European Journal of Combinatorics 94 (2021) 103314

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

On the stability of Baer subplanes ☆

Tamás Szőnyi [a,b,c], Zsuzsa Weiner [c,d]

[a] Department of Computer Science, Eötvös Loránd University, H-1117 Budapest, Pázmány Péter sétány 1/C, Hungary
[b] UP FAMNIT, University of Primorska, Glagoljaška 8, 6000, Koper, Slovenia
[c] MTA-ELTE Geometric and Algebraic Combinatorics Research Group, H-1117 Budapest, Pázmány Péter sétány 1/C, Hungary
[d] Prezi.com, H-1065 Budapest, Nagymező utca 54-56, Hungary

ARTICLE INFO

Article history:
Received 17 March 2020
Accepted 13 January 2021
Available online 10 February 2021

ABSTRACT

A blocking set in a projective plane is a point set intersecting each line. The smallest blocking sets are lines. The second smallest minimal blocking sets are Baer subplanes (subplanes of order $\sqrt{q}$). Our aim is to study the stability of Baer subplanes in PG(2, q). If we delete $\sqrt{q}+1-k$ points from a Baer subplane, then the resulting set has size $q+k$ and $(\sqrt{q}+1-k)(q-\sqrt{q})$ 0-secants. If we have somewhat more 0-secants, then our main theorem says that this point set can be obtained from a Baer subplane or from a line by deleting somewhat more than $\sqrt{q}+1-k$ points and adding some points. The motivation for this theorem comes from planes of square order, but our main result is valid also for non-square orders. Hence in this case the point set contains a relatively large collinear subset.

1. Introduction

A *blocking set* is a point set intersecting each line. It is easy to see that the smallest blocking sets of projective planes are lines. A blocking set is non-trivial if it contains no line. A blocking set is *minimal*, when no proper subset of it is a blocking set. Using combinatorial arguments Bruen

☆ The authors were partially supported by OTKA, Hungary Grant K 124950. In case of the first author Project No. ED18-1-2019-0030 (Application domain specific highly reliable IT solutions subprogramme) has been implemented with the support provided from the National Research, Development and Innovation Fund of Hungary, financed under the Thematic Excellence Programme funding scheme.

E-mail addresses: szonyi@cs.elte.hu (T. Szőnyi), zsuzsa.weiner@gmail.com (Z. Weiner).

https://doi.org/10.1016/j.ejc.2021.103314
0195-6698/

proved that the smallest non-trivial blocking sets of PG(2, q) have at least $q+\sqrt{q}+1$ points. When q is a square, minimal blocking sets of this size exist; they are the points of a Baer subplane, that is a subplane of order $\sqrt{q}$.

There are lots of interesting results on blocking sets, for a survey see [1,3] and [4,14]. For a set S, a line meeting S in i points is called an i-secant. Instead of 0-secants, we sometimes use the term skew lines or external lines.

The stability question for blocking sets would mean that sets having few 0-secants can be obtained from blocking sets by deleting a relatively small number of points. Some results of this type can be found in [11] and [10]. The next theorem of Erdős and Lovász shows the stability of lines.

Theorem 1.1 (*Erdős–Lovász [6]*)**.** *If S is a set of $q+k$ points, $0 \le k \le \sqrt{q}+1$, and the number of 0-secants is less than $(\lfloor\sqrt{q}\rfloor+1-k)(q-\lfloor\sqrt{q}\rfloor)$, where $k \le \sqrt{q}+1$, then the set contains at least $q+k-\lfloor\sqrt{q}\rfloor+1$ collinear points.*

Note that for $k=\sqrt{q}+1$, there is no such set (the number of 0-secants of S is expected to be less than 0). For $k<\sqrt{q}+1$, the result is sharp for q square: deleting $\sqrt{q}+1-k$ points from a Baer subplane gives this number of 0-secants. For the proof the reader is referred to [3].

The aim of this paper is to study the stability of Baer subplanes. So we have a set that has a little more 0-secants than what is guaranteed by the Erdős–Lovász bound. Then we wish to prove that it can be obtained from a Baer subplane (or a line) by deleting and adding some points.

The exact formulation of our main result is the following.

Theorem 1.2. *Let B be a point set in PG(2, q), with cardinality $q+k$, $0 \le k \le 0.6\sqrt{q}$ and $1600 \le q$. Assume that the number of skew lines of B is less than $(q-\sqrt{q}-c)(\sqrt{q}-k+c+1)$, where $0 \le c \le 0.05\sqrt{q}-2$. Then B contains more than $q+1-(\sqrt{q}-k+c+1)$ points from a line or more than $q+\sqrt{q}+1-(\sqrt{q}-k+c+1)$ points from a Baer subplane.*

2. Preliminaries

Here we collect some results from [11], which will be used later.

Lemma 2.1 (*[11]*)**.** *Let S be a point set of size less than $2q$ in PG(2, q), $q \ge 81$, and assume that the number of external lines δ of S is less than $(q^2-q)/2$. Denote by s the number of external lines of B passing through a point P. Then $(2q+1-|S|-s)s \le \delta$.* ■

When δ is relatively small, for example $O(q\sqrt{q})$, then after solving the second order inequality in the lemma above, we get that s is either relatively small ($O(\sqrt{q})$) or it is relatively large ($q-O(\sqrt{q})$). Note that if we delete few points from a minimal blocking set, then this is exactly the case; the number of skew lines through a deleted point is $O(q)$ and small otherwise (see [2]).

Theorem 2.2. *[11] Let B be a point set in PG(2, q), $q \ge 16$, of size less than $\frac{3}{2}(q+1)$. Denote the number of 0-secants of B by δ, and assume that*

$$\delta < \min\left((q-1)\frac{2q+1-|B|}{2(|B|-q)}, \frac{1}{2}(q-\sqrt{q})^{3/2}\right). \tag{1}$$

Then B can be obtained from a blocking set by deleting at most $\frac{\delta}{2q+1-|B|}+\frac{1}{2}$ points of it.

If $|B|=q$, then δ has to be at most the second term in (1). Observe that if the size of B is less than $q+\sqrt{q}/2$, then Theorem 1.1 is stronger than Theorem 2.2. Also note that when the size of B is around $q+\sqrt{q}$, Theorem 1.1 gives almost nothing, while the theorem above still gives some reasonable bound on the number of 0−secants. The situation is similar with our main theorem for $|B|>q+0.6\sqrt{q}$, that is our main theorem is weaker in this case than Theorem 2.2.

3. The stability of Baer subplanes

The aim of this section is to prove a stability version of the Erdős–Lovász bound. That is, we show that a set of $q+k$ points having at most $(q-\sqrt{q}-c)(\sqrt{q}-k+c+1)$ skew lines either contains a relatively large collinear set or it contains $q+k-c$ points from a Baer subplane. Note that if we delete $\sqrt{q}-k+c+1$ points from a Baer subplane and add c points, then our point set B' will have at least $(q-\sqrt{q}-c)(\sqrt{q}-k+c+1)$ and at most $(q-\sqrt{q})(\sqrt{q}-k+c+1)+1$ skew lines. (In the last bound we only need that $+1$, when we delete an entire Baer subline from the Baer subplane.) Throughout this paper, we will assume that $c \geq 0$ and so from $c \leq 0.05\sqrt{q}-2$ in Theorem 3.11, it follows that $q \geq 1600$.

Note that for the point set B', the points through which there pass at least $(q-\sqrt{q}-c)$ 0-secants are exactly the points that were deleted from the Baer subplane. In the first theorem we assume that there are no such points.

Theorem 3.1. *Let B be a point set in $PG(2, q)$, $1600 \leq q$, with cardinality $q+k$, $0 \leq k \leq 0.6\sqrt{q}$. Assume that the number δ, of skew lines of B is at most $(q-\sqrt{q}-c)(\sqrt{q}-k+c+1)$, where $0 \leq c \leq 0.05\sqrt{q}-2$. Furthermore, suppose that there is no point in $PG(2, q)$, through which the number of skew lines is at least $q-\sqrt{q}-c$. Then B contains a Baer subplane or a line.*

We will prove the theorem through a sequence of lemmas. The upper bound on δ and solving the quadratic inequality in Lemma 2.1 give that the number of skew lines through a point cannot be in a certain interval.

Lemma 3.2. *The number of skew lines to B through a point is either at most $\sqrt{q}-k+c+1$ or at least $q-\sqrt{q}-c$. If the assumptions of Theorem 3.1 hold, then the latter case cannot occur.* ■

Using a similar argument we can say something about the number of lines through a point P in B, which intersect B in at least two points. This will be called *the degree of P*.

Lemma 3.3. *The degree of a point in B is either at most $\sqrt{q}+c+2$ or at least $q-\sqrt{q}-c+k-2$.*

Proof. Let P be a point in B. The number of skew lines, δ' to the point set $B \setminus \{P\}$ is at most $\delta+q$ and hence by Lemma 2.1, the number of skew lines through P to $B \setminus \{P\}$ is less than $\sqrt{q}-k+c+3$ or larger than $q-\sqrt{q}-c-1$ and so the proof follows. ■

Lemma 3.4. *There are at most $2(\sqrt{q}+c+2)$ points in B with degree at least $q-\sqrt{q}-c+k-2$. Points with such degree will be called points with large degree.*

Proof. Let $L_1, L_2, \dots, L_{q^2+q+1}$ be the lines of $PG(2, q)$ and denote by n_i the number of points of B on the line L_i. Then

$$\sum_{n_i>1}(n_i-1) = |B|(q+1)-(q^2+q+1)+\delta = k(q+1)+\delta-1. \tag{2}$$

Note that $\sum_{n_i>1} n_i$ is the sum of the degrees of the points in B, and it is $\sum_{n_i>1}(n_i-1)+\sum_{n_i>1} 1$ and so $2\sum_{n_i>1}(n_i-1)$ is an upper bound on the sum of the degrees. Hence there can be at most $2(k(q+1)+\delta-1)/(q-\sqrt{q}-c+k-2)$ points in B with degree at least $q-\sqrt{q}-c+k-2$, which is at most $2(\sqrt{q}+c+2)$. ■

The next lemma summarizes some important properties of B.

Lemma 3.5.

(i) Every line intersects B in at most $\sqrt{q}+c+2$ points or there is a line contained in B.
(ii) The intersection of any two lines, each intersecting B in more than $\frac{\sqrt{q}+c+2}{2}$ points, lies in B.

Proof. Let ℓ be a line and let P be a point of $B \setminus \ell$. Then the degree of P is at least $|\ell \cap B|$. If there is a point $P \notin \ell$, which has degree at most $\sqrt{q}+c+2$, then $|B \cap \ell| \leq \sqrt{q}+c+2$. If each $P \notin \ell$ has large degree, then by Lemma 3.4 there are at most $2(\sqrt{q}+c+2)$ such points, hence $|B \cap \ell| \geq |B| - 2(\sqrt{q}+c+2)$. Now assume that ℓ is a line and $|\ell \cap B| \geq |B| - 2(\sqrt{q}+c+2)$. We show that $\ell \subseteq B$. Suppose to the contrary that $R \in \ell \setminus B$. Then the number of 0-secants through R is at least $q - 2(\sqrt{q}+c+2)$. By Lemma 3.2, through such a point there are at least $q - \sqrt{q} - c$ 0-secants, but this contradicts the assumption of Theorem 3.1; hence we proved (i).

To prove (ii), assume to the contrary that through a point $P \notin B$ there are two lines both intersecting B in more than $\frac{\sqrt{q}+c+2}{2}$ points. Then the number of skew lines through P is more than $q+1-2-(|B|-(\sqrt{q}+c+2))$, that contradicts Lemma 3.2 and the assumption of Theorem 3.1. ■

From now on we assume that there is no line contained in B.

Lemma 3.6. *Let P be a point in B with degree at most $\sqrt{q}+c+2$. Then there are more than $0.8(\sqrt{q}+c+2)$ lines through P intersecting B in more than $\frac{\sqrt{q}+c+2}{2}$ points.*

Proof. Assume to the contrary that there is a point P in B not satisfying the lemma. Using Lemma 3.5 (i) and counting the points of B on the lines through P, we get that B has at most $0.8(\sqrt{q}+c+2)(\sqrt{q}+c+2)+(1-0.8)(\sqrt{q}+c+2)\frac{\sqrt{q}+c+2}{2}$ points (here P was counted degree of P times), which is a contradiction since $(0.8+\frac{1-0.8}{2})(\sqrt{q}+c+2)^2 < q \leq q+k$. ■

Two lines meeting B in more than $\frac{\sqrt{q}+c+2}{2}$ points intersect in a point of B, hence if we take these lines through two points of B (and disregard the line joining the two points) then we get a relatively large "grid" R' inside B. Lemma 3.10 (2) will show that such grids can be embedded in a somewhat larger subgroup grid R, which has transitive automorphism group. Finally, we will show that there can be only few points that are not in the intersection of B and the subgroup grid and so it will follow that the subgroup grid is relatively large and it is contained in B. For the construction of the subgroup grid, Kneser's theorem is needed.

Result 3.7 (*Kneser [8]*). *Let $(G,+)$ be an Abelian group, $\emptyset \neq A, B$ be finite subsets of G. Then there is a subgroup H of G such that $A+B = A+B+H$ and $|A+B| \geq |A+H|+|B+H|-|H|$.* ■

Corollary 3.8. *Let M and N be subsets of the Abelian group $(G,+)$. Assume that $|M| = |N|$ and that $|M+N| < \frac{3}{2}|M|$. Then there exists a subgroup H, so that $M+N = M+N+H$ and both M and N are contained in a coset of the subgroup H (not necessarily in the same coset of H), that is $|M+H| = |N+H| = |H|$.*

A similar result can be found in [12].

Proof. Kneser's theorem assures that there is a subgroup H of G, so that $M+N = M+N+H$ and $|M+H|+|N+H|-|H| \leq |M+N|$. As $|M| \leq |M+H|$ and $|N| \leq |N+H|$, the above inequality and the assumption that $\frac{4}{3}|M+N| < 2|M|$ imply that $\frac{1}{3}|M+N| < |H|$. Since $M+N$ is the union of some cosets of H, the above inequality implies that $M+N$ is either one coset of H or the union of two cosets. The first case immediately yields the corollary. Now assume to the contrary that $|M+N| = 2|H|$. The condition $|M+N| < \frac{3}{2}|M|$ (and $|M+N| = 2|H|$) implies, $|M| = |N| > |H|$. Hence $|M+H| \geq 2|H|$ and $|N+H| \geq 2|H|$, so Kneser's theorem gives that $|M+N| \geq 3|H|$; a contradiction. ■

Lemma 3.9. *There exist three collinear points in B such that each of them has degree at most $\sqrt{q}+c+2$. Points with such degree will be called points with small degree.*

Proof. By Lemma 3.6, through a point with small degree, there are more than $0.8(\sqrt{q}+c+2)$ relatively long lines (these are lines intersecting B in more than $(\sqrt{q}+c+2)/2$ points) and by Lemma 3.4 there are only few points in total with large degree. Hence we can easily find a line intersecting B in more than $\frac{\sqrt{q}+c+2}{2}$ points and containing almost only small degree points. More precisely, through a point T with small degree there pass more than $0.8(\sqrt{q}+c+2)$ relatively long lines and in total there are at most $2(\sqrt{q}+c+2)$ points with large degree, hence by the pigeon hole principle there is a line through T containing at most 2 points with large degree. ■

Actually, the proof above gives that the three points lie on a relatively long line. But we will not use this fact in the next lemma.

Lemma 3.10. *Let P_0, P_1 and P_2 be three collinear points from B, so that each of them has degree at most $\sqrt{q}+c+2$. Let us choose our coordinate system so that these three points are $(0,1,0)$, $(0,0,1)$ and $(0,1,-1)$ and let ℓ be the line containing them. Consider the lines through P_0 and P_1 intersecting B in more than $(\sqrt{q}+c+2)/2$ points. These two sets of lines determine the grid R'. Then*

(1) $R' \subseteq B$.
(2) R' is contained in a subgroup grid $R = \{(x, y) : x \in a+H, y \in b+H\}$, where H is a subgroup of the additive group of $\mathrm{GF}(q)$ and $a, b \in \mathrm{GF}(q)$.
(3) R' contains at least $(\lceil 0.8(\sqrt{q}+c+2)\rceil - 1)^2$ points of $R \cap B$.
(4) The subgroup grid R has a transitive automorphism group.

Proof. (1) follows from Lemma 3.5. (2): The lines through P_0 intersecting B in more than $(\sqrt{q}+c+2)/2$ points have homogeneous coordinates $[c^*, 0, 1]$, $c \in C^*$. Similarly, the lines through P_1 intersecting B in more than $(\sqrt{q}+c+2)/2$ points have homogeneous coordinates $[d^*, 1, 0]$, $d \in D^*$. The lines through P_2 intersecting B in at least 2 points have homogeneous coordinates $[e, 1, 1]$, $e \in E$. Note that $|C^*|, |D^*| \geq 0.8(\sqrt{q}+c+2) - 1$ (the line ℓ might be a relatively long one so comes the -1) and $|E| \leq \sqrt{q}+c+2$. Observe also, that the lines $[c^*, 0, 1]$, $[d^*, 1, 0]$ and $[e, 1, 1]$ are concurrent if and only if $c^* + d^* = e$. So if we take any c^* and d^*, then the lines $[c^*, 0, 1]$ and $[d^*, 1, 0]$ meet in a point Q of B. So if the line QP_2 has coordinates $[e, 1, 1]$, then e is in E and so $C^* + D^* \subseteq E$. Now let G be the additive group of $\mathrm{GF}(q)$ and assume that $|C^*| \geq |D^*|$ and so disregard some lines from C^* in order to obtain the set C with $|C| = |D^*| \geq 0.8(\sqrt{q}+c+2) - 1$, hence we can apply the corollary of Kneser's theorem (Corollary 3.8) to deduce that there exists a subgroup H of the additive group of $\mathrm{GF}(q)$, so that C and D are contained in a coset of H. So $|E| = |H|$ and hence $|H| \leq \sqrt{q}+c+2$.

(3) follows from Lemma 3.6 (and from (2)). The automorphism group that acts regularly on R is just the group $\{\alpha_{h,k} : (x, y) \mapsto (h+x, k+y) \ : h, k \in H\}$, which proves (4). ■

Proof of Theorem 3.1. Finally, we will show that the subgroup grid R of Lemma 3.10 is a Baer subplane minus one line, which is contained in B and consequently the entire Baer subplane must be contained in B.

Note that the size of the subgroup H defining R should divide q and $|H| \leq \sqrt{q}+c+2$ implying $|H| \leq \sqrt{q}$. On the other hand, by Lemma 3.10, $|R'|$ is at least $(\lceil 0.8(\sqrt{q}+c+2)\rceil - 1)^2 > q/2$, which shows that $|H| = \sqrt{q}$.

Let P_0, P_1, P_2 be three collinear points having small degree (see Lemma 3.9). Construct the grid R' and the subgroup grid R containing it as in Lemma 3.10. First we will show that there are few points in $B \setminus R$ and not on the line ℓ containing P_0, P_1, P_2. Let P be a point in $B \setminus (R \cup \ell)$. Applying the automorphisms $\{\alpha_{h,k} : h, k \in H\}$ of R (see the proof of Lemma 3.10), shows that the orbit of P under the automorphism group of R has size at least $|R|$. Hence there must be a point Q in the orbit of P that is not in B, otherwise B would have at least $|R|+|R\cap B| \geq 2|R \cap B| > 2(\lceil 0.8(\sqrt{q}+c+2)\rceil - 1)^2 \geq 2(\lceil 0.8(\sqrt{q}+1)\rceil - 1)^2$ points, that is larger than $q+k$. It follows from Lemma 3.2, that through Q there are at least $q - \sqrt{q} + k - c$ lines intersecting B. Let x denote the number of lines through Q which intersect B in at least 2 points. Note that $x \leq |B| - (q - \sqrt{q} + k - c)$. Hence the total number of points on the lines passing through Q and having more than one point of B is at most $x + |B| - (q - \sqrt{q} + k - c)$. This shows that there are at least $|B \cap R| - (x + |B| - (q - \sqrt{q} + k - c))$ 1-secants of $B \cap R$ through Q. So the total number of lines through Q intersecting $B \cap R$ in at least one point is at least $|B \cap R| - (x + |B| - (q - \sqrt{q} + k - c)) + x$, which is $|B \cap R| - (|B| - (q - \sqrt{q} + k - c)) = |B \cap R| - (\sqrt{q} + c)$. This means that also through P there must be at least this many lines intersecting R. Hence at least

$(|B \cap R| - (\sqrt{q}+c)) - (|R| - |B \cap R|) = 2|B \cap R| - q - (\sqrt{q}+c)$ lines through P contain a point of $B \cap R$. So the degree of P is at least $2|B \cap R| - q - (\sqrt{q}+c)$. Using $|B \cap R| > (\lceil 0.8(\sqrt{q}+c+2)\rceil - 1)^2$, it can be easily checked that it is larger than $\sqrt{q}+c+2$ when $q \geq 1600$. So, by Lemma 3.3, it must be at least $q - \sqrt{q} - c + k$. By Lemma 3.4, there are at most $2(\sqrt{q}+c+2)$ such points, which gives $|B \setminus R| \leq 2(\sqrt{q}+c+2)$.

For simplicity, let ℓ be the line at infinity and consider the directions determined by the grid R (the points of ℓ that lie on lines joining two points of the grid R). Recall that, by Lemma 3.10, $R = \{(x, y) : x \in a + H, y \in b + H\}$.

If $P', P'' \in R$ and τ is a translation by the vector (h, k) then $\tau(P'), \tau(P'') \in R$ and they determine the same direction as P', P''. This shows that the lines of one parallel class are either 0-secants to R or they intersect R in the same number of points. So through a determined direction (hence on each line we see at least 2 points) there can be at most $q/2$ lines intersecting R. If such a determined infinite point was not in B, then the number of 0-secants of B through it would be at least $q/2 - |B \setminus R| \leq q/2 - 2(\sqrt{q}+c+2)$. By Lemma 3.2, through such a point there would pass at least $q - \sqrt{q} - c$ 0-secants of B, but as there are no such points by the assumption of Theorem 3.1, determined directions should be in B. Hence R determines at most $\sqrt{q}+c+2$ directions. It is well-known and also easy to prove (see beginning of [1, Section 3] or [9, Section 2.1]) that R together with the directions determined by R forms a blocking set. By Sziklai [13, Corollary 5.1], R together with its directions determined is either a Baer subplane or a line or R determines at least $1 + q^{1/3}\lceil \frac{q^{2/3}+1}{q^{1/3}+1} \rceil$ directions. It is easy to check that $\sqrt{q}+c+2 < 1 + q^{1/3}\lceil \frac{q^{2/3}+1}{q^{1/3}+1} \rceil$ and as R is a $\sqrt{q} \times \sqrt{q}$ grid, it must be an affine Baer subplane.

If there is a point in $R \setminus B$, then there are $\sqrt{q}+1$ lines through it which intersect R. We have seen that $|B \setminus R| \leq 2(\sqrt{q}+c+2)$, so through a point of $R \setminus B$ there would pass at least $q - \sqrt{q} - 2(\sqrt{q}+c+2)$ 0-secants of B. A point with so many 0-secants does not exist by Lemma 3.2. So together with the points of B on ℓ, we see that B contains a Baer subplane. ■

Instead of the result by Sziklai we could have used some results of Rédei [7], but the newer results are sharper and so they are easier to use.

Theorem 3.11. *Let B be a point set in $PG(2, q)$, $1600 \leq q$, with cardinality $q + k$, $0 \leq k \leq 0.6\sqrt{q}$. Assume that the number, δ of skew lines of B is less than $(q - \sqrt{q} - c)(\sqrt{q} - k + c + 1)$, where $0 \leq c \leq 0.05\sqrt{q} - 2$. Then B contains more than $q + 1 - (\sqrt{q} - k + c + 1)$ points from a line or more than $q + \sqrt{q} + 1 - (\sqrt{q} - k + c + 1)$ points from a Baer subplane.*

Proof. Lemma 3.2 implies that the number of skew lines through a point not in B is either at most $\sqrt{q} - k + c + 1$ or at least $q - \sqrt{q} - c$. Let A be the set of points with at least $q - \sqrt{q} - c$ skew lines through it. Let $\delta(k) := (q - \sqrt{q} - c)(\sqrt{q} - k + c + 1)$. Let us add a point P of A to B. Then $B \cup \{P\}$ has $q + k'$ points, where $k' = k + 1$ and the number of skew lines is at most $\delta(k')$. So again Lemma 3.2 implies that the number of skew lines through a point not in B is either at most $\sqrt{q} - k' + c + 1$ or at least $q - \sqrt{q} - c$. Thus we can continue adding the points of A one by one to B. When k' reaches $0.6\sqrt{q}$, denote the resulting set by B'. Then by Theorem 2.2, this point set can be obtained from a blocking set by deleting at most $\frac{\delta(k')}{2q+1-|B'|} + \frac{1}{2} < \frac{\delta(k')}{2q+1-(q+0.6\sqrt{q}+1)} + \frac{1}{2} < 0.5\sqrt{q}$ points. Note that the size of this blocking set is less than $q + 1 + q^{1/3}\lceil \frac{q^{2/3}+1}{q^{1/3}+1} \rceil$, when $q > 1600$; hence by Sziklai Corollary 5.1 [13], each line intersects it in 1 mod $\sqrt{q}$ points and so by Bruen [5] this blocking set is either a Baer subplane or a line. If k' does not reach $0.6\sqrt{q}$ at all, then applying Theorem 3.1 for the set $B \cup A$, we get that it contains a line or a Baer subplane. Hence in both cases there is a set A', such that $B \cup A'$ is either a Baer subplane or a line. Note that if we delete $\sqrt{q} - k + x + 1$ points from a Baer subplane and add x points outside of this Baer subplane, then the number of skew lines is at least $(q - \sqrt{q} - x)(\sqrt{q} - k + x + 1)$. Hence $x < c$. Similarly, if we delete x points from a line and add $k - x + 1$ points, then the number of skew lines will be at least $(q - k + 1 - x)x$. Hence $x < \sqrt{q} - k + c + 1$. ■

Remark 3.12. We may extend our main result to negative k, that is for sets with size less than q. To prove Lemma 3.6 we need that

$$(0.8 + \frac{1-0.8}{2})(\sqrt{q} + c + 2)^2 < q + k \tag{3}$$

So for example if $c < 0.02\sqrt{q} - 2$ and $-0.06q < k$, then the previous inequality remains true. As c has to be at least 0, this means that $q > 10000$.

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