



# New values for the bipartite Ramsey number of the four-cycle versus stars

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## ARTICLE INFO

### Article history:

Received 26 September 2019

Received in revised form 12 January 2021

Accepted 18 January 2021

Available online 17 February 2021

### Keywords:

Ramsey number  
 Quadrilateral-free  
 Projective plane  
 Incidence graph  
 Marginal set

## ABSTRACT

We provide new values of the bipartite Ramsey number  $R_B(C_4, K_{1,n})$  using induced subgraphs of the incidence graph of a projective plane. The approach, based on deleting subplanes of projective planes, has been used in related extremal problems and allows us to unify previous results and extend them. More importantly, using deep stability results on  $2 \bmod p$  sets and double blocking sets, we can show some of the limits of this technique when the projective plane is Desarguesian of large enough square order. Finally, we also disprove two conjectures about  $R_B(C_4, K_{1,n})$ .

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## 1. Introduction

Finite geometry, and in particular projective planes, has had many successful applications to (extremal) graph theory. Among the questions that have been investigated are the Zarankiewicz problem [24], Turán numbers [16,17] and Ramsey theory [31,32]. The common thread in these problems is that they revolve around excluding the quadrilateral or four-cycle  $C_4$  as a subgraph. One such problem is the following. By  $K_{n,m}$  we denote the complete bipartite graph with  $n$  vertices in one of its vertex classes and  $m$  in the other one.

**Definition 1.1.** The **bipartite Ramsey number**  $R_B(C_4, K_{1,n})$  is the smallest  $b$  such that for every red-blue edge-coloring of  $K_{b,b}$  there exists either a red  $C_4$  or a blue  $K_{1,n}$ .

We will investigate  $R_B(C_4, K_{1,n})$  as a function in  $n$  and denote this shortly by  $b(n)$ . It is worthwhile to mention that this is a variation of the usual Ramsey numbers  $R(C_4, K_{1,n})$ , which were recently the subject of study of Zhang, Cheng and Chen [31,32]. The bipartite counterpart had been investigated earlier by Carnielli, Gonçalves and Monte Carmelo in [10,21] where they obtained the following results.

**Lemma 1.2** ([10]). For all  $n \geq 2$ , we have

$$b(n) \leq n + \lceil \sqrt{n} \rceil.$$

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<https://doi.org/10.1016/j.disc.2021.112320>

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Using the incidence graphs of projective planes, and subgraphs thereof, they managed to obtain the following exact results. Note that the upper bound is attained for these values.

**Theorem 1.3** ([10,21]). *Let  $q$  be the order of a projective plane and  $1 \leq t \leq q + 1$ , then*

$$b(q^2 + 1) = q^2 + q + 2,$$

$$b(q^2 + 1 - t) = q^2 + q + 1 - t.$$

In Section 2, we obtain more exact values for the function  $b(n)$ , using a more general method to obtain suitable subgraphs of incidence graphs of projective planes. This involves introducing a seemingly new (leastwise rather unknown) concept to graph theory which we call *marginal sets* (Definition 2.7), and may be regarded as the counterpart of the well-known concept of dominating sets. Our method covers all of the above given values, and also provides several new instances; see Theorem 2.12 for details. Remark that all of these results attain the upper bound in Lemma 1.2.

In Section 3 we investigate the method introduced in Section 2 to see whether we have exploited all of its capabilities. The answer is partially affirmative: roughly speaking, our main result is that Desarguesian projective planes of large enough square order can provide only such examples of  $b(n) = n + \lceil \sqrt{n} \rceil$  that are also established by Theorem 2.12.

The values of  $n$  covered by Theorem 2.12 support the following conjecture of Carnielli and Monte Carmelo.

**Conjecture 1.4** (Conjecture 16, [10]). *For all  $n$ ,  $b(n) = n + \lceil \sqrt{n} \rceil$ .*

Having formulated the above conjecture, the authors of [10] remark that ‘we do not expect to find any possible counterexample among the first values of the function  $b(K_{2,2}; K_{1,n})$ , which is  $b(n)$  in their notation. A weaker conjecture was also formulated in [10].

**Conjecture 1.5** (Conjecture 17, [10]). *For all  $n$ ,  $b(n) < b(n + 1)$ .*

Clearly, if Conjecture 1.4 holds, then so does Conjecture 1.5. However, in Section 4 we show that both conjectures fail, and we also provide new exact values of  $b(n)$  that do not meet the upper bound of Lemma 1.2, the smallest such instance being  $b(18) = 22$ .

In Section 5 we sum up the findings of the present paper, relate it to some other areas of (extremal) graph theory, and provide a table of the known exact values of  $b(n)$  for small  $n$ .

## 2. Constructions giving $b(n) = n + \lceil \sqrt{n} \rceil$

To prove  $b(n) = n + \lceil \sqrt{n} \rceil$  for some  $n$ , one has to ensure the existence of a bipartite  $C_4$ -free graph  $H$  with  $n + \lceil \sqrt{n} \rceil - 1$  vertices in each part, not containing  $K_{1,n}$  in its bipartite complement. Equivalently,  $H$  needs to have minimum degree at least  $\lceil \sqrt{n} \rceil$ . Then we may embed  $H$  in  $K_{n+\lceil \sqrt{n} \rceil-1, n+\lceil \sqrt{n} \rceil-1}$  and color all the edges of  $H$  red, and all others blue. This shows that  $b(n) > n + \lceil \sqrt{n} \rceil - 1$  which, by the upper bound Lemma 1.2, proves the desired result.

### 2.1. Preliminaries

We will construct such graphs based on (finite) projective planes. We recall its definition and some notion here; for all others, such as the order of a projective plane, Baer subplanes, etc. we refer to [25]. All projective planes we discuss in the remainder will be finite, so we do not explicitly refer to them as such in what follows.

**Definition 2.1.** *A projective plane  $\Pi$  is a finite set of points  $\mathfrak{P}$  and lines  $\mathfrak{L}$  such that*

- every two lines meet in one point;
- every two points determine one line;
- there exist four points of which no three are collinear.

If we omit the last condition in the definition of a projective plane, we obtain so-called degenerate projective planes. An example of such a degenerate plane is the *near pencil*, which contains three but no four points in general position. A near pencil may also be defined as follows.

**Definition 2.2.** *A near pencil of size  $t + 1$  ( $t \geq 2$ ) consists of  $t + 1$  points  $P_0, P_1, \dots, P_t$  and the  $t + 1$  lines  $\{P_1, P_2, \dots, P_t\}, \{P_0, P_i\}$  ( $1 \leq i \leq t$ ). The point  $P_0$  will be called the *point of concurrency*.*

In the following,  $\Pi_q$  and  $\text{PG}(2, q)$  will refer to an arbitrary, or to the Desarguesian projective plane of order  $q$ , respectively, whereas  $\Pi$  will usually refer to a projective plane of arbitrary order.

**Definition 2.3.** *Let  $\Pi$  be a projective plane with point set  $\mathfrak{P}$  and line set  $\mathfrak{L}$ . Let  $\mathfrak{P}' \subset \mathfrak{P}$  and  $\mathfrak{L}' \subset \mathfrak{L}$ . Then  $(\mathfrak{P}', \mathfrak{L}')$  is a *closed system* of  $\Pi$  if for any two points of  $\mathfrak{P}'$ , the line of  $\Pi$  joining them is in  $\mathfrak{L}'$  and, dually, for any two lines of  $\mathfrak{L}'$ , their intersection point is in  $\mathfrak{P}'$ .*

If we restrict the lines of a closed system to the point set of the system, then we obtain a (possibly degenerate) projective plane, hence we may call such a substructure a (degenerate) subplane. A closed system  $(\mathfrak{P}', \mathcal{L}')$  of  $\Pi$  may be of the following four types:

1.  $|\mathfrak{P}'| \leq 1$  or  $|\mathcal{L}'| \leq 1$ .
2. There exists an incident point-line pair  $P \in \mathfrak{P}'$  and  $\ell \in \mathcal{L}'$ , and a non-empty set  $\mathcal{L}'$  of lines through  $P$  such that  $\ell \notin \mathcal{L}'$ , and a non-empty set  $\mathcal{P}'$  of points on  $\ell$  such that  $P \notin \mathcal{P}'$ , for which  $\mathfrak{P}' = \{P\} \cup \mathcal{P}'$  and  $\mathcal{L}' = \{\ell\} \cup \mathcal{L}'$ .
3. There exists a non-incident point-line pair  $P \in \mathfrak{P}'$  and  $\ell \in \mathcal{L}'$ , and a set  $\mathcal{L}'$  of two or more lines through  $P$ , for which  $\mathcal{L}' = \{\ell\} \cup \mathcal{L}'$  and  $\mathfrak{P}' = \{P\} \cup \{P \cap \ell' : \ell' \in \mathcal{L}'\}$ .
4. There exist four points in general position in  $\mathfrak{P}'$ , hence  $(\mathfrak{P}', \mathcal{L}')$  is a non-degenerate subplane.

In the first two types above, the number of points and lines may be different, while in the last two types they are equal. Let us note that closed systems of the third type are near pencils.

**Definition 2.4.** A  $t$ -fold blocking set is a set of points that meets every line of the plane in at least  $t$  points. Usually, 1-fold and 2-fold blocking sets are also called blocking and double blocking sets respectively. The dual concept for line sets are called  $(t$ -fold) covering sets.

**Remark 2.5.** Clearly, near pencils of size  $t + 1$  (i.e. containing  $t + 1$  points and  $t + 1$  lines) are contained in  $\Pi_q$  as long as  $t \leq q + 1$ . We recall that a non-degenerate subplane  $\Pi'$  with point set  $\mathfrak{P}'$  and line set  $\mathcal{L}'$  of a projective plane  $\Pi_q$  has an order  $s$  for which either  $s = \sqrt{q}$ , or  $s^2 + s \leq q$  holds (this observation is attributed to Bruck). In the first case,  $\Pi'$  is a Baer subplane and  $|\mathfrak{P}'| = |\mathcal{L}'| = q + \sqrt{q} + 1$ ; in the latter case,  $s^2 + s + 1 = |\mathfrak{P}'| = |\mathcal{L}'| \leq q + 1$ . We say that a line belongs to a subplane if they have at least two points in common. If  $p$  is a prime power and  $t \mid h$ , then  $\text{PG}(2, p^h)$  contains  $\text{PG}(2, p^t)$  as a subplane. Consequently, if  $q$  is a square prime power, then  $\text{PG}(2, q)$  contains Baer subplanes; moreover, it is well-known that  $\text{PG}(2, q)$  can be partitioned into  $q - \sqrt{q} + 1$  pairwise disjoint Baer subplanes (so each point or line of  $\text{PG}(2, q)$  belongs to exactly one Baer subplane of the partition). Finally, let us remark that the point set of a possibly degenerate subplane  $\Pi'$  of  $\Pi$  is a blocking set in  $\Pi$  if and only if either  $\Pi'$  is degenerate and contains all points of a line of  $\Pi$ , or  $\Pi'$  is a Baer subplane of  $\Pi$ .

**Definition 2.6.** The incidence graph  $I(\Pi_q)$  of a projective plane  $\Pi_q$  with point set  $\mathfrak{P}$  and line set  $\mathcal{L}$  is the bipartite graph with parts  $\mathfrak{P}$  and  $\mathcal{L}$ , where two vertices are adjacent if and only if they are incident in the projective plane. We will refer to the vertices in the two classes as points and lines as well.

Clearly,  $I(\Pi_q)$  has  $q^2 + q + 1$  vertices in each part, it is  $C_4$ -free, every vertex in it has degree  $q + 1$ , and hence it does not contain  $K_{1, q^2+1}$  in its bipartite complement. This shows already that  $b(q^2 + 1) > q^2 + q + 1$ , and by checking with Lemma 1.2, we see that this indeed gives

$$b(q^2 + 1) = q^2 + q + 2.$$

## 2.2. Constructions via marginal sets

We will construct induced subgraphs of  $I(\Pi)$  by deleting some vertices of it so that the minimum degree remains high, and we reach the upper bound of Lemma 1.2. As we want bipartite graphs with an equal amount of vertices in both parts, we need to delete as many points as lines. Hence we introduce the following concept, which will be central to our investigations.

**Definition 2.7.** Let  $G$  be a graph with vertex set  $V$ . A proper subset  $M$  of vertices is called  $t$ -marginal if for all vertices  $u \in V \setminus M$ ,  $u$  has at most  $t$  neighbors in  $M$ , and  $M$  is called strictly  $t$ -marginal if equality holds for some vertex not in  $M$ . If  $G$  is bipartite with parts  $A$  and  $B$ , then  $M$  is called balanced if  $|M \cap A| = |M \cap B|$ . In this case, the size of  $M$  is defined as  $|M \cap A|$ , and usually it will be denoted by  $v$ .

Note that  $t$ -marginal sets are the counterpart of  $t$ -fold dominating sets, where each vertex not in the set is required to have at least  $t$  neighbors in the set. We may call a  $t$ -marginal set  $M$  perfect if each vertex not in  $M$  has exactly  $t$  neighbors in  $M$ . Such sets are also called perfect  $t$ -fold dominating sets, which have their extremal graph theoretic applications as well; see Section 5 for further remarks. Let us point out a significant difference between  $t$ -marginal and  $t$ -fold dominating sets: the family of  $t$ -marginal sets of a given graph is neither upward nor downward closed in general (with respect to containment), while any superset of  $t$ -fold dominating set is again a  $t$ -fold dominating set.

Clearly, if  $M$  is a  $t$ -marginal set of a  $k$ -regular graph  $G$  with vertex set  $V$ , then  $V \setminus M$  induces a subgraph with minimum degree at least  $k - t$ . In the following, we will refer to the incidence graph of a plane  $\Pi$  only by referring to the plane itself and, unless explicitly claimed otherwise, we will always consider non-empty and balanced  $t$ -marginal sets of  $\Pi$ . For sake of convenience, we will refer to them shortly by ' $t$ -marginal sets of  $\Pi$ '. By writing  $M = \mathcal{P} \cup \mathcal{L}$ , we will always mean that  $\mathcal{P}$  is the set of points and  $\mathcal{L}$  is the set of lines contained in  $M$ . Let us note that  $t$ -marginal sets of  $\Pi_q$  are not interesting if  $t \geq q + 1$ .

We will use the following observation without explicitly referring to it.

**Remark 2.8.** A set  $M = \mathcal{P} \cup \mathcal{L}$  is  $t$ -marginal in  $\Pi$  if and only if every line of  $\Pi$  not in  $\mathcal{L}$  meets  $\mathcal{P}$  in at most  $t$  points, and every point of  $\Pi$  not in  $\mathcal{P}$  is covered by at most  $t$  lines of  $\mathcal{L}$ .

**Lemma 2.9.** Suppose that  $M = \mathcal{P} \cup \mathcal{L}$  is a strictly  $t$ -marginal set of  $\Pi_q$ ,  $1 \leq t \leq q$ , and let  $v = |\mathcal{P}| = |\mathcal{L}|$ . Then

$$b(q^2 - v + t + 1) \geq q^2 + q + 2 - v,$$

and

$$v \leq 2tq - t^2 + t.$$

Furthermore, the points and lines of  $\Pi_q$  not in  $M$  induce a subgraph proving equality in Lemma 1.2 if and only if

$$v \geq 2(t - 1)q - t^2 + 3t,$$

and in this case we have  $b(q^2 - v + t + 1) = q^2 + q + 2 - v$ .

**Proof.** The graph  $G$  spanned by the  $q^2 + q + 1 - v$  points and lines not in  $M$  is  $C_4$ -free and its minimum degree is  $q + 1 - t$ . Consequently, there is no  $K_{1, q^2 + q + 1 - v - (q + 1 - t) + 1} = K_{1, q^2 - v + t + 1}$  in the bipartite complement of  $G$ , thus we obtain  $b(q^2 - v + t + 1) \geq q^2 + q + 2 - v$ . Lemma 1.2 asserts  $b(n) \leq n + \lceil \sqrt{n} \rceil$ , whence

$$q^2 + q + 2 - v \leq q^2 - v + t + 1 + \lceil \sqrt{q^2 - v + t + 1} \rceil.$$

On the one hand, this means that  $q - t < \sqrt{q^2 - v + t + 1}$  or, equivalently,  $(q - t)^2 + 1 \leq q^2 - v + t + 1$  must hold in all cases, which yields the upper bound stated. On the other hand, we obtain equality if and only if  $\sqrt{q^2 - v + t + 1} \leq q - t + 1$  or, equivalently,  $q^2 - v + t + 1 \leq q^2 + t^2 + 1 - 2qt + 2q - 2t$ , which leads to the asserted lower bound.  $\square$

In the next two lemmas, we give constructions for  $t$ -marginal sets.

**Lemma 2.10.** Let  $M = \mathcal{P} \cup \mathcal{L}$  be a proper subset of  $\Pi$ . Then  $M$  is a 1-marginal set of  $\Pi$  if and only if  $(\mathcal{P}, \mathcal{L})$  is a closed system of  $\Pi$ .

**Proof.** If  $M$  is 1-marginal, then the line connecting any two points of  $\mathcal{P}$  must be in  $\mathcal{L}$  and, dually, the intersection point of any two lines of  $\mathcal{L}$  must be in  $\mathcal{P}$ , hence  $M$  is a closed system. Conversely, if  $M$  is a closed system, then for any point  $P \notin \mathcal{P}$ , there can be at most one line of  $\mathcal{L}$  through  $P$ , otherwise  $P$  would be the intersection point of two lines of  $\mathcal{L}$  whence, as  $M$  is a closed system,  $P \in \mathcal{P}$  would follow, a contradiction. Dually, any line not in  $\mathcal{L}$  may contain at most one point of  $\mathcal{P}$ .  $\square$

**Lemma 2.11.** Let  $M_1, \dots, M_t$  be 1-marginal sets of an arbitrary graph  $G$ . Then  $M_1 \cup \dots \cup M_t$  is  $t$ -marginal. In particular, the union of  $t$  closed systems in  $\Pi$  is  $t$ -marginal (not necessarily balanced or strict).

**Proof.** Trivial.  $\square$

In  $\Pi_q$ , a near pencil of size  $v$  exists for all  $v \in \{1, \dots, q + 2\}$ , which is a 1-marginal set in  $\Pi_q$  by Lemma 2.11 and, clearly, it is strict. To satisfy the bounds of Lemma 2.9 we need  $2 \leq v \leq 2q$ ; hence near pencils of size  $2 \leq v \leq q + 2$  yield  $b(q^2 + 2 - v) = q^2 + q + 2 - v$ , which gives back Theorem 1.3. However, we may use not only near pencils but the other types of closed systems of  $\Pi_q$ , in particular non-degenerate subplanes. In this way we also get graphs proving  $b(q^2 + 2 - v) = q^2 + q + 2 - v$  which are different from the previously found ones (except for  $v = 2$ ), but as the only closed systems containing more than  $q + 2$  points are Baer subplanes (see Remark 2.5), we obtain a new exact bipartite Ramsey number only in particular cases. Let us remark that the idea of removing a closed system from a projective plane is not new: it results in a so-called semi-plane, which is discussed already by Dembowski [13, Section 7.2]. However, we may take more than one closed systems to find new bipartite Ramsey numbers. In this way, we obtain the following theorem.

**Theorem 2.12.** If there is a projective plane of order  $q$  (in particular, if  $q$  is a prime power), then

$$b(q^2 - 2q) = q^2 - q - 1, \text{ and}$$

$$b(q^2 - 2q + 1) = q^2 - q.$$

If there is a projective plane of order  $q$  that contains a Baer subplane (in particular, if  $q$  is a square prime power), then

$$b(q^2 - q - \sqrt{q} + 1) = q^2 - \sqrt{q} + 1, \text{ and}$$

$$b(q^2 - q - \sqrt{q} + 1 - s) = q^2 - \sqrt{q} - s \text{ for all } q - \sqrt{q} \leq s \leq q.$$

If there is a projective plane of order  $q$  that contains two distinct Baer subplanes that have  $s \leq 2\sqrt{q}$  points in common (in particular, if  $q$  is a square prime power and  $s \in \{0, 1, 2, 3, \sqrt{q} + 1, \sqrt{q} + 2\}$ ), then

$$b(q^2 - 2q - 2\sqrt{q} + 1 + s) = q^2 - q - 2\sqrt{q} + s.$$

Finally,

$$b(34) = 40.$$

**Proof.** In all but the last case, we take one or the union of two closed systems of  $\Pi_q$  as a 1- or 2-marginal set  $M$  (cf. Lemma 2.11); all instances will be clearly strict. Let us denote the number of the points of  $M$  by  $v$ . Checking Lemma 2.9 we see that  $M$  fits our needs if it is 1-marginal and  $v \geq 2$ , or 2-marginal and  $v \geq 2q + 2$ .

Let  $\ell_1, \ell_2$  be two distinct lines, and let  $P_1 \notin \ell_1$  and  $P_2 \notin \ell_2$  be two distinct points in  $\Pi_q$ . Then  $P_i$  and  $\ell_i$  define a near pencil of size  $q + 2$  ( $i = 1, 2$ ). As  $\ell_1$  and  $\ell_2$  have one point in common, the union of these near pencils have size  $v = 2q + 3 - c$ , where  $0 \leq c \leq 2$  is the number of incidences occurring among  $P_1 \in \ell_2$  and  $P_2 \in \ell_1$ ; note that we need  $c \leq 1$ . Then Lemma 2.9 asserts  $b(q^2 - (2q + 3 - c) + 2 + 1) = b(q^2 - 2q + c) = q^2 + q + 2 - (2q + 3 - c) = q^2 - q - 1 + c$ . This verifies the first two bipartite Ramsey numbers.

Suppose now that there exists a projective plane  $\Pi_q$  containing a Baer subplane  $\Pi'$  (this is the case when  $q$  is a square prime power, see Remark 2.5). Then  $\Pi'$  itself is a strictly 1-marginal set of size  $v = q + \sqrt{q} + 1$ , whence Lemma 2.9 yields  $b(q^2 - (q + \sqrt{q} + 1) + 2) = q^2 + q + 2 - (q + \sqrt{q} + 1)$ , affirming the third assertion. Let  $\ell$  be a line meeting  $\Pi'$  in a single point  $Q$ , and let  $P$  be a point not in  $\Pi'$  such that the line  $PQ$  belongs to  $\Pi'$  (as there are  $\sqrt{q} + 1$  lines through  $Q$  belonging to  $\Pi'$ , each containing  $q - \sqrt{q}$  points not in  $\Pi'$ , we can choose such a point). Then  $P$  and  $s$  arbitrarily chosen points of  $\ell$  different from  $Q$  define a near pencil  $S$  of size  $s + 1$  such that  $S$  and  $\Pi'$  have no points or lines in common, so the strictly 2-marginal set  $\Pi' \cup S$  has  $v = q + \sqrt{q} + 1 + s + 1$  points and lines. As  $s$  may take any value between  $q - \sqrt{q}$  and  $q$  (so that  $v \geq 2q + 2$ ), by Lemma 2.9 this proves  $b(q^2 - (q + \sqrt{q} + 2 + s) + 2 + 1) = q^2 + q + 2 - (q + \sqrt{q} + 2 + s)$ , our fourth assertion.

Suppose now that  $\Pi_q$  contains two distinct Baer subplanes  $\Pi'_1, \Pi'_2$  that have  $s \leq 2\sqrt{q}$  points in common. Bose, Freeman and Glynn [7] proved that the common points and the common lines of two Baer subplanes always form a closed system with equally many points and lines, hence  $\Pi'_1 \cup \Pi'_2$  has the same number  $v = 2(q + \sqrt{q} + 1) - s \geq 2q + 2$  points and lines, and it is a strictly 2-marginal set. Lemma 2.9 then yields  $b(q^2 - (2q + 2\sqrt{q} + 2 - s) + 3) = q^2 + q + 2 - (2q + 2\sqrt{q} + 2 - s)$ , our last assertion. If  $q$  is a square prime power, then  $\text{PG}(2, q)$  is well-known to contain disjoint Baer subplanes; moreover, any four points in general position define a unique Baer subplane containing them, and if two Baer subplanes have three common points on a line, then all their  $\sqrt{q} + 1$  points on that line are common. This means that the intersection of two Baer subplanes, which is a closed system as mentioned before, cannot contain four points in general position, hence it must be a degenerate subplane. If this degenerate subplane contains three collinear points, then it must contain  $\sqrt{q} + 1$  collinear points; thus it may contain at most three points, or  $\sqrt{q} + 1$  collinear points, or a near pencil of size  $\sqrt{q} + 2$ . It is well-known that all these cases actually occur.

Finally, if we take four disjoint Baer subplanes of  $\text{PG}(2, 9)$  (recall that  $\text{PG}(2, 9)$  can be partitioned into 7 disjoint Baer subplanes), we obtain a 4-marginal set of size  $52 \geq 2(4 - 1)9 - 4^2 + 3 \cdot 4 = 50$ , hence Lemma 2.9 gives  $b(81 - 52 + 4 + 1) = 81 + 9 + 2 - 52$ , as asserted.  $\square$

Note that the union of  $t$  closed systems of  $\Pi_q$  has at most  $v = t(q + \sqrt{q} + 1)$  points, whereas we need  $v \geq 2(t - 1)q - t^2 + 3t$  to meet the upper bound of Lemma 1.2, which clearly does not hold for  $t \geq 3$  if  $q$  is large enough compared to  $t$  (precisely, if  $q \geq 15$  and  $3 \leq t \leq q - \sqrt{q} - 1$ ); but, as seen, we do get a bipartite Ramsey number not covered by the other constructions when  $q = 9$ .

### 3. Characterization

In this section our aim is to determine whether we have found all possible values of  $n$  for which we can get an exact value of  $b(n)$  by finding a subgraph of a projective plane that yields equality in Lemma 1.2. We start with some results for arbitrary planes, then proceed with Desarguesian ones by utilizing deep stability results on  $2 \bmod p$  sets and double blocking sets.

#### 3.1. Combinatorial arguments for general planes

Recall that 1-marginal sets are completely characterized by Lemma 2.10. Let us now give a general upper bound on the size of  $t$ -marginal sets that shows that taking  $t$  disjoint Baer subplanes is the best we can do in order to obtain a large enough  $t$ -marginal set (to meet the lower bound of Lemma 2.9). We use the so-called incidence bound [28], which originally can be found in [23]. Recall that a set  $X$  along with a family  $\mathcal{B}$  of its subsets, called blocks, is a  $2$ - $(w, k, \lambda)$  design if  $|X| = w, \forall B \in \mathcal{B}: |B| = k$ , and any two distinct elements of  $X$  are contained in precisely  $\lambda$  blocks. The replication number  $r$  of a design is the number of blocks containing a given point (this number does not depend on which point we take). Clearly,  $\Pi_q$  is a  $2$ - $(q^2 + q + 1, q + 1, 1)$  design with replication number  $q + 1$ .

**Lemma 3.1** ([23,28]). *Let  $(X, \mathcal{B})$  be a  $2$ - $(w, k, \lambda)$  design with total number of blocks  $b$  and replication number  $r$ . For  $S \subseteq X, T \subseteq \mathcal{B}$ , let  $i(S, T)$  denote the number of incidences between the sets  $S$  and  $T$ , i.e. the cardinality of  $\{(x, B) \in S \times T \mid x \in B\}$ . Then we have*

$$\left| i(S, T) - \frac{r|S||T|}{b} \right| \leq \sqrt{r - \lambda} \sqrt{|S||T| \left(1 - \frac{|S|}{w}\right) \left(1 - \frac{|T|}{b}\right)}.$$

If equality occurs, then for each  $S' \in \{S, X \setminus S\}$  and each  $T' \in \{T, \mathcal{B} \setminus T\}$ , every point of  $S'$  is contained in a constant number of blocks of  $T'$  and every block of  $T'$  contains a constant number of points of  $S'$ .

**Theorem 3.2.** Any  $t$ -marginal set  $M$  of  $\Pi_q$  has size at most  $t(q + \sqrt{q} + 1)$ . Furthermore, in case of equality, every line meets  $M$  in  $t$  or  $\sqrt{q} + t$  points and, dually, every point is covered by  $t$  or  $\sqrt{q} + t$  lines of  $M$ .

**Proof.** Let  $M = \mathcal{P} \cup \mathcal{L}$  be a  $t$ -marginal set of  $\Pi_q$ ,  $|\mathcal{P}| = |\mathcal{L}| = v$ . Let  $S = \mathcal{P}$  and  $T$  the set of all lines not in  $\mathcal{L}$ . Then  $|S| = v$ ,  $|T| = q^2 + q + 1 - v$  and  $i(S, T) \leq t(q^2 + q + 1 - v)$  by the definition of  $M$ . Apply the incidence bound (Lemma 3.1) with  $w = b = q^2 + q + 1$ ,  $r = k = q + 1$ ,  $\lambda = 1$  to get

$$\begin{aligned} \frac{(q + 1)v(q^2 + q + 1 - v)}{q^2 + q + 1} - t(q^2 + q + 1 - v) &\leq \\ \frac{r|S||T|}{b} - i(S, T) &\leq \sqrt{r - \lambda} \sqrt{|S||T| \left(1 - \frac{|S|}{w}\right) \left(1 - \frac{|T|}{b}\right)} = \\ \sqrt{q} \sqrt{v(q^2 + q + 1 - v) \left(1 - \frac{v}{q^2 + q + 1}\right) \left(1 - \frac{q^2 + q + 1 - v}{q^2 + q + 1}\right)}. \end{aligned}$$

After clearing denominators, this leads to the inequality

$$(q + 1)v(q^2 + q + 1 - v) - t(q^2 + q + 1)(q^2 + q + 1 - v) \leq \sqrt{q}v(q^2 + q + 1 - v),$$

from which  $v(q - \sqrt{q} + 1) \leq t(q^2 + q + 1) = t(q + \sqrt{q} + 1)(q - \sqrt{q} + 1)$  follows. If equality holds, then every line not in  $\mathcal{L}$  intersects  $\mathcal{P}$  in  $t$  points and every line of  $\mathcal{L}$  meets  $\mathcal{P}$  in some constant  $x$  points. Consider

$$t(q + \sqrt{q} + 1)(q + 1) = |\mathcal{P}|(q + 1) = t(q^2 + q + 1 - |\mathcal{L}|) + x|\mathcal{L}| = t(q^2 + q + 1 - t(q + \sqrt{q} + 1)) + xt(q + \sqrt{q} + 1).$$

Simplifying by  $t(q + \sqrt{q} + 1)$  and rearranging we get  $x = (q + 1) - (q - \sqrt{q} + 1 - t) = \sqrt{q} + t$ . The other statement follows dually.  $\square$

Since no  $t$ -marginal set of  $\Pi_q$  can be larger than the union of  $t$  disjoint Baer subplanes, then, as pointed out previously,  $t$ -marginal sets can lead to graphs attaining equality in Lemma 1.2 only if  $t \leq 2$  or  $t \geq q - \sqrt{q}$  (under  $q \geq 15$ ). Suppose that  $G$  is a  $C_4$ -free bipartite graph with minimum degree  $d \geq 2$ . Then  $G$  can be considered as the incidence graph of a so-called partial linear space, and it can be easily embedded into a linear space. A well-known, long open and supposedly very difficult conjecture (see [5]) claims that every finite linear space can be embedded into a finite projective plane. Assume that  $G$  can be embedded into  $\Pi_q$ . Then, clearly, points and lines of  $\Pi_q$  not belonging to  $G$  form a  $(q + 1 - d)$ -marginal set of  $\Pi_q$ . Thus characterizing  $t$ -marginal sets of  $\Pi_q$  for general values of  $t$ , or even for  $t \geq q - \sqrt{q}$ , seems very difficult. Indeed, if we consider a subplane  $\Pi_s$  of  $\Pi_q$ , its complement is a  $(q - s)$ -marginal set in  $\Pi_q$  with  $s \leq \sqrt{q}$ ; thus the aforementioned task includes describing all subplanes of projective planes. However, it is not known if every projective plane can be embedded into a larger (but still finite) one. Thus our aim in the following is to restrict ourselves to  $t = 2$  and study 2-marginal sets.

Lemma 2.9 and Theorem 3.2 show that using a 2-marginal set of  $\Pi_q$  to obtain graphs that attain equality in Lemma 1.2, we may get a bipartite Ramsey number  $b(n)$  only if  $q^2 - 2q - 2\sqrt{q} + 1 \leq n \leq q^2 - 2q + 1$  for some  $q$  that is the order of a projective plane. The next question is whether we find 2-marginal sets that prove equality for all values in this interval of length  $2\sqrt{q} + 1$ , or just for the specific ones given by our constructions in Theorems 2.12. We believe that the second case is the truth, at least in  $\text{PG}(2, q)$  if  $q$  is large enough. Before proceeding with Desarguesian planes, we give some technical lemmas that hold for arbitrary planes. An  $i$ -secant or  $(i+)$ -secant line with respect to a point set  $\mathcal{P}$  is a line that meets  $\mathcal{P}$  in exactly  $i$  or at least  $i$  points, respectively.

**Definition 3.3.** Let  $\mathcal{P}$  be a fixed set of points in  $\Pi_q$ .

- For a point  $P$ , let  $n_i^{\mathcal{P}}(P)$  denote the number of  $i$ -secant lines to  $\mathcal{P}$  through  $P$ , and let  $n_{i+}^{\mathcal{P}}(P)$  denote  $\sum_{j=i}^{q+1} n_j(P)$ .
- Define  $\mathcal{L}_i^{\mathcal{P}}$  as the set of all  $i$ -secant lines to  $\mathcal{P}$ , and let  $n_i^{\mathcal{P}} := |\mathcal{L}_i^{\mathcal{P}}|$ . Also,  $\mathcal{L}_{i+}^{\mathcal{P}}$  and  $n_{i+}^{\mathcal{P}}$  are defined analogously.

If  $\mathcal{P}$  is clear from the context, we will drop the superscripts and write  $n_i(P)$ ,  $n_{i+}(P)$ , etc.

**Lemma 3.4.** Fix an arbitrary point set  $\mathcal{P}$  of  $\Pi_q$ . If  $|\mathcal{P}| \geq 2q + 3$ , then  $\mathcal{L}_{3+}$  is a covering set.

**Proof.** Let  $P \in \mathcal{P}$ . Counting the points of  $\mathcal{P}$  on the lines through  $P$ , we get  $2q + 3 \leq |\mathcal{P}| \leq 1 + (q + 1 - n_{3+}(P)) + qn_{3+}(P)$ , which yields  $n_{3+}(P) \geq 2$ ; that is, at least two lines of  $\mathcal{L}_{3+}$  cover  $P$ . Let now  $P \notin \mathcal{P}$ . Considering the lines through  $P$ , we get  $2q + 3 \leq |\mathcal{P}| \leq 2(q + 1 - n_{3+}(P)) + qn_{3+}(P)$ , which yields  $n_{3+}(P) \geq 1$ ; that is, at least one line of  $\mathcal{L}_{3+}$  covers  $P$ .  $\square$

We note that this lemma is sharp. Let  $\ell_1$  and  $\ell_2$  be two distinct lines,  $P = \ell_1 \cap \ell_2$ , and let  $Q \notin \ell_1 \cup \ell_2$ . Then  $\mathcal{P} = \ell_1 \cup \ell_2 \cup \{Q\}$  has size  $2q + 2$ , and since  $\mathcal{L}_{3+}$  consists of  $\ell_1$ ,  $\ell_2$  and all lines through  $Q$  except  $PQ$ , the points of  $PQ \setminus \{P, Q\}$  are not covered by  $\mathcal{L}_{3+}$ .

**Corollary 3.5.** Let  $M = \mathcal{P} \cup \mathcal{L}$  be a 2-marginal set of  $\Pi_q$  of size  $v \geq 2q + 3$ . Then  $\mathcal{P}$  is a blocking set and  $\mathcal{L}$  is a covering set.

**Proof.** Since  $M$  is 2-marginal,  $\mathcal{P}_{3+} \subseteq \mathcal{P}$  and  $\mathcal{L}_{3+} \subseteq \mathcal{L}$  hold, hence the assertion follows immediately from Lemma 3.4 and its dual.  $\square$

**Lemma 3.6.** Let  $M = \mathcal{P} \cup \mathcal{L}$  be a 2-marginal set of  $\Pi_q$  with  $|\mathcal{P}| = |\mathcal{L}| = v \geq 2q + 3$ . Define  $c$  by  $|\mathcal{P}| = 2q + 2\sqrt{q} + 2 - c$ . Then

$$n_1 \leq \frac{c(2q\sqrt{q} - 2q - 2\sqrt{q} + c)}{2\sqrt{q} + 1}, \text{ and}$$

$$n_1 + n_{3+} \leq \frac{c(2q\sqrt{q} - 2q - 2\sqrt{q} + c)}{2\sqrt{q} + 1} + 2q + 2\sqrt{q} + 2 - c,$$

where  $n_i$  and  $n_{i+}$  are the quantities defined in Definition 3.3 with respect to  $\mathcal{P}$ .

**Proof.** By Theorem 3.2 and Corollary 3.5,  $c \geq 0$  and  $n_0 = 0$ . As  $M$  is 2-marginal,  $n_{3+} \leq v$  clearly holds. Let us write a slightly rearranged form of the so-called standard equations:

$$\sum_{i=3}^{q+1} n_i = q^2 + q + 1 - n_1 - n_2,$$

$$\sum_{i=3}^{q+1} in_i = v(q + 1) - n_1 - 2n_2,$$

$$\sum_{i=3}^{q+1} i(i - 1)n_i = v(v - 1) - 2n_2.$$

Then we have

$$0 \leq \sum_{i=3}^{q+1} (i - (\sqrt{q} + 2))^2 n_i = \sum_{i=3}^{q+1} i(i - 1)n_i - (2\sqrt{q} + 3) \sum_{i=3}^{q+1} in_i + (\sqrt{q} + 2)^2 \sum_{i=3}^{q+1} n_i$$

$$= v(v - 1) - 2n_2 - (2\sqrt{q} + 3)(v(q + 1) - n_1 - 2n_2) + (\sqrt{q} + 2)^2(q^2 + q + 1 - n_1 - n_2),$$

equivalently,

$$(q + 2\sqrt{q} + 1)n_1 + qn_2 \leq v(v - 1) - (2\sqrt{q} + 3)v(q + 1) + (\sqrt{q} + 2)^2(q^2 + q + 1).$$

It follows that

$$q(n_1 + n_2) + (2\sqrt{q} + 1)n_1 \leq v(v - 1) - (2\sqrt{q} + 3)v(q + 1) + (\sqrt{q} + 2)^2(q^2 + q + 1).$$

Let us use  $v \geq n_{3+} = q^2 + q + 1 - n_1 - n_2$  to find

$$q(q^2 + q + 1 - v) + (2\sqrt{q} + 1)n_1 \leq v(v - 1) - (2\sqrt{q} + 3)v(q + 1) + (\sqrt{q} + 2)^2(q^2 + q + 1),$$

that is,

$$n_1 \leq \frac{v(v - 1) - (2\sqrt{q} + 3)v(q + 1) + vq + 4(\sqrt{q} + 1)(q^2 + q + 1)}{2\sqrt{q} + 1}.$$

If we write  $v = 2q + 2\sqrt{q} + 2 - c$ , we obtain

$$n_1 \leq \frac{c(2q\sqrt{q} - 2q - 2\sqrt{q} + c)}{2\sqrt{q} + 1}.$$

Furthermore, as  $n_{3+} \leq v = 2q + 2\sqrt{q} + 2 - c$ ,

$$n_1 + n_{3+} \leq \frac{c(2q\sqrt{q} - 2q - 2\sqrt{q} + c)}{2\sqrt{q} + 1} + 2q + 2\sqrt{q} + 2 - c. \quad \square$$

Note that the upper bound found on  $n_1 + n_{3+}$  is increasing in  $c$  under  $0 \leq c \leq 2\sqrt{q}$  if  $q \geq 7$ . Let us remark that Theorem 3.2 can also be proved with the help of the standard equations, yet the proof is much more compact if we use the incidence bound.

### 3.2. Arguments for PG(2, q)

Throughout this part,  $M = \mathcal{P} \cup \mathcal{L}$  denotes a 2-marginal set of PG(2, q) with  $|\mathcal{P}| = |\mathcal{L}| = v \geq 2q + 3$ . By [Corollary 3.5](#) and [Lemma 3.6](#), we see that if  $v$  is not too small, then almost all lines intersect  $\mathcal{P}$  in exactly two points. Our strategy is to slightly modify  $\mathcal{P}$  so that it becomes a weighted double blocking set (see definition below), and then use characterization results on these types of sets.

**Definition 3.7.** A multiset or weighted set  $S$  will be considered as a set of points equipped with non-negative integer weights. The size or total weight of  $S$ , in notation  $|S|$ , is the sum of the weights of the points in  $S$ . The weighted intersection of a line  $\ell$  with  $S$  is the sum of the weights of the points in  $\ell \cap S$ . If we say that a line intersects a multiset  $S$  in  $k$  points, we mean that its weighted intersection with  $S$  is  $k$ . A weighted  $k$ -fold blocking set is a weighted set which intersects every line in at least  $k$  points. Weighted 2-fold blocking sets are also called weighted double blocking sets.

First we will use a theorem of [\[30\]](#) about multisets of PG(2, q) which intersect almost all lines in  $k \pmod p$  points, where  $p$  is the characteristic of the underlying field GF(q), and  $k$  is an arbitrary integer.

**Theorem 3.8** ([\[30\]](#), [Theorem 3.6](#)). Let  $S$  be a multiset in PG(2, q),  $17 < q$ ,  $q = p^h$ , where  $p$  is prime. Let  $0 \leq k < p$  be an arbitrary integer. Assume that the number of lines intersecting  $S$  in non- $(k \pmod p)$  points is  $\delta$ , where  $\delta < (\lfloor \sqrt{q} \rfloor + 1)(q + 1 - \lfloor \sqrt{q} \rfloor)$ . Assume furthermore that the following property holds: if through a point  $P$  there are more than  $q/2$  lines intersecting  $S$  in non- $k \pmod p$  points, then there exists a value  $r(P)$  such that the weighted intersection of more than  $\frac{2\delta}{q+1} + 5$  of these lines with  $S$  is  $r(P) \pmod p$ . Then there exists a multiset  $S'$  with the property that it intersects every line in  $k \pmod p$  points and the number of different points in  $(S \cup S') \setminus (S \cap S')$  is exactly  $\left\lceil \frac{\delta}{q+1} \right\rceil$ .

The statement of the above theorem may be rephrased in the following way: there exists a set  $X$  of  $\left\lceil \frac{\delta}{q+1} \right\rceil$  points such that we can obtain the multiset  $S'$  from  $S$  by giving appropriate weights of the points in  $X$ , and leaving the weights in  $S \setminus X$  untouched. (Note that  $X \subset S$  is not necessarily true.)

We set  $k = 2$  to apply [Theorem 3.8](#) for  $\mathcal{P}$ , considered as a multiset where every point has weight one. Let  $\delta$  denote the total number of non- $(2 \pmod p)$  secant lines to  $\mathcal{P}$ ; then  $\delta \leq n_1 + n_{3+}$  (recall  $n_0 = 0$ ), the quantity bounded from above by [Lemma 3.6](#). To meet the requirement

$$\delta < (\lfloor \sqrt{q} \rfloor + 1)(q + 1 - \lfloor \sqrt{q} \rfloor) \tag{1}$$

of [Theorem 3.8](#), we distinguish two cases. If  $q$  is a square, we need  $\delta < q\sqrt{q} + 1$ . If  $q$  is not a square, then substituting  $\lfloor \sqrt{q} \rfloor = \sqrt{q} - \varepsilon$  into (1) ( $0 \leq \varepsilon < 1$ ), it is easy to see that the right-hand side is strictly decreasing in  $\varepsilon$  if  $q \geq 4$ . Thus it is enough to satisfy the inequality for  $\varepsilon = 1$ , that is, to have  $\delta \leq q\sqrt{q} - q + 2\sqrt{q}$ .

Let  $|\mathcal{P}| = 2q + 2\sqrt{q} + 2 - c$  as in [Lemma 3.6](#). If  $q$  is a square, we assume  $c \leq \sqrt{q} - 1$  (that is,  $|\mathcal{P}| \geq 2q + \sqrt{q} + 3$ , which is the smallest possible size of a 2-marginal set not covered by the constructions in [Theorem 2.12](#)), in which case [Lemma 3.6](#) yields

$$\delta \leq n_1 + n_{3+} \leq \frac{(\sqrt{q} - 1)(2q\sqrt{q} - 2q - \sqrt{q} - 1)}{2\sqrt{q} + 1} + 2q + \sqrt{q} + 3 \leq q\sqrt{q},$$

provided that  $q \geq 49$ . If  $q$  is not a square, we assume  $c \leq \sqrt{q} - 2$  (that is,  $v \geq 2q + \sqrt{q} + 4$ ; this is needed to obtain a suitable upper bound on  $\delta$ ). Then, under  $q \geq 32$ , [Lemma 3.6](#) similarly yields

$$\delta < q\sqrt{q} - q + 2\sqrt{q}.$$

Thus the respective assumptions on  $c$  enable us to apply [Theorem 3.8](#) for  $\mathcal{P}$  as soon as it has the other property. The quantity involved therein,  $\frac{2\delta}{q+1} + 5$ , can be bounded above by  $2\sqrt{q} + 5$  in both cases. Suppose now that for a point  $P$ , the number of non- $(2 \pmod p)$  secants through  $P$  to  $\mathcal{P}$  is more than  $q/2$ , then clearly  $n_2(P) \leq (q + 1)/2$ . If  $P \notin \mathcal{P}$ , there are at most two  $(3+)$ -secants to  $\mathcal{P}$  on  $P$ . Recalling  $n_0(P) = 0$  by [Corollary 3.5](#), then  $n_1(P) \geq q + 1 - 2 - (q + 1)/2 = (q - 3)/2 > 2\sqrt{q} + 5$  holds if  $q \geq 61$ . Suppose now  $P \in \mathcal{P}$ . Then, for any integer  $t \geq 2$ , we have

$$|\mathcal{P}| \geq 1 + \sum_{i=1}^{t-1} n_i(P)(i - 1) + (t - 1)n_{t+}(P).$$

As  $n_{t+}(P) = q + 1 - \sum_{i=1}^{t-1} n_i(P)$ , this yields

$$|\mathcal{P}| \geq 1 + (t - 1)(q + 1) - \sum_{i=1}^{t-1} (t - i)n_i(P) = 1 + (t - 1)(q + 1) - (t - 2)n_2(P) - \sum_{\substack{i=1 \\ i \neq 2}}^{t-1} (t - i)n_i(P).$$

Suppose now and substitute  $n_i \leq 2\sqrt{q} + 5$  for all  $1 \leq i \leq t - 1$ ,  $i \neq 2$  to get

$$|\mathcal{P}| \geq 1 + (t - 1)(q + 1) - \frac{(t - 2)(q + 1)}{2} - \left( \binom{t}{2} - (t - 2) \right) (2\sqrt{q} + 5).$$



Easy calculations show that the right-hand side is increasing on the interval  $2 \leq t \leq \sqrt{q}/4 + 1$ , hence we may substitute  $t = \sqrt{q}/4$  to obtain the valid lower bound

$$|\mathcal{P}| \geq \frac{q^{3/2}}{16} + \frac{19q}{32} - 2\sqrt{q} - 9.$$

Compared with the upper bound  $|\mathcal{P}| \leq 2(q + \sqrt{q} + 1)$  of [Theorem 3.2](#), we get a contradiction for  $q \geq 641$ . (This bound on  $q$  could be lowered, but subsequent bounds would override it.) Thus we see that  $n_i(\mathcal{P}) > (2\sqrt{q} + 5)$  must hold for some  $1 \leq i \leq t - 1$  ( $i \neq 2$ ), which implies the property required in [Theorem 3.8](#). Thus, as remarked after [Theorem 3.8](#), there exists a set  $X$  of  $\left\lceil \frac{\delta}{q+1} \right\rceil \leq \sqrt{q}$  points such that it is possible to give appropriate weights to the points in  $X$ , while letting all points of  $\mathcal{P} \setminus X$  have weight one, in such a way that we obtain a multiset  $\mathcal{P}'$  which intersects every line in  $2 \pmod p$  points. Let us define another weighted point set  $\mathcal{P}^*$  as follows: we give weight 2 to the points of  $X \cap \mathcal{P}$  and weight 1 to the points of  $(X \setminus \mathcal{P}) \cup (\mathcal{P} \setminus X)$ . We will check that  $\mathcal{P}^*$  is a weighted double blocking set. As  $\mathcal{P}$  is a blocking set ([Corollary 3.5](#)), every line contains at least one point of  $\mathcal{P}$ . Consider an arbitrary line  $\ell$ . If  $\ell$  contains a point of  $X \cap \mathcal{P}$ , then its weighted intersection with  $\mathcal{P}^*$  has weight at least two. Suppose now  $\ell \cap X \cap \mathcal{P} = \emptyset$ . If  $|\ell \cap \mathcal{P}| \geq 2$ , then  $\ell$  intersects  $\mathcal{P}^*$  in at least two (different) points. If  $|\ell \cap \mathcal{P}| = 1$ , then the  $(2 \pmod p)$  property of  $\mathcal{P}'$  implies that  $\ell$  must contain a point of  $X \setminus \mathcal{P}$ , thus  $\ell$  intersects  $\mathcal{P}^*$  in at least two (different) points. Hence  $\mathcal{P}^*$  is a weighted double blocking set with total weight  $v + |X| \leq 2q + 3\sqrt{q} + 2$ , and it has at most  $|X| \leq \sqrt{q}$  points that are double (i.e., have weight two). We will use the following theorem that describes the structure of small enough weighted double blocking sets.

**Theorem 3.9** ([\[4\]](#), [Theorem 3.10](#)). *Let  $\mathcal{B}$  be a weighted  $k$ -fold blocking set in  $\text{PG}(2, q)$ ,  $q = p^h$ ,  $p$  prime,  $h > 1$ . Let  $C_2 = C_3 = 2^{-1/3}$  and  $C_p = 1$  for  $p > 3$ . Assume that  $|\mathcal{B}| = kq + k + C - (k - 1)(k - 2)/2$ , where*

- (1)  $C < C_p q^{2/3}$  and  $k < \min\{C_p q^{1/6}, q^{1/4}/2\}$ , or
- (2)  $q = p^2$ ,  $k < q^{1/4}/2$  and  $C < q^{3/4}/2$ .

*If the number of points with weight one is at least  $(k - 2)(q + \sqrt{q} + 1) + 16\sqrt{q} + 8q^{1/4}$  in (1) and at least  $(k - 2)(q + \sqrt{q} + 1) + 16\sqrt{q} + 16q^{1/6}$  in (2), then  $\mathcal{B}$  contains the weighted union of the point sets of  $k$  (not necessarily different) Baer subplanes and/or lines (considered as point sets).*

Let us check the conditions of [Theorem 3.9](#), case (1) for  $k = 2$  and  $C \leq 3\sqrt{q}$ . Since points of weight two are in  $X$ , there may be at most  $|X| \leq \sqrt{q}$  of them, so the number of points with weight one is at least  $v - |X| \geq 2q + \sqrt{q} + 3 - \sqrt{q} = 2q + 3 \geq 16\sqrt{q} + 8q^{1/4}$ , which satisfies the respective condition (under  $q \geq 89$ ). To satisfy  $2 < \min\{C_p q^{1/6}, q^{1/4}/2\}$ , we need  $q > 256$  (regardless whether  $p > 3$  or not), while  $3\sqrt{q} < C_p q^{2/3}$  requires  $q > 729$  if  $p > 3$  and  $q > 2916$  otherwise. Thus, for square prime powers, we may apply [Theorem 3.9](#) if  $q \geq 841$  and  $q \neq 1024$ . If  $q$  is a non-square power of a prime, then we need  $q \geq 787$  and  $q \notin \{2048, 2187\}$ .

Once we can apply [Theorem 3.9](#), we may conclude that  $\mathcal{P}^*$  contains the weighted union of two lines and/or Baer subplanes. Thus  $\mathcal{P} \cup X$  contains the union  $\mathcal{U}$  of two lines and/or Baer subplanes (without weights). Note that Baer subplanes exist in  $\text{PG}(2, q)$  if and only if  $q$  is a square.

Suppose that  $\mathcal{U}$  is the union of two (necessarily distinct) lines,  $\ell_1$  and  $\ell_2$ . Then  $|\mathcal{U}| = 2q + 1$ . Since  $v \geq 2q + \sqrt{q} + 3$ ,  $\mathcal{P}$  must contain at least  $v - (2q + 1) = \sqrt{q} + 2 \geq 3$  points not in  $\mathcal{U}$ . Let  $P \in \mathcal{P} \setminus \mathcal{U}$ . Then all but one lines through  $P$  meet  $\mathcal{U} = \ell_1 \cup \ell_2$  in two distinct points. As at most  $|X| \leq \sqrt{q}$  of these points may be not in  $\mathcal{P}$ , there must be at least  $q - \sqrt{q}$  lines on  $P$  that meet  $\mathcal{U} \cap \mathcal{P}$  in two points and, as  $P \in \mathcal{P}$ , these lines are  $(3+)$ -secants to  $\mathcal{P}$ . Since  $|\mathcal{P} \setminus \mathcal{U}| \geq 3$ , we see that  $v \geq n_{3+} \geq 3(q - \sqrt{q}) - 3 > 2q + 2\sqrt{q} + 2$ , a contradiction (as  $q \geq 29$ ).

Suppose now that  $\mathcal{U}$  is the union of a line  $\ell$  and a Baer subplane  $\Pi'$ . Since  $|\Pi' \cup \ell| \leq 2q + \sqrt{q} + 1$  and  $v \geq 2q + \sqrt{q} + 3$ ,  $|\mathcal{P} \setminus \mathcal{U}| \geq 2$ . Let  $P \in \mathcal{P} \setminus \mathcal{U}$ . Then every line through  $P$  intersecting  $\ell \setminus \Pi'$  is a  $(2+)$ -secant to  $\mathcal{U}$ , hence at least  $|\ell \setminus \Pi'| - |X| \geq q - \sqrt{q} - \sqrt{q} = q - 2\sqrt{q}$  of these are  $(3+)$ -secants to  $\mathcal{P}$ . One of these lines may be a line of  $\Pi'$  as well, but at least  $q - 2\sqrt{q} - 1$  of them are  $(3+)$ -secants through  $P$  that meet  $\Pi'$  in only one point. On two points of  $\mathcal{P} \setminus \mathcal{U}$  we find at least  $2(q - 2\sqrt{q} - 1) - 1$  such  $(3+)$ -secants to  $\mathcal{P}$ . Consider now the  $q + \sqrt{q} + 1$  lines of  $\Pi'$ . These are  $\sqrt{q} + 1$ -secants to  $\Pi'$ . At most one of them may contain more than  $\sqrt{q} - 2$  points of  $X$  (otherwise  $\sqrt{q} \geq |X| \geq 2(\sqrt{q} - 2) - 1$  would be a contradiction), so at least  $q + \sqrt{q}$  of them are  $(3+)$ -secants to  $\mathcal{P}$ . Altogether we see that  $v \geq n_{3+} \geq 2(q - 2\sqrt{q} - 1) - 1 + q + \sqrt{q} > 2q + 2\sqrt{q} + 2$ , a contradiction provided that  $q \geq 49$ .

Thus  $\mathcal{U}$  must be the union of two Baer subplanes,  $\Pi'_1$  and  $\Pi'_2$ . These can have at most  $\sqrt{q} + 2$  common lines, so they admit at least  $2q + \sqrt{q}$  distinct  $(\sqrt{q} + 2)$ -secants, of which only one may contain  $\sqrt{q} - 1$  or more points of  $X$ , so we find at least  $2q + \sqrt{q} - 1$   $(3+)$ -secant lines to  $\mathcal{P}$  which are also the lines of  $\Pi'_1$  or  $\Pi'_2$ . Suppose that there exists a point  $P \in \mathcal{P}$  not in  $\mathcal{U}$ . Then every line through  $P$  not meeting  $X \supseteq \Pi'_1 \cap \Pi'_2$  is a  $(3+)$ -secant to  $\mathcal{P}$ . At most two of these can be lines of  $\Pi'_1$  or  $\Pi'_2$ , so we see  $v \geq n_{3+} \geq 2q + \sqrt{q} - 1 + q + 1 - |X| - 2 \geq 3q - 2 > 2q + 2\sqrt{q} + 2$ , a contradiction under  $q \geq 16$ . Thus  $\mathcal{P} \subset \mathcal{U}$ . Suppose that there exists a point  $P \in \mathcal{U} \setminus \mathcal{P}$ ; say,  $P \in \Pi'_1$ . There pass  $\sqrt{q} + 1$  lines of  $\Pi'_1$  through  $P$ , which are  $\sqrt{q}$ -secants to  $\Pi'_1 \setminus \{P\}$ . As  $P \notin \mathcal{P}$ , there can be only two  $(3+)$ -secants on  $P$  to  $\mathcal{P}$ , so there must be at least  $(\sqrt{q} - 1)(\sqrt{q} - 2) = q - 3\sqrt{q} + 2$  further points of  $\Pi'_1$  missing from  $\mathcal{P}$ , whence  $v \leq |\mathcal{U}| - (q - 3\sqrt{q} + 2) \leq q + 5\sqrt{q}$  follows, in contradiction with our assumption  $v \geq 2q + \sqrt{q} + 3$ . Thus  $\mathcal{P} = \mathcal{U}$ , which case is indeed covered by [Theorem 2.12](#).

These arguments prove the following theorems.

**Theorem 3.10.** Suppose that  $q \geq 841$ ,  $q \neq 1024$  is a square prime power. Let  $M = \mathcal{P} \cup \mathcal{L}$  be a balanced 2-marginal set of  $\text{PG}(2, q)$ . Then either  $|\mathcal{P}| \leq 2q + \sqrt{q} + 2$ , or  $2q + 2\sqrt{q} - 1 \leq |\mathcal{P}| \leq 2q + 2\sqrt{q} + 2$ ; that is, the size of  $M$  is in the range covered by the constructions in [Theorem 2.12](#). Moreover, if  $|\mathcal{P}| \geq 2q + 2\sqrt{q} - 1$ , then  $M$  is the union of two Baer subplanes.

**Theorem 3.11.** Suppose that  $q = p^h$ ,  $p$  prime,  $h \geq 1$  odd. Assume that  $q \geq 787$ ,  $q \notin \{2048, 2187\}$ . Let  $M = \mathcal{P} \cup \mathcal{L}$  be a balanced 2-marginal set of  $\text{PG}(2, q)$ . Then  $|\mathcal{P}| < 2q + \sqrt{q} + 4$ .

Let us remark that if  $q$  is a prime, we can derive the conclusion of [Theorem 3.11](#) assuming  $q \geq 83$  only. To this end, we do not use [Theorem 3.8](#) directly but the techniques behind it as developed in [\[30\]](#), and an improved characterization of weighted  $k$ -fold blocking sets in  $\text{PG}(2, q)$ ,  $q$  prime, found in [\[15, Theorems 2.5, 2.13 and Proposition 2.15\]](#).

### 3.3. Planes of small order

[Theorems 3.10](#) and [3.11](#) limit the possible sizes of 2-marginal sets for planes with large enough orders. It is natural to ask whether or not an analogous result holds for planes of small order. We have performed a modest computer search to deliver some findings of this kind for Desarguesian planes. By [Theorems 2.12](#) and [3.2](#), the possible sizes not covered by our constructions are between  $2q + 4$  and  $\lfloor 2(q + \sqrt{q} + 1) \rfloor$  if  $q$  is not a square, and between  $2q + \sqrt{q} + 3$  and  $2q + 2\sqrt{q} - 2$  if  $q$  is a square.

$q$	l.b.	u.b.	$\exists$	$\nexists$
3	10	11	10	11
5	14	16	14, 16	15
7	18	21	18, 21	19, 20
8	20	23	20	21, 22, 23

In the table, l.b. and u.b. stand for the smallest and the largest size not covered by our general results, and the next two columns indicate the sizes for which we could verify the existence or non-existence of a 2-marginal set.

Let us remark that the 2-marginal sets of size 18 and 21 for  $q = 7$  yield  $b(31) = 37$  and  $b(34) = 40$ , two exact values not obtained by other results.

## 4. General results on $b(n)$

Let us give a new upper bound for  $b(n)$  for some particular values of  $n$ , which immediately refutes [Conjecture 1.4](#). In this section, given a graph  $G$ ,  $\delta(G)$  and  $\Delta(G)$  will refer to the smallest and largest degree in  $G$ , respectively, and  $\deg(u)$  will stand for the degree of the vertex  $u$ .

**Lemma 4.1.** If a projective plane of order  $q$  ( $q \geq 2$ ) does not exist then

$$b(q^2 + 1) \leq q^2 + q + 1 = q^2 + 1 + \lceil \sqrt{q^2 + 1} \rceil - 1.$$

**Proof.** Suppose, by contradiction, that  $b(q^2 + 1) > q^2 + q + 1$ , which implies that there exists a  $C_4$ -free bipartite graph  $H$  with vertex classes  $A$  and  $B$ , where  $|A| = |B| = q^2 + q + 1$  and  $\delta(H) \geq q + 1$ . Let  $T$  be the number of 3-vertex paths with middle vertex in  $A$ . On the one hand, we have  $T = \sum_{u \in A} \binom{\deg(u)}{2} \geq (q^2 + q + 1) \binom{q+1}{2}$ . On the other hand, as  $H$  is  $C_4$ -free, there is at most one path with two given vertices of  $B$  as endpoints, whence  $T \leq \binom{q^2 + q + 1}{2}$ . Since  $(q^2 + q + 1) \binom{q+1}{2} = \binom{q^2 + q + 1}{2}$ , it follows that each vertex in  $A$  has degree  $q + 1$ , and each pair of vertices in  $B$  have a unique common neighbor in  $A$ . Due to symmetry, we see that  $H$  is a  $(q + 1) \geq 3$ -regular graph, and every pair of vertices in the same vertex class has exactly one common neighbor, which yields that  $H$  is the incidence graph of a projective plane of order  $q$ , a contradiction.  $\square$

It is well-known that a projective plane of order 6 does not exist, hence  $b(37) \leq 43$ . More generally, the Bruck–Ryser Theorem [\[8\]](#), a celebrated non-existence result on symmetric block designs yields that if a projective plane of order  $q$  exists and  $q \equiv 1$  or  $2 \pmod{4}$ , then  $q$  is the sum of two squares. This gives infinitely many counterexamples to [Conjecture 1.4](#) as, for example, it follows that no projective plane of order  $q$  exists if  $q \equiv 6 \pmod{8}$ . The second smallest order for which a projective plane of order  $q$  does not exist (though the Bruck–Ryser Theorem does not exclude it) is 10 [\[27\]](#).

Besides refuting [Conjecture 1.4](#), [Lemma 4.1](#) can also be used to obtain new exact values of  $b(n)$  for particular values of  $n$ . In a projective plane of order  $q$ , we may take the union of two near pencils to construct a 2-marginal set of size  $2q + 1$ . This was seen in the proof of [Theorem 2.12](#), but it was not taken into account as its size does not match the requirements of [Lemma 2.9](#) to yield a construction meeting the upper bound in [Lemma 1.2](#). However, as [Lemma 4.1](#) improves [Lemma 1.2](#) in particular cases, such a 2-marginal set turns out to be useful.

**Theorem 4.2.** If a projective plane of order  $q$  does not exist but there is a projective plane of order  $q + 1$ , then

$$b(q^2 + 1) = q^2 + q + 1 = q^2 + 1 + \lceil \sqrt{q^2 + 1} \rceil - 1.$$

**Proof.** Lemma 4.1 gives the upper bound. Removing a 2-marginal set of size  $2(q + 1) + 1$  from a projective plane of order  $q + 1$  yields a balanced  $C_4$ -free bipartite graph on  $2((q + 1)^2 + (q + 1) + 1 - (2q + 3)) = 2(q^2 + q)$  vertices with minimum degree  $q$  (so its bipartite complement contains no  $K_{1,q^2+1}$ ), whence  $b(q^2 + 1) > q^2 + q$ .  $\square$

From our previous remarks it follows for example that  $q = 10$  and  $q = 7^{2k+1} - 1$  fit the requirements of Theorem 4.2 for all  $k \geq 0$ .

In addition, Lemma 4.1 enables us to obtain another new exact value of  $b(n)$  using a 3-marginal set.

**Proposition 4.3.**  $b(197) = 211 = 197 + \lceil \sqrt{197} \rceil - 1$ .

**Proof.** The key idea is that if a projective plane of order  $q$  does not exist but there is a projective plane of order  $Q = q + 2$  which contains a 3-marginal set of size  $4Q - 1$ , then we can get equality in Lemma 4.1. Indeed, Lemma 4.1 gives the upper bound  $b(q^2 + 1) \leq q^2 + q + 1$ , whereas removing a 3-marginal set of size  $4Q - 1 = 4q + 3$  from a projective plane of order  $Q = q + 2$  yields a balanced  $C_4$ -free bipartite graph on  $2(Q^2 + Q + 1 - (4Q - 1)) = 2(q^2 + 5q + 7) - (4q + 7) = 2(q^2 + q)$  vertices with minimum degree  $q$ , whence  $b(q^2 + 1) > q^2 + q$ . However, compared with the upper bound  $3(Q + \sqrt{Q} + 1)$  of Theorem 3.2 on the size of a 3-marginal set in  $\Pi_Q$ , we see that a suitably large 3-marginal set may exist only if  $Q \leq 16$ ; that is,  $q \leq 14$ . As the union of three disjoint Baer subplanes is a 3-marginal set of size  $3(16 + 4 + 1) = 63 = 4 \cdot 16 - 1$  in  $\text{PG}(2, 16)$ , we see that  $q = 14 \equiv 6 \pmod{8}$  fits all conditions in question, whence  $b(197) = 211$ . Let us remark that the two other possibly suitable values  $q = 6$  and  $q = 10$  are already covered by Theorem 4.2.  $\square$

Let us now turn to Conjecture 1.5. To proceed, we briefly recall the Zarankiewicz problem and some results on it. For positive integers  $n$  and  $m$ , let the Zarankiewicz number  $Z(n, m)$  denote the maximum number of edges a  $C_4$ -free bipartite graph may have provided that its classes contain  $n$  and  $m$  vertices. Clearly, if  $Z(m, m) < km$ , then every balanced  $C_4$ -free bipartite graph on  $2m$  vertices has minimum degree smaller than  $k$ , and thus its bipartite complement contains a  $K_{1,m-k+1}$ ; that is,  $b(m - k + 1) \leq m$ . Such straightforward connections of bipartite Ramsey numbers and Zarankiewicz numbers are well-known and often exploited; see [11,20], for example.

A counterexample for Conjecture 1.5 is based on the fact that  $Z(22, 22) = 108 < 5 \cdot 22$ , computed by Andrew Kay [26] and, independently, by Afzaly and McKay (unpublished; reported in [11]). Applying the observation above, this immediately gives  $b(18) \leq 22 = b(17)$  (Theorem 1.3); so  $b(17) = b(18) = 22$ . Let us note that  $n = 18$  is the smallest value of  $n$  for which Conjecture 1.4 fails. Let us formulate this situation in general.

**Lemma 4.4.**

1. If  $Z(q^2 + q + 2, q^2 + q + 2) < (q^2 + q + 2)(q + 1)$ , then  $b(q^2 + 2) \leq q^2 + q + 2 = q^2 + 2 + \lceil \sqrt{q^2 + 2} \rceil - 1$ .
2. If  $Z(q^2 + q + 2, q^2 + q + 2) < (q^2 + q + 2)(q + 1)$  and a projective plane of order  $q$  exists, then  $b(q^2 + 1) = b(q^2 + 2) = q^2 + q + 2$ .

To derive further counterexamples for both conjectures, we need the following lemma on  $Z(q^2 + q + 2, q^2 + q + 2)$ .

**Lemma 4.5.** If  $q \geq 5$ , then  $Z(q^2 + q + 2, q^2 + q + 2) \leq (q^2 + q + 2)(q + 1)$ . Furthermore, every graph providing equality is  $(q + 1)$ -regular.

**Proof.** We will need the following simple observation. Suppose that a balanced  $C_4$ -free bipartite graph has minimum degree  $\delta$  and maximum degree  $\Delta$ . Then, by taking a vertex  $u$  of degree  $\Delta$  and considering the vertices at distance 2 from  $u$ , we see that both classes contain at least  $1 + \Delta(\delta - 1)$  vertices.

Let  $G$  be a balanced  $C_4$ -free bipartite graph on  $2(q^2 + q + 2)$  vertices with  $e = (q^2 + q + 2)(q + 1) + \varepsilon$  edges. Let  $\delta(G) = \delta$  and  $\Delta(G) = \Delta$  for short. Removing a vertex of degree  $\delta$  from  $G$  we get  $Z(q^2 + q + 2, q^2 + q + 1) \geq e - \delta = (q^2 + q + 1)(q + 1) + 1 + q + \varepsilon - \delta$ . By Corollary 3.27 of [12], we know that  $Z(q^2 + q + 2, q^2 + q + 1) \leq (q^2 + q + 1)(q + 1) + 1$ , hence  $\delta \geq q + \varepsilon$  follows.

Let us prove the upper bound first. Suppose to the contrary that  $\varepsilon \geq 1$ . Then  $\delta \geq q + 1$  and, as  $e > (q^2 + q + 2)(q + 1)$ ,  $\Delta \geq q + 2$ . These give  $q^2 + q + 2 \geq 1 + \Delta(\delta - 1) \geq 1 + (q + 2)q = q^2 + 2q + 1$ , a contradiction. Hence  $\varepsilon \leq 0$ .

Suppose now  $\varepsilon = 0$ . Then  $\delta \geq q$ . Suppose to the contrary that  $\Delta \geq q + 2$ . As  $e = (q^2 + q + 2)(q + 1)$ , this yields  $\delta = q$ . Moreover,  $\Delta \geq q + 3$  would yield  $q^2 + q + 2 \geq 1 + \Delta(\delta - 1) \geq 1 + (q + 3)(q - 1) = q^2 + 2q - 2$ , in contradiction with  $q \geq 5$ ; hence  $\Delta = q + 2$ . Let  $A$  and  $B$  denote the vertex classes of  $G$ . Let  $w \in A$  be a vertex of degree  $q + 2$ , and let  $u \in A$  be of degree  $\delta = q$ . Let  $N(w) \subset B$  be the set of neighbors of  $w$ . As  $G$  is  $C_4$ -free, every vertex of  $A \setminus \{w\}$  may have at most one neighbor in  $N(w)$ ; furthermore, there may be at most  $Z(q^2 + q, q^2)$  edges between  $A \setminus \{w, u\}$  and  $B \setminus N(w)$ . Considering the edges incident with  $w, u$ , and the vertices of  $A \setminus \{w, u\}$ , we see that  $e \leq q + 2 + q + (q^2 + q) + Z(q^2 + q, q^2)$ . It is well-known that  $Z(q^2 + q, q^2) \leq q^2(q + 1)$ , and equality is reached only by the incidence graphs of affine planes of order  $q$ . This yields  $e \leq q^2 + 3q + 2 + q^2(q + 1) = (q^2 + q + 2)(q + 1) = e$ , which means that we have equality in all our estimates. Therefore,  $A \setminus \{w, u\}$  and  $B \setminus N(w)$  must span the incidence graph of an affine plane, so any two vertices of  $B \setminus N(w)$  (two points of the affine plane) have a unique neighbor in  $A \setminus \{w, u\}$  (a line of the plane). However, as  $u$  must have at least  $q - 1 \geq 2$  neighbors in  $B \setminus N(w)$ , we find a  $C_4$  in  $G$ , a contradiction. Hence  $\Delta = q + 1$  which, as  $e = (q^2 + q + 2)(q + 1) \leq (q^2 + q + 2)\Delta$ , proves the statement.  $\square$

Graphs attaining equality in [Lemma 4.5](#) are very rare: they exist for  $q = 2$  and  $3$ , but no other examples are known. Payne devoted the whole study [[29](#)] to a more general version of this question, and he formed a conjecture whose special case we rephrase here to fit the present setting.

**Conjecture 4.6** [[29](#)] p.268). *If  $q \geq 4$ , then  $C_4$ -free,  $(q + 1)$ -regular, balanced bipartite graphs on  $2(q^2 + q + 2)$  vertices do not exist.*

This conjecture is open in general, but Payne [[29](#), in-text remarks on p.277 and p.280, Theorems 5.1 and 5.2] could verify it for particular values of  $q$ . We formulate some of these results.

**Theorem 4.7** (Payne [[29](#)]). *If (1)  $q \equiv 1$  or  $2 \pmod{4}$  and  $q - 1$  is not a square, or (2) if  $q \equiv 0$  or  $3 \pmod{4}$  and  $q + 1$  is not a square, or (3) if  $q \leq 401$ , then [Conjecture 4.6](#) holds for  $q$ .*

Let us remark that Biggs and Ito [[6](#)] also proved [Conjecture 4.6](#) for  $q \equiv 4 \pmod{8}$  and  $q \equiv 6 \pmod{8}$ , which are special cases of [Theorem 4.7](#). If [Conjecture 4.6](#) holds for  $q$ , then [Lemma 4.5](#) assures  $Z(q^2 + q + 2, q^2 + q + 2) < (q^2 + q + 2)(q + 1)$ , hence [Lemma 4.4](#) can be applied. Thus [Theorem 4.7](#) immediately yields the following result, giving infinitely many counterexamples for [Conjectures 1.4](#) and [1.5](#).

**Corollary 4.8.** *If (1)  $q \equiv 1$  or  $2 \pmod{4}$  and  $q - 1$  is not a square, or (2)  $q \equiv 0$  or  $3 \pmod{4}$  and  $q + 1$  is not a square, or (3)  $4 \leq q \leq 401$ , then  $b(q^2 + 2) \leq q^2 + q + 2$ . Furthermore, if a projective plane of order  $q$  exists (and  $q$  satisfies one of the three preceding conditions), then  $b(q^2 + 1) = b(q^2 + 2) = q^2 + q + 2$ .*

## 5. Conclusion and remarks

We have found new values of  $n$  for which  $b(n)$  can be computed exactly. In most of these cases,  $b(n)$  reaches the known upper bound  $b(n) = n + \lceil \sqrt{n} \rceil$ , but for some particular values of  $n$ , we were able to improve this upper bound by one. Let us note that using the incidence graph of projective planes and subgraphs thereof, the values of  $n$  for which  $b(n)$  is known are not all consecutive. For example, the first unknown value for  $n$ , descending from  $q^2 + 1$ , remains  $n = q^2 - q - 1$ . The upper bound gives  $b(n) \leq q^2 - 1$ , so we would need to find a bipartite graph with  $q^2 - 2$  vertices in each part and minimum degree  $q$ . This could be done, for example, using a 1-marginal set of  $\Pi_q$  of size  $q + 3$ . Unfortunately, the only plane that possesses such a set is  $\text{PG}(2, 4)$ . However, it might be possible to take a subgraph of a projective plane that does not give a sharp lower bound but, applying some graph-theoretical adjustments such as adding and deleting edges in a clever way, one may obtain a suitable graph from it. This has been done for example in the case of the classic Ramsey number  $R(C_4, K_{1,n})$  by Zhang, Cheng and Chen [[31](#)].

[Theorem 3.10](#) assures that the spectrum of the sizes of 2-marginal sets of  $\text{PG}(2, q)$ , if  $q$  is a large enough square, is covered by the constructions of [Theorem 2.12](#). However, if  $q$  is not a square (and large enough), [Theorem 3.11](#) only excludes sizes at least  $2q + \sqrt{q} + 4$ , hence the existence of 2-marginal sets of size between  $2q + 4$  and  $2q + \sqrt{q} + 4$  remains an open problem in this case. In [Section 3.3](#) we have reported some sporadic sizes of 2-marginal sets; as seen, such examples may also give new exact values for  $b(n)$ .

Another way to obtain more exact bipartite Ramsey numbers with the method of this paper is to consider non-Desarguesian planes. The proof of [Theorem 3.10](#) relies heavily on results with a firm algebraic background, and it seems likely to fail for non-Desarguesian planes. A particularly interesting problem, also from a purely finite geometric viewpoint, is to find two Baer subplanes of a plane of order  $q$  that intersect in at least four but not more than  $\sqrt{q}$  points, which could yield new exact bipartite Ramsey numbers (cf. [Theorem 2.12](#)). The intersection types of two Baer subplanes are well understood in  $\text{PG}(2, q)$ , but we are not aware of any general results regarding this question apart from [[7](#)]. It could happen that two Baer subplanes intersect in a non-degenerate subplane.

Let us recall the classical De Bruijn–Erdős Theorem [[9](#)] claiming that a finite linear space on  $n$  points has at least  $n$  lines (unless all of its points are contained in a line), and in case of equality the space is a (possibly degenerate) projective plane. This yields that if a set  $\mathcal{P}$  of  $n$  points of a projective plane has not more than  $n$  lines intersecting it in at least two points, then  $\mathcal{P}$  is a (possibly degenerate) subplane. Is it true that if a point set  $\mathcal{P}$  in  $\Pi_q$  or  $\text{PG}(2, q)$  of size at least  $c(t)$  has no more than  $|\mathcal{P}|$  lines intersecting it in at least  $t + 1$  points, then  $\mathcal{P}$  is the union of  $t$  (possibly degenerate) subplanes (provided that  $q$  is large enough compared to  $t$ )? Some lower bound is clearly necessary here;  $c(t) = tq + t + 1$  would already be very interesting.

Essentially the same technique can be applied to construct small  $(k, g)$ -graphs. A  $(k, g)$ -graph is a  $k$ -regular graph of girth (length of its shortest cycle)  $g$ . The so-called cage problem asks for finding the smallest number  $c(k, g)$  of vertices a  $(k, g)$ -graph may have, and it has a large literature; see [[14](#)]. One possible approach to construct small  $(k, 6)$ -graphs is to consider regular subgraphs of projective planes. It is clear that finding a small  $(q + 1 - t)$ -regular (induced) subgraph of  $\Pi_q$  involves finding a large (perfect)  $t$ -marginal set of the plane. Induced subgraphs, and thus perfect  $t$ -marginal (or perfect  $t$ -fold dominating) sets of  $\Pi_q$  were studied in [[18, 19](#)] (although the respective substructures were called  $t$ -good structures); whereas non-induced subgraphs were proven useful in [[1](#)] (see [[24](#), Chapter 3] as well). [Theorem 3.2](#) shows that a  $(q + 1 - t)$ -regular subgraph of  $\Pi_q$ , induced or non-induced, may have at most  $2t(q + \sqrt{q} + 1)$  vertices less than  $\Pi_q$ . This was already known for induced subgraphs (see [[3, 18](#)]), but not for general ones. If one starts with the incidence

graph of a generalized quadrangle or a generalized hexagon of order  $(q, q)$ , it is also possible to obtain small  $(k, 8)$ -graphs or  $(k, 12)$ -graphs as their subgraphs, respectively; see [3] for a recent example.

Finally, we provide a table containing the currently known exact values of  $b(n)$  for small  $n$ . A bold number indicates a value of  $b(n)$  which, to the best of our knowledge, has not appeared in the literature explicitly prior to the present study. Let us note that with 2-marginal sets of  $\Pi_q$  we get a new exact value for  $b(n)$  only if there is no (known) projective plane of order  $q - 1$ , since otherwise Theorem 1.3 applied to  $\Pi_{q-1}$  covers a broader interval of exact values. In two cases, graphs known in the context of Zarankiewicz numbers or the cage problem were utilized to obtain the appropriate lower bound. Also, [10] reports that  $b(11) = 15$  was already found by Guy [22].

$n$	$b(n)$	$n$	$b(n)$	$n$	$b(n)$	$n$	$b(n)$	$n$	$b(n)$	$n$	$b(n)$
1	2	26	32 <sup>b</sup>	51	<b>58<sup>d</sup></b>	76	85 <sup>a</sup>	118	129 <sup>a</sup>	167	180 <sup>a</sup>
2	4 <sup>a</sup>	27	<b>32<sup>d</sup></b>	52		77	86 <sup>a</sup>	119	130 <sup>a</sup>	168	181 <sup>a</sup>
3	5 <sup>a</sup>	28		53		78	87 <sup>a</sup>	120	131 <sup>a</sup>	169	182 <sup>a</sup>
4	6 <sup>a</sup>	29		54		79	88 <sup>a</sup>	121	132 <sup>a</sup>	170	184 <sup>b</sup>
5	8 <sup>b</sup>	30		55		80	89 <sup>a</sup>	122	134 <sup>b</sup>	171	<b>184<sup>d</sup></b>
6	9 <sup>a</sup>	31	<b>37<sup>f</sup></b>	56	64 <sup>a</sup>	81	90 <sup>a</sup>	123	<b>134<sup>d</sup></b>	172	
7	10 <sup>a</sup>	32		57	65 <sup>a</sup>	82	92 <sup>b</sup>	124		(...)	
8	11 <sup>a</sup>	33		58	66 <sup>a</sup>	83	<b>92<sup>d</sup></b>	(...)		196	
9	12 <sup>a</sup>	34	<b>40<sup>f</sup></b>	59	67 <sup>a</sup>	84		142		197	<b>211<sup>j</sup></b>
10	14 <sup>b</sup>	35	<b>41<sup>g</sup></b>	60	68 <sup>a</sup>	(...)		143	<b>155<sup>g</sup></b>	198	
11	15 <sup>c</sup>	36	<b>42<sup>g</sup></b>	61	69 <sup>a</sup>	98		144	<b>156<sup>g</sup></b>	(...)	
12	16 <sup>a</sup>	37	<b>43<sup>i</sup></b>	62	70 <sup>a</sup>	99	<b>109<sup>g</sup></b>	145		216	
13	17 <sup>a</sup>	38		63	71 <sup>a</sup>	100	<b>110<sup>g</sup></b>	(...)		217	<b>232<sup>g</sup></b>
14	18 <sup>a</sup>	39	<b>46<sup>h</sup></b>	64	72 <sup>a</sup>	101	<b>111<sup>i</sup></b>	155		218	<b>233<sup>g</sup></b>
15	19 <sup>a</sup>	40		65	74 <sup>b</sup>	102		156	169 <sup>a</sup>	219	<b>234<sup>g</sup></b>
16	20 <sup>a</sup>	41		66	<b>74<sup>d</sup></b>	(...)		157	170 <sup>a</sup>	220	<b>235<sup>g</sup></b>
17	22 <sup>b</sup>	42	49 <sup>a</sup>	67		109		158	171 <sup>a</sup>	221	<b>236<sup>g</sup></b>
18	<b>22<sup>d</sup></b>	43	50 <sup>a</sup>	68		110	121 <sup>a</sup>	159	172 <sup>a</sup>	222	<b>237<sup>g</sup></b>
19	<b>24<sup>e</sup></b>	44	51 <sup>a</sup>	69		111	122 <sup>a</sup>	160	173 <sup>a</sup>	223	<b>238<sup>g</sup></b>
20	25 <sup>a</sup>	45	52 <sup>a</sup>	70	<b>79<sup>e</sup></b>	112	123 <sup>a</sup>	161	174 <sup>a</sup>	224	<b>239<sup>g</sup></b>
21	26 <sup>a</sup>	46	53 <sup>a</sup>	71		113	124 <sup>a</sup>	162	175 <sup>a</sup>	225	<b>240<sup>g</sup></b>
22	27 <sup>a</sup>	47	54 <sup>a</sup>	72	81 <sup>a</sup>	114	125 <sup>a</sup>	163	176 <sup>a</sup>	226	
23	28 <sup>a</sup>	48	55 <sup>a</sup>	73	82 <sup>a</sup>	115	126 <sup>a</sup>	164	177 <sup>a</sup>	(...)	
24	29 <sup>a</sup>	49	56 <sup>a</sup>	74	83 <sup>a</sup>	116	127 <sup>a</sup>	165	178 <sup>a</sup>	239	
25	30 <sup>a</sup>	50	58 <sup>b</sup>	75	84 <sup>a</sup>	117	128 <sup>a</sup>	166	179 <sup>a</sup>	240	256 <sup>a</sup>

Notation:

<sup>a</sup>[10,21], where a proper subgraph of a projective plane gives the lower bound.

<sup>b</sup>[10], where a projective plane gives the lower bound.

<sup>c</sup>Theorem 2.12 with a Baer subplane as a 1-marginal set.

<sup>d</sup>Corollary 4.8.

<sup>e</sup> $b(19) > 23$  shown by a  $C_4$ -free, 5-regular, bipartite graph on 46 vertices proving  $Z(23, 23) \geq 115$  [26].

<sup>f</sup>Lower bound shown by Lemma 2.9 with a sporadic 2-marginal set of size 18 or 21 in  $PG(2, 7)$ .

<sup>g</sup>Theorem 2.12 with a 2-marginal set.

<sup>h</sup> $b(39) > 45$  shown by the  $C_4$ -free, 7-regular incidence graph of an elliptic semiplane found by Baker [2], which also proves  $c(7, 6) \leq 90$  (see [14] as well).

<sup>i</sup>Theorem 4.2.

<sup>j</sup>Proposition 4.3.

(...)an interval where  $b(n)$  is not known.

### Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Acknowledgments

This work is the result of multiple research visits supported by the grants ‘Substructures of projective spaces’ and ‘Substructures in finite projective spaces: algebraic and extremal questions’ of the Fund for Scientific Research FWO Vlaanderen, Belgium and the Hungarian Academy of Sciences. Tamás Héger was also supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences, and by the National Research, Development and Innovation Office – NKFIH, Hungary Grant No. K124950. While preparing the final version of the manuscript, project no. ED\_18-1-2019-0030 (Application-specific highly reliable IT solutions) has been implemented with the support provided from the National Research, Development and Innovation Fund of Hungary, financed under the Thematic Excellence Programme funding scheme. We thank the anonymous referees for their valuable comments which helped us clarify the arguments and improve the presentation of the paper.

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