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Quantitative Fractional Helly and (p, q)-Theorems

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ABSTRACT

We consider quantitative versions of Helly-type questions, that is, instead of finding a point in the intersection, we bound the volume of the intersection. Our first main result is a quantitative version of the Fractional Helly Theorem of Katchalski and Liu, the second one is a quantitative version of the (p, q)-Theorem of Alon and Kleitman.

1. Introduction

The study of quantitative versions of Helly-type questions was initiated by Bárány, Katchalski and Pach in [6], where the ***Quantitative Volume Theorem*** is shown, stating the following. *Assume that the intersection of any 2d members of a finite family of convex sets in $\mathbb{R}^d$ is of volume at least one. Then the volume of the intersection of all members of the family is of volume at least $c(d)$, a constant depending on d only.*

In [6], it is proved that one can take $c(d) = d^{-2d^2}$ and conjectured that it should hold with $c(d) = d^{-cd}$ for an absolute constant $c > 0$. It was confirmed with $c(d) \approx d^{-2d}$ in [20], whose argument was refined by Brazitikos [8], who showed that one may take $c(d) \approx d^{-3d/2}$. For more on quantitative Helly-type results, see the surveys [10,15].

Quantitative variants of the Fractional Helly Theorem and the (p, q)-Theorem (see the statements below) have been obtained in [11,21,22]. In the present note, we prove such results as well. The main difference with those works is that we aim at low Helly numbers, but, in return, we obtain worse bounds on the volume. Thus, our work may be considered rough approximation of some intersections, while the earlier works are fine approximations.

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The ***Fractional Helly Theorem*** due to Katchalski and Liu [17] (see also [18, Chapter 8]) states the following. *Fix a dimension d, and an $\alpha \in (0, 1)$, and let $\mathcal{C}$ be a finite family of convex sets in $\mathbb{R}^d$ with the property that among the subfamilies of $\mathcal{C}$ of size $d+1$, there are at least $\alpha\binom{|\mathcal{C}|}{d+1}$ for which the intersection of the $d+1$ members is nonempty. Then, there is a subfamily $\mathcal{C}' \subset \mathcal{C}$ of size $|\mathcal{C}'| \geq \frac{\alpha}{d+1}|\mathcal{C}|$ such that the intersection of all members of $\mathcal{C}'$ is nonempty.*

Our first main result is the following, a Quantitative Fractional Helly Theorem (QFH in short).

Theorem 1.1 (*QFH*)**.** *For every dimension $d \geq 1$ and every $\alpha \in (0, 1)$, there is a $\beta \in (0, 1)$ such that the following holds.*

Let $\mathcal{C}$ be a finite family of convex sets in $\mathbb{R}^d$. Assume that among all subfamilies of size $3d+1$, there are at least $\alpha\binom{|\mathcal{C}|}{3d+1}$ for whom the intersection of the $3d+1$ members is of volume at least one.

Then, there is a subfamily $\mathcal{C}' \subset \mathcal{C}$ of size at least $\beta|\mathcal{C}|$ such that the intersection of all members of $\mathcal{C}'$ is of volume at least d^{-cd^2} with a universal constant $c > 0$.

In [21], using results on fine approximation of a convex body by polytopes, it is shown that one may find a subfamily whose members intersect in a set of volume at least $1-\varepsilon$ (in place of our d^{-cd^2}), if size $\varepsilon^{-O(d^2)}$ subfamilies are checked (in place of our $3d+1$). We may call such result a *'fine' Fractional Helly theorem*: the size (volume or surface area or some other quantity) of the intersection of the subfamily that we find is close to the size of the intersection of the subfamilies that need to be checked.

Remark 1.2 (*Ellipsoids and Volume*)**.** A well known consequence of John's Theorem, is that the volume of any compact convex set K in $\mathbb{R}^d$ with non-empty interior is at most d^d times larger than the volume of the largest volume ellipsoid contained in K. More precise bounds for this volume ratio are known (cf. [4]), but we will not need them. Thus, *from this point on, we phrase our results in terms of the volume of ellipsoids contained in intersections*. Its benefit is that this is how in the proofs we actually "find volume": we find ellipsoids of large volume.

In this spirit, we re-state our result in terms of ellipsoids. It immediately yields Theorem 1.1 by Remark 1.2.

Theorem 1.3 (*QFH in Terms of Ellipsoids*)**.** *For every dimension $d \geq 1$ and every $\alpha \in (0, 1)$, there is a $\beta \in (0, 1)$ such that the following holds.*

Let $\mathcal{C}$ be a finite family of convex sets in $\mathbb{R}^d$. Assume that among all subfamilies of size $3d+1$, there are at least $\alpha\binom{|\mathcal{C}|}{3d+1}$ for which the intersection of the $3d+1$ members contains an ellipsoid of volume one.

Then, there is a subfamily $\mathcal{C}' \subset \mathcal{C}$ of size at least $\beta|\mathcal{C}|$ such that the intersection of all members of $\mathcal{C}'$ contains an ellipsoid of volume at least $c^{d^2} d^{-5d^2/2+d}$, where c is the universal constant from Theorem 2.1.

Remark 1.4. The lower bound on β obtained by our proof is $\frac{\alpha}{C^d}$ with a universal constant $C > 1$. Note that in the case of the classical Fractional Helly Theorem, the sharp bound on β shown in [16] by Kalai is $\beta = 1-(1-\alpha)^{\frac{1}{d+1}}$, see also [12] by Eckhoff.

Holmsen [14] recently showed that, in an abstract setting, a colorful Helly-type result implies a fractional one. In [22], the authors combine Holmsen's result with a Quantitative Colorful Helly Theorem ([9, Proposition 1.4], [22, Corollary 1.0.3]) to obtain a quantitative fractional result, Proposition 1.5. We note that this approach works only when the size of subfamilies considered is large, $\frac{d(d+3)}{2}$. Theorem 1.3 is the first fractional Helly-type result on volume that concerns subfamilies of linear size and not quadratic in the dimension.

Proposition 1.5 (*QFH — Large Subfamilies, [22, Theorem 5.2.1]*)**.** *For every dimension $d \geq 1$ and every $\alpha \in (0, 1)$ the following holds.*

Let $\mathcal{C}$ be a finite family of convex bodies in $\mathbb{R}^d$. Assume that among all subfamilies of size $\frac{d(d+3)}{2}$, there are at least $\alpha\binom{|\mathcal{C}|}{\frac{d(d+3)}{2}}$ for which the intersection of the $\frac{d(d+3)}{2}$ members contains an ellipsoid of volume one.

Then, there is a subfamily $\mathcal{C}' \subset \mathcal{C}$ of size at least $\beta|\mathcal{C}|$ such that the intersection of all members of $\mathcal{C}'$ contains an ellipsoid of volume one, where $\beta = \frac{2\alpha}{d(d+3)}$.

To help compare with Theorem 1.3, we give a short proof of Proposition 1.5 in Section 3.

Remark 1.6. In general, the value of β we obtain is better than the one Holmsen's abstract result [14] would yield, which is roughly $\left(\frac{\alpha}{3d^4}\right)^{\left(d/\sqrt{2}\right)^{d^2}}$. On the other hand, in [14], it is shown that $\beta \to 1$ as $\alpha \to 1$.

Next, we turn to the (**p**, **q**)-**Theorem**, a strong generalization of Helly's Theorem, conjectured by Hadwiger and Debrunner in 1957 [13] and proved by Alon and Kleitman [3] in 1992. It states that *there is a function H such that if we have a family $\mathcal{C}$ of convex sets in $\mathbb{R}^d$, and among any p of them there are q with a nonempty intersection ($p \geq q \geq d+1$), then there is a set T with $|T| \leq H(p, q, d)$ such that every member of $\mathcal{C}$ contains at least one point from T.*

Observe that for any $p \geq q \geq d+1$, we have that $H(p, q, d) \leq H(p, d+1, d)$. Thus, to show the existence of the function H, it is sufficient to show that $H(p, d+1, d)$ is finite for all $p \geq d+1$.

Definition 1 (*Quantitative v-Transversal Number*)**.** For a $v > 0$, we say that a family T of ellipsoid of volume v is a *quantitative v-transversal* of a family $\mathcal{C}$ of convex bodies in $\mathbb{R}^d$, if every $C \in \mathcal{C}$ contains at least one ellipsoid from T. The quantitative v-transversal number $\tau = \tau(\mathcal{C}, v)$ for $\mathcal{C}$ is the smallest cardinality of a quantitative v-transversal.

If there is a $C \in \mathcal{C}$ that contains no ellipsoid of volume v, then we set $\tau(\mathcal{C}, v) = \infty$.

Our second main result follows. We phrase it without reference to q in the definition of H, in fact, we fix q at the smallest possible value for which we can prove the statement, $q = 3d+1$.

Theorem 1.7 (*Quantitative (p, q)-Theorem – Small Lower Bound on p*)**.** *For every dimension $d \geq 1$ and every positive integer $p \geq 3d+1$, there is an integer $H = H(p, d)$ such that the following holds.*

Let $\mathcal{C}$ be a finite family of convex sets in $\mathbb{R}^d$, each containing an ellipsoid of volume one, and assume that among any p members of $\mathcal{C}$, there exist $3d+1$ whose intersection contains an ellipsoid of volume one.

Then, $\mathcal{C}$ has quantitative v-transversal number at most $H(p, d)$ with $v = c^{d^2} d^{-5d^2/2+d}$, where c is the universal constant from Theorem 2.1.

In [21], it is shown in a very general setting that a 'fine' Fractional Helly Theorem implies a 'fine' (p, q)-Theorem. In [22], an 'intermediate range' result is obtained: p needs to be at least $\frac{d(d+3)}{2}$ (that is stronger than our assumption $p \geq 3d+1$), and, in return, we obtain a quantitative 1-transversal, that is, there is no loss in the volume of the ellipsoids. Theorem 1.7 is the first (p, q)-type result on the volume with a lower bound on p which is linear and not quadratic in the dimension.

Proposition 1.8 (*Quantitative (p, q)-Theorem – Large Lower Bound on p, [22, Theorem 5.0.1]*)**.** *For every dimension $d \geq 1$ and every positive integer $p \geq \frac{d(d+3)}{2}$, there is an integer $H = H(p, d)$ such that the following holds.*

Let $\mathcal{C}$ be a finite family of convex sets in $\mathbb{R}^d$, each containing an ellipsoid of volume one, and assume that among any p members of $\mathcal{C}$, there exist $\frac{d(d+3)}{2}$ whose intersection contains an ellipsoid of volume one.

Then, $\mathcal{C}$ has quantitative 1-transversal number at most $H(p, d)$.

For other 'fine' (p, q) theorems, that is, results where the number of selected sets is much larger (say, exponential in d) and, in return, one obtains a good approximation of the volume (and not just the volume of the largest ellipsoid contained in a set), see [19] and [11].

The structure of the paper is the following. In Section 2, we introduce some of our tools, mostly from [9]. We prove Proposition 1.5 in Section 3. Next, as a preparation for the (p, q) results, in Section 5, we prove a quantitative version of the **Selection Lemma** and the **Weak Epsilon Net Theorem** (see the statements of these two classical results in that section). Next, in Section 6, we give a short direct proof of Proposition 1.8. The arguments in these sections are modeled on proofs of the corresponding classical (i.e., non-quantitative) results: the combinatorial core of those arguments are extracted, and then combined with geometric facts from [9], which guarantee uniqueness of certain ellipsoids contained in a convex body (see Lemmas 2.4 and 2.5).

In contrast, the proofs of Theorems 1.3 and 1.7 require further ideas, that are presented in Sections 4 and 7. The difficulty in proving these results, where "rough approximation" is required, that is, the number of sets to be selected is $O(d)$ and not $O(d^2)$ (and, in return, there is a substantial loss of volume) is explained at the beginning of Section 7.

2. Preliminaries

We will rely on the following quantitative Helly theorem that is phrased in terms of the volume of an *ellipsoid* contained in a convex body.

Theorem 2.1 (*Quantitative Helly Theorem*)**.** *Let $C_1, \dots, C_n$ be convex sets in $\mathbb{R}^d$. Assume that the intersection of any 2d of them contains an ellipsoid of volume at least one. Then $\bigcap_{i=1}^n C_i$ contains an ellipsoid of volume at least $c^d d^{-3d/2}$ with an absolute constant $c > 0$.*

Theorem 2.1 was stated in [20] for volumes of intersections and not volumes of ellipsoids with the weaker bound $c^d d^{-2d}$, which was an improvement of the volume bound d^{-2d^2} given by Bárány, Katchalski, and Pach [6]. The proof in [20] clearly yields containment of ellipsoids as stated herein, and the argument was later refined by Brazitikos [8], who obtained the bound $c^d d^{-3d/2}$ as stated above.

Another one of our key tools is the following observation from [9].

Lemma 2.2 (*[9, Lemma 3.2]*)**.** *Assume that the origin centered Euclidean unit ball, $\mathbf{B}^d$ is the largest volume ellipsoid contained in the convex set C in $\mathbb{R}^d$. Let E be another ellipsoid in C of volume at least $\delta \mathrm{vol}\left(\mathbf{B}^d\right)$ with $0 < \delta < 1$. Then there is a translate of $\frac{\delta}{d^{d-1}} \mathbf{B}^d$ which is contained in E.*

We state an immediate corollary of the Fractional Helly Theorem (see page 1).

Proposition 2.3. *For every dimension $d \geq 1$ and every $\alpha \in (0, 1)$, the following holds.*

Let $\mathcal{C}$ be a finite family of convex sets in $\mathbb{R}^d$ and let L be a convex set in $\mathbb{R}^d$. Assume that among all subfamilies of size $d+1$, there are at least $\alpha\binom{|\mathcal{C}|}{d+1}$ for which the intersection of the $d+1$ members contains a translate of L.

Then, there is a subfamily $\mathcal{C}' \subset \mathcal{C}$ of size at least $\frac{\alpha}{d+1}|\mathcal{C}|$ such that the intersection of all members of $\mathcal{C}'$ contains a translate of L.

Proof of Proposition 2.3. We use the following operation, the *Minkowski difference* of two convex sets A and B:

$$A \sim B := \bigcap_{b \in B} (A - b).$$

It is easy to see that $A \sim B$ is the set of those vectors t such that $B + t \subseteq A$.

Now, replace each convex set C in $\mathcal{C}$ by $C \sim L$, and apply the Fractional Helly Theorem (see page 1). □

The following definition and two lemmas introduce the unique lowest ellipsoid of volume one contained in a convex body.

Definition 2. For an ellipsoid E, we define its *height* as the largest value of the orthogonal projection of E on the last coordinate axis.

Lemma 2.4 (*Uniqueness of Lowest Ellipsoid, [9, Lemma 2.5]*)**.** *Let C be a convex body, such that it contains an ellipsoid of volume one. Then there is a unique ellipsoid of volume one such that every other ellipsoid of volume one in C has larger height. We call this ellipsoid the* lowest ellipsoid *in C.*

Lemma 2.5 (*Lowest Ellipsoid Determined by $O(d^2)$ Members of an Intersection [9, Lemma 3.1]*). *Let $C_1, \dots, C_n$ be a finite family of convex bodies in $\mathbb{R}^d$ whose intersection contains an ellipsoid of volume one. Then, there are $d(d+3)/2-1$ indices $i_1, \dots, i_{d(d+3)/2-1} \in [n]$ such that $\bigcap_{i=1}^n C_i$, and $\bigcap_{j=1}^{d(d+3)/2-1} C_{i_j}$ have the same unique lowest ellipsoid.*

3. Proof of Proposition 1.5 – QFH for large subfamilies

The following argument follows closely the proof of Theorem 8.1.1. in [18], the only difference is the use of the unique lowest ellipsoid of a convex body (Lemma 2.4) instead of its lexicographic minimum.

Let $\mathcal{C} = \{C_1, \dots, C_n\}$. We call an index set $I \in \binom{[n]}{\frac{d(d+3)}{2}}$ *good*, if the corresponding intersection $\cap_{i\in I} C_i$ contains an ellipsoid of volume at least one. We say that a $\left(\frac{d(d+3)}{2}-1\right)$-element subset $S \subset I$ of a good index set $I \in \binom{[n]}{\frac{d(d+3)}{2}}$ is a *seed* of I, if $\cap_{i\in I} C_i$ and $\cap_{i\in S} C_i$ have the same lowest ellipsoid. By Lemma 2.5, all good index sets have a seed.

Since we have $\alpha \binom{n}{\frac{d(d+3)}{2}}$ good index sets and only $\binom{n}{\frac{d(d+3)}{2}-1}$ possible seeds, there is a $\left(\frac{d(d+3)}{2}-1\right)$-tuple $S \in \binom{[n]}{\frac{d(d+3)}{2}-1}$ which is the seed of at least

$$\frac{\alpha \binom{n}{\frac{d(d+3)}{2}}}{\binom{n}{\frac{d(d+3)}{2}-1}} = \alpha \frac{n - \frac{d(d+3)}{2} + 1}{\frac{d(d+3)}{2}}$$

good index sets. Every such good index set has the form $S \cup \{i\}$ for some $i \in [n] \setminus S$. So we have $\alpha \frac{n-\frac{d(d+3)}{2}+1}{d(d+3)/2}$ convex bodies containing the lowest ellipsoid of $\cap_{i\in S} C_i$, plus the $(\frac{d(d+3)}{2}-1)$ bodies from S. Hence, the lowest ellipsoid of $\cap_{i\in S} C_i$ is contained in at least

$$\alpha \frac{n+1-d(d+3)/2}{d(d+3)/2} + \frac{d(d+3)}{2} - 1 \geq \frac{2\alpha n}{d(d+3)}$$

convex bodies among the C_i, completing the proof of Proposition 1.5.

4. Proof of Theorem 1.3 – QFH for small subfamilies

Let $\mathcal{C} = \{C_1, \dots, C_n\}$. We call an index set $I \in \binom{[n]}{3d+1}$ *good*, if the corresponding intersection $\cap_{i\in I} C_i$ contains an ellipsoid of volume at least one. We say that a $2d$-element subset $S \subset I$ of a good index set $I \in \binom{[n]}{3d+1}$ is a *seed* of I, if the volume of the John ellipsoid of $\cap_{i\in S} C_i$ is at most $c^{-d} d^{3d/2}$ times the volume of the John ellipsoid of $\cap_{i\in I} C_i$, where c is the absolute constant from Theorem 2.1. By Theorem 2.1, all good index sets have a seed.

Since we have $\alpha \binom{n}{3d+1}$ good index sets and only $\binom{n}{2d}$ possible seeds, there is a $(2d)$-tuple $S \in \binom{[n]}{2d}$ which is the seed of at least

$$\frac{\alpha \binom{n}{3d+1}}{\binom{n}{2d}} \geq \gamma \binom{n}{d+1}$$

good index sets. Here γ depends on α and d, but not on n.

Let $I_1, \dots, I_{\gamma\binom{n}{d+1}}$ be good index sets whose seed is S. Denote the John ellipsoid of the intersection $\cap_{i\in S} C_i$ by $\mathcal{E}$ and the John ellipsoid of $\cap_{i\in I_j} C_i$ by $\mathcal{E}_j$. By Lemma 2.2, for every j, there is a $v_j \in \mathbb{R}^d$ such that $c^d d^{-5d/2+1} \mathcal{E} + v_j \subseteq \mathcal{E}_j$.

Thus, we have shown that at least $\gamma \binom{n}{d+1}$ of the $(d+1)$-wise intersections contain a translate of $c^d d^{-5d/2+1} \mathcal{E}$. We can apply Proposition 2.3 with $L = c^d d^{-5d/2+1} \mathcal{E}$, which implies that there are $\frac{\gamma}{d+1} n$ such C_i, that their intersection contains a translate of $c^d d^{-5d/2+1} \mathcal{E}$. And, since $\mathcal{E}$ has volume at least one, this ellipsoid has volume at least $c^{d^2} d^{-5d^2/2+d}$, completing the proof of Theorem 1.3.

5. Roadmap to (p, q): Selection lemma and weak epsilon net

In [2], partly by extracting the combinatorial arguments presented in earlier works, it is shown in an abstract setting (that is, working with hypergraphs in general, and not specifically with convex sets in $\mathbb{R}^d$) that a (p, q)-theorem may be obtained from a fractional Helly-type theorem in the following manner. First, combined with a Tverberg-type theorem, a fractional Helly-type theorem yields a selection lemma [2, Proposition 11], which in turn yields a weak epsilon-net theorem [2, Theorem 9] using a greedy algorithm. On the other hand, if a hypergraph satisfies the (p, q) condition, then the hypothesis of the fractional Helly-type theorem holds. It follows that the fractional transversal number of the hypergraph is bounded from above, which, combined with a weak epsilon-net, yields that its transversal number is bounded as well, which is what the (p, q)-theorem states. In the rest of the paper, we will mark where we follow this path, and where we do not.

In this section, we state the above listed classical (non-quantitative) results along with their quantitative analogs.

Tverberg's Theorem (see [23], and for a simpler proof [24]) states the following. *For every dimension $d \geq 1$ and integer $r \geq 1$, if $m \geq (r-1)(d+1)+1$ and $\{x_1, \dots, x_m\}$ is a set of points in $\mathbb{R}^d$ of size m, then there is a partition $\cup_{i=1}^r I_i = [m]$ such that $\cap_{i=1}^r \mathrm{conv}\left(\{x_j : j \in I_i\}\right)$ is not empty.*

It is shown in [2, Proposition 10] that in an abstract setting, a fractional Helly-type theorem yields a Tverberg-type theorem. However, the Tverberg number (the lower bound on m) obtained there is very large. Luckily, we have a quantitative Tverberg theorem due to Sarkar, Xue and Soberón with a much better Tverberg number.

Theorem 5.1 (*Quantitative Tverberg Theorem [22, Theorem 4.1.2]*). *For every dimension $d \geq 1$ and integer $r \geq 1$, if $m \geq (r-1)\left(\frac{d(d+3)}{2}+1\right)+1$ and $\{E_1, \dots, E_m\}$ is a multiset of ellipsoids of volume one, then there is a partition $\cup_{i=1}^r I_i = [m]$ such that $\cap_{i=1}^r \mathrm{conv}\left(\{E_j : j \in I_i\}\right)$ contains an ellipsoid of volume one.*

We will need the following form of the above result.

Corollary 5.2 (*Quantitative Tverberg Theorem with Equal Parts*). *For every dimension $d \geq 1$, if $a = \left(\frac{d(d+3)}{2}-1\right)\left(\frac{d(d+3)}{2}+1\right)+1$, $b = \frac{d(d+3)}{2}$ and $\{E_1, \dots, E_{ab}\}$ is a multiset of ellipsoids of volume one, then there is a partition $\cup_{i=1}^b I_i = [ab]$ such that $|I_1| = \cdots = |I_b| = a$ and $\cap_{i=1}^b \mathrm{conv}\left(\{E_j : j \in I_i\}\right)$ contains an ellipsoid of volume one.*

Proof of Corollary 5.2. Pick any a element subset $I \subset [ab]$ and use Theorem 5.1 with $r = b$. It yields a partition $\cup_{i=1}^b I_i' = I$ such that $\cap_{i=1}^b \mathrm{conv}\left(\{E_j : j \in I_i'\}\right)$ contains an ellipsoid of volume one. Now complete the parts of the obtained partition from the remaining $(ab-a)$ indexes into a-element disjoint index sets $I_1 \supset I_1', I_2 \supset I_2', \dots, I_b \supset I_b'$. Since $\mathrm{conv}\left(\{E_j : j \in I_i'\}\right) \subset \mathrm{conv}\left(\{E_j : j \in I_i\}\right)$, the partition $\cup_{i=1}^b I_i = [ab]$ will have the desired properties. □

The **Selection Lemma** (the planar version first proved by Boros and Füredi [7], the general version due to Bárány [5]) states the following. *For every dimension $d \geq 1$, there exists a $\lambda \in (0, 1)$ with the following property. If $\{x_1, \dots, x_n\}$ is a multiset of points in $\mathbb{R}^d$, then there is a subset $\mathcal{H} \subseteq \binom{[n]}{d+1}$ with $|\mathcal{H}| \geq \lambda \binom{n}{d+1}$ such that $\cap_{I \in \mathcal{H}} \mathrm{conv}\left(\{x_j : j \in I\}\right)$ is not empty. .*

Lemma 5.3 (*Quantitative Selection Lemma*). *For every dimension $d \geq 1$, there exists an integer $a(d)$ and a real number $\lambda(d) \in (0, 1)$ with the following property. If $n \geq a$ and $\{E_1, \dots, E_n\}$ is a multiset of volume one ellipsoids in $\mathbb{R}^d$, then there is a subset $\mathcal{H} \subseteq \binom{[n]}{a}$ with $|\mathcal{H}| \geq \lambda \binom{n}{a}$ such that $\cap_{I \in \mathcal{H}} \mathrm{conv}\left(\cup\{E_j : j \in I\}\right)$ contains an ellipsoid of volume one.*

The proof follows closely the proof of [2, Proposition 11].

Proof of Lemma 5.3.

Let $a = \left(\frac{d(d+3)}{2}-1\right)\left(\frac{d(d+3)}{2}+1\right)+1$ and $b = \frac{d(d+3)}{2}$ as in Corollary 5.2.

Let $\mathcal{S}=\left\{\operatorname{conv}\left(\cup\{E_i\ :\ i\in I\}\right)\ :\ I\in\binom{[n]}{a}\right\}$. Our plan is to show that a positive fraction of b-tuples in $\mathcal{S}$ has an intersection which contains an ellipsoid of volume one in order to apply Proposition 1.5 to $\mathcal{S}$. Let

$$T=\Bigg\{\{I_1,\dots,I_b\}\ :\ I_i\in\binom{[n]}{a}, I_i\cap I_j=\emptyset \text{ for } i\neq j \text{ and}$$
$$\cap_{i=1}^{b}\operatorname{conv}\left(\cup\{E_j\ :\ j\in I_i\}\right) \text{ contains an ellipsoid of volume } 1\Bigg\}.$$

Observe that $|T|$ is at least $\binom{n}{ab}$, since, according to Corollary 5.2, for each $J\in\binom{[n]}{ab}$ there exists pairwise disjoint $I_1,\dots,I_b\in\binom{J}{a}$ such that $\cap_{i=1}^{b}\operatorname{conv}\left(\cup\{E_j\ :\ j\in I_i\}\right)$ contains an ellipsoid of volume one, and so each J contributes a different b-tuple in T. Hence

$$|T|\geq\binom{n}{ab}\geq\left(\frac{n}{ab}\right)^{ab}\geq\frac{1}{(ab)^{(ab)}}\binom{\binom{n}{a}}{b},$$

which means that we can apply Proposition 1.5 to $\mathcal{S}$ and conclude that a $\lambda(d)=\beta\left(\frac{1}{(ab)^{(ab)}},d\right)$ fraction of the members of $\mathcal{S}$ has an intersection that contains an ellipsoid with volume 1. This completes the proof of Lemma 5.3. □

The **Weak Epsilon Net Theorem**, proved by Alon, Bárány, Füredi and Kleitman [1] states the following. *For every dimension $d\geq 1$ there exists a function $f:(0,1]\to\mathbb{R}$ with the following property. For any $\varepsilon\in(0,1]$, if $\mathcal{C}$ is a finite family of convex bodies in $\mathbb{R}^d$, and $w:\mathbb{R}^d\to[0,1]$ is a weight function such that $\sum_{x\in C}w(x)\geq\varepsilon$ for all $C\in\mathcal{C}$, and $\sum_{x\in\mathbb{R}^d}w(x)=1$, then there is a set $S\subset\mathbb{R}^d$ such that each $C\in\mathcal{C}$ contains an element of S, and S is of size at most $f(\varepsilon)$.*

Theorem 5.4 (*Existence of Quantitative Weak ε-Nets*)**.** *For every dimension $d\geq 1$ there exists a function $f:(0,1]\to\mathbb{R}$ with the following property.*

For any $\varepsilon\in(0,1]$, if $\mathcal{C}$ is a finite family of convex bodies in $\mathbb{R}^d$, and $\mathcal{E}$ is a finite family of volume one ellipsoids in $\mathbb{R}^d$, and $w:\mathcal{E}\to[0,1]$ is a rational weight function on this family of ellipsoids such that $\sum_{E\in\mathcal{E},E\subseteq C}w(E)\geq\varepsilon$ for all $C\in\mathcal{C}$, and $\sum_{E\in\mathcal{E}}w(E)=1$, then there is a family $\mathcal{S}$ of ellipsoids of volume one such that each $C\in\mathcal{C}$ contains a member of $\mathcal{S}$, and $\mathcal{S}$ is of size at most $f(\varepsilon)$.

Proof (*Sketch of the Proof of Theorem 5.4*)**.** The existence of quantitative weak epsilon nets follows from the Quantitative Selection Lemma (Lemma 5.3), using a greedy algorithm. A very similar proof can be found in [18, Theorem 10.4.2.]. We provide an outline of the proof for the readers convenience.

Since w is rational, there is an integer D and an integer valued function $\tilde{w}$ such that $w(E)=\frac{\tilde{w}(E)}{D}$ for all $E\in\mathcal{E}$. Let $\tilde{\mathcal{E}}$ be the multiset containing $\tilde{w}(E)$ copies from each $E\in\mathcal{E}$. We construct the set $\mathcal{S}$ greedily choosing ellipsoids one by one. We start with the empty set $\mathcal{S}_0=\emptyset$. Once $\mathcal{S}_i$ is constructed, if there is a $C\in\mathcal{C}$ containing no ellipsoid from $\mathcal{S}_i$, then let $\tilde{\mathcal{E}}_i$ be the ellipsoids from $\tilde{\mathcal{E}}$ contained in C. The size of $\tilde{\mathcal{E}}_i$ is at least εD. The Quantitative Selection Lemma, Lemma 5.3 guarantees an ellipsoid E in the convex hulls of at least $\lambda(d)\binom{|\tilde{\mathcal{E}}_i|}{a(d)}$ different $a(d)$-tuples from $\tilde{\mathcal{E}}_i$. Let $\mathcal{S}_{i+1}=\mathcal{S}_i\cup\{E\}$.

We obtain an upper bound on $f(\varepsilon)$ by observing that at the beginning, there are $\binom{|\tilde{\mathcal{E}}|}{a}$ a-tuple with no ellipsoid from $\mathcal{S}_0$ in their convex hulls, and this number of 'not hit' a-tuples decreases by at least $\lambda\binom{\varepsilon|\tilde{\mathcal{E}}|}{a}$ in each step. □

We note that by using Theorem 5.1, we obtain a smaller $f(\varepsilon)$ than the one guaranteed by [2, Theorem 9], where only combinatorial arguments are used.

6. Proof of Proposition 1.8 – quantitative (p,q)-theorem with a large bound on p

Let $\mathcal{C}$ be a finite family of convex bodies in $\mathbb{R}^d$ as in Proposition 1.8. We will represent $\mathcal{C}$ as a finite hypergraph. For each subfamily of $\mathcal{C}$, take the lowest volume one ellipsoid contained in the intersection of that subfamily, if there is such an ellipsoid. This way, we have a finite family of volume one ellipsoids. This family of ellipsoids, denote it by $\mathcal{E}$, will be the vertex set of our hypergraph. For each $C \in \mathcal{C}$, consider the subfamily of ellipsoids in $\mathcal{E}$ that are contained in C. The edges of the hypergraph will be subfamilies of ellipsoids obtained this way. We denote this hypergraph by $(\mathcal{E}, \mathcal{H})$.

A **fractional transversal** of a hypergraph is a function $\varphi : \mathcal{E} \to [0, 1]$ such that for every $H \in \mathcal{H}$, the sum $\sum_{E\in H} \varphi(E)$ is at least one. The fractional transversal number of $(\mathcal{E}, \mathcal{H})$ is just the infimum of $\sum_{E\in\mathcal{E}} \varphi(E)$ over all fractional transversals.

In [2, Theorem 8], the authors establish an upper bound for the fractional transversal number of all hypergraphs satisfying a certain fractional Helly property and a (p, q)-property.

Theorem 6.1 ([2]). *Let $(V, \mathcal{H})$ be a hypergraph with the following properties.*

There is a function $\beta : (0, 1) \to (0, 1)$ such that for every $H_1, \dots, H_n \in \mathcal{H}$, if at least $\alpha\binom{n}{q}$ of the q-tuples from $\{H_1, \dots, H_n\}$ have nonempty intersection with some $\alpha \in (0, 1)$, then there are a $\beta(\alpha)n$ members from $\{H_1, \dots, H_n\}$ having nonempty intersection.

There are positive integers $p \geq q$ such that among any p members of $\mathcal{H}$, some q have a nonempty intersection.

In this case, the fractional transversal number of $(V, \mathcal{H})$ can be bounded with a function depending on p, q and β only.

Proposition 1.5 guarantees that our $(\mathcal{E}, \mathcal{H})$ satisfies the fractional Helly property with $q = \frac{d(d+3)}{2}$. The (p, q)-property needed in Theorem 6.1 is assumed in Proposition 1.8. Hence by Theorem 6.1 and Proposition 1.5, the fractional transversal number of $(\mathcal{E}, \mathcal{H})$ is bounded from above by some $T > 0$ that depends on d only.

Note that since any fractional transversal is a solution of a linear program with integer coefficients, there is an optimal solution with rational φ. Our Quantitative Weak ε-Net Theorem, Theorem 5.4, applied with $\varepsilon = 1/T$ and a normalized φ now completes the proof of Proposition 1.8.

In summary, the proof of Proposition 1.8 mainly follows the classical line of reasoning that yields the (p, q)-Theorem. As we will see in the next section, to obtain Theorem 1.3, some other ideas are required.

7. Proof of Theorem 1.7 – quantitative (p, q)-theorem with a small bound on p

First, we discuss why the same argument as in the previous section cannot be repeated. The main idea in Section 6 was that we considered a hypergraph representing our convex sets, and studied properties of this hypergraph. In the setting of Theorem 1.7, however, there are *two* hypergraphs: we obtain one if we consider ellipsoids of volume one in our convex sets, and we obtain another, if ellipsoids of volume v are considered (with $v = c^{d^2} d^{-5d^2/2+d}$). Thus, some additional care is required.

Definition 3 (*Quantitative Fractional v-Transversal Number*). For a subset $S \subseteq \mathbb{R}^d$, let $E_v(S)$ be the set of volume v ellipsoids contained in S. Let $\mathcal{C}$ be a family of subsets of $\mathbb{R}^d$ and let $\varphi : E_v(\mathbb{R}^d) \to [0, 1]$ be a function that attains only finitely many nonzero values. We say that φ is a *quantitative fractional v-transversal* for $\mathcal{C}$, if $\sum_{E\in E_v(C)} \varphi(E) \geq 1$ for all $C \in \mathcal{C}$. The *quantitative fractional transversal number* of $\mathcal{C}$ is the infimum of $\sum_{E\in E_v(\mathbb{R}^d)} \varphi(x)$ over all quantitative fractional transversals φ of $\mathcal{C}$.

Lemma 7.1 (*Boundedness of Quantitative Fractional v-Transversal Number*). *For every d and $p \geq 3d+1$ there exists a $h = h(p, d) > 0$ with the following property.*

Let $\mathcal{C}$ be a finite family of convex sets in $\mathbb{R}^d$, each containing an ellipsoid of volume one, and assume that among any p members of $\mathcal{C}$, there exist $3d + 1$ whose intersection contains an ellipsoid of volume one.

Then $\mathcal{C}$ has quantitative fractional v-transversal number less than h with $v = c^{d^2} d^{-5d^2/2+d}$, where c is the universal constant from Theorem 2.1.

Proof. We are going to use linear programming duality as in [3], but first we need the definition of quantitative fractional v-matching numbers.

Definition 4. Let $\mathcal{C}$ be a finite family of convex sets in $\mathbb{R}^d$ and let $m : \mathcal{C} \to [0, 1]$ a function. We say that m is a *quantitative fractional v-matching* for $\mathcal{C}$, if for every ellipsoid $E \subset \mathbb{R}^d$ with volume v, the sum $\sum_{E\subset C\in\mathcal{C}} m(C)$ is at most 1. The *quantitative fractional v-matching number* of $\mathcal{C}$ is the supremum of $\sum_{C\in\mathcal{C}} m(C)$ over all quantitative fractional v-matchings of $\mathcal{C}$.

Let $v = c^{d^2} d^{-5d^2/2+d}$. Note that if we consider each member C of $\mathcal{C}$ as the collection of all volume v ellipsoids that it contains, then we obtain a hypergraph with finitely many edges, but an infinite vertex set. However, we can replace this infinite vertex set by a finite one, just like at the beginning of Section 6, and obtain another hypergraph with the same (fractional) transversal and matching numbers as those of the original hypergraph. Now, we may apply the duality of linear programming to see that the quantitative fractional v-matching number and the quantitative fractional v-transversal number are equal for any family of convex sets $\mathcal{C}$. We denote it by $\nu_v^*(\mathcal{C})$.

We know also, that being a solution of a linear program with integer coefficients, there is an optimal quantitative fractional v-matching taking only rational values. Let m be such a quantitative fractional v-matching and suppose, that $m(C) = \frac{\tilde{m}(C)}{D}$ for every C, where $\tilde{m}(C)$ and D are integers.

Let $\tilde{\mathcal{C}} = \{C_1, \ldots, C_N\}$ be the multiset that contains $\tilde{m}(C)$ copies of each $C \in \mathcal{C}$. Thus, we have

$$N = \sum_{C\in\mathcal{C}} \tilde{m}(C) = D\nu_v^*(\mathcal{C}) \tag{1}$$

Taking some sets with multiplicities does not change the quantitative fractional matching number, so $\nu_v^*(\tilde{\mathcal{C}}) = \nu_v^*(\mathcal{C})$. We know that among any p members of $\mathcal{C}$, there are $3d + 1$ whose intersection contains an ellipsoid with volume 1. So among any $\tilde{p} = 3d(p - 1) + 1$ members of $\tilde{\mathcal{C}}$ there are $3d + 1$ whose intersection contains an ellipsoid with volume 1, because the $\tilde{p}$ element multiset from $\tilde{\mathcal{C}}$ either contains $3d + 1$ copies of the same set, or p different sets from $\mathcal{C}$.

For every $I \in \binom{[N]}{\tilde{p}}$ there is a subset $J \subset I$ with $|J| = 3d + 1$ and $\cap_{j\in J} C_j$ containing an ellipsoid of volume one. So there are at least

$$\frac{\binom{N}{\tilde{p}}}{\binom{N-3d+1}{\tilde{p}-3d+1}} \geq \alpha \binom{N}{3d+1}$$

$3d + 1$-tuples from $\mathcal{C}$ whose intersection contains an ellipsoid of volume one. We can apply Theorem 1.3 and conclude that there are βN sets from $\tilde{\mathcal{C}}$ whose intersection contains an ellipsoid with volume v.

On the other hand, for any ellipsoid E of volume v, we have

$$\sum_{C\in\mathcal{C}:E\subseteq C} \frac{\tilde{m}(C)}{D} = \frac{1}{D}\#\{C \in \tilde{\mathcal{C}} \ : \ E \subseteq C\} \leq 1.$$

Thus, by (1), we have that no ellipsoid of volume v can be in more than $\frac{N}{\nu_v^*(\mathcal{C})}$ of the sets from $\tilde{\mathcal{C}}$, hence $\nu_v^*(\mathcal{C}) \leq \frac{1}{\beta}$, completing the proof of Lemma 7.1. □

Now we are ready to prove Theorem 1.7.

Proof of Theorem 1.7. If among any p members of $\mathcal{C}$ there are $3d + 1$ whose intersection contains an ellipsoid of volume 1, then from Lemma 7.1 it follows, that $\mathcal{C}$ has quantitative fractional v-transversal number at most $h(p, d)$ with $v = c^{d^2} d^{-5d^2/2+d}$. Let φ be a quantitative fractional v-transversal of size $h(p, d)$, and for every ellipsoid E with volume v let $w(E) = \frac{\varphi(E)}{h(p,d)}$. We can use Theorem 5.4 with w and $\mathcal{C}$ and conclude that there is a $\frac{1}{h}$-net S with volume v ellipsoids for $\mathcal{C}$ of size at most $f\left(\frac{1}{h}\right)$. Since for every $C \in \mathcal{C}$ the inequality $\sum_{E\subset C} w(E) \geq \frac{1}{h}$ holds, every $C \in \mathcal{C}$ contains at least one ellipsoid from S and Theorem 1.7 follows with $H(p, d) = |S| = f\left(\frac{1}{h(p,d)}\right)$. □

8. Concluding remarks

The following questions are left open. First, can the Helly number $3d + 1$ be replaced by $2d$ in Theorems 1.1, 1.3, 1.7? It would be interesting to see such result, even if the volume bound on the ellipsoid is worse than the d^{-cd^2}, which we obtained.

Second, can the volume bound in the above mentioned theorems be replaced by d^{-cd}?

Finally, as mentioned in Section 3, according to [14] by Holmsen, a colorful Helly type theorem yields a fractional Helly type theorem in the abstract setting, when one hypergraph is considered (see Section 7 for the explanation of one hypergraph vs. two). Can one give a quantitative analog of this result? More precisely, if we make some assumptions on a pair of hypergraphs, then does a colorful Helly type theorem that holds for the pair of hypergraphs imply the corresponding fractional result for the same pair? That would mean that Theorem 1.1 follows from [9] by an abstract argument.

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References

[1] Noga Alon, Imre Bárány, Zoltán Füredi, Daniel J. Kleitman, Point selections and weak ϵ-nets for convex hulls, Combin. Probab. Comput. 1 (3) (1992) 189–200.
[2] Noga Alon, Gil Kalai, Jiří Matoušek, Roy Meshulam, Transversal numbers for hypergraphs arising in geometry, Adv. Appl. Math. 29 (1) (2002) 79–101.
[3] Noga Alon, Daniel J. Kleitman, Piercing convex sets and the Hadwiger-Debrunner (p, q)-problem, Adv. Math. 96 (1) (1992) 103–112.
[4] Keith Ball, An elementary introduction to modern convex geometry, in: Flavors of Geometry, in: Math. Sci. Res. Inst. Publ., vol. 31, Cambridge Univ. Press, Cambridge, 1997, pp. 1–58.
[5] Imre Bárány, A generalization of Carathéodory's theorem, Discrete Math. 40 (2–3) (1982) 141–152.
[6] Imre Bárány, Meir Katchalski, János Pach, Quantitative Helly-type theorems, Proc. Amer. Math. Soc. 86 (1) (1982) 109–114.
[7] Endre Boros, Zoltán Füredi, The number of triangles covering the center of an n-set, Geom. Dedicata 17 (1) (1984) 69–77.
[8] Silouanos Brazitikos, Brascamp-Lieb inequality and quantitative versions of Helly's theorem, Mathematika 63 (1) (2017) 272–291.
[9] Gábor Damásdi, Viktória Földvári, Márton Naszódi, Colorful Helly-type theorems for the volume of intersections of convex bodies, J. Combin. Theory Ser. A 178 (2021) 105361.
[10] Jesús A. De Loera, Xavier Goaoc, Frédéric Meunier, Nabil H. Mustafa, The discrete yet ubiquitous theorems of Carathéodory, Helly, Sperner, Tucker, and Tverberg, Bull. Amer. Math. Soc. (N.S.) 56 (3) (2019) 415–511.
[11] Jesús A. De Loera, Reuben N. La Haye, David Rolnick, Pablo Soberón, Quantitative combinatorial geometry for continuous parameters, Discrete Comput. Geom. 57 (2) (2017) 318–334.
[12] Jürgen Eckhoff, An upper-bound theorem for families of convex sets, Geom. Dedicata 19 (1985) 217–227.
[13] Hugo Hadwiger, Hans Debrunner, Über eine Variante zum Hellyschen Satz, Arch. Der Math. 8 (4) (1957) 309–313.
[14] Andreas F. Holmsen, Large cliques in hypergraphs with forbidden substructures, Combinatorica 40 (4) (2020) 527–537.
[15] Andreas F. Holmsen, Rephael Wenger, Helly-type theorems and geometric transversals, in: Jacob E. Goodman, Joseph O'Rourke, Csaba D. Tóth (Eds.), Handbook of Discrete and Computational Geometry, CRC Press, Boca Raton, FL, 2018, pp. 91–123, Third edition.
[16] Gil Kalai, Intersection patterns of convex sets, Israel J. Math. 48 (2–3) (1984) 161–174.
[17] Meir Katchalski, Andrew Liu, A problem of geometry in R^n, Proc. Amer. Math. Soc. 75 (2) (1979) 284–288.
[18] Jiři Matousek, Lectures on Discrete Geometry, Graduate Texts in Mathematics, vol. 212, Springer-Verlag, New York, 2002, p. xvi+481.
[19] John A. Messina, Pablo Soberón, Isometric and affine copies of a set in volumetric helly results, 2020, http://arxiv.org/abs/2010.04135.
[20] Márton Naszódi, Proof of a conjecture of Bárány, Katchalski and Pach, Discrete Comput. Geom. 55 (1) (2016) 243–248.
[21] David Rolnick, Pablo Soberón, Quantitative (p,q) theorems in combinatorial geometry, Discrete Math. 340 (10) (2017) 2516–2527.
[22] Sherry Sarkar, Alexander Xue, Pablo Soberón, Quantitative combinatorial geometry for concave functions, 2019, ArXiv E-Prints, http://arxiv.org/abs/1908.04438.
[23] Helge Tverberg, A generalization of Radon's theorem, J. Lond. Math. Soc. 41 (1966) 123–128.
[24] Helge Tverberg, A generalization of Radon's theorem. II, Bull. Aust. Math. Soc. 24 (3) (1981) 321–325.