $\beta_2 \sum_{k=2}^{n} \nabla^2_{k,k}\lambda + \beta_1 \nabla^2_{1,1}\lambda$ by $\mathcal{L}\lambda$ and choose $i = 1$. Hence, we arrive to

$$\mathcal{L}\lambda - 2\frac{\beta_2}{\lambda - \mu}(e_1\mu)^2 - \Phi''(e_1\lambda - e_1\mu)^2 - \beta_2(\lambda - \mu)(\mu\lambda + 1) = 0.$$

Finally, substituting the identities (5.1) and (5.6) into the last equality, we deduce that

$$\mathcal{L}\lambda - \left(\frac{2\beta_1^2\beta_2}{(2\lambda - \Phi)\beta_2^2} + \frac{(\beta_1 + \beta_2)^2}{\beta_2^2}\Phi''\right)(e_1\lambda)^2 - \beta_2\big(2\lambda - \Phi\big)\big(\Phi\lambda - \lambda^2 + 1\big) = 0. \quad (5.10)$$

This equality confirms our proposition. □

5.3 Interior ball curvature and non-collapsing technique

This section involves the application of non-collapsing argument for the class of Weingarten hypersurfaces we are considering above in this chapter satisfying the curvature relation (5.1). More precisely, we derive a function that characterises the curvature of the largest ball in the enclosed region which touches the hypersurface at a given point. Then, we show that this inscribed ball curvature of these hypersurfaces satisfies a natural differential inequality in a viscosity sense by applying the maximum principle argument.

For convenience, we begin by illustrating the geometric picture of this function that refers to the interior ball curvature. Let $F : \Sigma \to \mathbb{S}^{n+1}$ be a compact embedded hypersurface which has two distinct principal curvatures λ and μ of multiplicities 1 and $n-1$, respectively. Assume that Σ satisfies the relation $\lambda + \mu = \Phi(\lambda - \mu)$. Let U be the enclosed region by Σ in $\mathbb{S}^{n+1}$ and choose the direction of the unit normal vector ν of Σ pointing out of U in such a way that $\lambda > 0 > \mu$ everywhere. For any two distinct points $x, y \in \Sigma$, there exists a ball $B_r(c)$ of center $c = x - r\nu(x)$ and radius r with the boundary curvature Γ in the region U which touches Σ at x. This is equivalent to the following:

$$
\begin{aligned}
B = B_r(c) \subset \Sigma \quad &\Longleftrightarrow \quad U \subset (B_r(c))^c, \\
&\Longleftrightarrow \quad |y-c|^2 \geq r^2, \quad \forall y \in \Sigma, r > 0; \\
&\Longleftrightarrow \quad |y - x + r\nu(x)|^2 \geq r^2; \\
&\Longleftrightarrow \quad |y-x|^2 + 2r\langle y-x, \nu(x)\rangle + |r\nu(x)|^2 \geq r^2; \\
&\Longleftrightarrow \quad |y-x|^2 + 2r\langle y-x, \nu(x)\rangle \geq 0; \\
&\Longleftrightarrow \quad \frac{2\langle x-y, \nu(x)\rangle}{|y-x|^2} \leq \frac{1}{r}; \\
&\Longleftrightarrow \quad k(x,y) = \frac{2\langle x-y, \nu(x)\rangle}{|y-x|^2} \leq \Gamma, \quad \forall y \in \Sigma;
\end{aligned}
$$

where ν is the unit normal vector at x. The largest ball in U that touches Σ at x has boundary curvature that is denoted by $\bar{k}(x)$. In other words, if the supremum is attained at some point y_0 such that $y_0 \neq x$, then there exists a ball B of boundary curvature $\bar{k}(x)$ contained in Σ that touches at x and y_0, where the center $c = x - \bar{k}^{-1}(x)\nu(x)$ and r is the radius of B. Hence, we define the inscribed ball curvature $\bar{k}(x)$ by

$$
\bar{k}(x) = \sup\left\{ \frac{2\langle x-y, \nu(x)\rangle}{|y-x|^2} : y \neq x \right\}.
$$

We now want to show that such interior ball curvature $\bar{k}$ is a viscosity solution of a differential inequality as follows:

Proposition 5.7. *Let $F : \Sigma^n \to \mathbb{S}^{n+1}$ be an embedded hypersurface with two distinct principal curvatures $\lambda > \mu$ of multiplicity 1 and n-1 respectively, satisfying the relation* (5.1)*. Define the set $\widetilde{\Sigma} = \{x : \bar{k}(x) > \lambda(x)\}$. Then $\bar{k}$ satisfies*

$$
\mathcal{L}\bar{k} - \frac{2\beta_1}{(\bar{k}-\lambda)}(e_1\bar{k})^2 - \Big(\beta_1(1-\lambda^2) + \beta_2(1-(\Phi-\lambda)^2)\Big)\bar{k} \geqslant 0,
$$

over $\widetilde{\Sigma}$ in a viscosity sense.

Proof. It is useful to start the proof by clarifying the geometry of our setting. Assume that $x_0 \in \widetilde{\Sigma}$ and $k(x,y)$ attains its maximum at y_0 that is $k(x_0,y_0) = \bar{k}(x_0)$ such that $x_0 \neq y_0$. Since Σ lies outside the ball of radius $r = \frac{1}{k}$ and center $c = x - r\nu$, then both x_0 and y_0 lie on the boundary of such ball. Hence, we have $c = x_0 - \bar{k}(x_0)^{-1}\nu(x_0) = y_0 - \bar{k}(x_0)^{-1}\nu(y_0)$ and thus we obtain the identity

$$
\nu(y_0) = \nu(x_0) + \bar{k}(x_0)(y_0 - x_0).
$$

Also, note that the tangent spaces of the hypersurface at x_0 and y_0 agree with the tangent spaces of the boundary sphere.

Since the sphere is symmetric, then the tangent spaces at x_0 and y_0 are related by a reflection map. In particular, let $\vec{w} = \frac{y_0 - x_0}{\|y_0 - x_0\|}$ be the unit vector from x_0 to y_0 and $d = \|y_0 - x_0\|$, then $T_{x_0}\Sigma$ and $T_{y_0}\Sigma$ are related by the reflection $R_{\vec{w}}$ in the hyperplane that is orthogonal to $\vec{w}$:

$$R_{\vec{w}}(\nu) = \nu - 2(\vec{w}, \nu)\vec{w}.$$

In other words, such reflection function takes tangent vectors at x_0 to tangent vectors at y_0 and the normal vector $\nu(x_0)$ to the normal vector $\nu(y_0)$.

In order to show that $\bar{k}$ satisfies the differential inequality in the viscosity sense, we consider a smooth function ϕ defined on a neighbourhood Ω of x_0 in $\widetilde{\Sigma}$, where $\phi(x) \geq \bar{k}(x)$ for x close to x_0 and equality holds at x_0. Thus, from the definition of $\bar{k}$ we also have $\phi(x) \geq k(x, y)$ for all $(x, y) \in \Sigma$ with x near x_0. We now want to prove that ϕ satisfies the following differential inequality

$$\mathcal{L}\phi\big|_{x_0} \geqslant 2\beta_1 \frac{(e_1\phi)^2}{(\phi - \lambda)} + \Big(\beta_1(1 - \lambda^2) + \beta_2(1 - (\Phi - \lambda)^2)\Big)\phi.$$

Let $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ be geodesic normal coordinates for Σ around x_0 and y_0 respectively. Since $x_0 \neq y_0$ such that $\phi(x_0) = k(x_0, y_0)$ and

$$\phi(x) \geq \bar{k}(x) \geq k(x, y),$$

for all $x \in \Omega$, the first derivatives must satisfy $\frac{\partial \phi}{\partial x^i} \geqslant \frac{\partial k}{\partial x^i}$ (with equality holding at (x_0, y_0)) and $\frac{\partial k}{\partial y^i} = 0$. Moreover, the second derivatives are given by

$$\Delta\phi\big|_{x_0}(u, u) \geqslant \Delta k|_{(x_0,y_0)}((u, v), (u, v)),$$

where u and v are tangent vectors of Σ at x_0 and y_0, respectively. Therefore, we obtain

$$\mathcal{L}\phi\big|_{x_0} \geqslant \mathcal{L}k(x, y)|_{(x_0,y_0)}. \tag{5.11}$$

In this step, our aim is to compute the term $\mathcal{L}k$ at (x_0, y_0) that is given by

$$\mathcal{L}k = a^{ij}(\partial_i^x + \partial_i^y)(\partial_j^x + \partial_j^y)k,$$

which we write as follows (see Section 5.1 for the definition of the operator $\mathcal{L}$):

$$\mathcal{L}k|_{(x_0,y_0)} \quad = \quad \sum_{i=1}^{n} a^{ii}(\partial_i^x + \partial_i^y)^2 k(x, y)|_{(x_0,y_0)}.$$

The first derivative of k can be computed as the following:

$$\begin{aligned}(\partial_i^x+\partial_i^y)k|_{(x_0,y_0)} &= (\partial_i^x+\partial_i^x)\left(\frac{2\langle x-y,\nu(x)\rangle}{|y-x|^2}\right)\\ &= \frac{2}{d^2}\left((\partial_i^x-\partial_i^x)\cdot\nu_x-(y-x)[(h^x)_i^p\partial_p^x-k(\partial_i^x-\partial_i^y)]\right)\\ &= \frac{2}{d^2}\left((\partial_i^x+\partial_i^x)\cdot(\nu_x+kd\vec{w})-d(h^x)_i^p\partial_p^x\cdot\vec{w}\right).\end{aligned}$$

Note that the derivatives of k in the direction of x with respect to e_1 is given by

$$e_1\phi=e_1k=\frac{2(k-\lambda)\partial_1^x\cdot\vec{w}}{d},$$

and with respect to e_i is $e_ik=0$ where $i\geq 2$, while the derivatives of k in the direction of y vanish.

We now differentiate the first derivatives of k again and multiply by a^{ii} and then take the sum of i in order to obtain the second derivative inequality as follows:

$$\begin{aligned}\mathcal{L}k &= \sum_{i=1}^n a^{ii}(\partial_i^x+\partial_i^y)^2k\\ &= \frac{2}{d^2}a^{ii}\Big((-\nabla^p(h_{ii})d\vec{w}\cdot\partial_p^x)-2(h^x)_i^p\partial_p^x(\partial_i^y-\partial_i^x)\\ &\quad+(h^x)_i^p d\vec{w}(h_{pi}^x v_x+g_{pi}^x)\\ &\quad+(v_x+kd\vec{w})\cdot(-h_{ii}^x v_x-g_{ii}^x+h_{ii}^y v_y+g_{ii}^y)\\ &\quad-k(\partial_i^x+\partial_i^y)^2+2\partial_ik(\partial_i^x+\partial_i^y)\cdot d\vec{w}\Big).\end{aligned}$$

We have the following useful identities:

$$\begin{aligned}a^{ij}g_{ij} &= \beta_1+\beta_2,\\ a^{ij}h_{ik}^x g^{kl}h_{lj}^x &= \lambda^2\beta_1+\mu^2\beta_2,\\ a^{11}h_{1k}^x g^{kl}h_{l1}^x &= \lambda^2\beta_1,\\ a^{ii}h_{ik}^x g^{kl}h_{li}^x &= \mu^2\beta_2.\end{aligned}$$

Substituting the previous identities into $\mathcal{L}k$ gives

$$\mathcal{L}k=2\beta_1\frac{(e_1k)^2}{(k-\lambda)}-k[\beta_1(\lambda^2-1)+\beta_2(\mu^2-1)]. \tag{5.12}$$

We now substitute the last equation (5.12) into (5.11) to obtain the following inequality:

$$\mathcal{L}\phi|_{x_0} \geqslant 2\beta_1 \frac{(e_1\phi)^2}{(\phi-\lambda)} + \left[\beta_1(1-\lambda^2) + \beta_2(1-(\Phi-\lambda)^2)\right]\phi.$$

Thus, we arrive to the required inequality:

$$\mathcal{L}\bar{k}|_{x_0} \geqslant 2\beta_1 \frac{(e_1\bar{k})^2}{(\bar{k}-\lambda)} - \bar{k}\Big(\beta_1(\lambda^2-1) + \beta_2((\Phi-\lambda)^2-1)\Big). \tag{5.13}$$

This completes the proof. □

5.4 Equality of maximum principal curvature and inscribed ball curvature

In this final section, we complete the proof of the Theorem 5.1 by considering Corollary 5.3 and providing the proof of following theorem:

Theorem 5.8. *Let Σ be a compact embedded hypersurface into $\mathbb{S}^{n+1}$ with two principal curvatures $\lambda \neq \mu$ of multiplicities 1 and $n-1$ respectively satisfying the relation* (5.1)*. Assume that the functions Φ and Φ' satisfy the conditions of Theorem 5.1. Then the principal curvatures are constant and thus Σ is congruent to the Clifford torus.*

Proof. In order to show this theorem, we first want to show the next result that arises from combining Propositions 5.6 and 5.7 as follows:

Lemma 5.9. *If $f = \frac{\bar{\kappa}}{\lambda}$, then we deduce that f satisfies the following inequality*

$$\begin{aligned} &\mathcal{L}f - \frac{2\beta_1}{f-1}(e_1 f)^2 - \frac{2(1-f)+4\beta_1 f}{\lambda(f-1)} e_1 f \cdot e_1\lambda \\ &+\left(\frac{2\beta_1^2\beta_2 f}{\beta_2^2(2\lambda-\Phi)\lambda} - \frac{2\beta_1 f^2}{(f-1)\lambda^2} + \frac{4f\Phi''}{\beta_2^2\lambda}\right)(e_1\lambda)^2 \\ &-\frac{(1-\lambda^2)f}{\lambda}\Big(\beta_2(\Phi-\lambda)+\beta_1\lambda\Big) \geq 0. \end{aligned}$$

in the viscosity sense on the set $V = \{f \geq 1\}$ in Σ.

Proof. Consider the function $f = \frac{\bar{\kappa}}{\lambda}$ and assume that ψ is a smooth function such that $\psi \geq f$ and the equality holds at some point x_0. By Proposition 5.7, we have

$$\mathcal{L}\phi - 2\beta_1 \frac{(e_1\phi)^2}{(\phi-\lambda)} - \Big(\beta_1(1-\lambda^2) + \beta_2\big(1-(\Phi-\lambda)^2\big)\Big)\phi\Big|_{x_0} \geq 0.$$

Since $\phi \geq \bar{k}$ with equality at x_0, then we have $\phi = \lambda\psi$. Differentiating this equation implies to

$$e_1\phi = (e_1\psi)\lambda + (e_1\lambda)\psi.$$

Also, we obtain

$$\mathcal{L}\phi = (\mathcal{L}\psi)\lambda + 2e_1\psi e_1\lambda + (\mathcal{L}\lambda)\psi.$$

Substituting these equations into the previous inequality gives

$$\begin{aligned}
&\mathcal{L}\psi - \frac{2\beta_1}{\psi-1}(e_1\psi)^2 - \frac{2(1-\psi)+4\beta_1\psi}{\lambda(\psi-1)} e_1\psi \cdot e_1\lambda \\
&+\left(\frac{2\beta_1^2\beta_2\psi}{\beta_2^2(2\lambda-\Phi)\lambda} - \frac{2\beta_1\psi^2}{(\psi-1)\lambda^2} + \frac{4\psi\Phi''}{\beta_2^2\lambda}\right)(e_1\lambda)^2 \\
&-\frac{(1-\lambda^2)\psi}{\lambda}\Big(\beta_2(\Phi-\lambda)+\beta_1\lambda\Big)\Big|_{x_0} \geq 0.
\end{aligned}$$

By combining the first and second terms of the coefficient of $(e_1\lambda)^2$, we deduce the following:

$$\begin{aligned}
&\mathcal{L}\psi - \frac{2\beta_1}{\psi-1}(e_1\psi)^2 - \frac{2(1-\psi)+4\beta_1\psi}{\lambda(\psi-1)} e_1\psi \cdot e_1\lambda \\
&+\left(\frac{4\psi\Phi''}{\beta_2^2\lambda} - 2\beta_1\beta_2\psi\frac{\psi\big(\beta_2(2\lambda-\Phi)-\lambda\big)+\beta_1\lambda}{\beta_2^2(2\lambda-\Phi)\lambda^2(\psi-1)}\right)(e_1\lambda)^2 \\
&-\frac{(1-\lambda^2)\psi}{\lambda}\Big(\beta_2(\Phi-\lambda)+\beta_1\lambda\Big)\Big|_{x_0} \geq 0.
\end{aligned}$$

In order to estimate the coefficient of $(e_1\lambda)^2$, we use the inequality $(2\lambda - \Phi)\Phi'' \leq \beta_1\beta_2$ and thus arrive at

$$\begin{aligned}
&\frac{4\psi\Phi''}{\beta_2^2\lambda} - 2\beta_1\beta_2\psi\frac{\psi\big(\beta_2(2\lambda-\Phi)-\lambda\big)+\beta_1\lambda}{\beta_2^2(2\lambda-\Phi)\lambda^2(\psi-1)} \\
&\leq \frac{2(\beta_1+\beta_2)\beta_1\beta_2\psi\lambda(\psi-1) - 2\beta_1\beta_2\psi\Big(\psi\big(\beta_2(2\lambda-\Phi)-\lambda\big)+\beta_1\lambda\Big)}{\beta_2^2(2\lambda-\Phi)\lambda^2(\psi-1)} \\
&= \frac{-2\beta_1\beta_2\psi}{\beta_2^2(2\lambda-\Phi)\lambda^2(\psi-1)}\Big((1+\beta_1)(2-\psi)\lambda - \beta_2\psi\mu\Big) \leq 0.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\mathcal{L}\psi - \frac{2\beta_1}{\psi-1}(e_1\psi)^2 - \frac{2(1-\psi)+4\beta_1\psi}{\lambda(\psi-1)}e_1\psi\cdot e_1\lambda \\
&-\frac{2\beta_1\beta_2\psi\Big((1+\beta_1)(2-\psi)\lambda-\beta_2\psi\mu\Big)}{\beta_2^2(2\lambda-\Phi)\lambda^2(\psi-1)}(e_1\lambda)^2 \\
&-\frac{(1-\lambda^2)\psi}{\lambda}\Big(\beta_2(\Phi-\lambda)+\beta_1\lambda\Big)\Big|_{x_0} \geq 0.
\end{aligned}$$

This confirms the lemma. □

Besides the above lemma, we need to show the following:

Corollary 5.10. *The equality of $\bar{k}$ and λ holds everywhere on Σ.*

Proof. To prove that $\bar{k}=\lambda$ everywhere on Σ we use a contradiction argument. The boundary curvature of the largest ball $\bar{k}$ and the maximum principal curvature λ satisfy the inequality $\bar{k}\geq\lambda$ everywhere on Σ . Assume that we have a point such that $\bar{k}>\lambda$. Therefore, $f=\frac{\bar{k}}{\lambda}>1$ on the open set V and $f=1$ on the boundary of V if it is non-empty. On the other hand, we have that f is a constant larger than one by considering Lemma 5.9 and the strong maximum principle argument for viscosity solutions of uniformly elliptic equations [27]. However, since $\bar{k}=\lambda$ at the point of the largest maximum principle curvature on the hypersurface, then we have $f=1$, This contradicts our assumption and hence $\bar{k}=\lambda$ for all points in Σ. □

We now want to prove that λ is constant by applying Brendle's argument [23]. We assume that $\alpha(s)$ is a geodesic with a parameter s on Σ such that $\alpha(0)=x_o$ and $\alpha'(0)=e_1$ for any point $x_0\in\Sigma$. Since $\bar{k}=\lambda$, then we have the inequality $k(x_0,\alpha(s))\leq\lambda(x)$. This can be written as

$$k(x_0,\alpha(s)) = Z(s) = \lambda(x_0)|x_0-\alpha(s)|^2 - 2\nu(x_0)\cdot(x_0-\alpha(s)) \geq 0,$$

where $Z(s)$ is a non-negative function and $Z(0)=0$.

We then compute the first derivative of Z

$$Z'(s) = 2(-\lambda(x_0)(x_0-\alpha(s))+\nu(x_0))\cdot\alpha'(s),$$

we also have $Z'(0)=0$. Differentiating again yields

$$Z''(s) = -2(\lambda_{\alpha(s)}\nu_{\alpha(s)}+\alpha(s))\cdot(-\lambda(x_0)(x_0-\alpha(s))+2\lambda(x_0)|\alpha'(s)|^2.$$

We then obtain that $Z''(0)$ vanishes. Finally, we have $Z'''(0) = 0$ since $Z(s)$ is non-negative. Therefore, we deduce that $-2e_1\lambda(x_0) = 0$. As we choose x_0 to be an arbitrary point in α and since Σ is a rotation hypersurface (Proposition 5.5), then λ is constant. Consequently, μ is also constant by using the relation (5.1). We conclude that Σ is congruent to the Clifford torus. □

Chapter 6

Non-collapsing of Spacelike MCF in Minkowski Space

The study of behavior and properties of embedded spacelike hypersurfaces into a Lorentzian manifold has attracted considerable interest among various aspects in mathematics and physics in recent years, see [46], [40], [39], [42], and [75]. It has been shown that distinct problems of the relativity theory are significantly connected to Lorentz-Minkowski space. From the physics perspective, Einstein's theory of general relativity is formulated in the setting of a 4-dimensional smooth manifold equipped with a Lorentzian metric.

In this chapter, we present the second main objective of the thesis concerning the flow of spacelike hypersurfaces evolving by their mean curvature in Lorentzian-Minkowski space $\mathbb{L}^{n,1}$. In particular, we extend the above result of B. Andrews [12] for compact embedded mean convex hypersurfaces in $\mathbb{R}^{n+1}$ (see section 3.2) to embedded mean convex spacelike hypersurfaces evolving by the MCF in $\mathbb{L}^{n,1}$. Specifically, we derive a non-collapsing estimate for our spacelike hypersurfaces by utilizing the generalized Omori-Yau maximum principle. More precisely, we deduce that the function given by the curvature of the largest future-pointing hyperboloid that touches our spacelike hypersurface at a given point is a viscosity subsolution of a differential equation corresponding to the linearisation of the mean curvature flow, analogous to the situation of Euclidean mean curvature flow.

The main result of this chapter is given by the following theorem:

Theorem 6.1. *Let $F : \Sigma^n \times [0, T) \to \mathbb{L}^{n,1}$ be a family of smooth spacelike embeddings with bounded principal curvatures deforming by the MCF in $\mathbb{L}^{n,1}$. Then, the future hyperboloid curvature function $\overline{Z}$ (defined for $(x, t) \in \Sigma \times [0, T)$ to* *be equal to the least curvature among all future timelike spheres in the future of* $\Sigma_t = F(\Sigma, t)$ *which touch* Σ_t *at* $F(x,t)$*) is a viscosity subsolution of the equation*

$$\frac{\partial \overline{Z}}{\partial t} = \Delta \overline{Z} - |A|^2 \overline{Z}. \tag{6.1}$$

Furthermore, if F *is a mean-convex solution of MCF, and* $\overline{Z}(x,0) \leq H(x,0)$ *for all* $x \in \Sigma$*, then* $\overline{Z}(x,t) \leq H(x,t)$ *for all* $(x,t) \in \Sigma \times [0,T)$*.*

Recall that the MCF for a spacelike hypersurface is formulated as follows:

$$\frac{\partial F(x,t)}{\partial t} = H(x,t)\nu(x,t),$$

for all $(x,t) \in \Sigma^n \times [0,T)$. Note that H and ν refer to the mean curvature and future-directed unit normal vector field on spacelike hypersurfaces, respectively. Also, the mean convexity condition indicates that the mean curvature is positive everywhere on these spacelike hypersurfaces. The definitions will be explained in the following sections.

This chapter is structured as follows: In the first section, we briefly introduce some notions of Semi-Riemannian manifolds. Then, we illustrate the geometry of spacelike hypersurfaces in $\mathbb{L}^{n,1}$ including the metric, the second fundamental form, and some basic and useful relations of this setting in section 6.2. Moreover, the evolution equations of geometric quantities such as the metric, the second fundamental form, the Weingarten map, and the mean curvature are obtained in the next section 6.3. Also, we represent the generalized Omori-Yau maximum principle in section 6.4. In the final and main section of this chapter, we provide the proof of a non-collapsing estimate for our evolving spacelike hypersurfaces by applying the Omori-Yau maximum principle to the boundary curvature of the touching hyperbola defined on these spacelike hypersurfaces.

6.1 Semi-Riemannian manifolds

We briefly introduce some basic concepts of the geometry of semi-Riemannian manifolds [81] in this section. In particular, we describe some notions of these manifolds with respect to their metrics. We describe the classification of tangent vectors according to the sign of their inner product. Finally, we mention some basic examples of semi-Riemannian manifolds including Minkowski space $\mathbb{L}^{n,1}$.

Let $\mathcal{N}$ be a smooth $(n+1)$-dimensional manifold. A smooth symmetric covariant 2-tensor $g_{\mathcal{N}}$ is called a *pseudo-Riemannian metric* if it is non-degenerate at each point $q \in \mathcal{N}$, so that there is no non-zero vector $u \in T_q\mathcal{N}$ for which $g_{\mathcal{N}}(u, v) = 0$ for every $v \in T_q\mathcal{N}$.

The pair $(\mathcal{N}^{n+1}, g_{\mathcal{N}})$ is called a pseudo-Riemannian or semi-Riemannian manifold. This is a generalisation of the concept of Riemannian manifold where the positive-definite condition of $g_{\mathcal{N}}$ is eased. Precisely, let ι be the signature of $g_{\mathcal{N}}$ (the maximal dimension of a subspace on which $g_{\mathcal{N}}$ is negative definite), taking value between 0 and $(n+1)$, the dimension of $\mathcal{N}$. Then, $(\mathcal{N}, g_{\mathcal{N}})$ is a Riemannian manifold if ι vanishes, that is, each $g_{\mathcal{N}}$ is positive-definite on $T_{\mathbf{q}}\mathcal{N}$. On the other hand, $(\mathcal{N}, g_{\mathcal{N}})$ is called a Lorentzian manifold when $\iota = 1$ and $n \geq 1$.

Let q be any point in the semi-Riemannian manifold $(\mathcal{N}^{n+1}, g_{\mathcal{N}})$. Then we can choose local coordinates $\{x^i\}_{i=1}^{n+1}$ for $\mathcal{N}$ near q. Then we can locally write the corresponding components of the metric as

$$g_{ij} = g_{\mathcal{N}}(\partial_i, \partial_j),$$

where $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_j = \frac{\partial}{\partial x^j}$ and $1 \leq i, j \leq n+1$. Consider that $\mathbf{u} = \sum \mathbf{u}^i \partial_i$ and $\mathbf{v} = \sum \mathbf{v}^j \partial_j$ are vector fields in $T_{\mathbf{q}}\mathcal{N}$, then the metric tensor $g_{\mathcal{N}}$ takes the form

$$g_{\mathcal{N}}(\mathbf{u}, \mathbf{v}) = \langle \mathbf{u}, \mathbf{v} \rangle = \sum g_{ij} \mathbf{u}^i \mathbf{v}^j,$$

also it is written as

$$g_{\mathcal{N}} = g_{ij} dx^i \otimes dx^j.$$

Since $g_{\mathcal{N}}$ is a smooth symmetric non-degenerate metric, then we have a smooth non-degenerate inverse metric g^{ij} such that $g^{ik} g_{kj} = \delta^i_j$, where g^{ij} is also symmetric that is $g^{ij} = g^{ji}$.

The causal characters for any tangent vector $\mathbf{u}$ can be defined with respect to the values of its scalar product. We say that $\mathbf{u}$ is spacelike if it satisfies $\langle \mathbf{u}, \mathbf{u} \rangle > 0$ or $\mathbf{u} = 0$, lightlike or null if $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ where $\mathbf{u} \neq 0$ and timelike if $\langle \mathbf{u}, \mathbf{u} \rangle < 0$ (see figure 6.1).

Let $\mathbb{R}^{n+1}_\iota$ be the semi-Euclidean manifold furnished with the semi-Euclidean metric g_{Eucl} where ι is the index of $\mathbb{R}^{n+1}$ such that $0 \leq \iota \leq n+1$. For each point $q \in \mathbb{R}^{n+1}_\iota$ and $u, v \in T\mathbb{R}^{n+1}$, the metric g_{Eucl} is given by

$$g_{Eucl}(u, v) = \langle u_q, v_q \rangle = \sum_{i=1}^{n+1-\iota} u^i v^i - \sum_{j=n+2-\iota}^{n+1} u^j v^j.$$

One of the fundamental examples of $(\mathbb{R}^{n+1}_\iota, g_{Eucl})$ is the Euclidean space $\mathbb{R}^{n+1}$ that is equipped by a Euclidean metric as follows:

$$g_{eucl}(u, v) = \sum_{i=1}^{n+1} u^i v^i,$$

where $\iota = 0$.

The second main example of interest in this chapter is the Minkowski space $\mathbb{L}^{n,1} = \mathbb{R}^n_1$ with a metric

$$g_{Mink}(u, v) = \sum_{i=1}^{n} u^i v^i - u^{n+1} v^{n+1},$$

where $\iota = 1$ and $n \geq 1$. In this case we can further classify timelike vectors into *future* and *past* according to the sign of their last component.

Other well known examples of Riemannian manifolds are the sphere $\mathbb{S}^{n+1}$ and the Hyperbolic space $\mathbb{H}^{n+1}$, whereas the de Sitter space $d\mathbb{S}^{n+1}$ and Anti de Sitter space $Ad\mathbb{S}^{n+1}$ are model spaces of Lorentzian manifolds [81].

The de Sitter space is the spacelike unit sphere $d\mathbb{S}^{n+1} = \{x \in \mathbb{L}^{n+1,1} = \mathbb{R}^{n+2}_1 : |x|^2 = 1\}$, with the induced Lorentzian metric which has signature $(n, 1)$. The Anti de Sitter space is (locally isometric to) the timelike unit sphere $\{x \in \mathbb{R}^{n+2}_2 : |x|^2 = -1\}$ in the Euclidean space with signature $(n, 2)$, and so again has signature $(n, 1)$.

6.2 Spacelike hypersurfaces in $\mathbb{L}^{n,1}$

This section contains some basic notions and definitions of the geometry of spacelike hypersurfaces in Minkowski space $\mathbb{L}^{n,1}$. Also, useful formulae of these hypersurfaces such as the structure equations of the ambient space $\mathbb{L}^{n,1}$ and spacelike hypersurfaces, the Gauss equation and the Weingarten relation will be included. We conclude this section by presenting the derivative of the Weingarten map and also the covariant derivatives and Laplacian of the second fundamental form of our hypersurfaces. The reader may refer to [40], [42], and [107] for more information on these materials.

We start by assuming that Σ is a *spacelike hypersurface*, which is a smooth manifold of dimension n which embedded in $\mathbb{L}^{n,1}$ by a spacelike embedding F: That is, $F : \Sigma \to \mathbb{L}^{n,1}$ is an embedding such that $DF|_x(v)$ is spacelike for every $x \in \Sigma$ and $v \in T_x\Sigma$. Equivalently, the induced metric g on Σ is Riemannian. For each point in Σ, there exists a unique past-directed timelike unit normal vector ν, so that $\langle \nu, \nu \rangle = -1$.

Note that a spacelike hypersurfaces can be described as a graph, by taking the $(n + 1)$ component to be given by a function f of the first n components. The spacelike condition then amounts to the requirement that $\|Df\| < 1$. One of the basic examples of these hypersurfaces of $\mathbb{L}^{n,1}$ is the hyberboloid model of

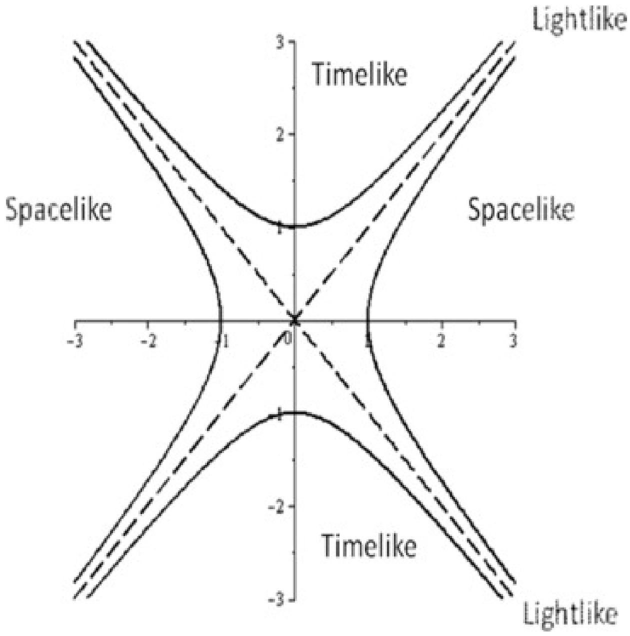

Figure 6.1: Minkowski space

the hyperbolic space $\mathbb{H}^n$ where $\mathbb{H}^n$ is the upper (future) sheet of the two-sheeted hyberboloid and is also called the future timelike unit sphere, given by

$$\mathbb{H}^n = \{u | \langle u, u \rangle = -1, u_{n+1} > 0\}.$$

We now illustrate the local geometry of spacelike hypersurfaces of $\mathbb{L}^{n,1}$ by using the method of moving frames. Let $e_1, ..., e_{n+1}$ be a local orthogonal moving frame of $\mathbb{L}^{n,1}$ around any point $q \in \Sigma$ such that the vectors $e_1, ..., e_n$ are tangent to Σ and e_{n+1} is the past-directed unit normal vector. Let $\omega_1, ..., \omega_{n+1}$ be the dual coframe, and $\omega_{\alpha\beta}$ the connection forms. Then the metric of $\mathbb{L}^{n,1}$ is $\overline{ds^2} = \sum_{\alpha=1}^{n+1} \epsilon_\alpha \omega_\alpha^2$ where $\epsilon_1 = ... = \epsilon_n = 1$ and $\epsilon_{n+1} = -1$.

We consider throughout this chapter the summation convention on the range of indices $1 \leq \alpha, \beta, ... \leq n+1$ and $1 \leq i, j, ... \leq n$. The structure equations of $\mathbb{L}^{n,1}$ are formulated as follows:

$$\begin{aligned} de_\alpha &= -\sum_\beta \epsilon_\beta \omega_{\alpha\beta} e_\beta, \\ d\omega_\alpha &= -\sum_\beta \epsilon_\beta \omega_{\alpha\beta} \wedge \omega_\beta, \quad \omega_{\alpha\beta} = -\omega_{\beta\alpha}, \\ d\omega_{\alpha\beta} &= -\sum_\gamma \epsilon_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta}. \end{aligned}$$

Restricted to the spacelike hypersurface Σ, we choose a local frame to be $e_1, ..., e_n$ where $\nu = e_{n+1}$. The induced Riemannian metric is defined by $ds^2 = \sum \omega_i^2$, then we have $\omega_{n+1} = 0$ and hence

$$0 = d\omega_{n+1} = -\sum_i \omega_{n+1i} \wedge \omega_i.$$

Also, by Cartan's lemma we deduce the following

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j,$$

where h is the second fundamental form of Σ and it is symmetric in the components i, j. If the trace of h vanishes identically, then Σ is called a maximal spacelike hypersurface. Also, Σ is said to be totally geodesic if h vanishes.

Therefore, the structure equations for Σ are given by

$$\begin{aligned} d\omega_i &= -\sum \omega_{ij} \wedge \omega_j, \quad \omega_{ij} = -\omega_{ji}, \\ d\omega_{ij} &= -\omega_{ik} \wedge \omega_{kj} + \frac{1}{2} R_{ijkl}\omega_k \wedge \omega_l, \end{aligned}$$

where R_{ijkl} refers to the Gauss equation that is expressed by

$$R_{ijkl} = -(h_{ik}h_{jl} - h_{il}h_{jk}).$$

Note that the Gauss equation in the context of $\mathbb{L}^{n,1}$ differs from the Euclidean Gauss equation since the sign of the curvature term is changed. Recall also that the curvature forms of the Lorentz-Minkowski space are zero.

We now dicuss some useful equations that will be frequently used in our computations in this chapter. The second fundamental form on Σ can be written as

$$h_{ij} = \langle \partial_i \partial_j F, \nu \rangle,$$

where ∂_i and ∂_j refer to the partial derivatives in the direction of e_i and e_j respectively. Also, the mean curvature H of Σ is given by $H = \frac{1}{n}\sum h_{ii}$ and the Codazzi identity is satisfied, that is $h_{ijk} = h_{ikj}$. Moreover, the following useful equation is called the Weingarten relation

$$\partial_i \partial_j F = -h_{ij}\nu + {\Gamma_{ij}}^k \partial_k F. \tag{6.2}$$

Differentiating the unit normal vector field ν of Σ (the Weingarten map) yields the following lemma:

Lemma 6.2. *The normal vector ν of Σ satisfies*

$$\partial_i \nu = -h_i^k \partial_k F. \tag{6.3}$$

Proof. Since $\partial_j F \cdot \nu = 0$ and $\nu \cdot \nu = -1$, then a straightforward computation using the previous relation 6.2 implies to

$$\begin{aligned} 0 &= \partial_i \partial_j F \cdot \nu + \partial_j F \cdot \partial_i \nu \\ &= \Big(-h_{ij}\nu + \Gamma_{ij}^k \partial_k F \Big) \cdot \nu + \partial_j F \cdot \partial_i \nu \\ &= h_{ij} + \partial_j F \cdot \partial_i \nu. \end{aligned}$$

Thus, $\partial_i \nu = -h_i^k \partial_k F$. □

Also, the covariant derivative of the second fundamental form h_{ij} is given by

$$h_{ijk}\omega_k = dh_{ij} - h_{kj}\omega_{ki} - h_{ik}\omega_{kj}.$$

Finally, the second fundamental form satisfies an indentity analogous to the Simons' identity (2.9) in the Euclidean setting:

$$\Delta h_{ij} = \nabla_i \nabla_j H + h_{ij}|A|^2 - H h_{ik} h_{kj}. \tag{6.4}$$

6.3 Evolution of geometric quantities

In this section, the evolution equations for some geometric quantities are derived for spacelike hypersurfaces of $\mathbb{L}^{n,1}$ evolving under mean curvature flow . In particular, we obtain the evolution equations for the metric, the Weingarten map, the second fundamental form and hence the mean curvature, see [40] and [42] for more information. These identities are very useful in order to simplify our computation in the proof of the main theorem of this chapter.

We start by computing the evolution of the metric g of the spacelike hypersurfaces as follows:

Lemma 6.3. *The induced metric g evolves under MCF according to*

$$\frac{\partial}{\partial t} g_{ij} = 2H h_{ij}.$$

Proof. Since $\frac{\partial F}{\partial x_i}$ is tangential to Σ, then we have $\frac{\partial F}{\partial x_i} \cdot \nu = 0$. Besides, we consider the definition of the second fundamental form $h_{ij} = -\langle \frac{\partial \nu}{\partial x_i}, \frac{\partial F}{\partial x_j} \rangle$ or instead $\frac{\partial \nu}{\partial x_i} = -h_i^j \partial_j^x$. Thus, by simple computation we obtain

$$\begin{aligned} \frac{\partial}{\partial t} g_{ij} &= \frac{\partial}{\partial t}\Big(\frac{\partial F}{\partial x_i} \cdot \frac{\partial F}{\partial x_j}\Big) \\ &= \frac{\partial}{\partial x_i}\Big(- H\nu\Big) \cdot \frac{\partial F}{\partial x_j} + \frac{\partial F}{\partial x_i} \cdot \frac{\partial}{\partial x_j}\Big(- H\nu\Big) \\ &= -H\Big(\frac{\partial \nu}{\partial x_i} \cdot \frac{\partial F}{\partial x_j}\Big) - H\Big(\frac{\partial F}{\partial x_i} \cdot \frac{\partial \nu}{\partial x_j}\Big) \\ &= 2Hh_{ij}. \end{aligned}$$ □

We now deduce the evolution equation of the unit normal vector to Σ:

Lemma 6.4. *The evolution of the unit normal ν to Σ is*

$$\frac{\partial \nu}{\partial t} = -\nabla H.$$

Proof. We can directly obtain this formula by basic computations using the identity $\frac{\partial \nu}{\partial t} \cdot \nu = 0$ and

$$\begin{aligned} \frac{\partial \nu}{\partial t} \cdot \frac{\partial F}{\partial x^i} &= \frac{\partial}{\partial t}\left(\nu \cdot \frac{\partial F}{\partial x^i}\right) - \nu \cdot \frac{\partial}{\partial x^i}\frac{\partial F}{\partial t} \\ &= -\nu \cdot \frac{\partial}{\partial x^i}(-H\nu) \\ &= \nu \cdot \left(-Hh_i^p \partial_p^x + \frac{\partial H}{\partial x^i}\nu\right) \\ &= -\frac{\partial H}{\partial x^i}. \end{aligned}$$

Therefore, we have $\frac{\partial \nu}{\partial t} = -\nabla H$. □

Also, computing the evolution equation of the second fundamental form h_{ij} and the mean curvature H give the following

Proposition 6.5. *The evolution equation of the second fundamental form h_{ij} is given by*

$$\frac{\partial}{\partial t} h_{ij} = \Delta h_{ij} + 2Hh_{ik}h_{jk} - h_{ij}|A|^2.$$

Proof.

$$\begin{aligned}
\frac{\partial}{\partial t} h_{ij} &= \frac{\partial}{\partial t}\left(-\frac{\partial \nu}{\partial x^i}\cdot\frac{\partial F}{\partial x^j}\right) \\
&= -\frac{\partial}{\partial x^i}\left(-\frac{\partial H}{\partial x^k} g^{kl}\frac{\partial F}{\partial x^l}\right)\cdot\frac{\partial F}{\partial x^j} - \frac{\partial \nu}{\partial x^i}\cdot\frac{\partial}{\partial x^j}(-H\nu) \\
&= \nabla_i\nabla_j H + H h_i^p h_j^q g_{pq}.
\end{aligned}$$

Using the above equation 6.4 yields to the required evolution equation. □

Consequently, we have the following evolution equation for H by taking the trace of the previous evolution equation for h_{ij}

$$\frac{\partial}{\partial t} H = \Delta H - |A|^2 H.$$

6.4 Parabolic Omori-Yau maximum principle

The maximum principle on compact Riemannian manifolds is an extremely useful tool applied to various settings in differential geometry and other related topics. The Omori-Yau maximum principle OYMP has been a significant and powerful analytical tool which extends the applicability of the maximum principle to the setting of non-compact Riemannian manifolds.

In this section, we are interested in the parabolic version of the OYMP for the mean curvature flow 2.14. We will first introduce the OYMP for the Laplace-Beltrami operator on a complete Riemannian manifold and its generalized form for a second order linear semi-elliptic operator. Then, we will discuss the parabolic analogue of the OYMP, and in particular a very general version which applies for the mean curvature flow 2.14 which will be used in our argument in the next section. For further information, you may refer to [8] [52], [65], [51], [80], [30], [108] and [87].

We start by explaining the OYMP for the Laplace-Beltrami operator on non-compact Riemannian manifolds. Let (Σ, g) be an n-dimensional smooth complete Riemannian manifold. Assume that $f : \Sigma \to \mathbb{R}$ is a real valued function defined on Σ such that $f \in C^2(\Sigma)$ and bounded from above, that is, $\bar{f} = \sup_\Sigma f < +\infty$. Then, under some geometric conditions on the manifold there exists a sequence of points $\{x_k\} \subset \Sigma$ where $k \in \mathbb{N}$ satisfying the following properties:

$$f(x_k) > \bar{f} - \frac{1}{k}, \qquad |\nabla f(x_k)| < \frac{1}{k}, \quad \textit{and} \quad \Delta f(x_k) < \frac{1}{k},$$

where ∇ and Δ refer to the gradient and Laplace-Beltrami operators, respectively.

The main idea of the proof of such result is based on considering a family of functions such that every function achieves its maximum at some point of Σ. Hence, utilizing the usual maximum principle implies to the above result, see for example [5].

In 1967, Omori [80] showed that if the sectional curvature of Σ is bounded from below, then for any smooth function f with $\bar{f} < +\infty$ there exists a sequence of points $\{x_k\}$ such that

$$f(x_k) > \bar{f} - \frac{1}{k}, \qquad |\nabla f(x_k)| < \frac{1}{k}, \quad \textit{and} \quad \mathrm{Hess}(f)(x_k) < \frac{1}{k} g,$$

where Hess is the Hessian operator. Soon after this work, Yau [108] deduced another form of the above result for complete Riemannian manifold Σ with Ricci curvature bounded from below. In particular, he proved that for any $f \in C^2(\Sigma)$ such that $\bar{f} < +\infty$ there exists $\{x_k\}$ satisfying the following conditions

$$f(x_k) > \bar{f} - \frac{1}{k}, \qquad |\nabla f(x_k)| < \frac{1}{k}, \quad \textit{and} \quad \Delta f(x_k) < \frac{1}{k}.$$

In other words, the OYMP for the Hessian and Laplacian operators holds on every complete Riemannian manifold with sectional and Ricci curvatures bounded from below, respectively. Precisely,

Theorem 6.6. *([80], [108]) Let Σ be a smooth complete Riemannian manifold with Ricci curvature bounded from below. Assume that $f \in C^2(\Sigma)$ has an upper bound, then there exists a sequence $\{x_k\}$ satisfying*

$$\lim_{k\to\infty} |\nabla f(x_k)| = 0, \qquad \limsup_{k\longrightarrow\infty} \Delta f(x_k) \leq 0, \quad \textit{and} \quad \lim_{k\to\infty} f(x_k) = \sup_{\Sigma} f. \quad (6.5)$$

The above theorem of Omori and Yau has emerged as a prominent result, leading to an extensive variety of applications in geometric analysis and partial differential equations.

There are some related results in which a sufficient condition in terms of the existence of an exhaustion function has been considered. The following definitions are needed in order to state some of these results.

Definition 6.7. Let $u : \Sigma \to \mathbb{R}$ be a continuous function and a point $x \in \Sigma$.

- a function u is proper, if the set $\{x : u(x) \leq r\}$ is compact for every $r \in \mathbb{R}$.
- a function v defined on a neighbourhood Ω_x of x is an upper-supporting function for u at x if $v \geq u$ on Ω_x, and equality holds at x.

Definition 6.8. Let $u : \Sigma \to \mathbb{R}$ be a proper continuous function. Then, u is a Δ-tamed exhaustion function if it satisfies the following conditions:

- $u \geq 0$.
- For every $x \in \Sigma$, u has a smooth, upper-supporting function v defined on an open neighborhood Ω_x such that $|\nabla v| \leq 1$ and $\Delta v \leq 1$ hold at x.

H.L. Royden ([91], Proposition 1) showed that every complete Riemannian manifold with Omori-Yau's condition, that is the Ricci curvature is bounded from below, admits a Δ-tamed exhaustion function. Motivated by this result, K.-T. Kim and H. Lee [65] provided a new sufficient condition for the validity of (6.5) on complete Riemannian manifolds in terms of the existence of an exhaustion function. More precisely,

Theorem 6.9. *[65] Let Σ be a manifold admitting a Δ-tamed exhaustion function. Then for every function $f : \Sigma \to \mathbb{R}$ bounded from above, there exists a sequence $\{x_k\}$ on Σ satisfying* (6.5)*.*

Now we want to introduce the generalized OYMP on (Σ, g) for a semi-elliptic operator. Let $L : C^\infty(\Sigma) \to C^\infty(\Sigma)$ be a second-order linear semi-elliptic operator with bounded coefficients and no zeroth order term. Assume that A is a symmetric positive semi-definite $(0,2)$-tensor field on Σ. The operator L can be expressed as

$$Lf = \mathrm{Tr}\left(A \circ \mathrm{Hess}(f)\right) + g(V, \nabla f),$$

where $A \in \Gamma(\mathrm{End}(T\Sigma))$ is self-adjoint with respect to the metric g, $\mathrm{Hess}(f) \in \Gamma(\mathrm{End}(T\Sigma))$ is the Hessian of f such that $\mathrm{Hess}(f)(X) = \nabla_X \nabla f$ where $X, V \in \Gamma(T\Sigma)$. Also, we assume that

$$\sup_\Sigma \mathrm{Tr}(A) + \sup_\Sigma |V| < \infty.$$

Similarly, we define an L-tamed exhaustion function by replacing the operator Δ by L in Definition 6.8. Also, H.L. Royden [16, Proposition 2] showed that every complete Riemannian manifold with its sectional curvature bounded below admits an L-tamed exhaustion function. Therefore, by considering the existence of an L-tamed exhaustion, K. Hong and C. Sung [52] generalized the OYMP for the Laplacian Δ to the above semi-elliptic operator L. More precisely, they deduced the following theorem

Theorem 6.10. *[52] Let Σ be an n-dimensional smooth complete Riemannian manifold admitting an L-tamed exhaustion function. Then for every real valued function f on Σ which is bounded above, there exists a sequence $\{x_k\}$ on Σ satisfying the following properties:*

$$\lim_{k\to\infty} |\nabla f(x_k)| = 0, \quad \limsup_{k\to\infty} Lf(x_k) \leq 0, \quad \textit{and} \quad \lim_{k\to\infty} f(x_k) = \sup_{\Sigma} f. \tag{6.6}$$

We now want to prove the following parabolic version of OYMP for the mean curvature flow 2.14 which provides a model for the main theorem 6.1 of this chapter. We start by demonstrating the theorem of OYMP in the parabolic context.

Theorem 6.11 (Parabolic Omori-Yau Maximum Principle for MCF)**.** *Let $\mathbb{L}^{n,1}$ be the Minkowski space and Σ be an n-dimensional smooth manifold. Let $F : \Sigma \times [0,T] \to \mathbb{L}^{n,1}$ be a smooth family of properly embedded complete spacelike hypersurfaces in $\mathbb{L}^{n,1}$ evolving by the MCF 2.14. Assume that there exists $a \geq 0$ and $y_0 \in \mathbb{L}^{n,1}$ such that the function $\rho(x,t) = a + |F(x,t) - y_0|^2$ is non-negative and proper on $\Sigma\times[0,T]$ (so that $\{(x,t) :\ \rho(x,t) \leq C\}$ is compact for each C), and satisfying $|\nabla\rho| \leq C(1+|\rho|)$. Let $Z : \Sigma\times[0,T] \to \mathbb{R}$ is a bounded continuous and twice differentiable function such that $\sup_{\Sigma\times[0,T)} Z > \sup Z_{\Sigma\times\{0\}} Z$. Then there is a sequence of points $(x_i,t_i) \in \Sigma\times[0,T]$ satisfying the following properties:*

$$Z(x_i,t_i) \to \sup Z, \quad |\nabla Z(x_i,t_i)| \to 0, \quad \textit{and} \quad \limsup_{i\to\infty}(\partial_t - \Delta)Z(x_i,t_i) \geq 0. \tag{6.7}$$

Before embarking on the proof, we make some remarks concerning the assumptions: The condition on the existence of a point y_0 satisfying the required condition is a mild restriction on the solution F of mean curvature flow, which is in particular satisfied if the evolving hypersurfaces have bounded principal curvatures at each time. It seems reasonable to conjecture that this assumption is true whenever the hypersurfaces are complete with respect to the induced metric, strictly spacelike and properly embedded.

Proof. We will produce the required sequence of points by maximising a sequence of modified functions: First, observe that there exists a sequence $(\bar{x}_i, s_i)$ such that $Z(\bar{x}_i, s_i)$ approaches $\sup_{\Sigma\times[0,T]} Z$, that is

$$\lim_{i\to\infty} Z(\bar{x}_i, s_i) = \sup_{\Sigma\times[0,T)} Z. \tag{6.8}$$

Now let y_0 be as assumed, and choose $\alpha_i > 0$ such that $\alpha_i < \frac{1}{i}$ and

$$\alpha_i \rho(\bar{x}_i, s_i) \leq \frac{1}{i}. \tag{6.9}$$

We now define a function $Z_i : \Sigma \times [0, T) \to \mathbb{R}$ by

$$Z_i(x,t) = Z(x,t) - \alpha_i(\rho(x,t)). \tag{6.10}$$

Note that

$$\sup_{\Sigma\times[0,T]} Z \geq \sup_{\Sigma\times[0,T]} Z_i \geq Z_i(\bar{x}_i, s_i) > \sup_{\Sigma\times[0,T]} Z - \frac{2}{i} > \sup_{\Sigma\times\{0\}} Z \geq \sup_{\Sigma\times\{0\}} Z_i$$

for sufficiently large i.

Furthermore $\rho > \frac{2}{i\alpha_i}$ implies $Z_i < \sup_{\Sigma\times[0,T]} Z - \frac{2}{i} < Z_i(\bar{x}_i, s_i) \leq \sup_{\Sigma\times[0,T]} Z_i$, so the supremum of Z_i is attained in the compact set $\{(x,t) : \ \rho(x,t) \leq \frac{2}{i\alpha_i}\}$. Let (x_i, t_i) be a point where $\sup_{\Sigma\times[0,T]} Z_i$ is attained, and observe that $t_i > 0$ since $\sup_{\Sigma\times\{0\}} Z_i < \sup_{\Sigma\times[0,T]} Z_i$.

By construction we have $Z(x_i, t_i) = \sup_{\Sigma\times[0,T]} Z_i > \sup_{\Sigma\times[0,T]} Z - \frac{2}{i}$. It remains to prove bounds for the derivatives of Z at (x_i, t_i): Since (x_i, t_i) is a spatial maximum of Z_i, we have at this point

$$0 = \nabla Z_i = \nabla Z - \alpha_i \nabla\rho \implies |\nabla Z_i| \leq \alpha_i |\nabla\rho| \leq \alpha_i C(a + |\rho|) \leq \frac{C}{i} + \frac{2}{i} \to 0$$

as $i \to \infty$. Furthermore, the spatial maximisation implies $\Delta Z_i \leq 0$, while the maximisation in the time direction (and the fact $t_i > 0$) implies $\partial_t Z_i \geq 0$ at (x_i, t_i), so we have

$$0 \leq (\partial_t - \Delta) Z_i = (\partial_t - \Delta) Z - \alpha_i (\partial_t - \Delta)\rho.$$

A direct computation gives

$$\partial_t \rho = 2H\nu \cdot (F(x,t) - y_0),$$

while

$$\Delta\rho = g^{ij}\nabla_i \left(2\partial_j F \cdot (F - y_0)\right) = 2H\nu \cdot (F - y_0) + 2n,$$

so that

$$(\partial_t - \Delta)\rho = -2n.$$

This gives at (x_i, t_i) the inequality

$$(\partial_t - \Delta) Z \geq \alpha_i (\partial_t - \Delta)\rho \geq -\frac{2n}{i}.$$

This completes the proof. □

We remark that essentially the same proof holds in the setting of hypersurfaces in Euclidean space evolving by mean curvature flow, where ρ can be taken to be the distance from any fixed $y_0 \in \mathbb{R}^{n+1}$. The complication in the Minkowski space setting arises only from the fact that the function ρ is not in general positive, and that the gradient of ρ cannot automatically be bounded in terms of ρ as in the Euclidean case.

In the following section we adapt the ideas of the proof to the more complicated setting of the non-collapsing estimate.

6.5 Non-collapsing of mean convex MCF in $\mathbb{L}^{n,1}$

This main section of this chapter involves an extension of Andrews' result [12], see also section 3.2, to mean convex embedded spacelike hypersurfaces evolving by the MCF of $\mathbb{L}^{n,1}$. In particular, we deduce a non-collapsing estimate by employing the parabolic version of OYMP (Theorem 6.11) to a quantity that relies on two points of these spacelike hypersurfaces. Precisely, we compare the radius of the largest hyperbola which touches the spacelike hypersurface at a given point to the curvature at that point. Also, we apply the parabolic version OYMP in order to control the behaviour of spacelike hypersurfaces at infinity and ensure that our argument is applicable to this setting.

The main purpose of this section is to provide the proof of Theorem 6.1. More precisely, we deduce that the interior boundary curvature of the touching hyperbola satisfies a differential inequality in the viscosity sense.

To explain this result, let $F : \Sigma^n \times [0, T) \to \mathbb{L}^{n,1}$ be a family of embedded spacelike hypersurfaces into $\mathbb{L}^{n,1}$, deforming by the MCF. We begin by defining a function on $\Sigma^n \times \Sigma^n$ which characterises the appropriate geometric notion of non-collapsing. In the Euclidean setting this quantity is obtained by touching each point of the hypersurface by interior or exterior spheres. However, in the setting of Minkowski space we consider future or past timelike hyperboloids which touch the evolving hypersurface at a given point but are disjoint elsewhere, and obtain a similar quantity. For concreteness, define the *future hyperboloid* of radius r centered at p to be the set

$$\mathcal{H}_+(p, r) = \{x \in \mathbb{L}^{n,1} : \ \langle x - p, x - p\rangle = -r^2, \ x^{n+1} > p^{n+1}\}.$$

Similarly, the *past hyperboloid* of radius r centered at p is given by

$$\mathcal{H}_-(p, r) = \{x \in \mathbb{L}^{n,1} : \ \langle x - p, x - p\rangle = -r^2, \ x^{n+1} < p^{n+1}\}.$$

In the following argument we define non-collapsing using future hyperboloids. The argument for past hyperboloids is similar.

Let x be an arbitrary point of Σ and U be the future of Σ, i.e. the set of all points that can be reached by following future timelike trajectories from points of Σ. We derive a condition for a future hyperboloid $\mathcal{H}_+(p,r)$ to touch Σ at $x \in \Sigma$, but be contained in U. If ν is the past timelike unit normal vector at x, then the touching condition and the first order condition are equivalent to the condition that $p = x + r\nu$. The fact that $\mathcal{H}_+(p,r)$ is equivalent to the statement $|y-p|^2 \geq -r^2$ for all $y \in \Sigma$, where we use the observation that since $y - x$ is spacelike, no point of Σ can intersect $\mathcal{H}_+(p,r)$. This gives

$$\begin{aligned}
\mathcal{H}_+(p,r) \subset U \quad &\Longleftrightarrow \quad d(p,y)^2 \geq -r^2 \quad \forall y \in \Sigma;\\
&\Longleftrightarrow \quad |y-p|^2 \geq -r^2;\\
&\Longleftrightarrow \quad |y-(x+r\nu)|^2 \geq -r^2;\\
&\Longleftrightarrow \quad |y-x|^2 - 2r\langle y-x,\nu\rangle - r^2 \geq -r^2;\\
&\Longleftrightarrow \quad \frac{2\langle y-x,\nu\rangle}{|y-x|^2} \leq \frac{1}{r}.
\end{aligned}$$

We now define the non-collapsing quantity by

$$Z(x,y,t) = \frac{2\langle F(y,t) - F(x,t), \nu(x,t)\rangle}{|F(y,t) - F(x,t)|^2}, \tag{6.11}$$

for all (x,y) belonging to $(\Sigma \times \Sigma) \setminus D$ and $t \in [0,T)$ where D is the diagonal set, that is, $D = \{(x,x) : x \in \Sigma\}$. Note that F is the embedding of the hypersurface into $\mathbb{L}^{n,1}$.

We will complete the proof of Theorem 6.1 in two stages: We will first prove the second statement comparing the future hyperboloid curvature with the mean curvature. The first statement (that the future hyperboloid curvature is a subsolution of the linearised MCF) requires a refinement of the argument of the Omori-Yau maximum principle, and will be presented subsequently.

Before we start the proof of Theorem 6.1, we should assume some conditions on our arbitrary spacelike hypersurface to ensure that the argument of maximum principle is applicable in this situation. Precisely, since an arbitrary spacelike hypersurface may grow rapidly at infinity, so that the non-collapsing quantity 6.11 may not attain a maximum value, so that the usual maximum principle argument will not apply.

Therefore, to rule out such obstacle we apply the parabolic version of generalized OYMP, refer to Theorem 6.11. In particular, we can obtain a sequence of points (y_k, t_k) where $k \in \mathbb{N}$ on our spacelike hypersurfaces such that

$$Z(y_k, t_k) \to \sup Z, \quad |\nabla Z(y_k, t_k)| \to 0, \quad \textit{and} \quad \limsup_{k\to\infty}(\partial_t - L)Z(y_k, t_k) \le 0. \tag{6.12}$$

In our situation we take a supremum of the quantity Z over y where ν is the past timelike unit normal vector such that $t_0 > 0$. Thus, we obtain the interior boundary curvature $\overline{Z}$ as follows:

$$\overline{Z}(x,t) = \sup_{y\in\Sigma\setminus\{x\}} \left\{ \frac{2\langle F(x,t) - F(y,t), \nu(x,t)\rangle}{|F(x,t) - F(y,t)|^2} \right\}.$$

Now we can start the proof of the main theorem 6.1 .

Proof. In the first stage of the proof, we will establish the preservation of inequalities of the form $\overline{Z} \le CH$ for solutions of the mean curvature flow with bounded curvature. Since $\overline{Z}(x,t) = \sup_{y\in\Sigma\setminus\{x\}} Z(x,y,t)$, this amounts to the preservation of inequalities $Z(x,y,t) - cH(x,t) \le 0$ in such flows. We prove this by applying the argument of the OYMP on the product manifold $(\Sigma \times \Sigma \setminus \Delta) \times [0,T]$ (we note that the function Z extends continuously from $\Sigma \times \Sigma \setminus \Delta$ to the completion $\mathcal{S}$ of given by appending the unit sphere bundle $\{(x,v) : x \in \Sigma,\ v \in T_x\Sigma,\ |v| = 1\}$ as in [19]).

By assumption, we have $Z(x,y,0) - cH(x,t) \le 0$ for all $y \ne x$. For the purposes of obtaining a contradiction, we suppose that there is $\bar{t} > 0$ and $\bar{x}, \bar{y} \in \Sigma$ such that $Z(\bar{x},\bar{y},\bar{t}) - cH(\bar{x},\bar{t}) > 0$. Let $\delta = \frac{Z(\bar{x},\bar{y},\bar{t}) - cH(\bar{x},\bar{t})}{2\bar{t}} > 0$. Then we consider the function $P(x,y,z) := \mathrm{e}^{-\beta t}(Z(x,y,t) - cH(x,t)) - \gamma t$, where β and γ will be chosen later, and note that $P \le 0$ on $\Sigma\times\Sigma\times\{0\}$ and $P(\bar{x},\bar{y},\bar{t}) \ge \mathrm{e}^{-\beta t_0}\delta_0 - \gamma t_0 > 0$ provided $\gamma > 0$ is chosen small enough for given β.

Following the method of the OYMP, we consider the modified functionals $P_\alpha(x,y,t) := P(x,y,t) - \alpha\rho(x,t) - \alpha\rho(y,t)$. The argument of Theorem 6.11 provides a sequence of points (x_i, y_i, t_i) such that $|\nabla P(x_i,y_i,t_i)| \le \frac{1}{i}$ and $(\partial_t - L)P(x_i,y_i,t_i) \ge (\partial_t - L)P - \frac{1}{i}$, where L is any weakly elliptic operator of the form $L = \sum_{i,j} g^{ij}_{(x,t)}\nabla^x_i\nabla^x_j + 2\sum_{i,j} a^{ij}(x,y,t)\nabla^x_i\nabla^y_j + \sum_{i,j} g^{ij}_{(y,t)}\nabla^y_i\nabla^y_j$.

In order to derive the computation of these derivatives, we choose $\{x^i\}$ and $\{y^i\}$ $(i = 1,...,n)$ to be local orthonormal coordinates of x and y, respectively. We also consider the following abbreviations to simplify the computations:

$$d = |F(x,t) - F(y,t)|,$$

$$w = \frac{F(x,t) - F(y,t)}{|F(x,t) - F(y,t)|},$$

$$dw = F(x,t) - F(y,t).$$

Also, we write $\partial_i^x = \frac{\partial F}{\partial x_i}$ and $\partial_i^y = \frac{\partial F}{\partial y_i}$.

Computing the first spatial derivative of P with respect to y yields

$$\begin{aligned}
e^{\beta t}\frac{\partial}{\partial y^i}P &= \frac{\partial}{\partial y^i}\left(\frac{2}{d^2}\langle dw, \nu_x\rangle - cH_x - \frac{\delta}{2}t\right) \\
&= -\frac{2}{d^2}\langle \partial_i^y, \nu_x\rangle + \frac{2}{d^2}Z\langle \partial_i^y, dw\rangle \\
&= -\frac{2}{d^2}\langle \partial_i^y, \nu_x - Zdw\rangle,
\end{aligned} \tag{6.13}$$

and the derivative in the x-direction is given by

$$\begin{aligned}
e^{\beta t}\frac{\partial}{\partial x^i}P &= \frac{2}{d^2}\langle dw, -\overset{xp}{h_i}\partial_p^x\rangle + \frac{2}{d^2}\langle \partial_i^x, dw\rangle Z - c\nabla_i H \\
&= -\frac{2}{d^2}\left(\langle \partial_i^x, dw\rangle Z + \langle \partial_p^x, dw\rangle \overset{xp}{h_i}\right) - c\nabla_i H.
\end{aligned} \tag{6.14}$$

In order to compute the second derivatives of P, we first differentiate the above equation (6.13) with respect to the y-coordinate. This is given by the following:

$$\begin{aligned}
e^{\beta t}\frac{\partial^2}{\partial y^i \partial y^j}P &= -\frac{2}{d^2}\langle -h_{ij}^y \nu_y, \nu_x - dwZ\rangle - \frac{2}{d^2}\langle \partial_i^y, -\frac{\partial Z}{\partial y^j}dw + Z\partial_j^y\rangle \\
&\quad -2\frac{\partial}{\partial y^j}(d^{-2})\langle \partial_i^y, \nu_x - dwZ\rangle \\
&= \frac{2}{d^2}\langle h_{ij}^y \nu_y, \nu_x - dwZ\rangle - \frac{2}{d^2}\langle \partial_i^y, \partial_j^y\rangle Z \\
&= -\frac{2}{d^2}\left(h_{ij}^y + Z\delta_{ij}\right).
\end{aligned}$$

We next differentiate the equation (6.14) in the direction of the x-coordinate and derive the following identity:

$$\begin{aligned}
e^{\beta t}\frac{\partial^2}{\partial x^i \partial x^j}P &= -\frac{\partial}{\partial x^j}\left(\frac{2}{d^2}\langle \partial_i^x, dw\rangle Z + \frac{2}{d^2}\langle \partial_p^x, dw\rangle \overset{xp}{h_i} + c\nabla_i H\right) \\
&= -\frac{2}{d^2}Z\delta_{ij} - \frac{2}{d^2}h_{ij}^x + Zh_{jp}^x h_{iq}^x \delta^{pq} - \frac{2}{d}\langle \partial_q^x, w\rangle \delta^{pq}\nabla_p h_{ij}^x \\
&\quad +Z^2 h_{ij}^x - \frac{2}{d}\langle \partial_i^x, w\rangle \frac{\partial Z}{\partial x^j} - \frac{2}{d}\langle \partial_j^x, w\rangle \frac{\partial Z}{\partial x^i} - c\frac{\partial^2 \phi}{\partial x^i \partial x^j}.
\end{aligned}$$

Moreover, the derivative of the equation (6.13) in the x-direction is given by

$$\begin{aligned} e^{\beta t}\frac{\partial^2}{\partial x^j \partial y^i}P &= -\frac{\partial}{\partial x^j}\left(\frac{2}{d^2}\langle \partial_i^y, \nu_x - Zdw\rangle\right) \\ &= -\frac{2}{d^2}\langle \partial_i^y, -\overset{xp}{h_j}\partial_p^x - Z(\partial_j^x) - \frac{\partial Z}{\partial x^j}dw\rangle \\ &= \frac{2}{d^2}\langle \partial_i^y \partial_p^x\rangle \left(\overset{xp}{h_j} + \delta_j^p Z\right) + \frac{2}{d}\frac{\partial \phi}{\partial x^j}\langle \partial_i^y, w\rangle. \end{aligned}$$

We now compute the derivative of P with respect to time and obtain the following:

$$\begin{aligned} e^{\beta t}\frac{\partial}{\partial t}P &= \frac{2}{d^2}H_x - \frac{2}{d^2}\langle H_y\nu_y, \nu_x - dwZ\rangle - \frac{2}{d}\langle w, \nabla H_x\rangle + Z^2 H_x - c\frac{\partial H}{\partial t} - \beta P - \gamma e^{\beta t} \\ &= \frac{2}{d^2}H_x - \frac{2}{d^2}H_y - \frac{2}{d}\langle w, \nabla H_x\rangle + Z^2 H_x - c(\Delta H - |A|^2 H) - \beta P - \gamma e^{\beta t}, \end{aligned}$$

where we used the expression for $\frac{\partial H}{\partial t}$ from Proposition 6.5.

Now we choose the coefficients a^{ij} in order to define the operator L: We define $a^{ij}(x,y,t) = g_x^{ip}g_y^{jq}\left\langle \partial_p^x, \partial_q^y\right\rangle$. Combining the last four identities gives

$$\begin{aligned} e^{\beta t}(\partial_t - L)P \leq\ & -\frac{2}{d^2}H_x + \frac{2}{d^2}H_y + \frac{2}{d}\langle w, \nabla H_x\rangle - Z^2 H_x \\ & -\frac{2}{d^2}g_x^{ij}h_{ij}^y - \frac{2}{d^2}g_x^{ij}Z\delta_{ij} \\ & -2g_x^{ij}\frac{2}{d^2}\langle \partial_i^y \partial_p^x\rangle\left(\overset{xp}{h_j} - \delta_j^p Z\right) + 2g_x^{ij}\frac{2}{d}\frac{\partial \phi}{\partial x^j}\langle \partial_i^y, w\rangle \\ & -g_x^{ij}\frac{2}{d^2}Z\delta_{ij} + \frac{2}{d^2}Zh_{ij}^x - Zh_{ij}^x h_{ij}^x\delta^{pq} + \frac{2}{d}\langle \partial_q^x, w\rangle\delta^{pq}\nabla_p h_{ij}^x \\ & +Z^2 h_{ij}^x - \frac{2}{d}\langle \partial_i^x, w\rangle\frac{\partial Z}{\partial x^j} - \frac{2}{d}\langle \partial_j^x, w\rangle\frac{\partial Z}{\partial x^i} + c|A|^2 H_x - \beta P - \gamma e^{\beta t}. \end{aligned}$$

Hence,

$$\begin{aligned} e^{\beta t}(\partial_t - L)P \leq\ & g_x^{ij}(P + \frac{\mu}{2}t)h_{ip}\delta^{pq}h_{jq} - \frac{4}{d^2}H_x + \frac{4}{d^2}g_x^{ij}h_{iq}\delta^{pq}\langle \partial_j^y, \partial_p^x\rangle \\ & -\frac{4}{d^2}Zg_x^{ij}\delta_{ij} + \frac{4}{d^2}Zg_x^{ij}\langle \partial_i^x, \partial_j^y\rangle - \frac{4}{d}g_x^{ij}\frac{\partial Z}{\partial x^i}\langle w, \partial_j^x - \partial_i^y\rangle - \beta P - \gamma e^{\beta t}. \end{aligned}$$

Note that the some of the terms can be rewritten as follows:

$$\frac{4}{d^2}H_x - \frac{4}{d^2}g_x^{ij}h_{iq}\delta^{pq}\langle \partial_j^y, \partial_p^x\rangle = \frac{4}{d^2}g_x^{ij}h_{iq}\delta^{pq}(\delta_{jp} - \langle \partial_j^y, \partial_p^x\rangle),$$

while the first two terms of the last line can be rewritten as follows:

$$\frac{4}{d^2}Zg_x^{ij}\delta_{ij} - \frac{4}{d^2}Zg_x^{ij}\langle\partial_i^x,\partial_j^y\rangle = \frac{4}{d^2}Zg_x^{ij}(\delta_{ij} - \langle\partial_i^x,\partial_j^y\rangle).$$

Thus, we arrive to

$$\begin{aligned} e^{\beta t}(\partial_t - L)P \ \leq\ & g_x^{ij}Ph_{ip}\delta^{pq}h_{jq} - \frac{4}{d^2}g_x^{ij}h_{iq}\delta^{pq}(\delta_{jp} - \langle\partial_j^y,\partial_p^x\rangle) \\ & -\frac{4}{d^2}Zg_x^{ij}(\delta_{ij} - \langle\partial_i^x,\partial_j^y\rangle) - \frac{4}{d}g_x^{ij}\frac{\partial Z}{\partial x^i}\langle w,\partial_j^x - \partial_i^y\rangle + \gamma(|A|^2t - e^{\beta t}) - \beta P \end{aligned}$$

By using the first derivative identity (6.14) for $\frac{\partial P}{\partial x^i}$, the previous inequality becomes

$$\begin{aligned} e^{\beta t}(\partial_t - L)P \ \leq\ & g_x^{ij}Ph_{ip}\delta^{pq}h_{jq} - \beta P \\ & +\frac{4}{d^2}g_x^{ij}\delta^{pq}(h_{iq} + Z\delta_{ip})(2\langle w,\partial_q^x\rangle\langle w,\partial_j^y - \partial_j^x\rangle) \\ & +\frac{4}{d^2}g_x^{ij}\delta^{pq}(h_{iq} + Z\delta_{ip})(\delta_{jq} - \langle\partial_j^y,\partial_q^x\rangle) + \gamma(|A|^2t - e^{\beta t}). \end{aligned}$$

If we choose $\beta > |A|^2$, this can be reformulated as

$$\begin{aligned} e^{\beta t}(\partial_t - L)P \ \leq\ & \gamma(|A|^2t - e^{\beta t}) \\ & -\frac{4}{d^2}g_x^{ij}\delta^{pq}(h_{iq} + Z\delta_{ip})(\delta_{qj} - \langle\partial_j^y,\partial_q^x\rangle + 2\langle w,\partial_q^x\rangle\langle w,\partial_j^y - \partial_j^x\rangle). \end{aligned}$$

The next Lemma controls the term $\delta_{qj} - \langle\partial_j^y,\partial_q^x\rangle + 2\langle w,\partial_q^x\rangle\langle w,\partial_j^y - \partial_j^x\rangle$:

Lemma 6.12. *We have*

$$\delta_{qj} - \langle\partial_j^y,\partial_q^x\rangle + 2\langle w,\partial_q^x\rangle\langle w,\partial_j^y - \partial_j^x\rangle \leq 0.$$

Proof. We consider the computation at the point (x_i, y_i, t_i). Let $\{x^i\}$ and $\{y^i\}$ be the local coordinates, we choose them so that $\partial_i^x = \partial_i^y$ for $i = 1, ..., n-1$. Hence ∂_n^x and ∂_n^x are coplanar with w. As a result, if $p = q = n$, then we find that $\delta_{qj} - \langle\partial_j^y,\partial_q^x\rangle + 2\langle w,\partial_q^x\rangle\langle w,\partial_j^y - \partial_j^x\rangle$ does not vanish.

Therefore, the proof will be divided into two parts: $\langle w,\nu_x\rangle \leq 0$ and $\langle w,\nu_x\rangle \geq 0$. If $\langle w,\nu_x\rangle \leq 0$, then we may assume that $\langle w,\nu_x\rangle = -\sin\theta$, where $\theta \in [0,\frac{\pi}{2})$. We can adjust the direction of ∂_n^x such that $\langle w,\partial_n^x\rangle = \cos\theta$. Assume that we have the conditions $\langle\partial_n^x,\partial_n^y\rangle = -\cos 2\alpha$ and $\langle\partial_n^y,\nu_x\rangle = \sin 2\alpha$ where $\alpha \in [0,\frac{\pi}{2})$ and the orientation of ∂_n^y is satisfied. By using the first derivatives of P, particularly equation (6.13) with respect to the y^n-direction, and the definition (6.11) of Z, we obtain

$$\begin{aligned}
\frac{\partial}{\partial y^n}P &= \frac{2}{d^2}\langle \partial^y_n, \nu_x - Zdw\rangle \\
&= 2\langle \partial^y_n, \nu_x\rangle - 2\langle \partial^y_n, Zdw\rangle \\
&= \langle \partial^y_n, \nu_x\rangle - 2\langle \partial^y_n, w\rangle\langle w, \nu_x\rangle \\
&= \sin 2\alpha \cos 2\theta - \sin 2\theta \cos 2\alpha \\
&= \sin(2\alpha - 2\theta) \to 0.
\end{aligned}$$

Thus, we have $\theta = \alpha$ and $\langle \partial^y_n, w\rangle = -\cos\alpha$. By substituting, we get

$$\begin{aligned}
\delta_{qj} - \langle \partial^y_j, \partial^x_q\rangle + 2\langle w, \partial^x_q\rangle\langle w, \partial^y_j - \partial^x_j\rangle &= 1 + \cos(2\alpha) - 2\cos\alpha(\cos\alpha + \cos\alpha) \\
&= 2\cos^2\alpha - 4\cos^2\alpha = -2\cos^2\alpha \leq 0.
\end{aligned}$$

In the case of $\langle w, \nu_x\rangle \geq 0$, this direction is similar to the previous case. We suppose that $\theta \in [0, \frac{\pi}{2})$ satisfying $\langle w, \nu_x\rangle = \sin\theta$. We have $\langle w, \partial^x_n\rangle = -\cos\theta$ by directing ∂^x_n . Also, we can choose the orientation of ∂^y_n and $\alpha \in [0, \frac{\pi}{2})$ to meet the conditions $\langle \partial^x_n, \partial^y_n\rangle = -\cos 2\alpha$ and $\langle \partial^y_n, \nu_x\rangle = \sin 2\alpha$. By using again the equations (6.13) and (6.11), we deduce that

$$\begin{aligned}
\langle \partial^y_n, \nu_x\rangle &= 2\langle \partial^y_n, w\rangle\langle w, \nu_x\rangle \\
\sin(2\alpha - 2\theta) &= 0,
\end{aligned}$$

this implies $\theta = \alpha$ and $\langle \partial^y_n, w\rangle = \cos\alpha$. The computation is given by the following:

$$\begin{aligned}
\delta_{qj} - \langle \partial^y_j, \partial^x_q\rangle + 2\langle w, \partial^x_q\rangle\langle w, \partial^y_j - \partial^x_j\rangle &= 1 + \cos(2\alpha) - 2\cos\alpha(-\cos\alpha - \cos\alpha) \\
&= 2\cos^2\alpha - 4\cos^2\alpha = -2\cos^2\alpha \leq 0.
\end{aligned}$$

□

Finally, the inequalities $\frac{1}{i} \leq (\partial_t - L)P \leq \gamma(|A|^2 t - \mathrm{e}^{\beta t})$ give a contradiction for T small enough, proving that P remains non-positive. Since $\gamma > 0$ is arbitrary, this implies that $Z \leq cH$ for all times, proving the second claim of Theorem 6.1. □

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