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On the global rigidity of tensegrity graphs

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ABSTRACT

A tensegrity graph is a graph with edges labeled as bars, cables and struts. A realization of a tensegrity graph T is a pair (T, p), where p maps the vertices of T into $\mathbb{R}^d$ for some $d \geq 1$. The realization is globally rigid if any realization (T, q) in $\mathbb{R}^d$ in which the bars have the same length and the cables and struts are not longer and not shorter, respectively, is an isometric image of (T, p). A tensegrity graph is weakly globally rigid in $\mathbb{R}^d$ if it has a generic globally rigid realization in $\mathbb{R}^d$, and strongly globally rigid in $\mathbb{R}^d$ if every generic realization in $\mathbb{R}^d$ is globally rigid.

In this paper we give a necessary condition for weak global rigidity in $\mathbb{R}^d$ and prove that in the $d = 1$ case the same condition is also sufficient. In particular, our results imply that a tensegrity graph has a generic globally rigid realization in $\mathbb{R}^1$ if and only if it has a generic universally rigid realization in $\mathbb{R}^1$. We also show that recognizing strongly globally rigid tensegrity graphs in $\mathbb{R}^d$ is co-NP-hard for all $d \geq 1$.

1. Introduction

A *tensegrity graph* $T = (V, B, C, S)$ is a simple graph with vertex set V and labeled edge set $E = B \cup C \cup S$, where B, C and S represent, respectively, rigid bars, inextensible cables and incompressible struts. We shall call the elements of E the *members* of T. A *tensegrity framework in* $\mathbb{R}^d$, or simply a *tensegrity*, is a pair (T, p), where T is a tensegrity graph and $p : V \to \mathbb{R}^d$ is an embedding of its vertices into Euclidean space. We also say that (T, p) is a *(d-dimensional) realization* of T. Tensegrity frameworks can be used to model various physical and biological systems (see e.g. [6,9]).

Of particular interest is the study of the rigidity properties of tensegrities. Intuitively, we say that a tensegrity is *rigid* if it cannot be continuously deformed so that the length of the bars remains constant and the length of the cables (struts, respectively) does not increase (decrease, resp.). A tensegrity is *globally rigid* if the pairwise distances of the vertices are uniquely determined under these length constraints. We shall give precise definitions in the next section. One approach to the study of the rigidity and global rigidity of tensegrities is to focus on the underlying tensegrity graph and explore the extent to which the combinatorial structure of the graph determines the rigidity of its realizations. This has been especially successful in the case of *bar frameworks*, where every member is a bar.[1] In this case it has been shown that for any fixed dimension d, either all realizations in $\mathbb{R}^d$ in sufficiently general position are rigid (globally rigid, respectively), or none of them are (see [1,2,7]). The most frequently used notion of general position in this setting is that of a *generic* framework, one in which the set of all vertex coordinates is algebraically independent over the rational numbers. Thus we can speak of *(generically) rigid* and *(generically) globally rigid* graphs in $\mathbb{R}^d$, for which every generic realization in $\mathbb{R}^d$ is rigid (globally rigid, respectively). A combinatorial characterization of these graphs is known in the case of $d = 1, 2$, while finding such a characterization in the $d \geq 3$ case is a major open problem.

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[1] In the case of bar frameworks we shall view the underlying tensegrity graph simply as a graph without any edge labeling.

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For general tensegrity graphs, the situation is different: it may happen that some generic realizations are rigid, while others are not. However, we can still explore the connection between the structure of the tensegrity graph and the rigidity of its generic realizations. For example, we may ask which tensegrity graphs have a generic rigid realization in $\mathbb{R}^d$; we shall call these graphs *weakly rigid* in $\mathbb{R}^d$. We may also require every generic realization in $\mathbb{R}^d$ to be rigid; in this case we shall say that the tensegrity graph is *strongly rigid* in $\mathbb{R}^d$. Weak and strong global rigidity may be defined analogously.

These notions are not very well understood. In fact, the only general results known are concerning weak and strong rigidity in $\mathbb{R}^1$. Recski and Shai [13] gave a polynomial-time checkable combinatorial characterization of weak rigidity in $\mathbb{R}^1$. In the case of strong rigidity in $\mathbb{R}^1$, Jackson, Jordán and Király gave a combinatorial characterization in terms of the so-called alternating cycle property but showed that recognizing these tensegrity graphs is co-NP-complete [10].

In this paper we consider the analogous problems in the case of global rigidity. The situation turns out to be similar to the case of weak and strong rigidity: we give a combinatorial characterization of weak global rigidity in $\mathbb{R}^1$ which can be checked in polynomial time (Theorem 7), while we show that recognizing strongly globally rigid tensegrity graphs in $\mathbb{R}^d$ is co-NP-hard (Corollary 12). For the latter result, we introduce the *d-dimensional odd cycle property* for tensegrity graphs and show that it is a necessary condition for strong global rigidity in $\mathbb{R}^d$ that is also sufficient for some special families of tensegrity graphs, though not in general.

The rest of the paper is laid out as follows. In Section 2 we recall the definitions and results that we use throughout the paper. In Section 3 we give a necessary condition for weak global rigidity in $\mathbb{R}^d$ and show that it is also sufficient in $\mathbb{R}^1$. Finally, in Section 4 we introduce and examine the d-dimensional odd cycle property and use it to show that recognizing strongly globally rigid tensegrity graphs is co-NP-hard in $\mathbb{R}^d$ for all fixed $d \geq 1$.

2. Preliminaries

Let (T, p) and (T, q) be two d-dimensional realizations of the tensegrity graph $T = (V, B, C, S)$. We say that (T, p) *dominates* (T, q) if we have

$$\begin{aligned} \|p(u) - p(v)\| &= \|q(u) - q(v)\| && \text{for all } uv \in B, \\ \|p(u) - p(v)\| &\geq \|q(u) - q(v)\| && \text{for all } uv \in C, \\ \|p(u) - p(v)\| &\leq \|q(u) - q(v)\| && \text{for all } uv \in S. \end{aligned}$$

In this case we also say that (T, q) *satisfies the member constraints of* (T, p), or, if every member of T is a bar, that (T, q) and (T, p) are *equivalent*. Here $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$. If we have

$$\|p(u) - p(v)\| = \|q(u) - q(v)\| \qquad \text{for all } u, v \in V,$$

then we say that (T, p) and (T, q) are *congruent*.

We say that a tensegrity (T, p) is *rigid* if there is some $\varepsilon > 0$ such that any other realization (T, q) with $\|p(v) - q(v)\| < \varepsilon$ for all $v \in V$ that satisfies the member constraints of (T, p) is, in fact, congruent to it. It can be shown that a tensegrity framework is not rigid if and only if it is *flexible*: there is a continuous motion (T, p_t), $0 \leq t \leq 1$ of frameworks satisfying the member constraints of $(T, p) = (T, p_0)$ such that (T, p_t) is not congruent to (T, p) for $t > 0$, see [14]. We say that the d-dimensional tensegrity framework (T, p) is *globally rigid* if every d-dimensional realization (T, q) that satisfies the member constraints of (T, p) is congruent to (T, p). We may also consider (T, p) as a framework in $\mathbb{R}^D$ for some $D > d$ by embedding $\mathbb{R}^d$ into $\mathbb{R}^D$ as the subspace defined by the equations $x_{d+1} = 0, \ldots, x_D = 0$. We say that (T, p) is *universally rigid* if it is globally rigid as a framework in $\mathbb{R}^D$, for all $D \geq d$.

Again, we say that a tensegrity graph is *weakly rigid* (*weakly globally rigid*, respectively) in $\mathbb{R}^d$ if it has a generic rigid (globally rigid, resp.) realization in $\mathbb{R}^d$. On the other hand, we say that a tensegrity graph is *strongly rigid* (*strongly globally rigid*, respectively) in $\mathbb{R}^d$ if every generic realization of the graph in $\mathbb{R}^d$ is rigid (globally rigid, resp.). As we mentioned before, for bar graphs, weak and strong rigidity are equivalent, as well as weak and strong global rigidity. In this case, we simply say that the graph is *rigid* in $\mathbb{R}^d$ (*globally rigid in* $\mathbb{R}^d$, respectively). Note that for any tensegrity graph $T = (V, B, C, S)$, if the bar graph $G = (V, B)$ is globally rigid in $\mathbb{R}^d$ then T is strongly globally rigid in $\mathbb{R}^d$. For a different example of a tensegrity graph that is strongly (and thus weakly) globally rigid in $\mathbb{R}^1$, see Fig. 1.

Let $E = B \cup C \cup S$ denote the members of the tensegrity graph T. We shall use the notation $\overline{T}$ to denote the *underlying graph* (V, E) of T. A *strict proper stress* of a tensegrity (T, p) is a mapping $\omega : E \to \mathbb{R}$ such that the value of ω is positive on cables and negative on struts and for every vertex $v \in V$ we have

$$\sum_{u: uv \in E} \omega(uv)(p(u) - p(v)) = 0.$$

We shall need the following result, which follows from Theorem 5.2 and Theorem 5.8 of [14].

Lemma 1. *If a generic tensegrity framework is rigid, then it has a strict proper stress.*

We shall also need some notions from graph theory. Let $G = (V, E)$ be a graph. An *open ear decomposition* of G is a partition of the edge set of G into $l \geq 0$ subsets $P_0, P_1, \ldots, P_l$, where P_0 is a simple cycle and for $1 \leq i \leq l$, P_i is a simple

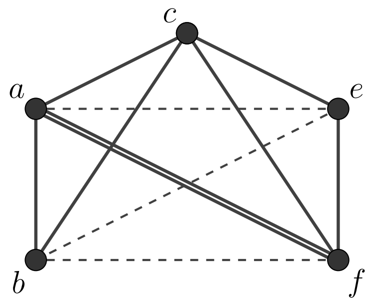

Fig. 1. An example of a tensegrity graph T that has the 1-dimensional odd cycle property. Bars, cables and struts are represented by solid, dashed and doubled lines, respectively. The bar graph of T has a unique 1-separation given by $X = \{a, b, c\}$, $Y = \{c, e, f\}$, and the cycle $ae, eb, bf, fa \in E_T(X - Y, Y - X)$ contains an odd number of cables. Lemma 9 implies that T is strongly globally rigid in $\mathbb{R}^1$.

path such that $V(\cup_{j<i}P_j) \cap V(P_i)$ consists of the two (distinct) endpoints of P_i. It is known that a graph is 2-connected if and only if it has an open ear decomposition, see [15, Theorem 19].

For subsets $X, Y \subseteq V$ we use $E_G(X, Y)$ to denote the set of edges in G with one endpoint in X and the other in Y. For an integer $d \geq 1$ and subsets $X, Y \subseteq V$ with $|X|, |Y| \geq d+1$, we say that the pair (X, Y) is a *d-separation* of G if $|X \cap Y| \leq d$, $X \cup Y = V$ and $E_G(X - Y, Y - X) = \varnothing$. We emphasize that, in contrast to some authors, we do not require $|X \cap Y| = d$ in the preceding definition. For a tensegrity graph $T = (V, B, C, S)$ and $X, Y \subseteq V$, we shall use the notation $E_T(X, Y)$ to denote the members of T with one endpoint in X and the other in Y.

If (G, p) is a bar framework in $\mathbb{R}^1$ and (X, Y) is a 1-separation of G with $X \cap Y = \{v\}$, then we may obtain an equivalent, non-congruent framework (G, p') by reflecting Y through v, so that $p'(y) = 2 \cdot p(v) - p(y)$ for $y \in Y$ and $p'(x) = p(x)$ for $x \in X$. It is folklore that if G is connected and (G, p) is generic, then, up to congruence, each realization equivalent to (G, p) may be obtained via a series of such reflections. In particular, if G has at least 3 vertices, then it is globally rigid in $\mathbb{R}^1$ if and only if it is 2-connected.

This can be generalized to higher dimensions as follows. Let (G, p) be a bar framework in $\mathbb{R}^d$ and (X, Y) a d-separation of G. Let $H \subseteq \mathbb{R}^d$ be an affine hyperplane containing the points $p(v), v \in X \cap Y$. Then we may obtain an equivalent framework (G, p') by reflecting the points $p(y), y \in Y$ through H. If neither $\{p(x), x \in X\}$ nor $\{p(y), y \in Y\}$ is contained in H (for example, if (G, p) is generic), then (G, p') and (G, p) are not congruent. In particular, we have the following theorem.

Theorem 2 (*[8, Theorem 3.1]*)**.** *Let $d \geq 1$ and let G be a graph on at least $d+2$ vertices. If G is globally rigid in $\mathbb{R}^d$, then it is $(d+1)$-connected.*

As a shorthand, we shall refer to the above construction of an equivalent framework as “reflecting Y through H”.

3. Existence of globally rigid realizations

In this section we examine the notion of weak global rigidity. We shall need the following observation regarding rigid tensegrities in $\mathbb{R}^d$. Recall that a point p of a convex set $C \subseteq \mathbb{R}^d$ is an *extreme point* of C if it cannot be written as a convex combination of points in $C - p$.

Lemma 3. *Let (T, p) be a generic rigid tensegrity in $\mathbb{R}^d$. Then every vertex v for which $p(v)$ is an extreme point of the convex hull of $\{p(u), u \in V\}$ is incident to at least one non-strut, as well as at least one non-cable member.*

Proof. By Lemma 1, (T, p) has a strict proper stress ω. Rearranging the equilibrium condition gives

$$\sum_{u:uv\in E} \omega(uv)p(u) = \sum_{u:uv\in E} \omega(uv)p(v),$$

where E denotes the members of T. Now ω cannot be positive on all members incident to v, since then dividing this equation by $\sum_{u:uv\in E} \omega(uv)$ would give that $p(v)$ is a convex combination of $p(u), uv \in E$, contradicting the assumption that $p(v)$ is an extreme point. For the same reason, ω cannot be negative on all members incident to v. This means that v is incident to at least one non-strut, as well as at least one non-cable member, as desired. □

We note that Lemma 3 remains true without the genericity assumption on (T, p) (provided that p is injective), but the proof is slightly more involved in that case.

The following lemma gives simple necessary conditions for weak global rigidity in $\mathbb{R}^d$.

Lemma 4. *If a tensegrity graph $T = (V, B, C, S)$ has a generic globally rigid realization in $\mathbb{R}^d$, then either it is a complete graph with only bar members, or it has at least $d+2$ vertices and satisfies the following conditions:*

- *The graph $\overline{T} = (V, B \cup C \cup S)$ obtained by replacing every member of T by bars is globally rigid in $\mathbb{R}^d$,*
- *T contains at least $(d+1)/2$ non-cable members,*
- *The graph $(V, B \cup C)$ is connected.*

Proof. First, suppose that T has at most $d+1$ vertices and has a generic globally rigid realization (T, p) in $\mathbb{R}^d$. By adding cables to T if necessary, we may assume that its underlying graph is complete. A globally rigid realization of T must also be rigid, so by Lemma 1 it has a strict proper stress ω. Now each vertex $v \in V$ has at most d neighbors and by genericity the vectors $p(u) - p(v)$, $uv \in B \cup C \cup S$ are linearly independent. But then the equilibrium condition for ω can only hold if $\omega(uv) = 0$ for all members uv, which implies that every member is a bar. This also means that the underlying graph of T must have been complete to begin with, as required.

Now assume that T has at least $d+2$ vertices and that T has a generic globally rigid realization (T, p). The fact that $\overline{T}$ must be globally rigid in $\mathbb{R}^d$ follows immediately from the definitions. The second condition follows from Lemma 3, since in a generic realization the convex hull must contain at least $d+1$ points from $\{p(u), u \in V\}$, all of which are extreme points. Finally, if $(V, B \cup C)$ were not connected, then we could translate in (T, p) one of its connected components by a sufficiently large amount to obtain a realization (T, q) satisfying the member constraints of (T, p) but not congruent to (T, p), contradicting global rigidity. □

Our goal in the rest of the section is to show that in the $d = 1$ case, the necessary conditions of Lemma 4 are also sufficient for weak global rigidity. The main ingredient in our proof is the following lemma. We note that this statement can be extended to (and follows from) a result regarding ear decompositions of matroids, based on e.g. [5, Theorem 5.2.9]. For the sake of completeness, we give a self-contained proof.

Lemma 5. *Let $G = (V, E)$ be a 2-connected graph. Then for any edge $e_0 \in E$ and any partition $E = E_1 \cup E_2$ of the edge set of G, where (V, E_1) is connected, there is an open ear decomposition $P_0, \ldots, P_l$ of G such that $e_0 \in P_0$ and P_i contains at most one edge from E_2 for $0 \le i \le l$.*

Proof. We may assume that $G_1 = (V, E_1)$ is a spanning tree by moving edges from E_1 to E_2. If $e_0 \in E_2$, then let the starting cycle P_0 of the ear decomposition be the unique cycle in the graph $G_1 + e_0$, and otherwise let P_0 be the unique cycle in $G_1 + e$ for any $e \in E_2$ such that $G_1 - e_0 + e$ is connected. In both cases $e_0 \in P_0$ and P_0 contains exactly one edge from E_2.

Suppose now that we have a suitable ear decomposition of some subgraph H of G. We shall show that if $V(H) \neq V$, then we can find an open ear extending H, i.e. a simple path whose end vertices are in $V(H)$ but its internal vertices are not in $V(H)$ and which contains exactly one edge from E_2. Starting from the subgraph induced by P_0 and applying this statement repeatedly yields a suitable ear decomposition of G.

Suppose, then, that $V(H) \neq V$. It follows from the connectedness of G_1 that there is an edge $uv \in E_1$ with $u \in V(H)$, $v \notin V(H)$. Observe that the existence of a suitable ear decomposition of H implies that $V(H)$ induces a connected subgraph of G_1. In particular, u cannot be a leaf vertex of G_1, so it is a cut-vertex of G_1. Let $A \subseteq V$ denote the set of vertices that are unreachable from $V(H) - u$ in $G_1 - u$. Clearly, $v \in A$, so, in particular, A is nonempty. Now, since G is 2-connected, there must be an edge $u'v'$, necessarily from E_2, such that $u' \in A$ and $u \neq v' \notin A$. Let P be the unique path in G_1 from u to u' and P' the path from v' to u. Concatenating P, $u'v'$ and P' gives a cycle in G with precisely one edge from E_2. This cycle must contain at least one other vertex from $V(H)$ besides u, since otherwise it would all lie in A, contradicting the choice of $u'v'$. Thus the cycle contains a suitable open ear as a subgraph. □

We shall use the following construction in the next two proofs. Let T be a tensegrity graph whose underlying graph is a cycle. We say that a realization (T, p) in $\mathbb{R}^1$ is a *stretched cycle* if there is some cyclic ordering $v_1, \ldots, v_n$ of the vertices of T such that $p(v_1) < p(v_2) < \cdots < p(v_n)$, $v_1 v_n$ is a strut or a bar, and $v_i v_{i+1}$ is a cable or a bar for $1 \le i \le n-1$. It is folklore that a stretched cycle is universally rigid: it is not difficult to see that it is globally rigid, and the triangle inequality ensures that for any $d \ge 1$ and any d-dimensional realization (T, q) that satisfies the member constraints of (T, p), the vertices of T all lie on a line in (T, q).

Lemma 6. *Let T be a tensegrity graph and (T, p) a generic universally rigid tensegrity in $\mathbb{R}^1$. Let T' be a tensegrity graph obtained from T by adding a path P on vertices $u = v_1, v_2, \ldots, v_k = v$ with $u, v \in V(T)$ and $v_i \notin V(T)$, $i = 2, \ldots, k-1$, and such that P contains at most one strut. Then there is a generic universally rigid tensegrity (T', p') extending (T, p).*

Proof. We may assume that $k \ge 3$ and $p(u) < p(v)$. We construct p' as follows. For every vertex $w \in V(T)$, let $p'(w) = p(w)$. If P consists only of cables and bars, then choose p' such that $p'(u) < p'(v_2) < \cdots < p'(v_{k-1}) < p'(v)$ holds. Otherwise, let $v_i v_{i+1}$ denote the single strut in P for some $1 \le i < k$ and choose p' such that $p'(v_i) < p'(v_{i-1}) < \cdots < p'(u) < p'(v) < p'(v_{k-1}) < \cdots < p'(v_{i+1})$ holds. Since these give open conditions on $p'(w)$, $w \notin V(T)$, we may choose p' to be generic.[2]

[2] Explicitly, this uses the fact that for any finite set $S \subseteq \mathbb{R}$ that is algebraically independent over the rational numbers, the numbers $x \in \mathbb{R}$ for which $S \cup \{x\}$ is also algebraically independent form a dense subset of $\mathbb{R}$.

We claim that the tensegrity (T', p') constructed in this way is universally rigid. Let (T', p'') be another tensegrity in $\mathbb{R}^d$ for some $d \geq 1$ that satisfies the member constraints of (T', p'). Since (T, p) is universally rigid, $(T, p''|_{V(T)})$ and (T, p) must be congruent, so by applying a rigid motion to (T', p'') we may assume that $p''(w) = p'(w)$ for all $w \in V(T)$. In particular, $\|p'(u) - p'(v)\| = \|p''(u) - p''(v)\|$ so we may as well assume that there is a bar connecting u and v in T'. Then $(P + uv, p'|_{V(P)})$ is a stretched cycle and its member constraints are satisfied by $(P + uv, p''|_{V(P)})$. It follows that the two tensegrity frameworks are congruent, and since $p'(u) = p''(u), p'(v) = p''(v)$, this implies $p'(w) = p''(w)$ for all $w = V(T')$, as desired. □

Theorem 7. *A tensegrity graph $T = (V, B, C, S)$ has a generic globally rigid realization in $\mathbb{R}^1$ if and only if the underlying graph $\overline{T} = (V, B \cup C \cup S)$ is 2-connected, not all members are cables and the graph $(V, B \cup C)$ is connected. In fact, if these conditions hold, then T has a generic universally rigid realization in $\mathbb{R}^1$.*

Proof. Necessity is given by Lemma 4 (recall that a bar graph is globally rigid in $\mathbb{R}^1$ if and only if it is 2-connected); we prove sufficiency by constructing a generic universally rigid realization of T. By Lemma 5, there is an open ear decomposition $P_0, P_1, \ldots, P_k$ of T such that each P_i, $0 \leq i \leq l$ contains at most one strut. Moreover, since T has a non-cable member, we may choose P_0 in such a way that it contains at least one bar or strut. We prove by induction on l that the sub-tensegrity graph of T induced by $\cup_{j \leq l} P_j$ has a generic universally rigid realization. For $l = 0$, we can realize P_0 as a generic stretched cycle. The induction step is given by Lemma 6. □

Since each of the three conditions in Theorem 7 can be checked in linear time in the number of members and vertices of T, this gives a polynomial-time checkable characterization of weak global rigidity in $\mathbb{R}^1$. We also note that it implies that a tensegrity graph in $\mathbb{R}^1$ has a generic globally rigid realization if and only if it has a generic universally rigid realization in $\mathbb{R}^1$. In the case of bar graphs this is known to hold in $\mathbb{R}^d$ for all $d \geq 1$, see [4]. It would be interesting to see whether this extends to tensegrity graphs in the $d \geq 2$ case as well.

In fact, the inductive proof strategy of Theorem 7 can be used to show that under the hypotheses of Theorem 7, the tensegrity graph has a generic realization in $\mathbb{R}^1$ that is *super stable*,[3] a property that implies universal rigidity (for the definition of super stability, see e.g. [3,11]). However, the existence of a generic super stable realization also follows from Theorem 7 and a recent result of Oba and Tanigawa [12] who showed that for generic tensegrity frameworks, universal rigidity is equivalent to super stability.

4. Strongly globally rigid tensegrity graphs

In this section we investigate strong global rigidity. As in the case of weak global rigidity, we first describe a necessary condition. Then we shall show that the same condition is sufficient for certain families of tensegrity graphs (although not in general), and use this result to prove that recognizing strongly globally rigid graphs in $\mathbb{R}^d$ is co-NP-hard.

Lemma 8. *Let $T = (V, B, C, S)$ be strongly globally rigid in $\mathbb{R}^d$. Then for any d-separation (X, Y) of the bar subgraph $G = (V, B)$, $E_T(X - Y, Y - X)$ contains a cycle with an odd number of cables.*

Proof. For clarity, we first focus on the case when $d = 1$ and the bar subgraph is connected. Suppose that the condition does not hold for some 1-separation (X, Y) of G with $X \cap Y = \{v\}$. We shall construct a realization (T, p) such that the framework (T, p') obtained by reflecting Y through v in (T, p) satisfies the member constraints of (T, p). Consider the bipartite graph H obtained from $(V, E_T(X - v, Y - v))$ by replacing each member with a bar. Let us choose an arbitrary root vertex $u_1, \ldots, u_k$ for each connected component of H. We define (T, p) in the following way: first, set $p(v) = 0$. For a vertex $u \neq v$, if u can be reached in H from one of the root vertices using an even number of cables, then let $p(u) < 0$ be an arbitrary negative value. Otherwise let $p(u) > 0$ be an arbitrary positive value. This is well-defined, for if some vertex u can be reached from some root vertex using both an even and an odd number of cables, then the concatenation of these walks is a closed walk containing an odd number of cables, and this would necessarily contain a cycle with an odd number of cables. Moreover, we may choose the position of the vertices so that (T, p) can be translated into a generic framework.

Now, in (H, p) the edges corresponding to cables have endpoints with different signs, while the edges corresponding to struts have endpoints with the same sign. It follows that by reflecting Y through v in (T, p), the length of each cable in $E_T(X - v, Y - v)$ decreases and the length of each strut increases, while the lengths of the members not in $E_T(X - v, Y - v)$ do not change. This means that the resulting tensegrity (T, p') satisfies the member constraints of (T, p). Thus, T is not strongly globally rigid, as needed.

The general case can be shown analogously, as follows. We say that a tensegrity is *quasi-generic* if it can be made generic by applying a suitable rigid motion. Clearly, to show that a tensegrity graph is not strongly globally rigid in $\mathbb{R}^d$, it is enough to construct a quasi-generic realization in $\mathbb{R}^d$ that is not globally rigid.

[3] We thank the anonymous reviewer for pointing this out.

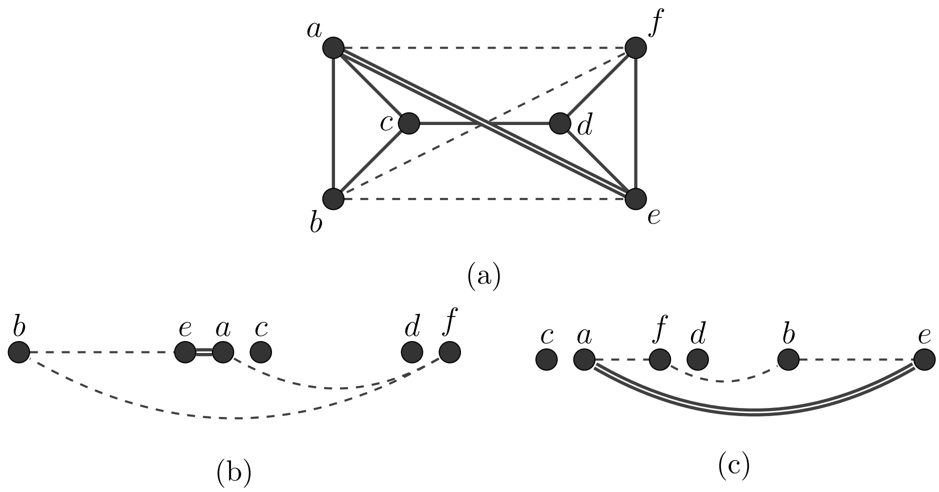

Fig. 2. An example of a tensegrity graph T which is strongly rigid in $\mathbb{R}^1$ and has the 1-dimensional odd cycle property, but is not strongly globally rigid in $\mathbb{R}^1$. In (b) and (c) the bars are not drawn. The tensegrity in (c) is obtained from the tensegrity in (b) by reflecting a and b through c, as well as e and f through d. It satisfies the member constraints of the tensegrity in (b), but is not congruent to it.

Suppose now that the for some d-separation (X, Y) of the bar subgraph G, $E_T(X - Y, Y - X)$ does not contain a cycle with an odd number of cables. To define (T, p), we first choose the points $p(v)$, $v \in X \cap Y$ such that they lie in the $x_1 = 0$ hyperplane in a quasi-generic position. Then we choose the first coordinates of $p(v)$, $v \notin X \cap Y$ as in the $d = 1$ case, and the rest of the coordinates arbitrarily, in such a way that the whole framework is in a quasi-generic position. Now the framework (T, p') obtained from (T, p) by reflecting Y through the $x_1 = 0$ hyperplane satisfies the member constraints of (T, p) but is not congruent to it, which shows that T is not strongly globally rigid in $\mathbb{R}^d$. □

We shall say that a tensegrity graph $T = (V, B, C, S)$ has the *d-dimensional odd cycle property* if it satisfies the condition given in Lemma 8, that is, if for any d-separation (X, Y) of the bar subgraph $G = (V, B)$, $E_T(X - Y, Y - X)$ contains a cycle with an odd number of cables. See Fig. 1 for an example in the $d = 1$ case. It follows from Lemma 8 that having the d-dimensional odd cycle property is a necessary condition for strong global rigidity in $\mathbb{R}^d$. Strong rigidity in $\mathbb{R}^d$ is also clearly necessary. As Fig. 2 shows, these conditions taken together are not sufficient in general, even in the $d = 1$ case; however, in the following lemmas we consider special cases where they do guarantee strong global rigidity.

Lemma 9. *Let $d \geq 1$ and let $T = (V, B, C, S)$ be a tensegrity graph such that the bar subgraph $G = (V, B)$ is obtained by taking some graphs $G_1, \dots, G_k$, each on at least $d+2$ vertices and globally rigid in $\mathbb{R}^d$, along with a sequence $v_1^i, \dots, v_d^i \in V(G_i)$ of d distinct vertices from each graph and identifying the vertices $v_j^1, \dots, v_j^k$ for each $j = 1, \dots, d$. Then T is strongly globally rigid in $\mathbb{R}^d$ if and only if it has the d-dimensional odd cycle property.*

Proof. Necessity follows from Lemma 8, so we only need to show sufficiency. Let (T, p) be a generic realization of T in $\mathbb{R}^d$ and for $1 \leq j \leq d$, let $v_j \in V$ denote the vertex in T corresponding to the vertices $v_j^1, \dots, v_j^k$. First, note that since each G_i is $(d + 1)$-connected by Theorem 2, in any d-separation (X, Y) of G, $X \cap Y$ must be $\{v_1, \dots, v_d\}$ and X must be the union of the vertex sets of some of $G_1, \dots, G_k$. Moreover, the frameworks equivalent to (G, p) are (up to congruence) precisely those frameworks that arise via taking a d-separation (X, Y) of G and reflecting Y through the affine hyperplane H spanned by $\{p(v), v \in X \cap Y\}$. This is because in an equivalent realization (G, p'), the sub-realization $(G_i, p'|_{V(G_i)})$ of each globally rigid subgraph G_i is determined up to reflection through H by the position of the d identified vertices.

We shall show that none of the tensegrities obtained in this way satisfy the member constraints of (T, p). Indeed, let (X, Y) be a d-separation of G, with H denoting, again, the affine hyperplane spanned by $\{p(v), v \in X \cap Y\}$. Then $E_T(X - Y, Y - X)$ contains a cycle with an odd number of cables. It follows that there is either a cable $u_1u_2 \in C$ in this cycle such that $p(u_1)$ and $p(u_2)$ lie on the same side of H, or a strut $u_1'u_2' \in S$ such that $p(u_1')$ and $p(u_2')$ lie on different sides of H. The length of this cable (strut, respectively) increases (decreases, resp.) if we reflect Y through H, so that the resulting tensegrity does not satisfy the member constraints of (T, p), which is what we wanted to show. □

The following lemma describes a different situation where the 1-dimensional odd cycle property and strong rigidity in $\mathbb{R}^1$ together characterize strong global rigidity in $\mathbb{R}^1$. As this lemma is not needed for our result on the hardness of recognizing strongly globally rigid tensegrity graphs, its proof, along with some discussion, is given in Appendix A.1.

Lemma 10. *Let $T = (V, B, C, S)$ be a tensegrity graph such that the bar subgraph $G = (V, B)$ has two connected components, each being globally rigid in $\mathbb{R}^1$. Then T is strongly globally rigid in $\mathbb{R}^1$ if and only if it is strongly rigid in $\mathbb{R}^1$ and it has the 1-dimensional odd cycle property.*

Finally, we consider the decision problem d-OCP in which, given a tensegrity graph $T = (V, B, C, S)$ we want to decide whether T has the d-dimensional odd cycle property.

Theorem 11. *d-OCP is co-NP-complete for every $d \geq 1$, even for tensegrity graphs $T = (V, B, C, S)$ in which the bar subgraph $G = (V, B)$ is obtained by taking some graphs $G_1, \ldots, G_k$ that are globally rigid in $\mathbb{R}^d$ along with a sequence $v_1^i, \ldots, v_d^i \in V(G_i)$ of d distinct vertices from each graph and identifying the vertices $v_j^1, \ldots, v_j^k$ for each $j = 1, \ldots, d$.*

Proof. The d-OCP problem is indeed in co-NP, since given a d-separation (X, Y) of G we can check whether the tensegrity graph H induced by the members $E_T(X - Y, Y - X)$ contains a cycle with an odd number of cables by subdividing each strut with a vertex and checking whether the resulting tensegrity graph T' contains a cycle of odd length, that is, whether the underlying graph of T' is bipartite.

In the following, let us call a d-separation (X, Y) of the bar graph of a tensegrity T *pure* if $E_T(X - Y, Y - X)$ does not contain a cycle with an odd number of cables. As a shorthand, we shall refer to cycles with an odd number of cables simply as *odd cycles*; this will not cause confusion, since for any d-separation (X, Y), $E_T(X - Y, Y - X)$ does not contain cycles that are odd in the usual sense (i.e. have odd length).

The hardness proof is along the same lines as [10, Theorem 3.1]. We shall show that there is a polynomial-time reduction from the complement of 3-SAT to d-OCP. Consider an instance of 3-SAT given by the formula φ containing n variables $x_1, \ldots, x_n$. For a variable x_i, we shall use the notation $x_i^1 = x_i, x_i^{-1} = \bar{x}_i$, where $\overline{x_i}$ denotes the negation of x_i.

Our aim is to construct a tensegrity graph $T = (V, B, C, S)$ whose pure separations are in a bijective correspondence with truth assignments satisfying φ. The bar subgraph $G = (V, B)$ will consist of $2n + 2$ sufficiently large vertex-disjoint complete graphs, glued along a sequence of d vertices from each graph, with the sizes of these complete graphs to be fixed later. This construction ensures that every d-separation of the bar subgraph has the set Z of glued vertices as the separating vertex set. We shall refer to the connected components of $G - Z$ as $\mathcal{T}, \mathcal{F}, P_1, \ldots, P_n, \overline{P}_1, \ldots, \overline{P}_n$. Here, $\mathcal{T}$ and $\mathcal{F}$ correspond to "true" and "false", while P_i and $\overline{P}_i$ correspond to the variable x_i and its negation $\bar{x}_i$. As with the variables, we shall use the notation $P_i^1 = P_i, P_i^{-1} = \overline{P}_i$.

Next, we add cables and struts between these components in such a way that the following will be true for the pure separations of G.

- In every pure d-separation (X, Y) of G, X contains exactly one of $\mathcal{T}$ and $\mathcal{F}$ and exactly one of P_i and $\overline{P}_i$ for each $i = 1, \ldots, n$.
- For every pure d-separation of G, the truth assignment in which each variable x_i is true if P_i and $\mathcal{T}$ are on the same side of the separation, and false otherwise, satisfies φ.
- Conversely, if a truth assignment satisfies φ and $\{x_i, i \in I\}$ denotes the set of true variables in this assignment, then (X, Y) is a pure d-separation of G, where $X = Z \cup V(\mathcal{T}) \cup_{i \in I} V(P_i) \cup_{i \notin I} V(\overline{P}_i)$ and $Y = Z \cup (V - X)$.

Thus we shall have a correspondence between pure d-separations of T and truth assignments satisfying φ. This can be done as follows. First, for each $i = 1, \ldots, n$ and $\varepsilon \in \{-1, 1\}$, take unused vertices (i.e. ones which have no incident cables and struts) $t \in \mathcal{T}, f \in \mathcal{F}, u_1, u_2 \in P_i^{\varepsilon}$ and add the odd cycle $u_1t, tu_2, u_2f \in C, fu_1 \in S$, see Fig. 3(a). This ensures that $\mathcal{T}$ and $\mathcal{F}$ must belong to different sides of a pure d-separation. Next, for each $i = 1, \ldots, n$, take unused vertices $u_1, u_2 \in P_i, \overline{u_1}, \overline{u_2} \in \overline{P}_i, t_1, t_2 \in \mathcal{T}, f_1, f_2 \in \mathcal{F}$, and add the odd cycles $u_1t_1, t_1\overline{u_1}, \overline{u_1}t_2 \in C$, $t_2u_1 \in S$ and $u_2f_1, f_1\overline{u_2}, \overline{u_2}f_2 \in C$, $f_2u_2 \in S$, see Fig. 3(b). These odd cycles make sure that P_i and $\overline{P_i}$ cannot belong to the same side of a pure d-separation. Finally, for each clause $\{x_i^{\varepsilon_i}, x_j^{\varepsilon_j}, x_k^{\varepsilon_k}\}$ of φ, where $1 \leq i \leq j \leq k \leq n$ and $\varepsilon_i, \varepsilon_j, \varepsilon_k \in \{-1, 1\}$, take unused vertices $t_1, t_2, t_3 \in \mathcal{T}, u_1 \in P_i^{\varepsilon_i}, u_2 \in P_j^{\varepsilon_j}, u_3 \in P_k^{\varepsilon_k}$, and add the odd cycle $t_1u_1, u_1t_2, t_2u_2, u_2t_3, t_3u_3 \in C$, $u_3t_3 \in S$ to T, see Fig. 3(c). Then in a pure d-separation at least one of the components corresponding to the literals in this clause must belong to the same side as $\mathcal{T}$. Note that since we always put the odd cycles onto previously unused vertices, the only odd cycles in the resulting tensegrity are the ones that we specified. It is not difficult to verify that the pure separations of the tensegrity constructed in this way have the three properties described above.

It follows that φ can be satisfied if and only if there exists a pure d-separation, which is the same as T not having odd cycle property. We only used linearly many vertices in the number of clauses and variables, so the size of T is polynomial in the size of the 3-SAT instance. □

Combining Lemma 9 and Theorem 11 yields the following hardness result regarding strong global rigidity.

Corollary 12. *For any $d \geq 1$, recognizing strongly globally rigid tensegrity graphs in $\mathbb{R}^d$ is co-NP-hard.* □

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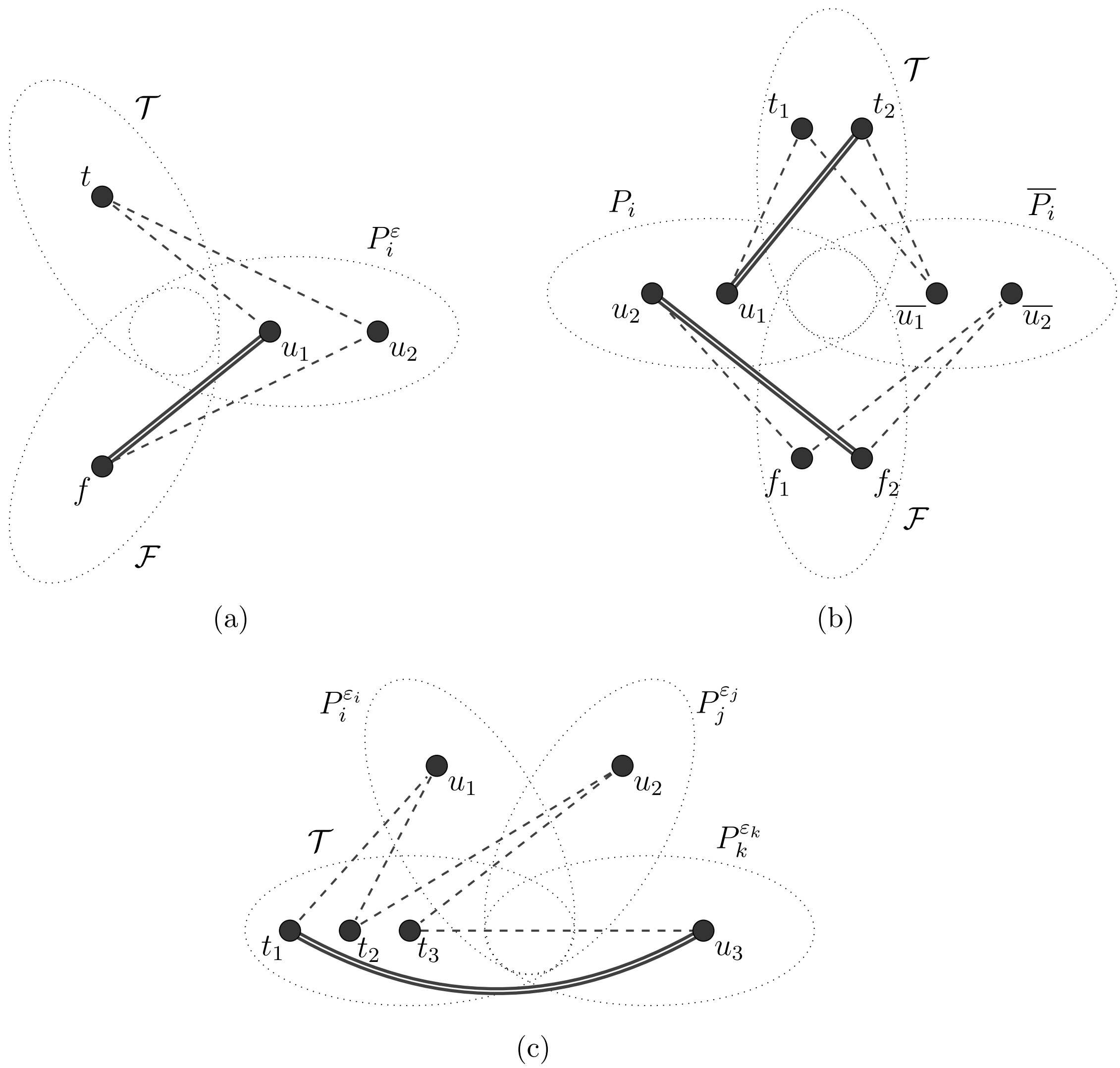

Fig. 3. The various types of odd cycles used in the proof of Theorem 11.

Appendix

A.1. Proof of Lemma 10

We recall the characterization of strong rigidity in $\mathbb{R}^1$ from [10]. We say that a tensegrity graph $T = (V, B, C, S)$ has the *alternating cycle property* if every bipartition $(U, V - U)$ of V is such that $E_T(U, V - U)$ contains either a bar or an *alternating cycle*, that is, a cycle in which cables and struts alternate.

Theorem A.1 ([10]). *A tensegrity graph is strongly rigid in $\mathbb{R}^1$ if and only if it has the alternating cycle property.*

The next lemma asserts that strong rigidity in $\mathbb{R}^1$ actually implies "orientation-preserving" global rigidity for all realizations.

Lemma A.2. *Suppose that $T = (V, B, C, S)$ is a strongly rigid tensegrity graph in $\mathbb{R}^1$ and let (T, p) be a realization in $\mathbb{R}^1$. Suppose that (T, p') is a realization satisfying the member constraints of (T, p) such that (T, p') can be obtained from (T, p) by translating each of the connected components of the bar subgraph $G = (V, B)$ by some amount. Then (T, p') is congruent to (T, p).*

Proof. By applying a translation to all of (T, p') we may suppose that each component of G is translated in the non-negative direction and that some component remains stationary. Let $\emptyset \neq U \subseteq V$ be the set of vertices that remain stationary. If $U = V$, then we are done, so suppose for contradiction that $U \subsetneq V$. Since T is strongly rigid, $E_T(U, V - U)$ contains an alternating cycle C. Let H denote the tensegrity graph on V with members $E_T(U, V - U)$ and for convenience, let us orient the cables in H from U to $V - U$, and the struts from $V - U$ to U, so that C becomes a directed cycle.

Let $l_s(p)$ and $l_s(p')$ denote the total length of the struts in C in (T, p) and (T, p'), respectively. Similarly, let $l_c(p)$ and $l_c(p')$ denote the total length of the cables in C in the respective realizations. Now each cable in (H, p) must point in the negative direction, since otherwise the length of the cable would increase as we apply the (positive) translations to (T, p), contradicting the fact that (T, p') satisfies the member constraints of (T, p). It follows that $l_s(p) \geq l_c(p)$. Similarly, each strut in (H, p') must point in the negative direction (since otherwise the translations would have shortened the strut) so that $l_c(p') \geq l_s(p')$. We also trivially have $l_c(p) \geq l_c(p')$ and $l_s(p) \leq l_s(p')$. Thus we have the following chain of inequalities:

$$l_c(p') \leq l_c(p) \leq l_s(p) \leq l_s(p') \leq l_c(p').$$

This implies that each member in C has the same length in (H, p) and (H, p'). But it is not difficult to see that either there is a strut in C which is longer in (H, p') than in (H, p), or a cable in C that is shorter in (H, p') than in (H, p), a contradiction. □

Proof of Lemma 10. Necessity follows from Lemma 8 and, in the case of strong rigidity, the definitions, so we only need to show sufficiency. Let us denote the connected components of G by X and Y and let (T, p) be a generic realization in $\mathbb{R}^1$. It is not difficult to see that the equivalent realizations of (G, p) arise, up to congruence, by either translating Y by some amount, or by reflecting Y through a point $x \in \mathbb{R}^1$. In the first case, proper translations are ruled out by Lemma A.2. In the second case, if there is a cable in $E_T(X, Y)$ with both endpoints on the same side of x (and different from x), then reflecting Y through x would lengthen this cable, thus the resulting tensegrity does not satisfy the member constraints of (T, p). Similarly, if there is a strut in $E_T(X, Y)$ with endpoints on different sides of x (and different from x), then the reflected image of this strut would be shorter. Now suppose that neither of these cases hold, so that each cable has endpoints on different sides of x and each strut has both endpoints on the same side of x, where in both cases x is allowed to be an endpoint of the given member. Then the same holds for a nearest vertex $p(v)$ instead of x. But the 1-dimensional odd cycle property applied to the 1-separation $(X + v, Y + v)$ implies that either there is cable in $E_T(X - v, Y - v)$ with both endpoints on one side of $p(v)$ and different from $p(v)$, or a strut with endpoints on different sides of $p(v)$ and different from $p(v)$, a contradiction. □

Lemmas 9 and 10 show that the 1-dimensional odd cycle property ensures, in effect, that if we start from a generic tensegrity in $\mathbb{R}^1$ and apply a reflection through some point in $\mathbb{R}^1$ to some of the 2-connected components of the bar subgraph, the resulting tensegrity does not satisfy the member constraints of the original one, except in the cases when we reflected all or none of the 2-connected components. Fig. 2 shows that if two or more reflections are involved then this is not true in general, i.e. the resulting tensegrity may satisfy the member constraints of the original one. This leaves open the case when the bar subgraph consists of three or more disjoint graphs that are globally rigid in $\mathbb{R}^1$. It is not difficult to see that in this case, given a generic realization of the tensegrity graph, the frameworks equivalent to the bar subgraph arise (up to congruence) by applying a translation to some of its connected components, and then reflecting some of the connected components through a point in $\mathbb{R}^1$. Note that, by Lemma A.2, if the tensegrity graph is strongly rigid in $\mathbb{R}^1$, then applying a proper translation to some of the connected components of the bar subgraph does not preserve the member constraints; still, it is unclear whether the composition of a translation and a reflection can yield a non-congruent tensegrity that satisfies the member constraints of the original one.

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