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MATHEMATISCHE NACHRICHTEN

Convergence of Vilenkin–Fourier series in variable Hardy spaces

Ferenc Weisz

Department of Numerical Analysis, Eötvös L. University, Pázmány P. sétány 1/C., Budapest H-1117, Hungary

Correspondence
Ferenc Weisz, Department of Numerical Analysis, Eötvös L. University, H-1117 Budapest, Pázmány P. sétány 1/C., Hungary.
Email: weisz@inf.elte.hu

Abstract

Let $p(\cdot) : [0, 1) \to (0, \infty)$ be a variable exponent function satisfying the log-Hölder condition and $0 < q \leq \infty$. We introduce the variable Hardy and Hardy–Lorentz spaces $H_{p(\cdot)}$ and $H_{p(\cdot),q}$ containing Vilenkin martingales. We prove that the partial sums of the Vilenkin–Fourier series converge to the original function in the $L_{p(\cdot)}$- and $L_{p(\cdot),q}$-norm if $1 < p_- < \infty$. We generalize this result for smaller $p(\cdot)$ as well. We show that the maximal operator of the Fejér means of the Vilenkin–Fourier series is bounded from $H_{p(\cdot)}$ to $L_{p(\cdot)}$ and from $H_{p(\cdot),q}$ to $L_{p(\cdot),q}$ if $1/2 < p_- < \infty$, $0 < q \leq \infty$ and $1/p_- - 1/p_+ < 1$. This last condition is surprising because the corresponding results for Fourier series or Fourier transforms hold without this condition. This implies some norm and almost everywhere convergence results for the Fejér means of the Vilenkin–Fourier series.

KEYWORDS
atomic decomposition, Fejér means, martingale Hardy and Hardy–Lorentz spaces, maximal Fejér operator variable exponent, Vilenkin–Fourier series, Vilenkin martingales

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1 | INTRODUCTION

It was proved by Lebesgue [22] that the Fejér means [7] of the trigonometric Fourier series of a one-dimensional integrable function converge almost everywhere to the function. Fine [8] extended this result to Walsh–Fourier series. Schipp [31] proved that the maximal operator σ_* of the Fejér means of Walsh–Fourier series is of weak type $(1, 1)$, that implies also the almost everywhere convergence. For Vilenkin–Fourier series, the weak type $(1, 1)$ inequality is due to Pál and Simon [28] (see also Weisz [41]). Fujii [9] and Simon [33] verified that σ_* is bounded from the Hardy space H_1 to L_1. Later the author [39, 40] generalized this result for both Walsh- and Vilenkin–Fourier series and proved the boundedness of σ_* from H_p to L_p for $1/2 < p \leq \infty$. Summability of Walsh- and Vilenkin–Fourier series is studied intensively in the literature (see, e.g., Gát [11, 12], Goginava [13–15], Persson, Tephnadze and Wall [26, 29], Simon [34–36] and Weisz [18, 38, 40, 42]).

In this paper, we generalize these theorems to Hardy and Lebesgue spaces with variable exponents. For a measurable function $p(\cdot)$, the variable Lebesgue space $L_{p(\cdot)}$ consists of all measurable functions f for which $\int_0^1 |f(x)|^{p(x)}\, dx < \infty$.

When $p(\cdot)$ is a constant, then we get back the usual L_p space. This topic needs essentially new ideas and is investigated very intensively in the literature nowadays (see, e.g., Cruz-Uribe and Fiorenza [5], Diening et al. [6], Kempka and Vybíral [21], Nakai and Sawano [27, 30], Jiao et al. [18–20], Yan et al. [47] and Liu et al. [23, 24]). Interest in the variable Lebesgue spaces has increased since the 1990s because of their use in a variety of applications (see the references in Jiao et al. [18]).

The log-Hölder continuity condition is a very common condition in this theory (see, e.g., Cruz-Uribe and Fiorenza [5] and Diening et al. [6]). Under this condition, the Hardy–Littlewood maximal operator is bounded on $L_{p(\cdot)}(\mathbb{R})$ $(p_- > 1)$. Nakai and Sawano [27] first introduced the Hardy space $H_{p(\cdot)}(\mathbb{R})$ with a variable exponent $p(\cdot)$ and established the atomic decompositions. Independently, Cruz-Uribe and Wang [4] also investigated the variable Hardy space $H_{p(\cdot)}(\mathbb{R})$. Sawano [30] improved the results in [27]. Ho [17] studied weighted Hardy spaces with variable exponents. Recently, Yan et al. [47] introduced the variable weak Hardy space $H_{p(\cdot),\infty}(\mathbb{R})$ and characterized these spaces via radial maximal functions, atoms and Littlewood–Paley functions. The Hardy–Lorentz spaces $H_{p(\cdot),q}(\mathbb{R})$ were investigated by Jiao et al. in [20]. Similar results for the anisotropic Hardy spaces $H_{p(\cdot)}(\mathbb{R})$ and $H_{p(\cdot),q}(\mathbb{R})$ can be found in Liu et al. [23, 24]. Martingale Musielak–Orlicz Hardy spaces were investigated in Xie et al. [44–46]. Very recently, these results were generalized for martingale Hardy spaces with variable exponent in Jiao et al. [18]. We generalized there the boundedness of σ_* from $H_{p(\cdot)}$ to $L_{p(\cdot)}$ for Walsh–Fourier series.

We introduce the variable Hardy and Hardy–Lorentz spaces $H_{p(\cdot)}$ and $H_{p(\cdot),q}$ containing Vilenkin martingales. We give the atomic decomposition of these two types of Hardy spaces. In this paper, we generalize the preceding theorems for Vilenkin–Fourier series. We calculate the partial sums in a new way and, with the help of this, we verify the convergence of the partial sums of the Vilenkin–Fourier series in the $L_{p(\cdot)}$- and $L_{p(\cdot),q}$-norm if $1 < p_- < \infty$. We prove also that σ_* is bounded from $H_{p(\cdot)}$ to $L_{p(\cdot)}$ and from $H_{p(\cdot),q}$ to $L_{p(\cdot),q}$ under the conditions $1/2 < p_- < \infty$, $0 < q \leq \infty$ and $1/p_- - 1/p_+ < 1$. It is also shown that these conditions are sharp. This last condition is surprising because the corresponding results for Fourier series or Fourier transforms hold without this condition (see Liu et al. [23, 24] and Weisz [43]). This gives a serious difference between the trigonometric Fourier analysis and Vilenkin–Fourier analysis. The proofs of these results need essentially new ideas. Finally, the boundedness of σ_* implies almost everywhere and norm convergence of the Fejér means as well.

2 | VARIABLE LEBESGUE AND LORENTZ SPACES

In this section, we recall some basic notations on variable Lebesgue spaces and variable Lorentz spaces and give some elementary and necessary facts about these spaces. Our main references are Cruz-Uribe and Fiorenza [5] and Diening et al. [6].

For a constant p, the L_p space is equipped with the quasi-norm

$$\|f\|_p := \left(\int_0^1 |f(x)|^p \, dx \right)^{1/p} \qquad (0 < p < \infty),$$

with the usual modification for $p = \infty$. Here we integrate with respect to the Lebesgue measure λ.

We are going to generalize these spaces. A measurable function $p(\cdot) : [0,1) \to (0,\infty)$ is called a *variable exponent*. For any variable exponent $p(\cdot)$ and any measurable set $A \subset [0,1)$, we will use the notation

$$p_-(A) := \operatorname*{ess\,inf}_{x\in A} p(x) \qquad \text{and} \qquad p_+(A) := \operatorname*{ess\,sup}_{x\in A} p(x).$$

If $A = [0,1)$, then the numbers $p_-(A)$ and $p_+(A)$ are denoted simply by p_- and p_+. Denote by $\mathcal{P}$ the *collection of all variable exponents*$p(\cdot)$ satisfying

$$0 < p_- \leq p_+ < \infty.$$

In what follows, we use the symbol

$$\underline{p} = \min\{p_-, 1\}.$$

For $p(\cdot) \in \mathcal{P}$ and a measurable function f, the *modular functional* $\varrho_{p(\cdot)}$ is defined by

$$\varrho_{p(\cdot)}(f) := \int_0^1 |f(x)|^{p(x)}\, dx$$

and the *Luxemburg quasi-norm* is given by setting

$$\|f\|_{p(\cdot)} := \inf\{\rho \in (0,\infty) : \varrho_{p(\cdot)}(f/\rho) \leq 1\}.$$

The *variable Lebesgue space* $L_{p(\cdot)}$ is defined to be the set of all measurable functions f such that $\varrho_{p(\cdot)}(f) < \infty$ and equipped with the quasi-norm $\|\cdot\|_{p(\cdot)}$. It is easy to see that if $p(\cdot)$ is a constant, then we get back the L_p spaces. It is known that $\|\rho f\|_{p(\cdot)} = |\rho|\|f\|_{p(\cdot)}$,

$$\||f|^s\|_{p(\cdot)} = \|f\|_{sp(\cdot)}^s$$

and

$$\|f+g\|_{p(\cdot)}^{\underline{p}} \leq \|f\|_{p(\cdot)}^{\underline{p}} + \|g\|_{p(\cdot)}^{\underline{p}},$$

where $p(\cdot) \in \mathcal{P}$, $s \in (0,\infty)$, $\rho \in \mathbb{C}$ and $f, g \in L_{p(\cdot)}$. Details can be found in the monographs Cruz-Uribe and Fiorenza [5] and Diening et al. [6]. The function $p'(\cdot)$ denotes the conjugate exponent function of $p(\cdot)$, i.e., $1/p(x) + 1/p'(x) = 1$ ($x \in [0,1)$). The well known Hölder's inequality can be generalized for variable Lebesgue spaces (see Cruz-Uribe and Fiorenza [5, p. 27] or Diening et al. [6, p. 74]).

Lemma 2.1. *Let $p(\cdot) \in \mathcal{P}$ with $p_- \geq 1$. For all $f \in L_{p(\cdot)}$ and $g \in L_{p'(\cdot)}$,*

$$\int_0^1 |fg|\, d\lambda \leq C_{p(\cdot)} \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)}.$$

In this paper the constants C are absolute constants and the constants $C_{p(\cdot)}$ are depending only on $p(\cdot)$ and may denote different constants in different contexts. For two positive numbers A and B, we use also the notation $A \lesssim B$, which means that there exists a constant C such that $A \leq CB$. The next lemma can be found, e.g., in Diening et al. [6, p. 77].

Lemma 2.2. *Let $p(\cdot) \in \mathcal{P}$ with $p_- \geq 1$. Then*

$$\frac{1}{2}\|f\|_{p(\cdot)} \leq \sup_{\|g\|_{p'(\cdot)} \leq 1} \int_0^1 |fg|\, d\lambda \leq 2\|f\|_{p(\cdot)}.$$

We denote by $C^{\log}$ the set of all functions $p(\cdot) \in \mathcal{P}$ satisfying the so-called *log-Hölder continuous condition*, namely, there exists a positive constant $C_{\log}(p)$ such that, for any $x, y \in [0,1)$,

$$|p(x) - p(y)| \leq \frac{C_{\log}(p)}{\log(e + 1/|x-y|)}. \tag{2.1}$$

The following two lemmas were proved in Cruze-Uribe and Fiorenza [5].

Lemma 2.3. *We have that $p(\cdot) \in C^{\log}$ if and only if for all intervals $I \subset [0,1)$,*

$$\lambda(I)^{p_-(I) - p_+(I)} \leq C. \tag{2.2}$$

Remark 2.4. There exist a lot of functions $p(\cdot)$ satisfying (2.1). For concrete examples we mention the function $a + cx$ for parameters a and c such that the function is positive ($x \in [0,1)$). All positive Lipschitz functions with order $0 < \alpha \leq 1$ also satisfy (2.1) and (2.2).

Lemma 2.5. *If $p(\cdot) \in C^{\log}$, then, for any interval $I \subset [0,1)$,*

$$\lambda(I)^{1/p_-(I)} \sim \lambda(I)^{1/p(x)} \sim \lambda(I)^{1/p_+(I)} \sim \|\chi_I\|_{p(\cdot)} \qquad (\forall x \in I),$$

where $\sim$ denotes the equivalence of the numbers.

The following lemma can be found in Jiao et al. [19] (see also [18]).

Lemma 2.6. *Let $p(\cdot) \in C^{\log}$ with $1 \leq p_- \leq p_+ < \infty$. Suppose that $f \in L_{p(\cdot)}$ with $\|f\|_{p(\cdot)} \leq 1/2$ and $f = f\chi_{\{|f|\geq 1\}}$. Then, for any interval $I \subset [0,1)$ and $x \in I$,*

$$\left(\frac{1}{\lambda(I)}\int_I |f(t)|\,dt\right)^{p(x)} \leq \left(\frac{C}{\lambda(I)}\int_I |f(t)|^{p(t)}\,dt\right).$$

The *variable Lorentz spaces* were introduced and investigated by Kempka and Vybíral [21]. $L_{p(\cdot),q}$ is defined to be the space of all measurable functions f such that

$$\|f\|_{p(\cdot),q} := \begin{cases} \left(\int_0^\infty \rho^q \left\|\chi_{\{x\in[0,1):\ |f(x)|>\rho\}}\right\|_{p(\cdot)}^q \frac{d\rho}{\rho}\right)^{1/q}, & \text{if } 0 < q < \infty, \\ \sup_{\rho\in(0,\infty)} \rho \left\|\chi_{\{x\in[0,1):\ |f(x)|>\rho\}}\right\|_{p(\cdot)}, & \text{if } q = \infty, \end{cases}$$

is finite. If $p(\cdot)$ is a constant, we get back the classical Lorentz spaces (see Lorentz [25] or Bergh and Löfström [1].

3 | VARIABLE MARTINGALE HARDY SPACES

Let $(p_n, n \in \mathbb{N})$ be a sequence of natural numbers with entries at least 2. We always suppose that the sequence (p_n) is bounded. Introduce the notations $P_0 = 1$ and

$$P_{n+1} := \prod_{k=0}^{n} p_k \qquad (n \in \mathbb{N}).$$

Let $\mathcal{F}_n$ be the σ-algebra

$$\mathcal{F}_n = \sigma\left\{\left[kP_n^{-1}, (k+1)P_n^{-1}\right) \ : 0 \leq k < P_n\right\},$$

where $\sigma(\mathcal{H})$ denotes the σ-algebra generated by an arbitrary set system $\mathcal{H}$. By a Vilenkin interval we mean one of the form $\left[kP_n^{-1}, (k+1)P_n^{-1}\right)$ for some $k, n \in \mathbb{N}, 0 \leq k < P_n$. The conditional expectation operators relative to $\mathcal{F}_n$ are denoted by E_n. An integrable sequence $f = (f_n)_{n\in\mathbb{N}}$ is said to be a martingale if f_n is $\mathcal{F}_n$-measurable for all $n \in \mathbb{N}$ and $E_n f_m = f_n$ in case $n \leq m$. Martingales with respect to $\left(\mathcal{F}_n, n \in \mathbb{N}\right)$ are called Vilenkin martingales. It is easy to show (see, e.g., Weisz [38]) that the sequence $\left(\mathcal{F}_n, n \in \mathbb{N}\right)$ is regular, i.e., $f_n \leq R f_{n-1}$ for all nonnegative Vilenkin martingales.

For a Vilenkin martingale $f = (f_n)_{n\in\mathbb{N}}$, the maximal function is defined by

$$M(f) := \sup_{n\in\mathbb{N}} |f_n|.$$

Now we can define the *variable martingale Hardy spaces* by

$$H_{p(\cdot)} := \left\{ f = (f_n)_{n\in\mathbb{N}} : \|f\|_{H_{p(\cdot)}} := \|M(f)\|_{p(\cdot)} < \infty \right\}.$$

The *variable martingale Hardy–Lorentz spaces* can be defined similarly:

$$H_{p(\cdot),q} := \left\{ f = (f_n)_{n\in\mathbb{N}} : \|f\|_{H_{p(\cdot),q}} := \|M(f)\|_{p(\cdot),q} < \infty \right\}.$$

The Hardy spaces can also be defined via equivalent norms. For a martingale $f = \left(f_n\right)_{n\geq 0}$ let

$$d_n f = f_n - f_{n-1} \qquad (n \geq 0)$$

denote the martingale differences, where $f_{-1} := 0$. The square function and the conditional square function of f are defined by

$$S(f) = \left(\sum_{n=0}^{\infty} |d_n f|^2 \right)^{1/2}, \qquad s(f) = \left(|d_0 f|^2 + \sum_{n=0}^{\infty} E_n |d_{n+1} f|^2 \right)^{1/2}.$$

We have shown the following theorem in [18].

Theorem 3.1. *Let $p(\cdot) \in C^{\log}$ and $0 < q \leq \infty$. Then*

$$\|M(f)\|_{p(\cdot)} \sim \|S(f)\|_{p(\cdot)} \sim \|s(f)\|_{p(\cdot)} \tag{3.1}$$

and

$$\|M(f)\|_{p(\cdot),q} \sim \|S(f)\|_{p(\cdot),q} \sim \|s(f)\|_{p(\cdot),q}. \tag{3.2}$$

If in addition $1 < p_- \leq p_+ < \infty$, then

$$\|M(f)\|_{p(\cdot)} \sim \|f\|_{p(\cdot)} \qquad \text{and} \qquad \|M(f)\|_{p(\cdot),q} \sim \|f\|_{p(\cdot),q}. \tag{3.3}$$

The atomic decomposition is a useful characterization of the Hardy spaces. A measurable function a is called a $p(\cdot)$-atom if there exists a stopping time τ such that

(a) $E_n(a) = 0$ for all $n \leq \tau$,
(b) $\|M(a)\|_\infty \leq \left\|\chi_{\{\tau<\infty\}}\right\|_{p(\cdot)}^{-1}$.

The atomic decomposition of the spaces $H_{p(\cdot)}$ and $H_{p(\cdot),q}$ were proved in Jiao et al. [18, 19]. The classical case can be found in [38, 42].

Theorem 3.2. *Let $p(\cdot) \in C^{\log}$ and let $0 < q \leq \infty$. Then the martingale $f = (f_n)_{n\in\mathbb{N}} \in H_{p(\cdot)}$ or $f = (f_n)_{n\in\mathbb{N}} \in H_{p(\cdot),q}$, respectively, if and only if there exists a sequence $\left(a^k\right)_{k\in\mathbb{Z}}$ of $p(\cdot)$-atoms such that for every $n \in \mathbb{N}$,*

$$f_n = \sum_{k\in\mathbb{Z}} \mu_k E_n a^k \qquad \text{almost everywhere,} \tag{3.4}$$

where $\mu_k = 3 \cdot 2^k \|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}$ *and* τ_k *is the stopping time associated with the* $p(\cdot)$*-atom* a^k*. Moreover,*

$$\|f\|_{H_{p(\cdot)}} \sim \inf \left\| \left(\sum_{k\in\mathbb{Z}} \left(\frac{\mu_k \chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}} \right)^t \right)^{1/t} \right\|_{p(\cdot)},$$

$$\|f\|_{H_{p(\cdot),q}} \sim \inf \left(\sum_{k\in\mathbb{Z}} 2^{kq} \|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}^q \right)^{1/q},$$

respectively, where $0 < t \le \underline{p}$ *is fixed and the infimum is taken over all decompositions of the form (3.4).*

4 | PARTIAL SUMS OF VILENKIN–FOURIER SERIES

Every point $x \in [0,1)$ can be written in the following way:

$$x = \sum_{k=0}^{\infty} \frac{x_k}{P_{k+1}} \qquad \left(0 \le x_k < p_k,\ x_k \in \mathbb{N}\right).$$

If there are two different forms, choose the one for which $\lim_{k\to\infty} x_k = 0$. The functions

$$r_n(x) := \exp \frac{2\pi \imath x_n}{p_n} \qquad (n \in \mathbb{N})$$

are called generalized Rademacher functions, where $\imath = \sqrt{-1}$. The functions corresponding to the sequence $(2, 2, \ldots)$ are called Rademacher functions.

The product system generated by the generalized Rademacher functions is the Vilenkin system:

$$w_n(x) := \prod_{k=0}^{\infty} r_k(x)^{n_k}$$

where

$$n = \sum_{k=0}^{\infty} n_k P_k \qquad \left(0 \le n_k < p_k\right).$$

The product system corresponding to the Rademacher functions is called Walsh system (see Vilenkin [37] or Schipp, Wade, Simon and Pál [32]).

Recall (see Pál and Simon [28] and Gát [10]) that the Vilenkin–Dirichlet kernels

$$D_n := \sum_{k=0}^{n-1} w_k$$

satisfy

$$D_{P_n}(x) = \begin{cases} P_n, & \text{if } x \in \left[0, P_n^{-1}\right), \\ 0, & \text{if } x \in \left[P_n^{-1}, 1\right), \end{cases} \qquad (n \in \mathbb{N}). \tag{4.1}$$

If $f \in L_1$, then the number

$$\widehat{f}(n) := \int_0^1 f\,\overline{w}_n\,d\lambda \qquad (n \in \mathbb{N})$$

is said to be the nth Vilenkin–Fourier coefficient of f. We can extend this definition to martingales as follows. If $f = (f_k)_{k\geq 0}$ is a martingale, then let

$$\widehat{f}(n) := \lim_{k\to\infty} \int_0^1 f_k\overline{w}_n\,d\lambda \qquad (n \in \mathbb{N}).$$

Since w_n is $\mathcal{F}_k$ measurable for $n < P_k$, it can immediately be seen that this limit does exist. We remember that if $f \in L_1$, then $E_k f \to f$ in the L_1-norm as $k \to \infty$, hence

$$\widehat{f}(n) = \lim_{k\to\infty} \int_0^1 (E_k f)\overline{w}_n\,d\lambda \qquad (n \in \mathbb{N}).$$

Thus the Vilenkin–Fourier coefficients of $f \in L_1$ are the same as the ones of the martingale $(E_k f)_{k\geq 0}$ obtained from f.

Denote by $s_n f$ the nth partial sum of the Vilenkin–Fourier series of a martingale f, namely,

$$s_n f := \sum_{k=0}^{n-1} \widehat{f}(k) w_k.$$

If $f \in L_1$, then

$$s_n f(x) = \int_0^1 f(t) D_n(x \dot{-} t)\,dt \qquad (n \in \mathbb{N}),$$

where $\dot{+}$ denotes the addition on the corresponding Vilenkin group and let $\dot{-}$ be its inverse (see Vilenkin [37] or Pál and Simon [28]). It is easy to see that

$$s_{P_n} f = f_n \qquad (n \in \mathbb{N})$$

and so, by martingale results,

$$\lim_{n\to\infty} s_{P_n} f = f \qquad \text{in the } L_p\text{-norm} \tag{4.2}$$

when $f \in L_p$ and $1 \leq p < \infty$. This theorem was extended in Pál and Simon [28] (for Walsh–Fourier series see Schipp, Wade, Simon and Pál [32] or Golubov, Efimov and Skvortsov [16]) for the partial sums $s_n f$ and for $1 < p < \infty$. More exactly,

$$\lim_{n\to\infty} s_n f = f \qquad \text{in the } L_p\text{-norm}$$

when $f \in L_p$ and $1 < p < \infty$. We generalize this theorem as follows.

Theorem 4.1. *Let $p(\cdot) \in C^{\log}$ with $1 < p_- \leq p_+ < \infty$. If $f \in L_{p(\cdot)}$, then*

$$\sup_{n\in\mathbb{N}} \|s_n f\|_{p(\cdot)} \lesssim \|f\|_{p(\cdot)}.$$

Proof. For $k, n \in \mathbb{N}$, let

$$I_k^n := \left[kP_n, (k+1)P_n\right) \cap \mathbb{N}$$

and

$$s_{I_k^n} f := \sum_{j \in I_k^n} \widehat{f}(j) w_j.$$

For simplicity, we suppose that $\widehat{f}(0) = 0$. It is easy to see that

$$\begin{aligned} s_{I_k^n} f &= \sum_{j \in I_k^n} \left(\int_0^1 f\, \overline{w}_j \, d\lambda \right) w_j \\ &= \sum_{i=0}^{P_n - 1} \left(\int_0^1 (f\, \overline{w}_{kP_n}) \overline{w}_i \, d\lambda \right) w_{kP_n} w_i \\ &= w_{kP_n} s_{P_n}\left(f\, \overline{w}_{kP_n}\right) = w_{kP_n} E_n\left(f\, \overline{w}_{kP_n}\right). \end{aligned}$$

It can be proved in the same way that, for an arbitrary $m \in I_k^n$,

$$s_{I_k^n} f = w_m E_n\left(f\, \overline{w}_m\right). \tag{4.3}$$

For

$$m = \sum_{k=0}^{\infty} m_k P_k \qquad \left(0 \le m_k < p_k\right),$$

we introduce

$$m(n) := \sum_{k=n}^{\infty} m_k P_k, \qquad I_n(m) := \left[m(n), m(n) + P_n\right) \qquad (n \in \mathbb{N}).$$

Notice that m is contained in $I_n(m)$ and $I_n(m) \subset I_{n+1}(m)$. Observe that

$$[0, m) = \bigcup_{n=0}^{\infty} \left[m(n+1), m(n)\right),$$

which implies

$$s_m f = \sum_{n=0}^{\infty} \sum_{k \in [m(n+1), m(n))} \widehat{f}(k) w_k. \tag{4.4}$$

Let

$$g_{m,n} := \sum_{k \in [m(n+1), m(n+1) + P_{n+1})} \widehat{f}(k) w_k - \sum_{k \in [m(n), m(n) + P_n)} \widehat{f}(k) w_k.$$

By (4.3),

$$g_{m,n} = w_m E_{n+1}\left(f\, \overline{w}_m\right) - w_m E_n\left(f\, \overline{w}_m\right) = w_m d_{n+1}\left(f\, \overline{w}_m\right). \tag{4.5}$$

Since

$$[m(n+1), m(n)) \subset \big[m(n+1), m(n+1)+P_{n+1}\big) \setminus \big[m(n), m(n)+P_n\big),$$

we have

$$\sum_{k\in[m(n+1),m(n))} \widehat{f}(k)w_k = \sum_{k\in[m(n+1),m(n))} \widehat{g_{m,n}}(k)w_k = \sum_{l=0}^{m_n-1} \sum_{k=m(n+1)+lP_n}^{m(n+1)+(l+1)P_n-1} \widehat{g_{m,n}}(k)w_k.$$

It is easy to see that

$$m-\big(m_n-l\big)P_n \in \big[m(n+1)+lP_n, m(n+1)+(l+1)P_n\big) \qquad \big(0\le l<m_n\big),$$

which implies by (4.3) that

$$\begin{aligned}\sum_{k\in[m(n+1),m(n))} \widehat{f}(k)w_k &= \sum_{l=0}^{m_n-1} w_{m-(m_n-l)P_n} E_n\big(g_{m,n}\overline{w}_{m-(m_n-l)P_n}\big)\\ &= w_m \sum_{l=0}^{m_n-1} \overline{w}_{(m_n-l)P_n} E_n\big(\big(g_{m,n}\overline{w}_m\big)w_{(m_n-l)P_n}\big)\\ &= w_m \sum_{l=0}^{m_n-1} \overline{r}_n^{m_n-l} E_n\Big(\big(g_{m,n}\overline{w}_m\big)r_n^{m_n-l}\Big).\end{aligned}$$

Taking into accout (4.4) and (4.5), we conclude that

$$\begin{aligned}s_m f &= \sum_{n=0}^{\infty} w_m \sum_{l=0}^{m_n-1} \overline{r}_n^{m_n-l} E_n\Big(\big(g_{m,n}\overline{w}_m\big)r_n^{m_n-l}\Big)\\ &= w_m \sum_{n=0}^{\infty}\sum_{l=0}^{m_n-1} \overline{r}_n^{m_n-l} E_n\Big(d_{n+1}\big(f\,\overline{w}_m\big)r_n^{m_n-l}\Big).\end{aligned}$$

Since r_n is $\mathcal{F}_{n+1}$ measurable and $E_n\big(r_n^i\big)=0$ for $i=1,\dots,p_n-1$, we can see that

$$E_n\left(\sum_{l=0}^{m_n-1} \overline{r}_n^{m_n-l} E_n\Big(d_{n+1}\big(f\,\overline{w}_m\big)r_n^{m_n-l}\Big)\right)=0,$$

hence

$$\left(\sum_{l=0}^{m_n-1} \overline{r}_n^{m_n-l} E_n\Big(d_{n+1}\big(f\,\overline{w}_m\big)r_n^{m_n-l}\Big)\right)_{n\in\mathbb{N}}$$

is a martingale difference sequence. Then, by (3.1) and (3.2),

$$\begin{aligned}\|s_m f\|_{p(\cdot)} = \big\|\overline{w}_m s_m f\big\|_p &= \left\|\sum_{n=0}^{\infty}\sum_{l=0}^{m_n-1} \overline{r}_n^{m_n-l} E_n\Big(d_{n+1}\big(f\,\overline{w}_m\big)r_n^{m_n-l}\Big)\right\|_{p(\cdot)}\\ &\lesssim \left\|\left(\sum_{n=0}^{\infty}\left|\sum_{l=0}^{m_n-1} \overline{r}_n^{m_n-l} E_n\Big(d_{n+1}\big(f\,\overline{w}_m\big)r_n^{m_n-l}\Big)\right|^2\right)^{1/2}\right\|_{p(\cdot)}\end{aligned}$$

$$\lesssim \left\| \left(\sum_{n=0}^{\infty} \sum_{l=0}^{m_n-1} \left| E_n \left(d_{n+1} (f \overline{w}_m) r_n^{m_n-l} \right) \right|^2 \right)^{1/2} \right\|_{p(\cdot)}$$

$$\lesssim \left\| \left(\sum_{n=0}^{\infty} E_n \left| d_{n+1} (f \overline{w}_m) \right|^2 \right)^{1/2} \right\|_{p(\cdot)}$$

$$\lesssim \| f \overline{w}_m \|_{p(\cdot)} = \| f \|_{p(\cdot)},$$

which finishes the proof. □

Since the Vilenkin polynomials are dense in $L_{p(\cdot)}$, Theorem 4.1 implies

Corollary 4.2. *Let $p(\cdot) \in C^{\log}$ with $1 < p_- \le p_+ < \infty$. If $f \in L_{p(\cdot)}$, then*

$$\lim_{n\to\infty} s_n f = f \qquad \text{in the } L_{p(\cdot)}\text{-norm.}$$

Similarly, for variable Lorentz spaces, we can prove the following two results. The Vilenkin polynomials are also dense in $L_{p(\cdot),q}$ with $0 < q < \infty$. For $q = \infty$, let us denote by $\mathscr{L}_{p(\cdot),\infty}$ the closure of the Vilenkin polynomials in $L_{p(\cdot),\infty}$. For simplicity, under $L_{p(\cdot),\infty}$, we mean $\mathscr{L}_{p(\cdot),\infty}$ in this paper.

Theorem 4.3. *Let $p(\cdot) \in C^{\log}$ with $1 < p_- \le p_+ < \infty$ and $0 < q \le \infty$. If $f \in L_{p(\cdot),q}$, then*

$$\sup_{n\in\mathbb{N}} \|s_n f\|_{L_{p(\cdot),q}} \lesssim \|f\|_{L_{p(\cdot),q}}.$$

Corollary 4.4. *Let $p(\cdot) \in C^{\log}$ with $1 < p_- \le p_+ < \infty$ and $0 < q \le \infty$. If $f \in L_{p(\cdot),q}$, then*

$$\lim_{n\to\infty} s_n f = f \qquad \text{in the } L_{p(\cdot),q}\text{-norm.}$$

5 | FEJÉR MEANS OF VILENKIN–FOURIER SERIES

The results of the preceding section are not true if $p_- \le 1$ (see, e.g., Schipp, Wade, Simon and Pál [32] or Example 5.4.2 in [16]). However, in this case we can consider a summability method. For $n \in \mathbb{N}$ and a martingale f, the Fejér mean of order n of the Vilenkin–Fourier series of f is given by

$$\sigma_n f := \frac{1}{n} \sum_{k=1}^{n} s_k f.$$

Of course, $\sigma_n f$ has better convergence properties than $s_k f$. It is simple to show that

$$\sigma_n f(x) = \int_0^1 f(t) K_n(x \dot{-} t)\, dt \qquad (n \in \mathbb{N})$$

if $f \in L_1$, where the Vilenkin–Fejér kernels are defined by

$$K_n := \frac{1}{n} \sum_{k=1}^{n} D_k \qquad (n \in \mathbb{N}).$$

The maximal operator σ_* is defined by

$$\sigma_* f = \sup_{n\in\mathbb{N}} |\sigma_n f|.$$

It is known (see Pál and Simon [28] and Gát [10]) that

$$|K_n(x)| \le C P_N^{-1} \sum_{j=0}^{N-1} P_j \sum_{i=j}^{N-1} \sum_{l=0}^{p_j-1} D_{P_i}\left(x \dot{+} l P_{j+1}^{-1}\right) \tag{5.1}$$

and

$$K_{P_n}(x) = \frac{1}{2}\left(P_n^{-1}+1\right) D_{P_n}(x) + \sum_{j=0}^{n-1} \sum_{l=1}^{p_j-1} \frac{P_j}{P_n} \frac{1}{1-e^{-2\pi\imath l/p_j}} D_{P_n}\left(x \dot{+} l P_{j+1}^{-1}\right), \tag{5.2}$$

where $x \in [0,1)$, $n, N \in \mathbb{N}$ and $P_{N-1} \le n < P_N$.

6 | THE MAXIMAL OPERATOR U

To be able to prove some boundedness results for σ_*, we have to investigate two versions of maximal operators. For a martingale $f = (f_n)$, the first one is given by

$$U_s f(x) := \sup_{x\in I} \sum_{j=0}^{n-1} \left(\frac{P_j}{P_n}\right)^s \sum_{l=0}^{p_j-1} \frac{1}{\lambda(I^{j,l})} \left| \int_{I^{j,l}} f_n \, d\lambda \right|,$$

where I is a Vilenkin interval with length P_n^{-1}, s is a positive constant and

$$I^{j,l} := I \dot{+} l P_{j+1}^{-1}.$$

Of course, if $f \in L_1$, then we can write in the definition f instead of f_n. Let us define $I_{k,n} := \left[k P_n^{-1}, (k+1) P_n^{-1}\right)$ with $0 \le k < P_n$, $n \in \mathbb{N}$. The definition can be rewritten to

$$U_s f := \sup_{n\in\mathbb{N}} \sum_{k=0}^{P_n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} \left(\frac{P_j}{P_n}\right)^s \sum_{l=0}^{p_j-1} \frac{1}{\lambda(I_{k,n}^{j,l})} \left| \int_{I_{k,n}^{j,l}} f_n \, d\lambda \right|,$$

where $I_{k,n}^{j,l} := \left(I_{k,n}\right)^{j,l}$.

An operator $T : X \to Y$ is called a σ-sublinear operator if for any $\alpha \in \mathbb{C}$ it satisfies

$$\left| T\left(\sum_{k=1}^{\infty} f_k\right) \right| \le \sum_{k=1}^{\infty} \left| T\left(f_k\right) \right| \quad \text{and} \quad |T(\alpha f)| = |\alpha| |T(f)|,$$

where X is a martingale space and Y is a measurable function space. We will apply the following well-known theorem in martingale theory (see, e.g., Weisz [38]).

Theorem 6.1. *Let p be a constant and $0 < p \le 1$. Suppose that $T : L_\infty \to L_\infty$ is a bounded σ-sublinear operator and*

$$\|T a \chi_{I^c}\|_p \le C_p \tag{6.1}$$

for all p-atoms a, where I is the support of a. Then we have

$$\|T f\|_p \lesssim \|f\|_{H_p} \qquad (f \in H_p).$$

Theorem 6.2. *For all* $0 < p \le \infty$ *and all* $0 < s < \infty$, *we have*

$$\|U_s f\|_p \le C_p \|f\|_{H_p} \qquad (f \in H_p). \tag{6.2}$$

Proof. Observe that (6.2) holds for $p = \infty$. Indeed,

$$\|U_s f\|_\infty \le \sup_{n\in\mathbb{N}} \sum_{j=0}^{n-1} \left(\frac{P_j}{P_n}\right)^s \sum_{l=0}^{p_j-1} \|f\|_\infty \le C\|f\|_\infty$$

because of

$$\sum_{j=0}^{n} \left(\frac{P_j}{P_n}\right)^s \le C. \tag{6.3}$$

By Theorem 6.1 and interpolation, the proof will be complete if we show that the operator U_s satisfies (6.1) for each $0 < p \le 1$. Choose a p-atom a with support I, where I is a Vilenkin interval with length $|I| = P_K^{-1}$ ($K \in \mathbb{N}$). We can assume that $I = [0, P_K^{-1})$. It is easy to see that

$$\sum_{j=0}^{n-1} \left(\frac{P_j}{P_n}\right)^s \sum_{l=0}^{p_j-1} \frac{1}{\lambda(J^{j,l})} \left| \int_{J^{j,l}} a \, d\lambda \right| = 0$$

if $n \le K$, where J is a Vilenkin interval with lenght P_n^{-1}. Therefore we can suppose that $n > K$. Observe that $x \notin [0, P_K^{-1})$ and $x \in J$ imply that $J^{j,l} \cap [0, P_K^{-1}) = \emptyset$ if $j \ge K$. Thus $\int_{J^{j,l}} a = 0$ for $j \ge K$. Hence, we may assume that $j < K$. The same holds if $j < K$ and $x \in [P_{j+1}^{-1} + P_K^{-1}, P_j^{-1})$, because $x{+}lP_{j+1}^{-1} \notin [0, 2^{-K})$. Hence

$$\begin{aligned}
|U_s a(x)| &\le \sup_{n>K} \chi_J(x) \sum_{j=0}^{K-1} \left(\frac{P_j}{P_n}\right)^s \chi_{[P_{j+1}^{-1}, P_{j+1}^{-1}+P_K^{-1})}(x) \sum_{l=0}^{p_j-1} \frac{1}{\lambda(J^{j,l})} \left| \int_{J^{j,l}} a \, d\lambda \right| \\
&\le P_K^{1/p} \sum_{j=0}^{K-1} \left(\frac{P_j}{P_K}\right)^s \chi_{[P_{j+1}^{-1}, P_{j+1}^{-1}+P_K^{-1})}(x)
\end{aligned}$$

and

$$\int_{I^c} |U_s a(x)|^p \le P_K \sum_{j=0}^{K-1} \left(\frac{P_j}{P_K}\right)^{sp} P_K^{-1} \le C_p,$$

which completes the proof of the theorem. □

Since H_p is equivalent to L_p when $1 < p \le \infty$, we obtain

Corollary 6.3. *For all* $1 < p \le \infty$ *and all* $0 < s < \infty$, *we have*

$$\|U_s f\|_p \le C_p \|f\|_p \qquad (f \in L_p).$$

We generalize this result to Lebesgue spaces with variable exponents.

Theorem 6.4. *Let $p(\cdot) \in C^{\log}$, $1 < p_- \le p_+ < \infty$ and $0 < s < \infty$. If*

$$\frac{1}{p_-} - \frac{1}{p_+} < s, \tag{6.4}$$

then

$$\|U_s f\|_{p(\cdot)} \le C_{p(\cdot)} \|f\|_{p(\cdot)} \qquad \big(f \in L_{p(\cdot)}\big).$$

Proof. We assume that $\|f\|_{p(\cdot)} \le 1/2$. Then

$$\begin{aligned}\int_0^1 |U_s f(x)|^{p(x)}\,dx &\lesssim \int_0^1 \left|U_s\big(f\chi_{|f|\ge 1}\big)(x)\right|^{p(x)} dx + \int_0^1 \left|U_s\big(f\chi_{|f|<1}\big)(x)\right|^{p(x)} dx \\ &\lesssim \int_0^1 \left|U_s\big(f\chi_{|f|\ge 1}\big)(x)\right|^{p(x)} dx + C.\end{aligned}$$

So it is enough to prove that

$$\int_0^1 \left|U_s\big(f\chi_{|f|\ge 1}\big)(x)\right|^{p(x)} dx \lesssim C.$$

For a fixed k, n, let us denote by $\Lambda_{k,n}$ those pairs (j, l) for which $0 \le j < n, 0 \le l < p_j$ and

$$\frac{1}{\lambda(I_{k,n}^{j,l})} \int_{I_{k,n}^{j,l}} |f(t)|\,dt \le 1.$$

Then

$$\begin{aligned}&\int_0^1 \left(\sup_{n\in\mathbb{N}} \sum_{k=0}^{P_n-1} \chi_{I_{k,n}}(x) \sum_{(j,l)\in\Lambda_{k,n}} \left(\frac{P_j}{P_n}\right)^s \frac{1}{\lambda(I_{k,n}^{j,l})} \int_{I_{k,n}^{j,l}} |f|\,d\lambda \right)^{p(x)} dx \\ &\lesssim \int_0^1 \left(\sup_{n\in\mathbb{N}} \sum_{k=0}^{P_n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} \sum_{l=0}^{p_j-1} \left(\frac{P_j}{P_n}\right)^s \right)^{p(x)} dx \le C.\end{aligned}$$

Hence, we may suppose that $\|f\|_{p(\cdot)} \le 1/2$, $|f| \ge 1$ or $f = 0$ and

$$\frac{1}{\lambda(I_{k,n}^{j,l})} \int_{I_{k,n}^{j,l}} |f(t)|\,dt > 1 \tag{6.5}$$

for all $j = 0, \dots, n-1$, $k = 0, \dots, P_n - 1$, $n \in \mathbb{N}$. Let us denote by $I_{k,n,j,l,1}$ $\big($resp. $I_{k,n,j,l,2}\big)$ those points $x \in I_{k,n}$ for which $p(x) \le p_+\big(I_{k,n}^{j,l}\big)$ $\big($resp. $p(x) > p_+\big(I_{k,n}^{j,l}\big)\big)$. Then

$$\begin{aligned}\int_\Omega |U_s f(x)|^{p(x)}\,dx &\lesssim \sum_{m=1}^{2} \int_0^1 \left(\sup_{n\in\mathbb{N}} \sum_{k=0}^{P_n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} \left(\frac{P_j}{P_n}\right)^s \sum_{l=0}^{p_j-1} \frac{\chi_{I_{k,n,j,l,m}}(x)}{\lambda(I_{k,n}^{j,l})} \int_{I_{k,n}^{j,l}} |f(t)|\,dt \right)^{p(x)} dx \\ &=: (A) + (B).\end{aligned}$$

Let $q(x) := p(x)/p_0 > 1$ for some $1 < p_0 < p_-$. Since the sets $I_{k,n}$ are disjoint for a fixed n and the function $t \mapsto t^{q(x)}$ is convex (x is fixed), we conclude

$$(A) \lesssim \int_0^1 \left(\sup_{n\in\mathbb{N}} \sum_{k=0}^{P_n-1} \chi_{I_{k,n}}(x) \left(\sum_{j=0}^{n-1} \left(\frac{P_j}{P_n}\right)^s \sum_{l=0}^{p_j-1} \frac{\chi_{I_{k,n,j,l,1}}(x)}{\lambda(I_{k,n}^{j,l})} \int_{I_{k,n}^{j,l}} |f(t)|\, dt \right)^{q(x)} \right)^{p_0} dx$$

$$\lesssim \int_0^1 \left(\sup_{n\in\mathbb{N}} \sum_{k=0}^{P_n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} \left(\frac{P_j}{P_n}\right)^s \sum_{l=0}^{p_j-1} \left(\frac{\chi_{I_{k,n,j,l,1}}(x)}{\lambda(I_{k,n}^{j,l})} \int_{I_{k,n}^{j,l}} |f(t)|\, dt \right)^{q(x)} \right)^{p_0} dx$$

$$\lesssim \int_0^1 \left(\sup_{n\in\mathbb{N}} \sum_{k=0}^{P_n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} \left(\frac{P_j}{P_n}\right)^s \sum_{l=0}^{p_j-1} \left(\frac{\chi_{I_{k,n,j,l,1}}(x)}{\lambda\left(I_{k,n}^{j,l}\right)} \int_{I_{k,n}^{j,l}} |f(t)|\, dt \right)^{q_+\left(I_{k,n}^{j,l}\right)} \right)^{p_0} dx.$$

Here we have used (6.5), the boundedness of (p_j) and the fact that $q(x) \le q_+\left(I_{k,n}^{j}\right)$ on $I_{k,n,j,l,1}$. By Lemma 2.6,

$$(A) \lesssim \int_0^1 \left(\sup_{n\in\mathbb{N}} \sum_{k=0}^{P_n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} \left(\frac{P_j}{P_n}\right)^s \sum_{l=0}^{p_j-1} \frac{\chi_{I_{k,n,j,l,1}}(x)}{\lambda(I_{k,n}^{j,l})} \int_{I_{k,n}^{j,l}} |f(t)|^{q(t)}\, dt \right)^{p_0} dx$$

$$\lesssim \left\| U_s(|f|^{q(\cdot)}) \right\|_{p_0}^{p_0} \lesssim \left\| |f|^{q(\cdot)} \right\|_{p_0}^{p_0} \le C.$$

Choose $0 < r < s$ and observe that

$$(B) \lesssim \int_0^1 \left(\sup_{n\in\mathbb{N}} \sum_{k=0}^{P_n-1} \chi_{I_{k,n}}(x) \left(\sum_{j=0}^{n-1} \left(\frac{P_j}{P_n}\right)^{s-r} \sum_{l=0}^{p_j-1} \left(\frac{P_j}{P_n}\right)^r \frac{\chi_{I_{k,n,j,l,2}}(x)}{\lambda(I_{k,n}^{j,l})} \int_{I_{k,n}^{j,l}} |f(t)|\, dt \right)^{q(x)} \right)^{p_0} dx$$

$$\lesssim \int_0^1 \left(\sup_{n\in\mathbb{N}} \sum_{k=0}^{P_n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} \left(\frac{P_j}{P_n}\right)^{s-r} \sum_{l=0}^{p_j-1} \left(\left(\frac{P_j}{P_n}\right)^r \frac{\chi_{I_{k,n,j,l,2}}(x)}{\lambda(I_{k,n}^{j,l})} \int_{I_{k,n}^{j,l}} |f(t)|\, dt \right)^{q(x)} \right)^{p_0} dx.$$

By Hölder's inequality,

$$(B) \lesssim \int_0^1 \left(\sup_{n\in\mathbb{N}} \sum_{k=0}^{P_n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} \left(\frac{P_j}{P_n}\right)^{s-r} \sum_{l=0}^{p_j-1} \left(\frac{P_j}{P_n}\right)^{rq(x)} \left(\frac{\chi_{I_{k,n,j,l,2}}(x)}{\lambda(I_{k,n}^{j,l})} \int_{I_{k,n}^{j,l}} |f(t)|^{q_-\left(I_{k,n}^{j,l}\right)}\, dt \right)^{q(x)/q_-\left(I_{k,n}^{j,l}\right)} \right)^{p_0} dx$$

$$\lesssim \int_0^1 \left(\sup_{n\in\mathbb{N}} \sum_{k=0}^{P_n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} \left(\frac{P_j}{P_n}\right)^{s-r} \sum_{l=0}^{p_j-1} \left(\frac{P_j}{P_n}\right)^{rq(x)} P_n^{q(x)/q_-\left(I_{k,n}^{j,l}\right)} \chi_{I_{k,n,j,l,2}}(x) \left(\int_{I_{k,n}^{j,l}} |f(t)|^{q_-\left(I_{k,n}^{j,l}\right)}\, dt \right)^{q(x)/q_-\left(I_{k,n}^{j,l}\right)} \right)^{p_0} dx.$$

Since $|f| \ge 1$ or $f = 0$, $q(x) > q_-(I_{k,n}^{j,l})$ on $I_{k,n,j,l,2}$, $q_-(I_{k,n}^{j,l}) \le q(t) < p(t)$ for all $t \in I_{k,n}^{j,l}$ and

$$\int_{I_{k,n}^{j,l}} |f(t)|^{q_-\left(I_{k,n}^{j,l}\right)}\, dt \le \int_{I_{k,n}^{j,l}} |f(t)|^{p(t)}\, dt \le \frac{1}{2},$$

we conclude

$$(B) \lesssim \int_0^1 \left(\sup_{n\in\mathbb{N}} \sum_{k=0}^{P_n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} \left(\frac{P_j}{P_n}\right)^{s-r} \sum_{l=0}^{p_j-1} \left(\frac{P_j}{P_n}\right)^{rq(x)} P_n^{q(x)/q_-\left(I_{k,n}^{j,l}\right)} \chi_{I_{k,n,j,l,2}}(x) \int_{I_{k,n}^{j,l}} |f(t)|^{q_-\left(I_{k,n}^{j,l}\right)}\, dt \right)^{p_0} dx$$

$$\lesssim \int_0^1 \left(\sup_{n\in\mathbb{N}} \sum_{k=0}^{P_n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} \left(\frac{P_j}{P_n}\right)^{s-r} \sum_{l=0}^{p_j-1} P_j^{rq(x)} P_n^{-rq(x)+q(x)/q_-\left(I_{k,n}^{j,l}\right)-1} \chi_{I_{k,n,j,l,2}}(x) \frac{1}{\lambda(I_{k,n}^{j,l})} \int_{I_{k,n}^{j,l}} |f(t)|^{q(t)}\, dt \right)^{p_0} dx.$$

For fixed k, n let J_j denote the Vilenkin interval with length P_j^{-1} and $I_{k,n} \subset J_j$. Then $I_{k,n}^{j,l} \subset J_j \dotplus 2^{-j-1} = J_j$. Inequality (2.2) implies that $P_j^{-p(x)} \sim P_j^{-p_-\left(I_{k,n}^{j,l}\right)}$ for $x \in I_{k,n}$. It is easy to check that for $x \in I_{k,n,j,l,2}$,

$$P_j^{rq(x)} = P_j^{rq(x)} P_j^{q(x)} P_j^{-q(x)} \lesssim P_j^{rq(x)} P_j^{q_-\left(I_{k,n}^{j,l}\right)} P_j^{-q(x)}$$

$$< P_j^{\left(rq(x) - \frac{q(x)-q_-\left(I_{k,n}^{j,l}\right)}{q_-\left(I_{k,n}^{j,l}\right)}\right)} = P_j^{\left(rq(x) - \frac{q(x)}{q_-\left(I_{k,n}^{j,l}\right)} + 1\right)}.$$

Furthermore,

$$rq(x) - \frac{q(x)}{q_-(I_{k,n}^{j})} + 1 \geq q(x)\left(r - \frac{1}{q_-}\right) + 1 \geq \begin{cases} 1, & \text{if } r - \frac{1}{q_-} \geq 0, \\ q_+\left(r - \frac{1}{q_-}\right) + 1, & \text{if } r - \frac{1}{q_-} < 0. \end{cases}$$

Let $r_0 := \min\left(1, q_+\left(r - \frac{1}{q_-}\right) + 1\right)$. Then $r_0 > 0$ if and only if

$$\frac{1}{q_-} - \frac{1}{q_+} < r. \tag{6.6}$$

Hence

$$(B) \lesssim \int_0^1 \left(\sup_{n\in\mathbb{N}} \sum_{k=0}^{P_n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} \left(\frac{P_j}{P_n}\right)^{s-r} \sum_{l=0}^{p_j-1} \left(\frac{P_j}{P_n}\right)^{rq(x)-q(x)/q_-\left(I_{k,n}^{j,l}\right)+1} \chi_{I_{k,n,j,l,2}}(x) \frac{1}{\lambda(I_{k,n}^{j,l})} \int_{I_{k,n}^{j,l}} |f(t)|^{q(t)}\, dt \right)^{p_0} dx$$

$$\lesssim \int_0^1 \left(\sup_{n\in\mathbb{N}} \sum_{k=0}^{P_n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} \left(\frac{P_j}{P_n}\right)^{s-r+r_0} \sum_{l=0}^{p_j-1} \frac{1}{\lambda(I_{k,n}^{j,l})} \int_{I_{k,n}^{j,l}} |f(t)|^{q(t)}\, dt \right)^{p_0} dx$$

$$\lesssim \left\| U_{s-r+r_0}\left(|f|^{q(\cdot)}\right) \right\|_{p_0}^{p_0} \lesssim \left\| |f|^{q(\cdot)} \right\|_{p_0}^{p_0} \leq C.$$

Since p_0 can be arbitrarily near to 1 and r to s, inequality (6.6) proves the theorem with the range (6.4). □

Remark 6.5. Inequality (6.4) and Theorem 6.4 hold if $s \geq 1$, or more generally if $p_- > \max(1/s, 1)$.

The operator U_s is not bounded on $L_{p(\cdot)}$ when (6.4) is not satisfied. More exactly, the following theorem holds.

Theorem 6.6. *Let $p(\cdot) \in C^{\log}$, $1 < p_- \le p_+ < \infty$ and $0 < s < \infty$. If*

$$\frac{1}{p_-\left(I_{0,n} \dotplus P_1^{-1}\right)} - \frac{1}{p_+\left(I_{0,n}\right)} > s \tag{6.7}$$

for all $n \in \mathbb{N}$, then U_s is not bounded on $L_{p(\cdot)}$.

Proof. It is easy to see that

$$\int_0^1 |U_s f(x)|^{p(x)}\, dx \ge \int_0^1 \chi_{I_{0,n}}(x) \left(P_n^{-s} \frac{1}{\lambda(I_{0,n}^{0,0})} \int_{I_{0,n}^{0,0}} f(t)\, dt \right)^{p(x)} dx.$$

Defining

$$f(t) := \chi_{I_{0,n}^{0,0}}(t) P_n^{1/p_-\left(I_{0,n}^{0,0}\right)},$$

we have

$$\|f\|_{p(\cdot)} = P_n^{1/p_-\left(I_{0,n}^{0,0}\right)} \left\| \chi_{I_{0,n}^{0,0}} \right\|_{p(\cdot)} \le C$$

by Lemma 2.5. Using (2.2), we can see that

$$\begin{aligned}\int_0^1 |U_s f(x)|^{p(x)}\, dx &\ge \int_{I_{0,n}} P_n^{-sp(x)} P_n^{p(x)/p_-\left(I_{0,n}^{0,0}\right)}\, dx \\ &\ge C \int_{I_{0,n}} P_n^{p_+\left(I_{0,n}\right)\left(1/p_-\left(I_{0,n}^{0,0}\right)-s\right)}\, dx = C P_n^{p_+\left(I_{0,n}\right)\left(1/p_-\left(I_{0,n}^{0,0}\right)-s\right)} P_n^{-1}\end{aligned}$$

which tends to infinity as $n \to \infty$ if (6.7) holds. □

7 | THE MAXIMAL OPERATOR V

For a martingale $f = (f_n)$, we define the second version of maximal operator by

$$V_{\alpha,s} f(x) := \sup_{x \in I} \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} \left(\frac{P_j}{P_n}\right)^{\alpha} \left(\frac{P_j}{P_i}\right)^{s} \sum_{l=0}^{p_j - 1} \frac{1}{\lambda(I^{j,i,l})} \left| \int_{I^{j,i,l}} f_n \, d\lambda \right|,$$

where I is a Vilenkin interval with length P_n^{-1}, α, s are positive constants and

$$I^{j,i,l} := I \dotplus \left[l P_{j+1}^{-1}, l P_{j+1}^{-1} \dotplus P_i^{-1} \right).$$

Obviously,

$$V_{\alpha,s} f := \sup_{n \in \mathbb{N}} \sum_{k=0}^{P_n - 1} \chi_{I_{k,n}} \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} \left(\frac{P_j}{P_n}\right)^{\alpha} \left(\frac{P_j}{P_i}\right)^{s} \sum_{l=0}^{p_j - 1} \frac{1}{\lambda(I_{k,n}^{j,i,l})} \left| \int_{I_{k,n}^{j,i,l}} f_n \, d\lambda \right|,$$

where $I_{k,n}^{j,i,l} := \left(I_{k,n}\right)^{j,i,l}$.

Theorem 7.1. *Suppose that* $0 < p \leq \infty$ *and* $0 < \alpha, s < \infty$. *Then*

$$\|V_{\alpha,s} f\|_p \leq C_p \|f\|_{H_p} \qquad (f \in H_p).$$

Proof. The inequality holds for $p = \infty$ because

$$\left\|V_{\alpha,s} f\right\|_\infty \leq \sup_{n\in\mathbb{N}} \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} \left(\frac{P_j}{P_n}\right)^\alpha \left(\frac{P_j}{P_i}\right)^s \sum_{l=0}^{p_j-1} \|f\|_\infty \leq C\|f\|_\infty.$$

We are going to show (6.1) for $V_{\alpha,s}$ and for each $0 < p \leq 1$. Let a be a p-atom with support $I = \left[0, P_K^{-1}\right)$. If $i \leq K$, then

$$\int_{I^{j,i,l}} a \, d\lambda = 0.$$

Thus $i > K$ and so $n > K$. Similarly to the proof of Theorem 6.2, $j < K$ and $x \in \left[P_{j+1}^{-1}, P_{j+1}^{-1} + P_K^{-1}\right)$. Hence, in case $x \notin \left[0, P_K^{-1}\right)$,

$$\begin{aligned}
\left|V_{\alpha,s} a(x)\right| &\leq \sup_{n>K} \chi_J(x) \sum_{j=0}^{K-1} \sum_{i=K}^{n-1} \left(\frac{P_j}{P_n}\right)^\alpha \left(\frac{P_j}{P_i}\right)^s \sum_{l=0}^{p_j-1} \frac{1}{\lambda(J^{j,i,l})} \left|\int_{J^{j,i,l}} a \, d\lambda\right| \chi_{\left[P_{j+1}^{-1}, P_{j+1}^{-1} + P_K^{-1}\right)}(x) \\
&\leq P_K^{1/p} \chi_J(x) \sum_{j=0}^{K-1} \sum_{i=K}^{\infty} \left(\frac{P_j}{P_K}\right)^\alpha \left(\frac{P_j}{P_i}\right)^s \chi_{\left[P_{j+1}^{-1}, P_{j+1}^{-1} + P_K^{-1}\right)}(x) \\
&\leq P_K^{1/p} \sum_{j=0}^{K-1} \left(\frac{P_j}{P_K}\right)^\alpha \chi_{\left[P_{j+1}^{-1}, P_{j+1}^{-1} + P_K^{-1}\right)}(x).
\end{aligned}$$

where J is a Vilenkin interval with length P_n^{-1}. Consequently,

$$\int_{I^c} \left|V_{\alpha,s} a(x)\right|^p \leq P_K \sum_{j=0}^{K-1} \left(\frac{P_j}{P_K}\right)^{\alpha p} P_K^{-1} \leq C_p,$$

which finishes the proof. □

Corollary 7.2. *For all* $1 < p \leq \infty$ *and all* $0 < \alpha, s < \infty$, *we have*

$$\|V_{\alpha,s} f\|_p \leq C_p \|f\|_p \qquad (f \in L_p).$$

Theorem 7.3. *Let* $p(\cdot) \in C^{\log}$, $1 < p_- \leq p_+ < \infty$ *and* $0 < \alpha, s < \infty$. *If*

$$\frac{1}{p_-} - \frac{1}{p_+} < \alpha + s, \tag{7.1}$$

then

$$\|V_{\alpha,s} f\|_{p(\cdot)} \leq C_{p(\cdot)} \|f\|_{p(\cdot)} \qquad (f \in L_{p(\cdot)}).$$

Proof. Similarly to the proof of Theorem 6.4, we may suppose again that $\|f\|_{p(\cdot)} \leq 1/2$, $|f| \geq 1$ or $f = 0$ and

$$\frac{1}{\lambda(I_{k,n}^{j,i,l})} \int_{I_{k,n}^{j,i,l}} |f(t)| \, dt > 1.$$

We denote by $I_{k,n,j,i,l,1}$ (resp. $I_{k,n,j,i,l,2}$) those points $x \in I_{k,n}$ for which $p(x) \le p_+(I_{k,n}^{j,i,l})$ (resp. $p(x) > p_+(I_{k,n}^{j,i,l})$). Then

$$\int_0^1 |V_{\alpha,s} f(x)|^{p(x)}\, dx \lesssim \sum_{m=1}^{2} \int_0^1 \left(\sup_{n\in\mathbb{N}} \sum_{k=0}^{P_n-1} \chi_{I_{k,n}} \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} \left(\frac{P_j}{P_n}\right)^{\alpha} \left(\frac{P_j}{P_i}\right)^{s} \sum_{l=0}^{p_j-1} \frac{\chi_{I_{k,n,j,i,l,m}}(x)}{\lambda(I_{k,n}^{j,i,l})} \int_{I_{k,n}^{j,i,l}} |f|\, dt \right)^{p(x)} dx$$

$$=: (C) + (D).$$

Again, let $q(x) := p(x)/p_0 > 1$ for some $1 < p_0 < p_-$. By convexity and Lemma 2.6,

$$(C) \lesssim \int_0^1 \left(\sup_{n\in\mathbb{N}} \sum_{k=0}^{P_n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} \left(\frac{P_j}{P_n}\right)^{\alpha} \left(\frac{P_j}{P_i}\right)^{s} \sum_{l=0}^{p_j-1} \left(\frac{\chi_{I_{k,n,j,i,l,1}}(x)}{\lambda(I_{k,n}^{j,i,l})} \int_{I_{k,n}^{j,i,l}} |f(t)|\, dt \right)^{q(x)} \right)^{p_0} dx$$

$$\lesssim \int_0^1 \left(\sup_{n\in\mathbb{N}} \sum_{k=0}^{P_n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} \left(\frac{P_j}{P_n}\right)^{\alpha} \left(\frac{P_j}{P_i}\right)^{s} \sum_{l=0}^{p_j-1} \left(\frac{\chi_{I_{k,n,j,i,l,1}}(x)}{\lambda(I_{k,n}^{j,i,l})} \int_{I_{k,n}^{j,i,l}} |f(t)|\, dt \right)^{q_+(I_{k,n}^{j,i,l})} \right)^{p_0} dx$$

$$\lesssim \int_0^1 \left(\sup_{n\in\mathbb{N}} \sum_{k=0}^{P_n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} \left(\frac{P_j}{P_n}\right)^{\alpha} \left(\frac{P_j}{P_i}\right)^{s} \sum_{l=0}^{p_j-1} \frac{\chi_{I_{k,n,j,i,l,1}}(x)}{\lambda(I_{k,n}^{j,i,l})} \int_{I_{k,n}^{j,i,l}} |f(t)|^{q(t)}\, dt \right)^{p_0} dx$$

$$\lesssim \left\| V_{\alpha,s}(|f|^{q(\cdot)}) \right\|_{p_0}^{p_0} \lesssim \left\| |f|^{q(\cdot)} \right\|_{p_0}^{p_0} \le C.$$

We obtain for some $0 < \alpha_0 < \alpha$ and $0 < r < s + \alpha_0$ that

$$(D) \lesssim \int_0^1 \left(\sup_{n\in\mathbb{N}} \sum_{k=0}^{P_n-1} \chi_{I_{k,n}}(x) \left(\sum_{j=0}^{n-1} \sum_{i=j}^{n-1} \left(\frac{P_j}{P_n}\right)^{\alpha} \left(\frac{P_j}{P_i}\right)^{s} \sum_{l=0}^{p_j-1} \frac{\chi_{I_{k,n,j,i,l,2}}(x)}{\lambda(I_{k,n}^{j,i,l})} \int_{I_{k,n}^{j,i,l}} |f(t)|\, dt \right)^{q(x)} \right)^{p_0} dx$$

$$\lesssim \int_0^1 \left(\sup_{n\in\mathbb{N}} \sum_{k=0}^{P_n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} \left(\frac{P_j}{P_n}\right)^{\alpha-\alpha_0} \left(\frac{P_j}{P_i}\right)^{\alpha_0+s-r} \sum_{l=0}^{p_j-1} \left(\left(\frac{P_j}{P_i}\right)^{r} \frac{\chi_{I_{k,n,j,i,l,2}}(x)}{\lambda(I_{k,n}^{j,i,l})} \int_{I_{k,n}^{j,i,l}} |f(t)|\, dt \right)^{q(x)} \right)^{p_0} dx$$

and so

$$(D) \lesssim \int_0^1 \left(\sup_{n\in\mathbb{N}} \sum_{k=0}^{P_n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} \left(\frac{P_j}{P_n}\right)^{\alpha-\alpha_0} \left(\frac{P_j}{P_i}\right)^{\alpha_0+s-r} \sum_{l=0}^{p_j-1} \left(\frac{P_j}{P_i}\right)^{rq(x)} \right.$$

$$\left. \left(\frac{\chi_{I_{k,n,j,i,l,2}}(x)}{\mathbb{P}(I_{k,n}^{j,i,l})} \int_{I_{k,n}^{j,i,l}} |f(t)|^{q_-(I_{k,n}^{j,i,l})}\, dt \right)^{q(x)/q_-(I_{k,n}^{j,i,l})} \right)^{p_0} dx$$

$$\lesssim \int_0^1 \left(\sup_{n\in\mathbb{N}} \sum_{k=0}^{P_n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} \left(\frac{P_j}{P_n}\right)^{\alpha-\alpha_0} \left(\frac{P_j}{P_i}\right)^{\alpha_0+s-r} \sum_{l=0}^{p_j-1} \left(\frac{P_j}{P_i}\right)^{rq(x)} \right.$$

$$\left. P_i^{q(x)/q_-(I_{k,n}^{j,i,l})} \chi_{I_{k,n,j,i,l,2}}(x) \left(\int_{I_{k,n}^{j,i,l}} |f(t)|^{q_-(I_{k,n}^{j,i,l})}\, dt \right)^{q(x)/q_-(I_{k,n}^{j,i,l})} \right)^{p_0} dx.$$

Since

$$\int_{I_{k,n}^{j,i,l}} |f(t)|^{q_-\left(I_{k,n}^{j,i,l}\right)}\,dt \le \int_{I_{k,n}^{j}} |f(t)|^{p(t)}\,dt \le \frac{1}{2},$$

we conlude that

$$(D) \lesssim \int_0^1 \left(\sup_{n\in\mathbb{N}} \sum_{k=0}^{P_n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1}\sum_{i=j}^{n-1} \left(\frac{P_j}{P_n}\right)^{\alpha-\alpha_0} \left(\frac{P_j}{P_i}\right)^{\alpha_0+s-r} \sum_{l=0}^{p_j-1} \left(\frac{P_j}{P_i}\right)^{rq(x)} \right.$$

$$\left. P_i^{q(x)/q_-\left(I_{k,n}^{j,i,l}\right)-1} \chi_{I_{k,n,j,i,l,2}}(x) \frac{1}{\lambda\left(I_{k,n}^{j,i,l}\right)} \int_{I_{k,n}^{j,i,l}} |f(t)|^{q_-\left(I_{k,n}^{j,i,l}\right)}\,dt \right)^{p_0} dx.$$

Similarly to the proof of Theorem 6.4, we get

$$(D) \lesssim \int_0^1 \left(\sup_{n\in\mathbb{N}} \sum_{k=0}^{P_n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1}\sum_{i=j}^{n-1} \left(\frac{P_j}{P_n}\right)^{\alpha-\alpha_0} \left(\frac{P_j}{P_i}\right)^{\alpha_0+s-r} \sum_{l=0}^{p_j-1} \left(\frac{P_j}{P_i}\right)^{\left(rq(x)-\frac{q(x)}{q_-\left(I_{k,n}^{j,i}\right)}+1\right)} \frac{1}{\lambda\left(I_{k,n}^{j,i,l}\right)} \int_{I_{k,n}^{j,i,l}} |f(t)|^{q(t)}\,dt \right)^{p_0} dx$$

$$\lesssim \int_0^1 \left(\sup_{n\in\mathbb{N}} \sum_{k=0}^{P_n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1}\sum_{i=j}^{n-1} \left(\frac{P_j}{P_n}\right)^{\alpha-\alpha_0} \left(\frac{P_j}{P_i}\right)^{\alpha_0+s-r+r_0} \sum_{l=0}^{p_j-1} \frac{1}{\lambda\left(I_{k,n}^{j,i,l}\right)} \int_{I_{k,n}^{j,i,l}} |f(t)|^{q(t)}\,dt \right)^{p_0} dx$$

$$\lesssim \left\| V_{\alpha-\alpha_0,\alpha_0+s-r+r_0}\left(|f|^{q(\cdot)}\right)\right\|_{p_0}^{p_0} \lesssim \left\| |f|^{q(\cdot)} \right\|_{p_0}^{p_0} \le C,$$

whenever (6.6) holds. Note that r_0 was defined just before (6.6). Since r can be arbitrarily near to $s+\alpha_0$ and α_0 to α, this completes the proof. □

Remark 7.4. Inequality (7.1) and Theorem 7.3 hold if $p_- > \max(1/(\alpha+s), 1)$.

The operator $V_{\alpha,s}$ is not bounded on $L_{p(\cdot)}$ if (7.1) is not true.

Theorem 7.5. *Let $p(\cdot) \in C^{\log}$, $1 < p_- \le p_+ < \infty$ and $0 < \alpha, s < \infty$. If*

$$\frac{1}{p_-\left(I_{0,n} \dot{+} P_1^{-1}\right)} - \frac{1}{p_+\left(I_{0,n}\right)} > \alpha + s$$

for all $n \in \mathbb{N}$, then $V_{\alpha,s}$ is not bounded on $L_{p(\cdot)}$.

Proof. Choosing $j = 0$ and $i = n-1$, the theorem can be shown in the same way as Theorem 6.6. □

8 | THE MAXIMAL FEJÉR OPERATOR ON $H_{p(\cdot)}$

In this section, we apply the atomic characterization to prove the boundedness of σ_* from $H_{p(\cdot)}$ to $L_{p(\cdot)}$. The following generalization of Theorem 6.1 can be proved similarly to the case of dyadic martingales (see Jiao et al. [18]).

Theorem 8.1. *Let $p(\cdot) \in C^{\log}$ and $0 < t < \underline{p}$. Suppose that the σ-sublinear operator $T : L_\infty \to L_\infty$ is bounded and*

$$\left\| \sum_{k\in\mathbb{Z}} \mu_k^t T\left(a^k\right)^t \chi_{\{\tau_k=\infty\}} \right\|_{\frac{p(\cdot)}{t}} \lesssim \left\| \sum_{k\in\mathbb{Z}} 2^{kt} \chi_{\{\tau_k<\infty\}} \right\|_{\frac{p(\cdot)}{t}}, \tag{8.1}$$

where $\mu_k = 3 \cdot 2^k \|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}$ *and* τ_k *is the stopping time associated with the* $p(\cdot)$*-atom* a^k*. Then we have*

$$\|Tf\|_{p(\cdot)} \lesssim \|f\|_{H_{p(\cdot)}} \qquad (f \in H_{p(\cdot)}).$$

Note that the σ-sublinearity cannot be omitted (see Bownik, Li, Yang and Zhou [2, 3, 48]).

Theorem 8.2. *Let* $p(\cdot) \in C^{\log}$ *and* $1/2 < t < \underline{p}$*. If*

$$\frac{1}{p_-} - \frac{1}{p_+} < 1, \tag{8.2}$$

then

$$\left\| \sum_{k\in\mathbb{Z}} \mu_k^t \sigma_*\left(a^k\right)^t \chi_{\{\tau_k=\infty\}} \right\|_{\frac{p(\cdot)}{t}} \lesssim \left\| \sum_{k\in\mathbb{Z}} 2^{kt} \chi_{\{\tau_k<\infty\}} \right\|_{\frac{p(\cdot)}{t}}, \tag{8.3}$$

where $\mu_k = 3 \cdot 2^k \|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}$ *and* τ_k *is the stopping time associated with the* $p(\cdot)$*-atom* a^k*.*

Proof. For a fixed $k \in \mathbb{Z}$, the sets $\{\tau_k = K\}$ are disjoint and there exist disjoint Vilenkin intervals $I_{k,K,m} \in \mathcal{F}_K$ such that

$$\{\tau_k = K\} = \bigcup_m I_{k,K,m} \qquad (K \in \mathbb{N}),$$

where the union in m is finite and $|I_{k,K,m}| = P_K^{-1}$. Thus

$$\{\tau_k < \infty\} = \bigcup_{K\in\mathbb{N}} \bigcup_m I_{k,K,m},$$

where the Vilenkin intervals $I_{k,K,m}$ are disjoint for a fixed $k \in \mathbb{Z}$. Then

$$a^k = \sum_{K\in\mathbb{N}} \sum_m a^k \chi_{I_{k,K,m}}.$$

Since $\int_{I_{k,K,m}} a^k \, d\lambda = 0$, it is easy to see that $\widehat{a^k \chi_{I_{k,K,m}}}(n) = 0$ if $n < P_K$ and in this case $\sigma_n\left(a^k \chi_{I_{k,K,m}}\right) = 0$. Therefore we can suppose that $n \geq P_K$. If $j \geq K$, $0 \leq l \leq p_j - 1$ and $x \notin I_{k,K,m}$ then $x \dot{+} lP_{j+1}^{-1} \notin I_{k,K,m}$. Hence for $x \notin I_{k,K,m}$ and $i \geq j \geq K$, $0 \leq l \leq p_j - 1$, we have

$$a^k(t)\chi_{I_{k,K,m}}(t) D_{P_i}\left(x \dot{+} lP_{j+1}^{-1} \dot{-} t\right) dt = 0.$$

Since $n \geq P_K$ and $P_N > n \geq P_{N-1}$, one has $N - 1 \geq K$. By (5.1) we obtain for $x \notin I_{k,K,m}$ that

$$\begin{aligned}
\left|\sigma_n\left(a^k \chi_{I_{k,K,m}}\right)(x)\right| &\leq C P_N^{-1} \sum_{j=0}^{N-1} P_j \sum_{i=j}^{N-1} \sum_{l=0}^{p_j-1} \int_{I_{k,K,m}} |a^k(t)| D_{P_i}\left(x \dot{+} lP_{j+1}^{-1} \dot{-} t\right) dt \\
&\lesssim \|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}^{-1} P_K^{-1} \sum_{j=0}^{K-1} P_j \sum_{i=j}^{K-1} \sum_{l=0}^{p_j-1} \int_{I_{k,K,m}} D_{P_i}\left(x \dot{+} lP_{j+1}^{-1} \dot{-} t\right) dt \\
&+ \|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}^{-1} \sum_{j=0}^{K-1} P_j \sum_{i=K}^{\infty} P_i^{-1} \sum_{l=0}^{p_j-1} \int_{I_{k,K,m}} D_{P_i}\left(x \dot{+} lP_{j+1}^{-1} \dot{-} t\right) dt.
\end{aligned}$$

Observe that the right hand side is independent of n. Using (4.1), we can verify that for $x \notin I_{k,K,m}$,

$$\int_{I_{k,K,m}} D_{P_i}\left(x\dot{+}lP_{j+1}^{-1}\dot{-}t\right) dt = P_i P_K^{-1} \chi_{I_{k,K,m}\dot{+}\left[lP_{j+1}^{-1}, lP_{j+1}^{-1}\dot{+}P_i^{-1}\right)}(x) = P_i P_K^{-1} \chi_{I_{k,K,m}^{j,i,l}}(x)$$

if $j \le i \le K-1$, $0 \le l \le p_j - 1$ and

$$\int_{I_{k,K,m}} D_{P_i}\left(x\dot{+}lP_{j+1}^{-1}\dot{-}t\right) dt = \chi_{I_{k,K,m}\dot{+}\left[lP_{j+1}^{-1}, lP_{j+1}^{-1}\dot{+}P_K^{-1}\right)}(x) = \chi_{I_{k,K,m}^{j,l}}(x)$$

if $i \ge K_l$ and $0 \le l \le p_j - 1$. Therefore, for $x \notin I_{k,K,m}$,

$$\sigma_*\left(a^k \chi_{I_{k,K,m}}\right)(x) \lesssim \|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}^{-1} \sum_{j=0}^{K-1} P_j \sum_{i=K}^{\infty} P_i^{-1} \sum_{l=0}^{p_j-1} \chi_{I_{k,K,m}^{j,l}}(x) + \|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}^{-1} P_K^{-1} \sum_{j=0}^{K-1} P_j \sum_{i=j}^{K-1} \sum_{l=0}^{p_j-1} P_i P_K^{-1} \chi_{I_{k,K,m}^{j,i,l}}(x).$$

Since

$$\sum_{i=K}^{\infty} \frac{P_K}{P_i} \le C,$$

we have

$$\sigma_*\left(a^k \chi_{I_{k,K,m}}\right)(x) \lesssim \|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}^{-1} P_K^{-1} \sum_{j=0}^{K-1} P_j \sum_{l=0}^{p_j-1} \chi_{I_{k,K,m}^{j,l}}(x) + \|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}^{-1} P_K^{-2} \sum_{j=0}^{K-1} P_j \sum_{i=j}^{K-1} \sum_{l=0}^{p_j-1} P_i \chi_{I_{k,K,m}^{j,i,l}}(x).$$

Consequently, for $x \in \{\tau_k = \infty\}$,

$$\begin{aligned}\sigma_*\left(a^k\right)(x) &\lesssim \|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}^{-1} \left(\sum_{K\in\mathbb{N}} \sum_m P_K^{-1} \sum_{j=0}^{K-1} P_j \sum_{l=0}^{p_j-1} \chi_{I_{k,K,m}^{j,l}}(x) + \sum_{K\in\mathbb{N}} \sum_m P_K^{-2} \sum_{j=0}^{K-1} P_j \sum_{i=j}^{K-1} \sum_{l=0}^{p_j-1} P_i \chi_{I_{k,K,m}^{j,i,l}}(x) \right) \\ &=: \|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}^{-1} (A_k(x) + B_k(x)).\end{aligned}$$

Then

$$\left\| \sum_{k\in\mathbb{Z}} \mu_k^t \sigma_*\left(a^k\right)^t \chi_{\{\tau_k=\infty\}} \right\|_{\frac{p(\cdot)}{t}} \lesssim \left\| \sum_{k\in\mathbb{Z}} 2^{kt} A_k^t \right\|_{\frac{p(\cdot)}{t}} + \left\| \sum_{k\in\mathbb{Z}} 2^{kt} B_k^t \right\|_{\frac{p(\cdot)}{t}} =: Z_1 + Z_2.$$

Let us choose $\max(1, p_+) < r < \infty$. By Lemma 2.2, there is $g \in L_{\left(\frac{p(\cdot)}{t}\right)'}$ with norm less than 1 such that

$$\begin{aligned} Z_1 &\lesssim \int_0^1 \sum_{k\in\mathbb{Z}} 2^{kt} \sum_{K\in\mathbb{N}} \sum_m P_K^{-t} \sum_{j=0}^{K-1} P_j^t \sum_{l=0}^{p_j-1} \chi_{I_{k,K,m}^{j,l}} |g| \, d\lambda \\ &\le \sum_{k\in\mathbb{Z}} 2^{kt} \sum_{K\in\mathbb{N}} \sum_m P_K^{-t} \sum_{j=0}^{K-1} P_j^t \sum_{l=0}^{p_j-1} \left\| \chi_{I_{k,K,m}^{j,l}} \right\|_{\frac{r}{t}} \left\| \chi_{I_{k,K,m}^{j,l}} g \right\|_{\left(\frac{r}{t}\right)'} \\ &\lesssim \sum_{k\in\mathbb{Z}} 2^{kt} \sum_{K\in\mathbb{N}} \sum_m P_K^{-t} \sum_{j=0}^{K-1} P_j^t \sum_{l=0}^{p_j-1} \int_0^1 \chi_{I_{k,K,m}} \left(\frac{1}{\lambda(I_{k,K,m}^{j,l})} \int_{I_{k,K,m}^{j,l}} |g|^{\left(\frac{r}{t}\right)'} \right)^{1/\left(\frac{r}{t}\right)'} d\lambda \end{aligned}$$

because $\lambda(I_{k,K,m}) = \lambda(I^{j,l}_{k,K,m}) = P_K^{-1}$. Applying Hölder's inequality again and the boundedness of the Vilenkin system, we conclude

$$Z_1 \lesssim \int_0^1 \sum_{k\in\mathbb{Z}} 2^{kt} \sum_{K\in\mathbb{N}} \sum_m \chi_{I_{k,K,m}} \sum_{j=0}^{K-1} \left(\frac{P_j}{P_K}\right)^{t(1/(\frac{r}{t})+1/(\frac{r}{t})')} \sum_{l=0}^{p_j-1} \left(\frac{1}{\lambda(I^{j,l}_{k,K,m})} \int_{I^{j,l}_{k,K,m}} |g|^{(\frac{r}{t})'}\right)^{1/(\frac{r}{t})'} d\lambda$$

$$\lesssim \int_0^1 \sum_{k\in\mathbb{Z}} 2^{kt} \sum_{K\in\mathbb{N}} \sum_m \chi_{I_{k,K,m}} \left(\sum_{j=0}^{K-1} \left(\frac{P_j}{P_K}\right)^t\right)^{1/(\frac{r}{t})} \left(\sum_{j=0}^{K-1} \left(\frac{P_j}{P_K}\right)^t \sum_{l=0}^{p_j-1} \left(\frac{1}{\lambda(I^{j,l}_{k,K,m})} \int_{I^{j,l}_{k,K,m}} |g|^{(\frac{r}{t})'}\right)\right)^{1/(\frac{r}{t})'} d\lambda$$

By (6.3),

$$Z_1 \lesssim \int_0^1 \sum_{k\in\mathbb{Z}} 2^{kt} \sum_{K\in\mathbb{N}} \sum_m \chi_{I_{k,K,m}} \left(\sum_{j=0}^{K-1} \left(\frac{P_j}{P_K}\right)^t \sum_{l=0}^{p_j-1} \left(\frac{1}{\lambda(I^{j,l}_{k,K,m})} \int_{I^{j,l}_{k,K,m}} |g|^{(\frac{r}{t})'}\right)\right)^{1/(\frac{r}{t})'} d\lambda$$

$$\leq \int_0^1 \sum_{k\in\mathbb{Z}} 2^{kt} \sum_{K\in\mathbb{N}} \sum_m \chi_{I_{k,K,m}} \left(U_t\left(|g|^{(\frac{r}{t})'}\right)\right)^{1/(\frac{r}{t})'} d\lambda$$

$$\leq \left\| \sum_{k\in\mathbb{Z}} 2^{kt} \sum_{K\in\mathbb{N}} \sum_m \chi_{I_{k,K,m}} \right\|_{p(\cdot)/t} \left\| \left(U_t\left(|g|^{(\frac{r}{t})'}\right)\right)^{1/(\frac{r}{t})'} \right\|_{(p(\cdot)/t)'}.$$

Using Theorem 6.4, we get

$$Z_1 \lesssim \left\| \sum_{k\in\mathbb{Z}} 2^{kt} \sum_{K\in\mathbb{N}} \sum_m \chi_{I_{k,K,m}} \right\|_{\frac{p(\cdot)}{t}} \|g\|_{L_{\left(\frac{p(\cdot)}{t}\right)'}} \lesssim \left\| \sum_{k\in\mathbb{Z}} 2^{kt} \chi_{\{\tau_k<\infty\}} \right\|_{\frac{p(\cdot)}{t}},$$

whenever

$$\frac{1}{((p(\cdot)/t)'/(r/t)')_-} - \frac{1}{((p(\cdot)/t)'/(r/t)')_+} = \frac{r/(r-t)}{p_+/(p_+-t)} - \frac{r/(r-t)}{p_-/(p_--t)} < t. \tag{8.4}$$

Since r can be arbitrarily large, this means that

$$\frac{p_+-t}{p_+} - \frac{p_--t}{p_-} < t,$$

which is exactly (8.2).

In the estimation of Z_2, we have to use $p_- > t > 1/2$. Choose $\max(1, p_+) < r < \infty$ large enough such that $2t > r/(r-t)$. Moreover, choose again a function $g \in L_{\left(\frac{p(\cdot)}{\varepsilon}\right)'}$ with $\|g\|_{L_{\left(\frac{p(\cdot)}{\varepsilon}\right)'}} \leq 1$ such that

$$Z_2 \lesssim \int_0^1 \sum_{k\in\mathbb{Z}} 2^{kt} \sum_{K\in\mathbb{N}} \sum_m P_K^{-2t} \sum_{j=0}^{K-1} P_j^t \sum_{i=j}^{K-1} \sum_{l=0}^{p_j-1} P_i^t \chi_{I^{j,i,l}_{k,K,m}} |g|\, d\lambda$$

$$\lesssim \sum_{k\in\mathbb{Z}} 2^{kt} \sum_{K\in\mathbb{N}} \sum_m \sum_{j=0}^{K-1} \left(\frac{P_j}{P_K}\right)^t \sum_{i=j}^{K-1} \sum_{l=0}^{p_j-1} \left(\frac{P_i}{P_K}\right)^t \left\|\chi_{I^{j,i,l}_{k,K,m}}\right\|_{\frac{r}{t}} \left\|\chi_{I^{j,i,l}_{k,K,m}} g\right\|_{(\frac{r}{t})'}$$

$$\lesssim \sum_{k\in\mathbb{Z}} 2^{kt} \sum_{K\in\mathbb{N}} \sum_{m} \sum_{j=0}^{K-1} \left(\frac{P_j}{P_K}\right)^t \sum_{i=j}^{K-1} \sum_{l=0}^{p_j-1} \left(\frac{P_i}{P_K}\right)^t \left(\frac{P_i}{P_K}\right)^{-1} \int_0^1 \chi_{I_{k,K,m}} \left(\frac{1}{\lambda(I_{k,K,m}^{j,i,l})} \int_{I_{k,K,m}^{j,i,l}} |g|^{\left(\frac{r}{t}\right)'}\right)^{1/\left(\frac{r}{t}\right)'} d\lambda.$$

Moreover,

$$Z_2 \lesssim \int_0^1 \sum_{k\in\mathbb{Z}} 2^{kt} \sum_{K\in\mathbb{N}} \sum_{m} \chi_{I_{k,K,m}} \sum_{j=0}^{K-1} \sum_{i=j}^{K-1} \left(\frac{P_j P_i}{P_K^2}\right)^{t\left(1/\left(\frac{r}{t}\right)+1/\left(\frac{r}{t}\right)'\right)} \left(\frac{P_i}{P_K}\right)^{-1} \sum_{l=0}^{p_j-1} \left(\frac{1}{\lambda(I_{k,K,m}^{j,i,l})} \int_{I_{k,K,m}^{j,i,l}} |g|^{\left(\frac{r}{t}\right)'}\right)^{1/\left(\frac{r}{t}\right)'} d\lambda$$

$$\lesssim \int_0^1 \sum_{k\in\mathbb{Z}} 2^{kt} \sum_{K\in\mathbb{N}} \sum_{m} \chi_{I_{k,K,m}} \left(\sum_{j=0}^{K-1} \sum_{i=j}^{K-1} \left(\frac{P_j}{P_K}\right)^t \left(\frac{P_i}{P_K}\right)^t\right)^{\frac{t}{r}}$$

$$\left(\sum_{j=0}^{K-1} \sum_{i=j}^{K-1} \left(\frac{P_j}{P_K}\right)^t \left(\frac{P_i}{P_K}\right)^t \left(\frac{P_i}{P_K}\right)^{-\left(\frac{r}{t}\right)'} \sum_{l=0}^{p_j-1} \frac{1}{\lambda(I_{k,K,m}^{j,i,l})} \int_{I_{k,K,m}^{j,i,l}} |g|^{\left(\frac{r}{t}\right)'}\right)^{1/\left(\frac{r}{t}\right)'} d\lambda$$

$$\lesssim \int_0^1 \sum_{k\in\mathbb{Z}} 2^{kt} \sum_{K\in\mathbb{N}} \sum_{m} \chi_{I_{k,K,m}} \left(\sum_{j=0}^{K-1} \sum_{i=j}^{K-1} \left(\frac{P_j}{P_K}\right)^{2t-r/(r-t)} \left(\frac{P_j}{P_i}\right)^{r/(r-t)-t} \sum_{l=0}^{p_j-1} \frac{1}{\lambda(I_{k,K,m}^{j,i,l})} \int_{I_{k,K,m}^{j,i,l}} |g|^{\left(\frac{r}{t}\right)'}\right)^{1/\left(\frac{r}{t}\right)'} d\lambda.$$

Note that $((p(\cdot)/t)')_+ < \infty$ and $\left(\frac{r}{t}\right)' < (p(\cdot)/t)'$. By Theorem 7.3,

$$Z_2 \le \int_0^1 \sum_{k\in\mathbb{Z}} 2^{kt} \sum_{K\in\mathbb{N}} \sum_{m} \chi_{I_{k,K,m}} \left(V_{2t-r/(r-t),r/(r-t)-t}\left(|g|^{\left(\frac{r}{t}\right)'}\right)\right)^{1/\left(\frac{r}{t}\right)'} d\lambda$$

$$\le \left\| \int_0^1 \sum_{k\in\mathbb{Z}} 2^{kt} \sum_{K\in\mathbb{N}} \sum_{m} \chi_{I_{k,K,m}} \right\|_{p(\cdot)/t} \left\| \left(V_{2t-r/(r-t),r/(r-t)-t}\left(|g|^{\left(\frac{r}{t}\right)'}\right)\right)^{1/\left(\frac{r}{t}\right)'} \right\|_{(p(\cdot)/t)'}$$

$$\lesssim \left\| \sum_{k\in\mathbb{Z}} 2^{kt} \chi_{\{\tau<\infty\}} \right\|_{\frac{p(\cdot)}{t}},$$

whenever (8.4) and (8.2) hold. Combining the estimates of Z_1 and Z_2, we finish the proof. □

Remark 8.3. If $1 \le p_- < \infty$, then (8.2) holds for all p_+. If $1/2 < p_- < 1$, then p_+ can be chosen such that $p_+ > 1$ and (8.2) holds.

We immediately get the boundedness of σ_* from $H_{p(\cdot)}$ to $L_{p(\cdot)}$ by the above theorems.

Theorem 8.4. *Let $p(\cdot) \in C^{\log}$ satisfy (8.2). If $1/2 < p_- < \infty$, then*

$$\|\sigma_* f\|_{p(\cdot)} \lesssim \|f\|_{H_{p(\cdot)}} \qquad (f \in H_{p(\cdot)}).$$

For $p = 1$, this theorem is due to Simon [33]. For other constant p's with $1/2 < p \le \infty$, this theorem was proved by the author in [40]. If $p(\cdot) = p$ and $p \le 1/2$, then the theorem is not true anymore (see Simon and Weisz [34, 36] and Goginava [15]). If (8.2) does not hold, then a counterexample can be found in Theorem 8.10 below. This theorem implies the next consequences about the convergence of $\sigma_n f$ in the usual way (see, e.g., [18, 42]).

Corollary 8.5. *Let $p(\cdot) \in C^{\log}$ satisfy (8.2). If $1/2 < p_- < \infty$ and $f \in H_{p(\cdot)}$, then $\sigma_n f$ converges almost everywhere on $[0,1)$ and in the $L_{p(\cdot)}$-norm.*

Let I be a Vilenkin interval with length P_k^{-1}. The restriction of a martingale f to I is defined by

$$f\chi_I := \left(E_n f\chi_I, n \ge k\right).$$

Corollary 8.6. *Let $p(\cdot) \in C^{\log}$ satisfy (8.2), $1/2 < p_- < \infty$ and $f \in H_{p(\cdot)}$. If there exists a Vilenkin interval I such that the restriction $f\chi_I \in L_1(I)$, then*

$$\lim_{n\to\infty} \sigma_n f(x) = f(x) \qquad \text{for a.e. } x \in I \text{ and in the } L_{p(\cdot)}\text{-norm.}$$

Since $f \in H_{p(\cdot)}$ with $1 \le p_- < \infty$ implies that f is integrable, we obtain the next corollary.

Corollary 8.7. *Let $p(\cdot) \in C^{\log}$ satisfy (8.2), $1 \le p_- < \infty$ and $f \in H_{p(\cdot)}$. Then*

$$\lim_{n\to\infty} \sigma_n f(x) = f(x) \qquad \text{for a.e. } x \in [0,1) \text{ and in the } L_{p(\cdot)}\text{-norm.}$$

Note that $H_{p(\cdot)}$ is equivalent to $L_{p(\cdot)}$ if $1 < p_- < \infty$. Considering only $\sigma_{P_n} f$, we do not need the restriction $1/2 < p_-$ about $p(\cdot) \in \mathcal{P}$.

Theorem 8.8. *Let $p(\cdot) \in C^{\log}$ satisfy (8.2) and $0 < t < \underline{p}$. Then*

$$\left\| \sum_k \mu_k^t \sup_{n\in\mathbb{N}} \left|\sigma_{P_n}\left(a^k\right)\right|^t \chi_{\{\tau_k=\infty\}} \right\|_{\frac{p(\cdot)}{t}} \lesssim \left\| \sum_k 2^{kt} \chi_{\{\tau_k<\infty\}} \right\|_{\frac{p(\cdot)}{t}}, \tag{8.5}$$

where $\mu_k = 3 \cdot 2^k \|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}$ and τ_k is the stopping time associated with the $p(\cdot)$-atom a^k.

Proof. Taking into account (5.2) and the proof of Theorem 8.2, we can suppose that $n \ge K$. For $x \notin I_{k,K,m}$, we obtain

$$\begin{aligned}
\left|\sigma_{P_n}\left(a^k \chi_{I_{k,K,m}}\right)(x)\right| &\le C \sum_{j=0}^{n} \frac{P_j}{P_n} \sum_{l=1}^{p_j-1} \int_{I_{k,K,m}} |a^k(t)| D_{P_n}\left(x \dot{+} l P_{j+1}^{-1} \dot{-} t\right) dt \\
&\lesssim \|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}^{-1} P_K^{-1} \sum_{j=0}^{K-1} P_j \sum_{l=0}^{p_j-1} \int_{I_{k,K,m}} D_{P_n}\left(x \dot{+} l P_{j+1}^{-1} \dot{-} t\right) dt \\
&\lesssim \|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}^{-1} P_K^{-1} \sum_{j=0}^{K-1} P_j \sum_{l=0}^{p_j-1} \chi_{I_{k,K,m}^{j,l}}(x).
\end{aligned}$$

If $x \in \{\tau = \infty\}$, then

$$\begin{aligned}
\sup_{n\in\mathbb{N}} \left|\sigma_{P_n}\left(a^k\right)\right| &\lesssim \|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}^{-1} \sum_{K\in\mathbb{N}} \sum_m P_K^{-1} \sum_{j=0}^{K-1} P_j \sum_{l=0}^{p_j-1} \chi_{I_{k,K,m}^{j,l}}(x) \\
&= \|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}^{-1} A_k(x)
\end{aligned}$$

and the proof can be finished as in Theorem 8.2. □

We deduce the next result from this and Theorem 8.1.

Theorem 8.9. *If $p(\cdot) \in C^{\log}$ and (8.2) is satisfied, then*

$$\left\| \sup_{n\in\mathbb{N}} |\sigma_{P_n} f| \right\|_{p(\cdot)} \lesssim \|f\|_{H_{p(\cdot)}} \qquad (f \in H_{p(\cdot)}).$$

Neither Theorem 8.9 nor 8.4 hold if (8.2) is not satisfied. More exactly, we show

Theorem 8.10. *Let $p(\cdot) \in C^{\log}$. If*

$$\frac{1}{p_-(I_{0,n-1})} - \frac{1}{p_+(I_{0,n}^{0,1})} > 1 \tag{8.6}$$

for all $n \in \mathbb{N}$, then σ_* as well as $\sup_{n\in\mathbb{N}} |\sigma_{P_n}|$ are not bounded from $H_{p(\cdot)}$ to $L_{p(\cdot)}$.

Proof. Let

$$a_{n-1}(t) = P_{n-1}^{1/p_-(I_{0,n-1})} \left(p_{n-1}\chi_{I_{0,n}} - \chi_{I_{0,n-1}}\right)$$

and $x \notin I_{0,n-1}$. One can show that a_{n-1} is an atom multiplied by a constant for all $n \geq 1$, and so $\|a_{n-1}\|_{H_{p(\cdot)}} \leq C$. As in Theorem 8.8,

$$\begin{aligned}
\left|\sigma_{P_n} a_{n-1}(x)\right| &= \left| \sum_{j=0}^{n} \frac{P_j}{P_n} \sum_{l=1}^{p_j-1} c_{j,l} \int_0^1 a_{n-1}(t) D_{P_n}\left(x \dot{+} l P_{j+1}^{-1} \dot{-} t\right) dt \right| \\
&= \left| \sum_{j=0}^{n-2} \frac{P_j}{P_n} \sum_{l=1}^{p_j-1} c_{j,l} \chi_{I_{0,n-1}^{j,l}}(x) \int_{I_{0,n-1}} a_{n-1}(t) D_{P_n}\left(x \dot{+} l P_{j+1}^{-1} \dot{-} t\right) dt \right| \\
&= \sum_{j=0}^{n-2} \frac{P_j}{P_n} \sum_{l=1}^{p_j-1} |c_{j,l}| \chi_{I_{0,n-1}^{j,l}}(x) \left| \int_{I_{0,n-1}} a_{n-1}(t) D_{P_n}\left(x \dot{+} l P_{j+1}^{-1} \dot{-} t\right) dt \right|,
\end{aligned}$$

where the numbers $c_{j,l}$ are given in (5.2). Observe that $|c_{j,l}| \geq C > 0$. We choose $j = 0$, $l = 1$ and $x \in I_{0,n}^{0,1}$ to obtain

$$|\sigma_{P_n} a_{n-1}(x)| \geq C \chi_{I_{0,n}^{0,1}}(x) P_n^{-1} P_{n-1}^{1/p_-(I_{0,n-1})}.$$

Then

$$\begin{aligned}
\int_0^1 \sup_{k\in\mathbb{N}} |\sigma_{P_k} a_{n-1}(x)|^{p(x)}\, dx &\geq \int_0^1 |\sigma_{P_n} a_{n-1}(x)|^{p(x)}\, dx \\
&\geq \int_{I_{0,n}^{0,1}} P_n^{-p(x)} P_{n-1}^{p(x)/p_-(I_{0,n-1})}\, dx \\
&\geq C \int_{I_{0,n}^{0,1}} P_n^{p_+\left(I_{0,n}^{0,1}\right)\left(1/p_-(I_{0,n-1})-1\right)}\, dx \\
&= C P_n^{p_+\left(I_{0,n}^{0,1}\right)\left(1/p_-(I_{0,n-1})-1\right)} P_n^{-1}
\end{aligned}$$

which tends to infinity as $n \to \infty$ if (8.6) holds. □

The following corollaries can be shown as above.

Corollary 8.11. *Let $p(\cdot) \in C^{\log}$ satisfy (8.2). If $f \in H_{p(\cdot)}$, then $\sigma_{P_n} f$ converges almost everywhere on $[0,1)$ and in the $L_{p(\cdot)}$-norm.*

Corollary 8.12. *Let $p(\cdot) \in C^{\log}$ satisfy (8.2). If $f \in H_{p(\cdot)}$ and there exists a Vilenkin interval I such that the restriction $f\chi_I \in L_1(I)$, then*

$$\lim_{n\to\infty} \sigma_{P_n} f(x) = f(x) \qquad \text{for a.e. } x \in I \text{ and in the } L_{p(\cdot)}\text{-norm.}$$

9 | THE MAXIMAL FEJÉR OPERATOR ON $H_{P(\cdot),Q}$

In this section, we extend the main results in Section 8 to the variable Hardy–Lorentz space setting. Using the ideas of Section 8, the theorems can be proved similarly to the corresponding results for the Walsh system, so we omit the proofs. The results for Walsh–Fourier series are due to Jiao et al. [18].

Theorem 9.1. *Let $p(\cdot) \in C^{\log}$ and $0 < q \le \infty$. Suppose that the σ-sublinear operator $T : L_\infty \to L_\infty$ is bounded and*

$$\big\| |Ta|^\beta \chi_{\{\tau=\infty\}} \big\|_{p(\cdot)} \le C \big\| \chi_{\{\tau<\infty\}} \big\|_{p(\cdot)}^{1-\beta}$$

for some $0 < \beta < 1$ and all $p(\cdot)$-atoms a, where τ is the stopping time associated with a. Then we have

$$\|Tf\|_{L_{p(\cdot),q}} \lesssim \|f\|_{H_{p(\cdot),q}} \qquad (f \in H_{p(\cdot),q}).$$

Applying this theorem, we can prove the next one.

Theorem 9.2. *Let $p(\cdot) \in C^{\log}$ satisfy (8.2). If $1/2 < p_- < \infty$, then*

$$\big\| |\sigma_* a|^\beta \chi_{\{\tau=\infty\}} \big\|_{p(\cdot)} \le C \big\| \chi_{\{\tau<\infty\}} \big\|_{p(\cdot)}^{1-\beta}$$

for some $0 < \beta < 1$ and for all $p(\cdot)$-atoms a, where τ is the stopping time associated with a.

Theorem 9.3. *Let $p(\cdot) \in C^{\log}$ satisfy (8.2). If $0 < q \le \infty$, then*

$$\Big\| \sup_{n\in\mathbb{N}} |\sigma_{P_n} f| \Big\|_{L_{p(\cdot),q}} \lesssim \|f\|_{H_{p(\cdot),q}} \qquad (f \in H_{p(\cdot),q}).$$

If in addition $1/2 < p_- < \infty$, then

$$\|\sigma_* f\|_{L_{p(\cdot),q}} \lesssim \|f\|_{H_{p(\cdot),q}} \qquad (f \in H_{p(\cdot),q}).$$

This theorem was proved by the author in [40] for $H_{p,q}$ with constant p. For a constant p with $p \le 1/2$, the second inequality does not hold (see the remark after Theorem 8.4).

The Vilenkin polynomials are dense in $H_{p(\cdot),q}$ with $0 < q < \infty$. For $q = \infty$, we denote by $\mathscr{H}_{p(\cdot),\infty}$ the closure of the Vilenkin polynomials in $H_{p(\cdot),\infty}$. Under $H_{p(\cdot),\infty}$, we mean $\mathscr{H}_{p(\cdot),\infty}$.

Corollary 9.4. *Let $p(\cdot) \in C^{\log}$ satisfy (8.2) and $0 < q \le \infty$. If $f \in H_{p(\cdot),q}$, then $\sigma_{P_n} f$ converges almost everywhere on $[0,1)$ and in the $L_{p(\cdot),q}$-norm. If in addition $1/2 < p_- < \infty$, then the convergence holds for $\sigma_n f$, too.*

Corollary 9.5. *Let $p(\cdot) \in C^{\log}$ satisfy (8.2), $0 < q \le \infty$ and $f \in H_{p(\cdot),q}$. If there exists a Vilenkin interval I such that the restriction $f\chi_I \in L_1(I)$, then*

$$\lim_{n\to\infty} \sigma_{P_n} f(x) = f(x) \qquad \text{for a.e. } x \in I \text{ and in the } L_{p(\cdot),q}(I)\text{-norm.}$$

If in addition $1/2 < p_- < \infty$, then the convergence holds for $\sigma_n f$, too.

Corollary 9.6. *Let $p(\cdot) \in C^{\log}$ satisfy (8.2), $1 \le p_- < \infty$, $0 < q \le \infty$ and $f \in H_{p(\cdot),q}$. Then*

$$\lim_{n\to\infty} \sigma_n f(x) = f(x) \qquad \text{for a.e. } x \in [0,1) \text{ and in the } L_{p(\cdot),q}\text{-norm.}$$

Now we prove that for integrable functions the limit of $\sigma_n f$ is exactly the function. Since $L_1 \subset H_{1,\infty}$, more exactly,

$$\|f\|_{H_{1,\infty}} = \sup_{\rho>0} \rho\, \lambda(M(f) > \rho) \le C\|f\|_1 \qquad (f \in L_1),$$

(see, e.g., Weisz [38]), we obtain the next corollary, which was shown for Walsh–Fourier series by Fine [8], Schipp [31] and Weisz [39] and for Vilenkin–Fourier series by Pál and Simon [28] and Weisz [41].

Corollary 9.7. *If $f \in L_1$, then*

$$\lim_{n\to\infty} \sigma_n f(x) = f(x) \qquad \text{for a.e. } x \in [0,1).$$

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ORCID

Ferenc Weisz https://orcid.org/0000-0002-7766-2745

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