msp

Algebraic & Geometric Topology 22 (2022) 709–738

The topological slice genus of satellite knots

PETER FELLER
ALLISON N MILLER
JUANITA PINZÓN-CAICEDO

We present evidence supporting the conjecture that, in the topological category, the slice genus of a satellite knot $P(K)$ is bounded above by the sum of the slice genera of K and $P(U)$. Our main result establishes this conjecture for a variant of the topological slice genus, the $\mathbb{Z}$–slice genus. Notably, the conjectured upper bound does not involve the algebraic winding number of the pattern P. This stands in stark contrast with the smooth category, where, for example, there are many genus 1 knots whose $(n,1)$–cables have arbitrarily large smooth 4–genera. As an application, we show that the $(n,1)$–cable of any knot of 3–genus 1 (eg the figure-eight or trefoil knot) has topological slice genus at most 1, regardless of the value of $n \in \mathbb{N}$. Further, we show that the lower bounds on the slice genus coming from the Tristram–Levine and Casson–Gordon signatures cannot be used to disprove the conjecture.

57M25, 57N70

1 Introduction

The satellite operation associates to a pattern knot P in a solid torus and a knot K in S^3, the satellite knot $P(K)$ in S^3 obtained as the image of P under the identification of the solid torus with a 0–framed neighborhood of K. See Figure 1 for an example and Section 2 for precise definitions.

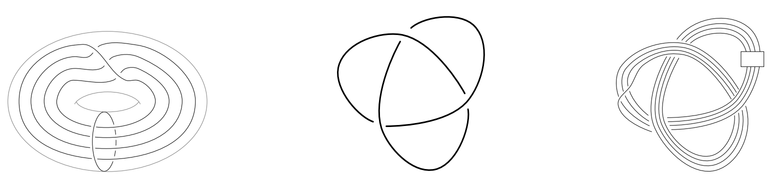

Figure 1: A pattern $P = C_{4,1}$ (left), a knot K (center), and the satellite $P(K)$ (right). The box on the right contains three negative full twists correcting for the $+3$ writhe in the diagram of K.

Published: 9 August 2022 DOI: 10.2140/agt.2022.22.709

It is natural to ask how the complexity of the knot $P(K)$ is determined by those of its constituent pieces P and K. One of the best-studied measures of complexity of a knot J is the Seifert or 3–genus $g_3(J)$, an invariant defined as the minimal genus of an embedded oriented surface in S^3 with boundary J. Classical work of Schubert [28] gives an exact formula for the Seifert genus of a satellite knot: for any nontrivial knot K,

$$g_3(P(K)) = g_3(P) + |w| g_3(K),$$

where w denotes the (algebraic) winding number of P in the solid torus, and $g_3(P)$ is a version of the Seifert genus for patterns.

In the context of knot concordance and 4–manifold topology, instead of considering spanning surfaces embedded in S^3 one considers surfaces lying in 4–dimensional space. The topological 4–genus of K, denoted by $g_4^{\mathrm{top}}(K)$, is thus the minimal genus of any surface properly embedded in B^4 with boundary K via a locally flat embedding. The smooth 4–genus $g_4^{\mathrm{sm}}(K)$ is analogously defined. These 4–dimensional genera exhibit more complicated behavior than the Seifert genus and one cannot hope for an exact formula à la Schubert. To illustrate this, pick a knot J and consider the connected sum pattern P_J. This pattern has geometric winding number 1 and satisfies $P_J(K) = J \# K$ for all knots K. Even in this simple case, in either category we see there is no formula for $g_4(P_J(K))$ in terms of $g_4(P_J)$ and $g_4(K)$. Despite the fact that $g_4(J) = g_4(-J)$, the 4–genera of $P_J(-J) = J \# {-J}$ and $P_J(J) = J \# J$ are quite different: one always has that $g_4(J \# {-J}) = 0$ and often that $g_4(J \# J) = 2g_4(J)$. Nevertheless, Schubert's methods do extend to show that an inequality

$$g_4(P(K)) \le g_4(P) + |w| g_4(K) \tag{1}$$

holds in both categories, where, as before, $g_4(P)$ is a version of the 4–genus for patterns and is generally strictly larger than $g_4(P(U))$.

Distinguishing between the smooth and topological 4–dimensional properties of knots is an area of active research in low-dimensional topology, in part motivated by the fact that the difference between smooth and topologically slice knots is related to the existence of exotic differentiable structures on 4–manifolds. Adding to the body of work that investigates the disparity between the smooth and topological categories in dimension 4, we show that satellite operations seem to affect g_4^{sm} and g_4^{top} very differently. In the smooth category, the naive expectation that $g_4^{\mathrm{sm}}(P(K))$ is approximately $|w| g_4^{\mathrm{sm}}(K)$ often holds. For example, if n is a nonnegative natural number and $C_{n,1}(T_{2,3})$ denotes the $(n, 1)$ cable of the trefoil, it is known that

$$g_4^{\mathrm{sm}}(C_{n,1}(T_{2,3})) = n. \tag{2}$$

More generally, for any pattern P with winding number w, we have

$$\lim_{n\to\infty} \frac{g_4^{\mathrm{sm}}(P(T_{2,2n+1}))}{g_4^{\mathrm{sm}}(T_{2,2n+1})} = |w|. \tag{3}$$

We even see that $g_4^{\mathrm{sm}}(P(K))$ can be much larger than $|w|g_4^{\mathrm{sm}}(K)$: for any $w, m \in \mathbb{N}$ there exists a pattern $Q = Q_{w,m}$ of winding number w and infinitely many knots J such that

$$g_4^{\mathrm{sm}}(Q(J)) = g_4^{\mathrm{sm}}(Q(U)) + |w|g_4^{\mathrm{sm}}(J) + m. \tag{4}$$

We expect that these observations are known to the experts, but for completeness we prove them in Section 4.

In contrast to the smooth case, we present evidence for the counterintuitive idea that the winding number of P essentially does not contribute to $g_4^{\mathrm{top}}(P(K))$. We do this by working with the $\mathbb{Z}$*–slice genus*, a version of g_4^{top} inspired by the work of Freedman. The definition (see Feller and Lewark [7]) is

$$g_{\mathbb{Z}}(K) := \min\bigl\{\mathrm{genus}(F) \mid F \hookrightarrow B^4 \text{ is an oriented locally flat surface with } \partial F = K \text{ and } \pi_1(B^4 \setminus F) \cong \mathbb{Z}\bigr\}.$$

Without going too far afield, we note that Freedman's work can be roughly described as showing that the high-dimensional surgery-theoretic slogan that algebra governs topology does apply to topology in dimension 4, under certain conditions [9]. The constraint that $\pi_1(B^4 \setminus F) \cong \mathbb{Z}$ in the definition of $g_{\mathbb{Z}}(K)$ guarantees that we are in such a situation.

Note that definitionally $g_4^{\mathrm{top}} \le g_{\mathbb{Z}}$ and, since the complement of a Seifert surface with interior properly pushed into the 4–ball has fundamental group $\mathbb{Z}$, we also have $g_{\mathbb{Z}} \le g_3$ (see Gompf and Stipsicz [12, Proposition 6.2.1] or Feller and Lewark [7, Proof of Theorem 1] for more details). The statement of [9, Theorem 1.13] can be rewritten in terms of the $\mathbb{Z}$–slice genus to say that $g_{\mathbb{Z}}(K) = 0$ if and only if $\Delta_K(t) = 1$.

Our main theorem then reads as follows:

Theorem 1.1 *For any pattern P and knot K, $g_{\mathbb{Z}}(P(K)) \le g_{\mathbb{Z}}(P(U)) + g_{\mathbb{Z}}(K)$.*

The first step in establishing Theorem 1.1 is to reduce the topological problem to an algebraic one by applying a result of Feller and Lewark [8], that $g_{\mathbb{Z}}$ can be identified with the algebraic genus g_{alg}, an invariant defined purely in terms of Seifert matrices [7]. We then use Schubert's construction of Seifert surfaces for satellite knots to embark on a careful analysis and manipulation of certain Seifert matrices for $P(U)$, K and $P(K)$, proving the analogue of Theorem 1.1 for g_{alg} in Proposition 2.4.

Duncan McCoy has an alternative proof of Theorem 1.1 which relies on his recent work [23] analyzing the behavior of g_{alg} under so-called "null homologous twisting operations". A third way to prove Theorem 1.1 is obtained by combining a result of Livingston and Melvin [22] about the Blanchfield pairing and the recent characterization of $g_{\mathbb{Z}}$ in terms of the Blanchfield pairing given in [8, Theorem 1.1]. This is discussed in more detail at the end of Section 2.

Although it is not present in the statement of Theorem 1.1, our proof implies that a satellite knot $P(K)$ shares a Seifert matrix with the connected sum $P(U)\,\#\,C_{w,1}(K)$, where $w = w(P)$ is, as usual, the winding number of P. In particular, when $w(P) = 0$ we obtain $g_{\mathbb{Z}}(P(K)) = g_{\mathbb{Z}}(P(U))$ and when $w(P) = \pm 1$ we have $g_{\mathbb{Z}}(P(K)) = g_{\mathbb{Z}}(P(U)\,\#\,K)$. This second fact is interesting given that it is an open problem whether $P(K)$ and $P(U)\,\#\,K$ must be topologically concordant for $w(P) = +1$. However, we think that Theorem 1.1 is most striking for $|w(P)| > 1$, where its consequences stand in contrast with smooth results such as (2), (3) and (4). For example, the following result is radically different from (2):

Example 1.2 (the $(n,1)$–cable of the trefoil) For a knot K and $n > 0$, Theorem 1.1 implies

$$g_4^{\mathrm{top}}(C_{n,1}(K)) \le g_{\mathbb{Z}}(C_{n,1}(K)) \le g_{\mathbb{Z}}(C_{n,1}(U)) + g_{\mathbb{Z}}(K) \le g_3(K).$$

Additionally, a simple Tristram–Levine signature computation at an appropriate $z \in S^1$ shows that $C_{n,1}(T_{2,3})$ is never slice (see the proof of (2)) and hence that

$$g_4^{\mathrm{top}}(C_{n,1}(T_{2,3})) = 1 \quad \text{for all } n > 0.$$

In Section 4 we establish the following contrast with (3):

Corollary 1.3 *Let P be a pattern of winding number w. Then*

$$\lim_{n\to\infty} \frac{g_4^{\mathrm{top}}(P(T_{2,2n+1}))}{g_4^{\mathrm{top}}(T_{2,2n+1})} = \begin{cases} 1 & \text{if } w \neq 0, \\ 0 & \text{if } w = 0. \end{cases}$$

In Example 4.6, we consider the family of knots $K_p = C_{2,2p+1}(T_{2,p})$ indexed by odd positive integers p and offer a quantitative measure of the difference between their topological and smooth 4–genera. Theorem 1.1, together with information on $g_4^{\mathrm{sm}}(K_p)$ using that K_p is the closure of a positive braid, can be used to show that

$$\limsup_{p\to\infty} \frac{g_4^{\mathrm{top}}(K_p)}{g_4^{\mathrm{sm}}(K_p)} \le \frac{2}{3}.$$

The novelty here is not just the improvement of previous bounds, but also the fact that we do so without relying on explicit example-based calculations, contrasting with Rudolph [25] and Baader, Feller, Lewark and Liechti [1].

With all this in mind we make the following conjecture about the topological 4–genera of satellite knots:

Conjecture 1.4 *For any pattern P and knot K, $g_4^{\mathrm{top}}(P(K)) \leq g_4^{\mathrm{top}}(P(U)) + g_4^{\mathrm{top}}(K)$.*

As additional evidence, we show that the known lower bounds on the topological 4–genus are not capable of disproving Conjecture 1.4. In Section 3, we observe that the satellite formula of Litherland [20] implies that the lower bound for g_4^{top} given by Tristram–Levine signatures cannot be used to establish that a pair P and K fails to satisfy Conjecture 1.4. Moreover, we consider Gilmer's lower bound for g_4^{top} [11] in terms of Casson–Gordon signature invariants [3; 4], and in Theorem 3.4 we show that this bound cannot be used to disprove Conjecture 1.4.

Our main result also enables the precise computation of g_4^{top} for certain families of satellite knots.

Corollary 1.5 *Let P be a pattern with $\Delta_{P(U)}(t) = 1$.*

(1) *For every knot K, $g_4^{\mathrm{top}}(P(K)) \leq g_3(K)$.*

(2) *If $w(P) \neq 0$ and K is a knot such that $2g_3(K) = |\sigma_z(K)|$ for some $z \in S^1$ with $\Delta_K(z) \neq 0$, then*

$$g_4^{\mathrm{top}}(P(K)) = g_3(K) = g_4^{\mathrm{top}}(K).$$

Note the hypothesis on the winding number of P in (2) is necessary: if P has winding number 0 and $\Delta_{P(U)}(t) = 1$, then $P(K)$ has Alexander polynomial 1 and so $g_4^{\mathrm{top}}(P(K)) = 0$ for any knot K.

Proof Since $\Delta_{P(U)}(t) = 1$, we have that $g_{\mathbb{Z}}(P(U)) = 0$, and therefore

$$g_4^{\mathrm{top}}(P(K)) \leq g_{\mathbb{Z}}(P(K)) \leq g_{\mathbb{Z}}(P(U)) + g_{\mathbb{Z}}(K) = g_{\mathbb{Z}}(K) \leq g_3(K),$$

establishing our first claim. The stated assumption on $\sigma_z(K)$ implies that $g_4^{\mathrm{top}}(K) = g_{\mathbb{Z}}(K) = g_3(K)$. Now let $\xi \in S^1$ be a prime-power root of unity such that no root of $\Delta_K(t)$ lies between ξ^n and z, where $n = |w|$ is the absolute value of the winding number of P. This, together with Litherland's formula for the Tristram–Levine signature of a satellite knot [20], shows that

$$|\sigma_\xi(P(K))| = |\sigma_\xi(P(U)) + \sigma_{\xi^n}(K)| = |0 + \sigma_{\xi^n}(K)| = |\sigma_z(K)|.$$

This combines with Taylor's result [29] that $2g_4^{\text{top}}(P(K)) \geq |\sigma_\xi(P(K))|$ to show, as desired, that

$$2g_4^{\text{top}}(P(K)) \geq |\sigma_z(K)| = 2g_4^{\text{top}}(K) = 2g_3(K). \qquad \square$$

Acknowledgements Feller gratefully acknowledges support by the SNSF grant 181199. Pinzón-Caicedo is grateful to the Max Planck Institute for Mathematics in Bonn for its hospitality and financial support while a portion of this work was prepared for publication, Pinzón-Caicedo was partially supported by NSF grant DMS-1664567. We are very thankful for our exchange with Duncan McCoy, who, immediately after learning of Theorem 1.1, observed an alternative proof for it based on the behavior of the algebraic genus under null homologous twisting operations; compare [23]. We also gratefully acknowledge very helpful comments from the anonymous referee.

2 Definitions and main result for the algebraic genus

In this section we establish an inequality relating the $\mathbb{Z}$–genera of $P(U)$, K and $P(K)$. We do so by establishing an inequality between their algebraic genera, defined below. Since $g_{\mathbb{Z}}$ and g_{alg} are the same for knots, this will translate back to an inequality for $g_{\mathbb{Z}}$ when P is a one-component pattern, as in the case of interest. The advantage of working with g_{alg} is that one can work with algebraic manipulations of Seifert matrices, which we will see can be taken to have a particular form for satellites.

We start by recalling the relevant definitions and properties.

Definition 2.1 For a link $L \subseteq S^3$ with r components, one defines its *algebraic genus* as

$$g_{\text{alg}}(L) = \min\Big\{\tfrac{1}{2}(m-2n-r+1) \,\Big|\, \text{there exists a Seifert surface } F \text{ for } L \text{ with an } m\times m \text{ Seifert matrix of the form } \left[\begin{smallmatrix} B & * \\ * & * \end{smallmatrix}\right], \text{ where } B \text{ is a } 2n\times 2n \text{ matrix satisfying } \det(tB-B^T)=t^n\Big\}.$$

A Seifert surface F for L is said to *realize* the algebraic genus $g_{\text{alg}}(L)$ if it has a Seifert matrix as above such that $\frac{1}{2}(m-2n-r+1) = g_{\text{alg}}(L)$.

The definition is chosen so that a knot K has $g_{\text{alg}}(K) = 0$ if and only if it has Alexander polynomial 1. Indeed, $2n \times 2n$ matrices B with $\det(tB - B^T) = t^n$ for some $n \in \mathbb{Z}$ are

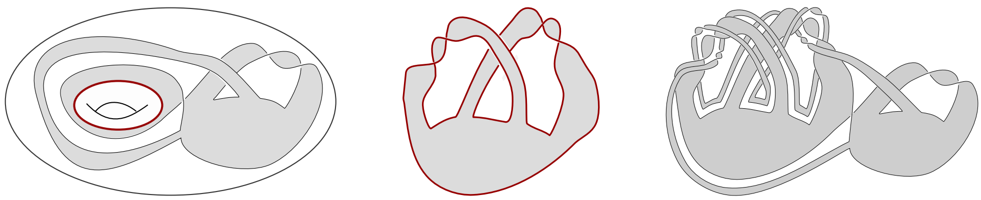

Figure 2: A Seifert surface for a pattern P (left) and a Seifert surface for a knot K (center) combine to give a Seifert surface for $P(K)$ (right). The highlighted curve on the left represents the curve l from Definition 2.2.

exactly the matrices that occur as Seifert matrices of knots with Alexander polynomial 1. We call such a B an *Alexander trivial matrix* or, if it is a diagonal subblock of a larger matrix, an *Alexander trivial submatrix*. A key feature of the algebraic genus is that $g_{\mathbb{Z}}(L) \leq g_{\text{alg}}(L)$ for all links L, so g_{alg} provides an upper bound on $g_{\mathbb{Z}}$ and thus g_4^{top} — see [7] — obtained entirely from Seifert matrices. Furthermore, $g_{\mathbb{Z}}(K) = g_{\text{alg}}(K)$ for all knots K [8, Corollary 1.5], which is what we use to translate statements about g_{alg} to ones about $g_{\mathbb{Z}}$.

Before formally defining the satellite operation, we establish some notation: given a submanifold Y of X, we let $N(Y)$ denote a small open tubular neighborhood of Y.

Definition 2.2 Let $P \sqcup \eta \subset S^3$ be a link of $r+1 \geq 2$ components with η an unknot such that P is contained in the interior of the solid torus $V = S^3 \setminus N(\eta)$. Let l be a simple closed curve in ∂V such that l is isotopic in S^3 to a meridian of η. Let $K \subset S^3$ be a knot and let $h\colon V \to \overline{N(K)} \subset S^3$ be an orientation-preserving homeomorphism taking l to a 0–framed longitude of K and a 0–framed longitude of η to a meridian of K. The image of P under h, denoted by $P(K)$, is the *satellite link* with *pattern* P and *companion* K. The (*algebraic*) *winding number* of P is defined as $w = \text{lk}(P, \eta)$.

The reader may be used to requiring P to be a connected pattern, ie restricting to $r = 1$. In this section, we consider general patterns with $r \geq 1$, where $P(K)$ is a link rather than a knot. However, in all other sections we only consider:

Remark 2.3 Without loss of generality, it is enough to consider patterns with nonnegative winding number. Indeed, if P is a pattern with negative winding number then $w = \text{lk}(P, \eta) < 0$, and so $\text{lk}(P^{\text{rev}}, \eta) = -w > 0$ and P^{rev} has positive winding number. Furthermore, since $P^{\text{rev}}(K) = P(K)^{\text{rev}}$, any notion of genus agrees on $P(K)$ and $P^{\text{rev}}(K)$.

Main result about the algebraic genus of satellites

Our main theorem about the algebraic genus of satellites is the following:

Proposition 2.4 *For a satellite link $P(K)$ with pattern P and companion K,*

$$g_{\mathrm{alg}}(P(K)) \leq g_{\mathrm{alg}}(P(U)) + \min\{|w|, 1\} g_{\mathrm{alg}}(K).$$

In fact, for $|w| = 1$ and $w = 0$, we have that $P(K)$ is S-equivalent to $P(U) \# K$ and $P(U)$, respectively.

Before we provide the proof of Proposition 2.4, we derive Theorem 1.1 from it.

Proof of Theorem 1.1 Let P be a one-component pattern and K be a knot. Then $g_{\mathbb{Z}} = g_{\mathrm{alg}}$ for $P(K)$, $P(U)$ and K, since they are all knots [8, Corollary 1.5]. Using these equalities, Theorem 1.1 follows immediately from Proposition 2.4. □

Our proof of Proposition 2.4 uses a construction of a Seifert surface for $P(K)$ similar to the one in [18, Chapter 6, Theorem 6.15], and illustrated below, with some additional attention paid to realizing g_{alg}.

Lemma 2.5 *Let $P \sqcup \eta$ be a pattern with winding number $w \geq 0$, and let l denote a chosen 0–framed longitude in the boundary of $V = S^3 \setminus N(\eta)$. There exists a Seifert surface $G \subset S^3 \setminus N(\eta)$ for the link $P \sqcup wl$ such that $G \cup_{wl} wD^2$ is a Seifert surface for $P(U)$ that realizes $g_{\mathrm{alg}}(P(U))$. Here wl and wD^2 denote w parallel copies of l and D^2, respectively.*

Proof The link $P(U)$ is obtained by regarding the pattern P as a link in S^3, forgetting about the effect of the unknotted component η. Let F be a Seifert surface for $P(U)$ whose Seifert form realizes $g_{\mathrm{alg}}(P(U))$. Using general position, assume that η intersects F transversely k times, so that the intersection of $\overline{N(\eta)}$ and the surface F consists of a collection of k disjoint disks. Denote by p and n the number of disks that intersect η positively and negatively, respectively, and note that $w = p - n$. To prove the lemma, it is enough to modify F so that $k = w$, or equivalently that $n = 0$, without losing the property that the Seifert form of F realizes $g_{\mathrm{alg}}(P(U))$. In the following paragraph we will prove that this modification can be achieved by stabilizations. Once we establish this the proof will be finished, since, by [7, Lemma 11], the property of realizing the algebraic genus is preserved by stabilization.

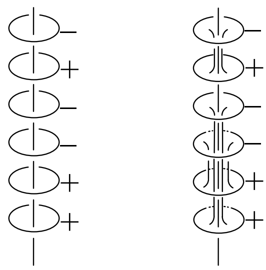

Figure 3: The unknotted component η and some disks in $F \cap \overline{N(\eta)}$ (left) and the annulus obtained after stabilizing (right).

Assume that $n > 0$, so that, since $w \geq 0$, it must be that $p > 0$. Choose a disk $D_i^- \subset F$ intersecting η negatively and a disk $D_i^+ \subset F$ intersecting η positively that are adjacent on η (ie they are connected by an arc on η that is disjoint from all the other disks).

Let a be an arc in η joining D_i^+ to D_i^- such that a is disjoint from all the other disks. Stabilize F using a tube surrounding the arc a to find a new Seifert surface that has two fewer intersections with η. Iterate this procedure of choosing two disks and stabilizing until a total of n stabilizations have happened. Call the result of these stabilizations F' and notice that F' intersects η only with positive sign, and so, if k' denotes the number of disks in the intersection $F' \cap \overline{N(\eta)}$, then $k' = w$, as sought. For a local picture of this procedure, see Figure 3. Finally, as noted above, Lemma 11 of [7] shows that stabilization of a Seifert surface preserves the property of realizing g_{alg} and so F' also realizes $g_{\text{alg}}(P(U))$. We then let $G = F' \cap V$. □

With the previous lemma in place, we are now ready to prove Proposition 2.4.

Proof of Proposition 2.4 Fix a knot K and a pattern P with $r \geq 1$ components and algebraic winding number w. Without loss of generality, assume $w \geq 0$. Let G be a Seifert surface for $P \sqcup wl$ as in Lemma 2.5, and let V_1 be a Seifert matrix for $P(U)$ corresponding to a choice of a basis for the first homology of $G \cup wD^2$. Similarly, let S be a Seifert surface for K that realizes $g_{\text{alg}}(K)$ and let V_2 be a Seifert matrix corresponding to a choice of a basis for the first homology of S. Assume that we have picked our bases for the first homology of $G \cup wD^2$ and S so that

$$V_1 = \begin{bmatrix} A_1 & * \\ * & * \end{bmatrix} \quad \text{and} \quad V_2 = \begin{bmatrix} A_2 & B & C \\ B^T & & \\ C^T & & D \end{bmatrix},$$

where the matrices V_1 and V_2 are of size $(2m_1+r-1)\times(2m_1+r-1)$ and $2m_2\times 2m_2$, respectively, for some nonnegative integers m_1 and m_2, and for $i=1,2$ the matrix A_i is an Alexander trivial matrix of size $2(m_i-g_i)\times 2(m_i-g_i)$ for $g_1=g_{\mathrm{alg}}(P(U))$ and $g_2=g_{\mathrm{alg}}(K)$. We note that B and C are $2(m_2-g_2)\times g_2$ matrices, and we may further choose our basis for $H_1(S)$ so that

$$D=\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22}\end{bmatrix}$$

is a $2g_2\times 2g_2$ matrix such that

$$D-D^T=\begin{bmatrix} 0 & I_{g_2} \\ -I_{g_2} & 0\end{bmatrix}.$$

Let $G(K)$ denote $h(G)$ for $h\colon V\to\overline{N(K)}$ as in Definition 2.2, in other words the image of G when we tie V into the knot K. Let $\widetilde{F}$ be the Seifert surface of $P(K)$ given as

$$\widetilde{F}=G(K)\cup wF,$$

where wF denotes $|w|$ many parallel copies of F with boundaries equal to the boundaries of $G(K)$ and $\widetilde{F}=G(K)\cup wF$ gets the orientation induced by $G(K)$. Then, pushing forward the basis of $H_1(G\cup wD,\mathbb{Z})$ via h_* and taking parallel copies of the basis of $H_1(F,\mathbb{Z})$ chosen earlier, we obtain a basis for $H_1(\widetilde{F};\mathbb{Z})$ and the following Seifert matrix for $P(K)$:

$$V=\left[\begin{array}{c|c} V_1 & 0 \\ \hline 0 & |w|V_2\end{array}\right],\quad \text{where } |w|V_2:=\begin{bmatrix} V_2 & V_2 & \cdots & V_2 \\ V_2^T & V_2 & \cdots & V_2 \\ \vdots & \vdots & \ddots & \vdots \\ V_2^T & V_2^T & \cdots & V_2\end{bmatrix}.$$

Compare also with the construction in [18, Chapter 6], where this calculation is given for a particular, similarly constructed Seifert surface for $P(K)$. Note that if $|w|$ is 1 or 0, then V is a Seifert matrix for $P(U)\,\#\,K$ or $P(U)$, respectively. This establishes the "in fact" part of Proposition 2.4.

Next, observe that a $2m\times 2m$ Alexander trivial submatrix M_0 of a matrix M and a $2n\times 2n$ Alexander trivial submatrix N_0 of a matrix N automatically combine to give a $2(m+n)\times 2(m+n)$ Alexander trivial submatrix $M_0\oplus N_0$ of $M\oplus N$. Since $V=V_1\oplus|w|V_2$, it therefore suffices to show that there exists a submatrix X_Δ of $|w|V_2$ that is Alexander trivial and of size $2(|w|m_2-g_2)\times 2(|w|m_2-g_2)$. To simplify the matrix manipulation, notice that a simple matrix congruence transforms $|w|V_2$ into the following matrix:

$$X = \begin{bmatrix} V_2 & 0 & \cdots & 0 \\ -(V_2-V_2^T) & V_2-V_2^T & \cdots & 0 \\ 0 & -(V_2-V_2^T) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & V_2-V_2^T \end{bmatrix}$$

$$= \begin{bmatrix} \begin{array}{|ccc|} \hline A_2 & B & C \\ B^T & D_{11} & D_{12} \\ C^T & D_{21} & D_{22} \\ \hline \end{array} & & \cdots & & \\ \begin{array}{|ccc|} A_2^T-A_2 & 0 & 0 \\ 0 & 0 & -I_g \\ 0 & I_g & 0 \\ \hline \end{array} & \begin{array}{|ccc|} \hline A_2-A_2^T & 0 & 0 \\ 0 & 0 & I_g \\ 0 & -I_g & 0 \\ \hline \end{array} & \cdots & & \\ & & \ddots & & \\ & & \cdots & \begin{array}{|ccc|} \hline A_2-A_2^T & 0 & 0 \\ 0 & 0 & I_g \\ 0 & -I_g & 0 \\ \hline \end{array} & \\ & & \cdots & \begin{array}{|ccc|} A_2^T-A_2 & 0 & 0 \\ 0 & 0 & -I_g \\ 0 & I_g & 0 \\ \hline \end{array} & \begin{array}{|ccc|} \hline A_2-A_2^T & 0 & 0 \\ 0 & 0 & I_g \\ 0 & -I_g & 0 \\ \hline \end{array} \end{bmatrix}.$$

That is, $|w|V_2$ is congruent to a $|w| \times |w|$ block matrix X with (i, j) block entry given by the $2m_2 \times 2m_2$ matrix V_2 if $i = j = 1$, by $V_2 - V_2^T$ if $i = j > 1$, by $V_2^T - V_2$ if $i = j + 1$, and by 0 otherwise. Then, replacing X by QXQ^t, where Q is a permutation matrix, we obtain

$$X' = \begin{bmatrix} \begin{array}{|cc|cc|} \hline \multicolumn{2}{|c|}{\multirow{4}{*}{Y}} & B & C \\ \cline{3-4} & & 0 & 0 \\ & & \vdots & \vdots \\ & & 0 & 0 \\ \hline B^T & 0 \cdots 0 & D_{11} & D_{12} \\ C^T & 0 \cdots 0 & D_{21} & D_{22} \\ \hline \end{array} & & \cdots & & \\ \begin{array}{|cc|} 0 & -I_g \\ I_g & 0 \\ \hline \end{array} & \begin{array}{|cc|} \hline 0 & I_g \\ -I_g & 0 \\ \hline \end{array} & \cdots & & \\ & & \ddots & & \\ & & \cdots & \begin{array}{|cc|} \hline 0 & I_g \\ -I_g & 0 \\ \hline \end{array} & \\ & & \cdots & \begin{array}{|cc|} 0 & -I_g \\ I_g & 0 \\ \hline \end{array} & \begin{array}{|cc|} \hline 0 & I_g \\ -I_g & 0 \\ \hline \end{array} \end{bmatrix},$$

where

$$Y = \begin{bmatrix} A_2 & 0 & \dots & 0 \\ -(A_2 - A_2^T) & A_2 - A_2^T & \dots & 0 \\ 0 & -(A_2 - A_2^T) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_2 - A_2^T \end{bmatrix},$$

ie Y is a $|w| \times |w|$ block matrix with (i,j) block entry equal to A_2 if $i = j = 1$, $A_2 - A_2^T$ if $i = j > 1$, $A_2^T - A_2$ if $i = j + 1$, and 0 otherwise.

We will show that X_Δ, the matrix obtained from X' by deleting the first block row and column after Y and the last block row and column, is Alexander trivial.

Indeed, note that the matrix $X_\Delta - t(X_\Delta)^T$ is given by

$$\left[\begin{array}{c|c|c|cc|c|cc|cc|c}
\multicolumn{2}{c|}{\multirow{2}{*}{Y - tY^T}} & & (1-t)C & & & & & & & \\
\multicolumn{2}{c|}{} & & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & & & \cdots & & & & \\
\hline
(1-t)C^T & 0 \;\cdots\; 0 & & D_{22} - tD_{22}^T & & tI_g & \cdots & & & & \\
\hline
& & & -I_g & & \begin{matrix}0 & \Psi\\ -\Psi & 0\end{matrix} & \cdots & & & & \\
& & & & & & \ddots & & & & \\
& & & & & & \cdots & 0 & \Psi & 0 & -tI_g \;\; 0 \\
& & & & & & \cdots & -\Psi & 0 & tI_g & 0 \;\; 0 \\
& & & & & & \cdots & 0 & -I_g & 0 & \Psi \;\; 0 \\
& & & & & & \cdots & I_g & 0 & -\Psi & 0 \;\; tI_g \\
& & & & & & & 0 & 0 & 0 & -I_g \;\; 0
\end{array}\right],$$

where, for layout, we have used the notation $\Psi = (1+t)I_g$.

Thus, the only nonzero entry in its final block row is $-I_g$ in the penultimate block column, and similarly the only nonzero entry in its final block column is tI_g in the penultimate block row.

We can therefore delete the final two rows and columns of $X_\Delta - t(X_\Delta)^T$ without changing its determinant. Thus, $\det(X_\Delta - t(X_\Delta)^T)$ is given by the following matrix:

$$\det\left[\begin{array}{ccc|c|cc|c|ccc}
\multicolumn{3}{c|}{\multirow{2}{*}{$Y-tY^T$}} & (1-t)C & & & & & & \\
 & & & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & & & \cdots & & & \\
\hline
(1-t)C^T & 0\ \cdots\ 0 & & D_{22}-tD_{22}^T & tI_g & & \cdots & & & \\
\hline
 & & & -I_g & 0 & (1+t)I_g & & & & \\
 & & & & -(1+t)I_g & 0 & \cdots & & & \\
 & & & & & & \ddots & & & \\
 & & & & & & & 0 & (1+t)I_g & 0 \\
 & & & & & & \cdots & -(1+t)I_g & 0 & tI_g \\
 & & & & & & \cdots & 0 & -I_g & 0
\end{array}\right]$$

and repeating this procedure one observes that

$$\det(X_\Delta - t(X_\Delta)^T) = \det\left[\begin{array}{cc|c|c}
\multicolumn{2}{c|}{\multirow{2}{*}{$Y-tY^T$}} & (1-t)C & 0 \\
 & & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & \vdots \\
\hline
(1-t)C^T & 0\ \cdots\ 0 & D_{22}-tD_{22}^T & tI_g \\
\hline
0 & \cdots & -I_g & 0
\end{array}\right] = \det(Y - tY^T).$$

By reversing the row and column moves we performed on $|w|V_2$ at the beginning of this argument, we see that Y is congruent to $|w|A_2$, and hence

$$\det(X_\Delta - t(X_\Delta)^T) = \det(Y - tY^T) = \det\bigl(|w|A_2 - t(|w|A_2)^T\bigr).$$

To see that $|w|A_2$ is Alexander trivial, notice that if J is a knot with Seifert form A_2, then $|w|A_2$ is a Seifert form for $C_{|w|,1}(J)$. A formula of Litherland [20] states that, for all patterns P and knots K,

$$\Delta_{P(K)}(t) = \Delta_{P(U)}(t)\Delta_K(t^{|w|}). \tag{5}$$

and this formula implies that

$$\det\bigl(|w|A_2 - t(|w|A_2)^T\bigr) = \Delta_{C_{|w|,1}(J)}(t) = \Delta_J(t^{|w|}) = 1. \qquad \square$$

The following example demonstrates that the inequality from Proposition 2.4 can be sharp and, moreover, can sometimes be attained in a nice geometric way.

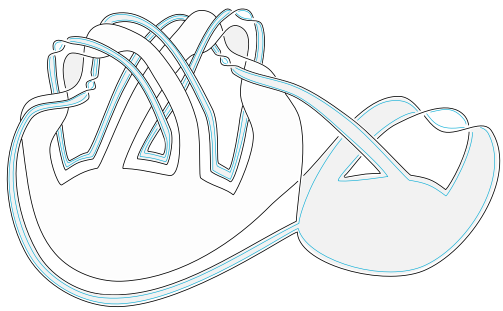

Figure 4: A Seifert surface for $M(4_1)$ with separating curve γ isotopic to $D(4_1)$.

Example 2.6 (the Mazur pattern) The Mazur satellite of the figure-eight knot, $M(4_1)$, has a genus 2 Seifert surface F illustrated in Figure 2 that comes from two genus 1 surfaces realizing the algebraic genera of $M(U)$ and of 4_1, respectively. The proof of Proposition 2.4 implies that there is some curve γ which bounds a genus 1 subsurface of F and, when considered as a knot in S^3, has $\Delta_\gamma(t) = 1$. In fact, as illustrated in Figure 4, we can pick γ to be isotopic to the positive Whitehead double $D(4_1)$.

The Blanchfield pairing perspective We end this section with an alternative proof of Theorem 1.1.

Recall that for a knot $K \subset S^3$ the Alexander module A_K is the first integer homology of the infinite cyclic cover of the knot complement viewed as a $\mathbb{Z}[t^{\pm 1}]$–module via the deck group action. The Blanchfield pairing $\mathrm{Bl}(K)$ of K is a nonsingular, hermitian, sesquilinear form

$$\mathrm{Bl}(K)\colon A_K \times A_K \mapsto \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$$

that is linear in the first variable and antilinear in the second variable with respect to the involution induced by $t \mapsto t^{-1}$. The Blanchfield form $\mathrm{Bl}(K)$ can be expressed in terms of a Seifert matrix for K. Moreover, two Seifert matrices are S-equivalent if and only if they determine isomorphic Blanchfield forms. Here, isomorphic means that, for two knots K and K', there exists a $\mathbb{Z}[t^{\pm 1}]$–module isomorphism $\phi\colon A_K \to A_{K'}$ such that $\mathrm{Bl}(K')(\phi(x), \phi(y)) = \mathrm{Bl}(K)(x, y)$ for all $x, y \in A_K$. See [2; 14; 15; 10] for more details.

In [22, Theorem 2], Livingston and Melvin show that

$$\mathrm{Bl}(P(K))(t) \cong \mathrm{Bl}(P(U))(t) \oplus \mathrm{Bl}(K)(t^w),$$

generalizing a result of Litherland [20], where this was established for $\mathbb{Q}[t^{\pm 1}]$ coefficients.

This enables another proof of Theorem 1.1. Namely, the recent characterization of $g_{\mathbb{Z}} = g_{\mathrm{alg}}$ in terms of the Blanchfield pairing from [8, Theorem 1.1] implies that the inequality $g_{\mathbb{Z}}(P(K)) \leq g_{\mathbb{Z}}(P(U)) + g_{\mathbb{Z}}(K)$ follows from $\mathrm{Bl}(P(K))(t) \cong \mathrm{Bl}(P(U))(t) \oplus \mathrm{Bl}(K)(t^w)$. We do not provide details of this here as we have an elementary matrix–based proof, which (at least when stated for the algebraic genus), works more generally for satellites of multiple components.

3 Lower bounds on g_4^{top} and satellite operations

In this section, we discuss lower bounds for the topological 4–genera of knots, namely Tristram–Levine signatures and Casson–Gordon signatures, and explain why these invariants cannot be used to disprove Conjecture 1.4. While this is immediate from classical formulas in the case of Tristram–Levine signatures, we consider it a priori somewhat surprising that Casson–Gordon signatures fail to disprove Conjecture 1.4. All patterns P in this section are connected, ie they are knots in a solid torus V.

The Tristram–Levine signatures σ_z are classical knot invariants [30; 17], which have a simple behavior with respect to satellite operations. Namely, for a pattern P with winding number w one has

(6) $\sigma_z(P(K)) = \sigma_z(P(U)) + \sigma_{z^w}(K)$ for all knots K and $z \in S^1$; see [19].

Furthermore, a classical result [29; 21] establishes that signatures give a lower bound for g_4^{top}:

(7) $$|\sigma_z(K)| \leq 2g_4^{\mathrm{top}}(K) \quad \text{for all knots } K \text{ and regular } z \in S^1.$$

Here, $z \in S^1$ is said to be regular if it does not arise as the root of an Alexander polynomial of a knot. For example, all prime-power-order roots of unity are regular.

As a consequence, one has that

$$\begin{aligned} \max_{\text{regular } z\in S^1} |\sigma_z(P(K))| &\leq \max_{\text{regular } z\in S^1} |\sigma_z(P(U))| + \max_{\text{regular } z\in S^1} |\sigma_z(K)| \\ &\leq 2g_4^{\mathrm{top}}(P(U)) + 2g_4^{\mathrm{top}}(K), \end{aligned}$$

which shows the lower bound for $g_4^{\mathrm{top}}(P(K))$ given by the Levine–Tristram signatures of $P(K)$ cannot be used to establish that a pair P and K fails to satisfy the inequality of Conjecture 1.4.

The next family of slice genus bounds come from Casson–Gordon signatures by work of Gilmer. Our main result of this section, Theorem 3.4, can be informally paraphrased as "one cannot use Casson–Gordon signatures to prove $g_4^{\mathrm{top}}(P(K)) > g_4^{\mathrm{top}}(P(U)) + g_4^{\mathrm{top}}(K)$".

Casson–Gordon à la Gilmer

We will be working with torsion abelian groups G equipped with linking forms $\lambda\colon G \times G \to \mathbb{Q}/\mathbb{Z}$. In particular, when we write $G \cong G_1 \oplus G_2$ we are implicitly decomposing the pair $(G, \lambda) \cong (G_1, \lambda_1) \oplus (G_2, \lambda_2)$. Our main examples of such pairs (G, λ) will be $G = H_1(\Sigma_n(K))$, the first homology of the n^{th} cyclic branched cover of a knot K for n a prime power and $\lambda = \lambda_n^K$ the so-called torsion linking form.

Definition 3.1 Given a subgroup $G \leq H_1(\Sigma_n(K))$, we call $H \leq G$ an *invariant metabolizer* of G if

- H is a metabolizer for $\lambda_n|_G$, ie $|H|^2 = |G|$ and $\lambda_n|_{H \times H} = 0$;[1]
- H is preserved by the $\mathbb{Z}_n$–action induced by the covering transformation of $\Sigma_n(K)$.

To a knot K, a prime power n and a prime-power-order character $\chi\colon H_1(\Sigma_n(K)) \to \mathbb{Z}_q$, Casson and Gordon associate a collection of rational numbers $\{\sigma_r \tau(K, \chi)\}_{r=1}^{q}$, called Casson–Gordon signatures [3; 4]. These signatures were employed to give the first examples of nonslice yet algebraically slice knots. Work of Gilmer extended the sliceness obstruction of [3; 4] to give lower bounds on g_4^{top} [11], stated here in the reformulation and mild strengthening of [24]. From now on, for $n \in \mathbb{N}$ we fix a primitive n^{th} root of unity denoted by z_n.

Theorem 3.2 [24; 11] *Let K be a knot and suppose that $g_4^{\mathrm{top}}(K) \leq g$. Then, for any prime power n, there is a decomposition of $H_1(\Sigma_n(K)) = A_1 \oplus A_2$ such that the following properties hold*:

(I) *A_1 has an even presentation of rank $2(n-1)g$ with signature $\sum_{i=1}^{n} \sigma_K(z_n^i)$.*

[1] We warn the reader that the traditional definition of a metabolizer M of G, ie a subgroup satisfying $M = M^{\perp} := \{g \in G : \lambda_n(g, m) = 0 \text{ for all } m \in M\}$, coincides with this definition only when $\lambda_n|_{G \times G}$ is nonsingular.

(II) *A_2 has an invariant metabolizer B such that, given any prime-power-order character χ which vanishes on $A_1 \oplus B$, we have*

$$\left|\sigma_1\tau(K,\chi) + \sum_{i=1}^{n} \sigma_K(z_n^i)\right| \leq 2ng.$$

(III) *$A_1 \oplus B$ is also covering transformation–invariant.*

An equivalent formulation of Theorem 3.2 states that, for χ an order q character as above, we have $|\sigma_r\tau(K,\chi) + \sum_{i=1}^{n} \sigma_K(z_n^i)| \leq 2ng$ for any $r = 1, \dots q$, since $\sigma_r\tau(K,\chi) = \sigma_1\tau(K, r'\chi)$ for some r' and $\chi|_H = 0$ implies that $r'\chi|_H = 0$ as well.

Given a knot K and some $g \geq 0$, we say that (K,n,g) satisfies the Gilmer 4–genus bounds if the conclusions of Theorem 3.2 hold. If (K,n,g) satisfies the Gilmer bound for all prime powers n, we say that (K,g) satisfies the Gilmer bound.

Casson–Gordon signatures of a satellite knot

We will need the following general formula for the Casson–Gordon signatures of a satellite knot. Recall that, given a map $\chi\colon H_1(\Sigma_n(K)) \to \mathbb{Z}_q$, we denote by $\sigma_r\tau(K,\chi)$ the r^{th} Casson–Gordon signature of (K,χ). In the exceptional case when $n = 1$ and so $\Sigma_1(K) = S^3$ and χ must be trivial, we somewhat abusively let $\sigma_r\tau(K,\chi)$ denote the Tristram–Levine signature $\sigma_K(z_q^r)$.

Theorem 3.3 [20] *Let P be a pattern described by an unknotted curve η in the complement of $P(U)$, ie the solid torus V is $S^3 \smallsetminus N(\eta)$. Suppose P has winding number m, let $n \in \mathbb{N}$ and define $d = \gcd(m,n)$. Then there is a canonical isomorphism*

$$\alpha\colon H_1\big(\Sigma_n(P(K))\big) \to H_1\big(\Sigma_n(P(U))\big) \oplus \bigoplus_{i=1}^{d} H_1(\Sigma_{n/d}(K)).$$

Moreover, letting $t_n^{P(K)}$, $t_n^{P(U)}$ and $t_{n/d}^{K}$ refer to the maps induced on $H_1\big(\Sigma_n(P(K))\big)$, $H_1\big(\Sigma_n(P(U))\big)$ and $H_1(\Sigma_{n/d}(K))$ by the appropriate covering transformations and writing $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_d)$, we have that, for any $x \in H_1\big(\Sigma_n(P(K))\big)$,

$$\alpha(t_n^{P(K)} \cdot x) = (t_n^{P(U)} \cdot \alpha_0(x), t_{n/d}^{K} \cdot \alpha_d(x), \alpha_1(x), \dots, \alpha_{d-1}(x)).$$

Furthermore, for n and q prime powers, let

$$\chi = (\chi_0, \chi_1, \dots, \chi_d)\colon H_1\big(\Sigma_n(P(U))\big) \oplus \bigoplus_{i=1}^{d} H_1(\Sigma_{n/d}(K)) \to \mathbb{Z}_q$$

be a tuple of characters and denote by $\eta_1, \dots \eta_d$ the homology classes of the d lifts of η to $\Sigma_n(P(U))$. Then the Casson–Gordon signature $\sigma_1\tau(P(K), \chi \circ \alpha)$ is given by

$$\sigma_1\tau(P(K), \chi \circ \alpha) = \sigma_1\tau(P(U), \chi_0) + \sum_{i=1}^{d} \sigma_{\chi_0(\eta_i)}\tau(K, \chi_i).$$

Theorem 3.4 *Let P be a pattern, K be any knot, $g \geq g_4^{\text{top}}(P(U)) + g_4^{\text{top}}(K)$ and n be a prime power. Then there is a decomposition of $H_1(\Sigma_n(K)) = A_1 \oplus A_2$ such that the following properties hold*:

- (I) *A_1 has an even presentation of rank $2(n-1)g$ with signature $\sum_{i=1}^{n} \sigma_K(z_n^i)$.*
- (II) *A_2 has an invariant metabolizer B such that, given any prime-power-order character χ which vanishes on $A_1 \oplus B$, we have*

$$\left|\sigma_1\tau(K, \chi) + \sum_{i=1}^{n} \sigma_K(z_n^i)\right| \leq 2ng.$$

- (III) *$A_1 \oplus B$ is also covering transformation–invariant.*

The proof of Theorem 3.4 is obtained using Litherland's formula for Casson–Gordon signatures and Gilmer's bounds for $P(U)$ and K.

Proof Let $g_K = g_4^{\text{top}}(K)$, $g_P = g_4^{\text{top}}(P(U))$, and let n be an arbitrary prime power. We show that $(P(K), n, g)$ satisfies the Gilmer bounds for $g \geq g_P + g_K$.

By Theorem 3.2, there is a decomposition of $H_1\bigl(\Sigma_n(P(U))\bigr) = A_1^P \oplus A_2^P$ with the following properties:

- (PI) A_1^P has an even rank $2(n-1)g_P$ presentation of signature $\sum_{i=1}^{n} \sigma_{P(U)}(z_n^i)$.
- (PII) A_2^P has an invariant metabolizer B^P such that, if $\chi\colon H_1\bigl(\Sigma_n(P(U))\bigr) \to \mathbb{Z}_q$ is a character of prime-power order vanishing on $A_1^P \oplus B^P$, then

$$\left|\sigma_1\tau(P(U), \chi) + \sum_{i=1}^{n} \sigma_{P(U)}(z_n^i)\right| \leq 2ng_P.$$

- (PIII) $A_1^P \oplus B^P$ is also covering transformation–invariant.

Write the algebraic winding number of P as $m = p^a m'$, where $p^a = \gcd(m, n)$. So $n = p^b$ for $b \geq a \geq 0$. Note that when $a = b$, ie $n = p^a$ divides m, we have that η lifts to n distinct curves in $\Sigma_n(P(U))$, and when $a<b$ we have that η lifts to strictly fewer than n curves in $\Sigma_n(P(U))$.

Case 1 ($a=b$, so $n=p^a$ divides m) Decompose

$$H_1\bigl(\Sigma_n(P(K))\bigr)\cong H_1\bigl(\Sigma_n(P(U))\bigr)\oplus 0$$

using α from Theorem 3.3 and take $A_1=\alpha^{-1}(A_1^P)$, $A_2=\alpha^{-1}(A_2^P)$ and $B=\alpha^{-1}(B^P)$. To check (I), we observe that

$$\begin{aligned}(8)\quad \sum_{i=1}^n \sigma_{P(K)}(z_n^i) \overset{(6)}{=} \sum_{i=1}^n(\sigma_{P(U)}(z_n^i)+\sigma_K(z_n^{im})) &= \sum_{i=1}^n(\sigma_{P(U)}(z_n^i)+\sigma_K(z_n^{inm'}))\\ &= \sum_{i=1}^n \sigma_{P(U)}(z_n^i).\end{aligned}$$

Thus, A_1 has an even presentation of rank $2(n-1)g_K$ with signature $\sum_{i=1}^n \sigma_{P(K)}(z_n^i)$ by (PI). Noting that the trivial group 0 certainly has an even presentation of rank $2(n-1)(g-g_K)$ and signature 0, we have that A_1 has an even presentation of rank $2(n-1)g_K+2(n-1)(g-g_K)$ with signature $\sum_{i=1}^n \sigma_{P(K)}(z_n^i)+0$. This concludes the proof of (I).

To check (II), we let $\chi\colon H_1\bigl(\Sigma_n(P(U))\bigr)\to\mathbb{Z}_q$ be of prime-power order with $\chi|_{A_1\oplus B}=0$ and bound

$$(*)=\left|\sigma_1\tau(P(K),\chi\circ\alpha)+\sum_{i=1}^n\sigma_{P(K)}(z_n^i)\right|$$

as follows:

$$\begin{aligned}(*) &= \left|\sigma_1\tau(P(K),\chi\circ\alpha)+\sum_{i=1}^n(\sigma_{P(U)}(z_n^i))\right| && \text{(by (8))}\\
&= \left|\sigma_1\tau(P(U),\chi)+\sum_{i=1}^n\sigma_K(z_q^{\chi(\eta_i)})+\sum_{i=1}^n(\sigma_{P(U)}(z_n^i))\right| && \text{(by Theorem 3.3)}\\
&\le \left|\sigma_1\tau(P(U),\chi)+\sum_{i=1}^n(\sigma_{P(U)}(z_n^i))\right|+\left|\sum_{i=1}^n\sigma_K(z_q^{\chi(\eta_i)})\right|\\
&\le 2ng_P+2ng_K && \text{(by (PII) and (7))}\\
&\le 2ng.\end{aligned}$$

Finally, the invariance of B and $A_1\oplus B$ under $t_n^{P(K)}$ is immediate from (PIII) and the fact that in this case we have $\alpha=\alpha_0$ and so $\alpha(t_n^{P(K)}\cdot x)=t_n^{P(U)}\cdot\alpha(x)$ for all $x\in H_1\bigl(\Sigma_n(P(K))\bigr)$.

Case 2 $(b > a)$ By Theorem 3.2, there is a decomposition of $H_1(\Sigma_{p^{b-a}}(K)) = A_1^K \oplus A_2^K$ with the following properties:

(KI) A_1^K has an even rank $2(p^{b-a}-1)g_K$ presentation of signature

$$s = \sum_{i=1}^{p^{b-a}} \sigma_K(z_{p^{b-a}}^i).$$

(KII) A_2^K has an invariant metabolizer B^K such that, if $\chi\colon H_1(\Sigma_{p^{b-a}}(K)) \to \mathbb{Z}_q$ is a character of prime-power order q vanishing on $A_1 \oplus B$, then

$$|\sigma_1\tau(K,\chi) + s| \le 2(p^{b-a})g_K.$$

(KIII) $A_1 \oplus B$ is also covering transformation–invariant.

Decompose $H_1\big(\Sigma_n(P(K))\big) \cong H_1\big(\Sigma_n(P(U))\big) \oplus \bigoplus_{i=1}^{p^a} H_1(\Sigma_{p^{b-a}}(K))$ using α from Theorem 3.3 and take

$$A_1 = \alpha^{-1}\bigg(A_1^P \oplus \bigoplus_{i=1}^{p^a} A_1^K\bigg) \quad\text{and}\quad A_2 = \alpha^{-1}\bigg(A_2^P \oplus \bigoplus_{i=1}^{p^a} A_2^K\bigg).$$

By taking the direct sum of our assumed presentations for A_1^P and A_1^K from (PI) and (KI), respectively, we have that $A_1 \cong A_1^P \oplus \bigoplus_{i=1}^{p^a} A_1^K$ has an even presentation of rank

$$\begin{aligned} 2(p^b-1)g_P + p^a 2(p^{b-a}-1)g_K &= 2(p^b-1)g_P + 2(p^b-p^a)g_K \\ &\le 2(p^b-1)g = 2(n-1)g \end{aligned}$$

and signature $\sum_{i=1}^n \sigma_{P(U)}(z_n^i) + p^a s$. However, since $p^a = \gcd(p^b, p^a m')$, we know that $(p,m') = 1$ and so $\{z_{p^{b-a}}^{m'j} : j = 1,\dots,p^{b-a}\} = \{z_{p^{b-a}}^i : i = 1,\dots,p^{b-a}\}$. It follows that

$$\begin{aligned} p^a s = p^a \sum_{i=1}^{p^{b-a}} \sigma_K(z_{p^{b-a}}^i) &= p^a \sum_{j=1}^{p^{b-a}} \sigma_K(z_{p^{b-a}}^{m'j}) = \sum_{j=1}^{p^b} \sigma_K(z_{p^{b-a}}^{m'j}) \\ &= \sum_{j=1}^{p^b} \sigma_K(z_{p^b}^{p^a m' j}) = \sum_{i=1}^{n} \sigma_K(z_n^{mi}) \end{aligned}$$

and thus

$$\sum_{i=1}^n \sigma_{P(U)}(z_n^i) + p^a s = \sum_{i=1}^n (\sigma_{P(U)}(z_n^i) + \sigma_K(z_n^{mi})) = \sum_{i=1}^n \sigma_{P(K)}(z_n^i). \tag{9}$$

This concludes the proof of (I) since the even presentation of A_1 of rank $2(p^b-1)g_P + 2(p^b - p^a)g_K$ and signature $\sum_{i=1}^n \sigma_{P(K)}(z_n^i)$ just described can be increased if necessary to have rank $2(n-1)g = 2(p^b-1)g$ by connect-sum with an even presentation of the trivial group with signature 0 and appropriate rank.

Now, note that $B^P \oplus \bigoplus_{i=1}^{p^a} B^K$ is an metabolizer for $A_1^P \oplus \bigoplus_{i=1}^{p^a} A_2^K$ and set

$$B = \alpha^{-1}\biggl(B^P \oplus \bigoplus_{i=1}^{p^a} B^K\biggr).$$

Since B^P and B_K are invariant under $t_n^{P(U)}$ and $t_{n/d}^K$, respectively, by (KII) and (PII), we have that $B^P \oplus \bigoplus_{i=1}^{p^a} B^K$ is invariant under

$$(x_0, x_1, \dots, x_d) \mapsto (t_n^{P(U)} \cdot x_0, t_{n/p^a}^K \cdot x_{p^a}, x_1, \dots, x_{p^a-1}).$$

Therefore, B is invariant under $t_n^{P(K)}$. Since $A_1^P \oplus B^P$ is invariant under $t_n^{P(U)}$ by (PIII), each $A_1^K \oplus B^K$ is invariant under t_{n/p^a}^K by (KIII) and

$$\begin{aligned}\alpha(A_1 \oplus B) = \alpha(A_1) \oplus \alpha(B) &= \biggl(A_1^P \oplus \bigoplus_{i=1}^{p^a} A_1^K\biggr) \oplus \biggl(B^P \oplus \bigoplus_{i=1}^{p^a} B^K\biggr)\\ &= (A_1^P \oplus B^P) \oplus \bigoplus_{i=1}^{p^a} (A_1^K \oplus B^K),\end{aligned}$$

we can similarly observe that $A_1 \oplus B$ is invariant under the action of $t_n^{P(K)}$, thereby establishing (III).

To check (II), let

$$\chi = (\chi_0, \chi_1, \dots, \chi_{p^a})\colon H_1\bigl(\Sigma_n(P(U))\bigr) \oplus \bigoplus_{i=1}^{p^a} H_1(\Sigma_{p^{b-a}}(K)) \to \mathbb{Z}_q$$

be a character of prime-power order and suppose that χ vanishes on

$$\alpha(A_1 \oplus B) = (A_1^P \oplus B^P) \oplus \bigoplus_{i=1}^{p^a} (A_1^K \oplus B^K).$$

In particular, χ_0 vanishes on $A_1^P \oplus B^P$ and χ_i vanishes on the i^{th} copy of $A_1^K \oplus B^K$.

We are now interested in bounding

$$(*) := \biggl|\sigma_1 \tau(P(K), \chi \circ \alpha) + \sum_{i=1}^{p^b} \sigma_{P(K)}(z_{p^b}^i)\biggr|.$$

By Theorem 3.3 and (9), we have that

$$\begin{aligned}(*) &= \left|\sigma_1\tau(P(U),\chi_0)+\sum_{i=1}^{p^a}\sigma_{\chi_0(\eta_i)}\tau(K,\chi_i)+\sum_{i=1}^{n}\sigma_{P(U)}(z_n^i)+p^a s\right| \\ &\leq \left|\sigma_1\tau(P(U),\chi_0)+\sum_{i=1}^{n}\sigma_{P(U)}(z_n^i)\right|+\left|\sum_{i=1}^{p^a}(\sigma_{\chi_0(\eta_i)}\tau(K,\chi_i)+s)\right| \\ &\leq \left|\sigma_1\tau(P(U),\chi_0)+\sum_{i=1}^{n}\sigma_{P(U)}(z_n^i)\right|+\sum_{i=1}^{p^a}\left|\sigma_{\chi_0(\eta_i)}\tau(K,\chi_i)+s\right| \\ &\leq 2ng_P+\sum_{i=1}^{p^a}2p^{b-a}g_K \quad \text{(by (KII) and (PII))} \\ &= 2ng_P+2ng_K \leq 2ng. \end{aligned}$$ □

Besides Tristram–Levine signatures and Gilmer's Casson–Gordon obstruction, the only known obstruction to being a knot with small g_4^{top} comes from recent work of Cha, Miller and Powell [5]. This work uses certain $L^{(2)}$ ρ–invariants to show that certain families of knots with vanishing Tristram–Levine signature functions and vanishing Casson–Gordon sliceness obstructions still have members with arbitrarily large g_4^{top}. Moreover, their constructions are all of the form $J = \#_{i=1}^{n} P(K_i)$ for P a winding number 0 satellite with $P(U)$ slice. However, these techniques only show $g_4^{\mathrm{top}}(J) \geq g$ for g orders of magnitude smaller than $\sum_{i=1}^{n} g_4^{\mathrm{top}}(K_i)$, and hence seem ill-suited to trying to disprove Conjecture 1.4.

4 New computations and contrast with the smooth setting

We will use the following result of Hom [13] on how the Heegaard Floer invariant τ behaves under cabling:

Theorem 4.1 [13] *Let K be a knot with $g_4^{\mathrm{sm}}(K) = \tau(K) > 0$; then, for any $w > 0$ we have*

$$g_4^{\mathrm{sm}}(C_{w,1}(K)) = \tau(C_{w,1}(K)) = w\tau(K) = wg_4^{\mathrm{sm}}(K)$$

and $\varepsilon(C_{w,1}(K)) = \varepsilon(K) = +1$.

Proposition 4.2 *For P a winding number w pattern,*

$$\lim_{n\to\infty}\frac{g_4^{\mathrm{sm}}(P(T_{2,2n+1}))}{g_4^{\mathrm{sm}}(T_{2,2n+1})} = |w|.$$

Proof Let P be a winding number w pattern. Since P and $C_{w,1}$ are homologous in V, there exists a surface F in $V \times I$ with boundary $P \times \{1\} \sqcup -C_{w,1} \times \{0\}$. One can use an argument analogous to the one that shows that patterns have a well-defined action on concordance — see Cochran and Harvey [6] — to show that, for any knot K,

$$|g_4^{\mathrm{sm}}(P(K)) - g_4^{\mathrm{sm}}(C_{w,1}(K))| \leq g_4^{\mathrm{sm}}(P(K) \# -C_{w,1}(K)) \leq g(F).$$

Therefore, since $\lim_{n\to\infty} g_4^{\mathrm{sm}}(T_{2,2n+1}) = \infty$, we have, as desired, that

$$\lim_{n\to\infty} \frac{g_4^{\mathrm{sm}}(P(T_{2,2n+1}))}{g_4^{\mathrm{sm}}(T_{2,2n+1})} = \lim_{n\to\infty} \frac{g_4^{\mathrm{sm}}(C_{w,1}(T_{2,2n+1}))}{g_4^{\mathrm{sm}}(T_{2,2n+1})} = \lim_{n\to\infty} \frac{wn}{n} = w. \qquad \square$$

Remark 4.3 This argument shows that, for any collection $\{K_n\}$ of quasipositive knots (or knots with $\tau(K_n) = g_4^{\mathrm{sm}}(K_n) \neq 0$ and $\varepsilon(K_n) = +1$) with $\lim_{n\to\infty} g_4(K_n) = \infty$, we have

$$\lim_{n\to\infty} \frac{g_4^{\mathrm{sm}}(P(K_n))}{g_4^{\mathrm{sm}}(K_n)} = |w|.$$

The following result, together with Proposition 2.4 in the winding number 0 case, immediately implies Corollary 1.3, since $\lim_{n\to\infty} g_4^{\mathrm{top}}(T_{2,2n+1}) = \infty$.

Proposition 4.4 *Let P be a winding number $w > 0$ pattern. Then*

$$-g_4^{\mathrm{top}}(P(U)) \leq g_4^{\mathrm{top}}(P(T_{2,2n+1})) - g_4^{\mathrm{top}}(T_{2,2n+1}) \leq g_{\mathbb{Z}}(P(U)).$$

Proof Let $K_n = T_{2,2n+1}$. We first observe that, for $t_n \in \left(\frac{2n-1}{2n+1}\pi, \frac{2n+3}{2n+1}\pi\right)$, we have

$$2n = |\sigma_{e^{it_n}}(K_n)| \leq 2g_4^{\mathrm{top}}(K_n) \leq 2g_{\mathbb{Z}}(K_n) \leq 2g_3(K_n) = 2n,$$

and hence we have equality throughout.

Now let P be a pattern of winding number $w > 0$ and observe by Theorem 1.1 that

$$g_4^{\mathrm{top}}(P(K_n)) \leq g_{\mathbb{Z}}(P(K_n)) \leq g_{\mathbb{Z}}(P(U)) + g_{\mathbb{Z}}(K_n) = g_{\mathbb{Z}}(P(U)) + g_4^{\mathrm{top}}(K_n).$$

We now need to obtain our lower bound on $g_4^{\mathrm{top}}(P(K_n))$. Let $s_n \in \left(\frac{(2n-1)\pi}{(2n+1)w}, \frac{(2n+3)\pi}{(2n+1)w}\right)$ be such that e^{is_n} is not a root of $\Delta_{P(U)}(t)$. It follows that e^{is_n} is not a root of $\Delta_{P(K_n)}(t) = \Delta_{P(U)}(t) \cdot \Delta_{K_n}(t^w)$ and so

$$\begin{aligned} 2g_4^{\mathrm{top}}(P(K_n)) \geq |\sigma_{e^{is_n}}(P(K_n))| &= |\sigma_{e^{is_n}}(P(U)) + \sigma_{e^{iws_n}}(K_n)| \\ &\geq 2g_4^{\mathrm{top}}(K_n) - |\sigma_{e^{is_n}}(P(U))| \\ &\geq 2g_4^{\mathrm{top}}(K_n) - 2g_4^{\mathrm{top}}(P(U)). \qquad \square \end{aligned}$$

Proposition 4.5 *For any $w, m \in \mathbb{N}$, there exists a winding number w pattern P such that, for any quasipositive knot K,*

$$g_4^{\mathrm{sm}}(P(K)) = g_4^{\mathrm{sm}}(P(U)) + |w| g_4^{\mathrm{sm}}(K) + m.$$

Proof Let $P_{m,w} = Q^m \circ C_{w,1}$, where Q denotes the Mazur pattern, $\circ$ denotes pattern composition and Q^m denotes the m–fold composition of Q, which is a winding number 1 pattern. Note that $P_{m,w}$ is a winding number w pattern. Let K be a quasipositive knot. By Levine [16], if J is any knot with $\varepsilon(J) = +1$, then $\tau(Q(J)) = \tau(J) + 1$ and $\varepsilon(Q(J)) = +1$. Applying this to $J = C_{w,1}(K)$ and using Theorem 4.1 gives us that

$$\begin{aligned} g_4^{\mathrm{sm}}(P_{m,w}(K)) \geq \tau(P_{m,w}(K)) = \tau\big(Q^m(C_{w,1}(K))\big) = \tau(C_{w,1}(K)) + m \\ = w g_4^{\mathrm{sm}}(K) + m. \end{aligned}$$

Since a single crossing change transforms Q to a core of the solid torus, we have that $g_4^{\mathrm{sm}}(Q(J)) \leq g_4^{\mathrm{sm}}(J) + 1$ for any knot J. It is also easy to check that $g_4^{\mathrm{sm}}(C_{w,1}(J)) \leq w g_4^{\mathrm{sm}}(J)$ for any knot J, and so

$$g_4^{\mathrm{sm}}(P_{m,w}(K)) = g_4^{\mathrm{sm}}\big(Q^m(C_{w,1}(K))\big) \leq g_4^{\mathrm{sm}}(C_{w,1}(K)) + m \leq w g_4^{\mathrm{sm}}(K) + m,$$

and we have the desired equality. □

Example 4.6 Let p and q be odd positive integers. We consider $C_{2,q}(T_{2,p})$, the $(2,q)$–cable of the $(2,p)$–torus knot. From another point of view, $C_{2,q}(T_{2,p})$ is the knot obtained as the closure of the 4–braid $(a_2 a_1 a_3 a_1)^p {a_1}^{q-2p}$. Such a knot is strongly quasipositive[2] and as such has $g_3 = \tau = g_4^{\mathrm{sm}}$. Concretely,

$$g_4^{\mathrm{sm}}(C_{2,q}(T_{2,p})) = g_3(C_{2,q}(T_{2,p})) = \tfrac{1}{2}(q-1) + 2g_3(T_{2,p}) = \tfrac{1}{2}(q-1) + p - 1.$$

In contrast, as an application of Theorem 1.1 we have that

$$\begin{aligned} g_4^{\mathrm{top}}(C_{2,q}(T_{2,p})) \leq g_{\mathbb{Z}}(C_{2,q}(T_{2,p})) \leq g_{\mathbb{Z}}(C_{2,q}(U)) + g_{\mathbb{Z}}(T_{2,p}) \\ = \tfrac{1}{2}(q-1) + \tfrac{1}{2}(p-1), \end{aligned} \tag{10}$$

where the equality follows from $\frac{1}{2}|\sigma(T_{2,p})| = g_4^{\mathrm{top}}(T_{2,p}) = g_{\mathbb{Z}}(T_{2,p}) = g_3(T_{2,p}) = \frac{1}{2}(p-1)$ for $p > 1$ odd. This a priori seems unexpected for all values of q. We discuss some special cases.

[2]For $q \geq 0$, all $(2,q)$–cables of a nontrivial strongly quasipositive K are strongly quasipositive since each is the boundary of a quasipositive Seifert surface. Indeed, a Seifert surface is given as a q–fold positive Hopf plumbing on the 0–framed annulus with core K. This Seifert surface is quasipositive since positive Hopf plumbing preserves quasipositivity (see [27]) and the 0–framed annulus with core K is a quasipositive Seifert surface (see [26, Lemma 1 and its proof]).

Upper and lower bounds coincide on g_4^{top} For $q=1$, (10) is of course subsumed by (1), and the inequalities are equalities. Similarly, the upper and lower bounds agree for $p=1$, though this is less interesting since $C_{2,q}(T_{2,1})=T_{2,q}$. In fact, the lower bound for $g_4^{\mathrm{top}}(C_{2,q}(T_{2,p}))$ coming from Tristram–Levine signatures equals the upper bound of (10) when $q=1,3$ and any p, when $q=5$ and $p=3,5,7,9$, when $q=7$ and $p=3$, and for any q when $p=1$. Indeed, for $p,q\geq 3$, a Tristram–Levine signature calculation[3] yields

$$g_4^{\mathrm{top}}(C_{2,q}(T_{2,p}))\geq \frac{q-1}{2}+\frac{p-1}{2}-\min\Big\{\Big\lfloor \frac{q}{4}-\frac{q}{2p}\Big\rfloor,\Big\lfloor \frac{p}{2}-\frac{2p}{q}\Big\rfloor\Big\}, \tag{11}$$

and one easily checks $\lfloor \frac{q}{4}-\frac{q}{2p}\rfloor=0$ if and only if $\lfloor \frac{p}{2}-\frac{2p}{q}\rfloor=0$ if and only if $\frac{1}{2}<\frac{1}{p}+\frac{2}{q}$.

Positive braid knots For $q=2p+1$, $C_{2,2p+1}(T_{2,p})$ is the blackboard $+1$–framed cable of $T_{2,p}$ and as such the closure of a positive 4–braid. (Indeed, $(a_2a_1a_3a_1)^p a_1{}^{q-2p}$ is evidently a positive 4–braid for $q\geq 2p$.) We find

$$\begin{aligned} p+1\leq g_4^{\mathrm{top}}(C_{2,2p+1}(T_{2,p}))&\leq p+\frac{p-1}{2}=\frac{3p-1}{2}\\ &< g_4^{\mathrm{sm}}(C_{2,2p+1}(T_{2,p}))=2p-1, \end{aligned}\tag{12}$$

where the first inequality comes from (11) with $q=2p+1$. This constitutes a significant difference between g_4^{top} and g_4^{sm} for an infinite family of knots given as closures of a positive 4–braid: for large p the situation is

$$\frac{1}{2}\leq \limsup_{p\to\infty}\frac{g_4^{\mathrm{top}}}{g_3}(C_{2,2p+1}(T_{2,p}))\overset{(12)}{\leq}\frac{3}{4}<1=\frac{g_4^{\mathrm{sm}}}{g_3}.$$

We iterate the construction described above as follows. For any positive braid β of length c with closure a knot K, one may consider the cable $C_{2,2c+1}(K)$. This is the blackboard $+1$–framed cable of the standard diagram of K coming from β and hence is the closure of a positive braid of double the braid index of β and length $4c+1$. We consider the result of iterating this process n times, starting with $K=T_{2,p}$ for $p\geq 3$ odd, and defining the knot

$$K_{n,p}=C_{2,2c_{n-1}+1}\big(C_{2,2c_{n-1}+1}(\cdots C_{2,2c_0+1}(T_{2,p})\cdots)\big),$$

[3] Setting $z=e^{2\pi i t}$ with $t=\frac{1}{4}+\frac{1}{2p}-\varepsilon$ and $t=\frac{q-2}{2q}+\varepsilon$ for ε sufficiently small, we have, by (6), $\frac{1}{2}|\sigma_z(C_{2,q}(T_{2,p}))|=\lfloor\frac{q}{4}+\frac{q}{2p}+\frac{1}{2}\rfloor+\frac{p-1}{2}=\frac{q-1}{2}+\frac{p-1}{2}-\lfloor\frac{q}{4}-\frac{q}{2p}\rfloor$ and $\frac{1}{2}|\sigma_z(C_{2,q}(T_{2,p}))|=\frac{q-1}{2}+\lfloor\frac{2p}{q}+\frac{1}{2}\rfloor=\frac{p-1}{2}+\frac{q-1}{2}-\lfloor\frac{p}{2}-\frac{2p}{q}\rfloor$, respectively.

where $c_0 := p$ and for $k \geq 1$ we define

$$c_k = 4c_{k-1} + 1 = 4(4c_{k-2} + 1) + 1 = \cdots = 4^k p + \tfrac{1}{3}(4^k - 1).$$

Since $K_{n,p}$ is a positive knot, by applying Schubert's theorem for the 3–genus of a satellite knot we obtain

$$g_4^{\mathrm{sm}}(K_{n,p}) = g_3(K_{n,p}) = \sum_{k=0}^{n-1} c_k 2^{n-1-k} + \bigl(\tfrac{1}{2}(p-1)\bigr)2^n = 2^{2n-1}\bigl(p + \tfrac{1}{3}\bigr) - 2^n + \tfrac{1}{3}.$$

Iteratively applying Proposition 2.4, we find

$$g_4^{\mathrm{top}}(K_{n,p}) \leq g_{\mathbb{Z}}(K_{n,p}) \leq \sum_{0}^{n-1} c_k + \tfrac{1}{2}(p-1) = 2^{2n-1}\bigl(\tfrac{2}{3}p + \tfrac{2}{9}\bigr) - \tfrac{1}{9}(3n+1) + \tfrac{1}{6}(p-3).$$

Thus, we have

$$\limsup_{n\to\infty} \frac{g_4^{\mathrm{top}}(K_{n,p})}{g_3(K_{n,p})} \leq \frac{2}{3} < 1 = \frac{g_4^{\mathrm{sm}}(K_{n,p})}{g_3(K_{n,p})}.$$

Algebraic knots and torus knots For $q = 4p+1$, $C_{2,4p+1}(T_{2,p})$ is an algebraic knot, which is smooth cobordism distance 1 from the torus knot $T_{4,2p+1}$. Consequently,

$$\begin{aligned} g_4^{\mathrm{top}}(T_{4,2p+1}) \leq g_4^{\mathrm{top}}(C_{2,4p+1}(T_{2,p})) + 1 &\overset{(10)}{\leq} \tfrac{1}{2}(5p-1) + 1 \\ &< 3p = g_4^{\mathrm{sm}}(T_{4,2p+1}) = g_3(T_{4,2p+1}) \end{aligned}$$

for $p > 1$. This gives

$$\lim_{p\to\infty} \frac{g_4^{\mathrm{top}}(T_{4,2p+1})}{g_3(T_{4,2p+1})} \leq \frac{5}{6}.$$

A priori, this is not particularly interesting since better upper bounds for g_4^{top} of torus knots with braid index 4 were obtained in [1, Lemma 22(ii)]. However, we find it noteworthy for two reasons. Firstly, in contrast to the somewhat example-based nature of the upper bounds from [1], it is pleasant that no explicit Seifert matrix consideration for a specific knot is needed once Theorem 1.1 is available. Secondly, by considering iterated cables of torus knots, one can find bounds on the topological 4–genera of torus knots of larger braid index that significantly improve the main results of [1]. However, this does not yield better results than those obtained by McCoy [23], whose upper bounds on $g_4^{\mathrm{top}}(T_{p,q})$ for large p and q improve any previous work; we refer the reader to his text for said bounds.

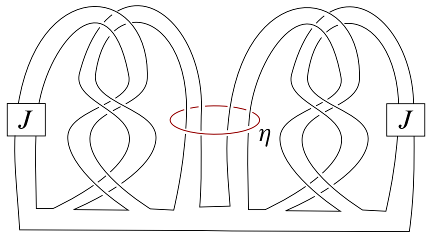

Figure 5: The pattern P_J, which depends on the choice of an auxiliary knot J and has algebraic winding number equal to 0.

As the final element to this article, we exhibit examples where the bounds on $g_4^{\mathrm{top}}(P(K))$ coming from Theorem 1.1 are far from sharp. For instance, if P is a pattern with geometric winding number 1 and such that $g_4^{\mathrm{top}}(P(U)) = n$, then

$$0 = g_4^{\mathrm{top}}(P(U)\,\#\,{-P(U)}) = g_4^{\mathrm{top}}\big(P(-P(U))\big) < g_4^{\mathrm{top}}(P(U)) + g_4^{\mathrm{top}}(-P(U)) = 2n.$$

There are also many examples of pairs (P, K) where the topological 4–genus of $P(K)$ cannot be determined by combining the upper bounds coming from Theorem 1.1 with the known lower bounds. We give a particularly interesting family that may relate to Conjecture 1.4.

Example 4.7 Let P_J be the pattern shown in Figure 5, described as a knot in the complement of the unknot η. Since $P_J(U)$ has $H_1\big(\Sigma_2(P_J(U))\big) \cong (\mathbb{Z}_3)^4$, we have that

$$2 = \tfrac{1}{2}(4) \le g_{\mathbb{Z}}(P_J(U)) \le g_3(P_J(U)) = 2,$$

where for the first inequality we used that half the minimal number of generators for the first homology of the double branched cover of a knot is a lower bound for $g_{\mathbb{Z}}$; see [7, Proposition 12(ii); 8, Corollary 1.5]. So the best algebraic bound we can obtain is $g_4^{\mathrm{top}}(P_J(K)) \le 2$, which also follows immediately from considering the genus 2 Seifert surface for $P_J(U)$ in the complement of η. Conjecture 1.4 suggests that, in fact,

$$g_4^{\mathrm{top}}(P_J(K)) \le \min\{2, g_4^{\mathrm{top}}(K)\}.$$

While for many choices of J (eg $J = \#^n T_{2,3}$ for large n) one can use Casson–Gordon signatures to prove that $P_J(T_{2,3})$ is not slice, Theorem 3.4 shows that it is not possible to use Casson–Gordon signatures to establish that $g_4^{\mathrm{top}}(P_J(T_{2,3})) > 1 = g_4^{\mathrm{top}}(P_J(U)) + g_4^{\mathrm{top}}(T_{2,3})$.

Given the gap between the known lower and upper bounds on g_4^{top} in the case of $P_J(T_{2,3})$, we propose the following as a stimulus for future work:

Problem 4.8 *For some nonslice knot J, determine* $g_4^{\mathrm{top}}(P_J(T_{2,3})) \in \{1,2\}$.

References

[1] **S Baader**, **P Feller**, **L Lewark**, **L Liechti**, *On the topological* 4*–genus of torus knots*, Trans. Amer. Math. Soc. 370 (2018) 2639–2656 MR Zbl

[2] **R C Blanchfield**, *Intersection theory of manifolds with operators with applications to knot theory*, Ann. of Math. 65 (1957) 340–356 MR Zbl

[3] **A J Casson**, **C M Gordon**, *On slice knots in dimension three*, from "Algebraic and geometric topology, II" (R J Milgram, editor), Proc. Sympos. Pure Math., 32, Amer. Math. Soc., Providence, RI (1978) 39–53 MR Zbl

[4] **A J Casson**, **C M Gordon**, *Cobordism of classical knots*, from "À la recherche de la topologie perdue" (L Guillou, A Marin, editors), Progr. Math. 62, Birkhäuser, Boston (1986) 181–199 MR Zbl

[5] **J C Cha**, **A N Miller**, **M Powell**, *Two-solvable and two-bipolar knots with large four-genera*, Math. Res. Lett. 28 (2021) 331–382 MR Zbl

[6] **T Cochran**, **S Harvey**, *The geometry of the knot concordance space*, Algebr. Geom. Topol. 18 (2018) 2509–2540 MR Zbl

[7] **P Feller**, **L Lewark**, *On classical upper bounds for slice genera*, Selecta Math. 24 (2018) 4885–4916 MR Zbl

[8] **P Feller**, **L Lewark**, *Balanced algebraic unknotting*, *linking forms*, *and surfaces in three- and four-space*, preprint (2019) arXiv 1905.08305

[9] **M H Freedman**, *The topology of four-dimensional manifolds*, J. Differential Geom. 17 (1982) 357–453 MR Zbl

[10] **S Friedl**, **M Powell**, *A calculation of Blanchfield pairings of* 3*–manifolds and knots*, Mosc. Math. J. 17 (2017) 59–77 MR Zbl

[11] **P M Gilmer**, *On the slice genus of knots*, Invent. Math. 66 (1982) 191–197 MR Zbl

[12] **R E Gompf**, **A I Stipsicz**, 4*–Manifolds and Kirby calculus*, Graduate Studies in Math. 20, Amer. Math. Soc., Providence, RI (1999) MR Zbl

[13] **J Hom**, *Bordered Heegaard Floer homology and the tau-invariant of cable knots*, J. Topol. 7 (2014) 287–326 MR Zbl

[14] **C Kearton**, *Cobordism of knots and Blanchfield duality*, J. Lond. Math. Soc. 10 (1975) 406–408 MR Zbl

[15] **K H Ko**, *A Seifert-matrix interpretation of Cappell and Shaneson's approach to link cobordisms*, Math. Proc. Cambridge Philos. Soc. 106 (1989) 531–545 MR Zbl

[16] **A S Levine**, *Nonsurjective satellite operators and piecewise-linear concordance*, Forum Math. Sigma 4 (2016) art. id. e34 MR Zbl

[17] **J Levine**, *Knot cobordism groups in codimension two*, Comment. Math. Helv. 44 (1969) 229–244 MR Zbl

[18] **W B R Lickorish**, *An introduction to knot theory*, Graduate Texts in Math. 175, Springer (1997) MR Zbl

[19] **R A Litherland**, *Signatures of iterated torus knots*, from "Topology of low-dimensional manifolds" (R A Fenn, editor), Lecture Notes in Math. 722, Springer (1979) 71–84 MR Zbl

[20] **R A Litherland**, *Cobordism of satellite knots*, from "Four-manifold theory" (C Gordon, R Kirby, editors), Contemp. Math. 35, Amer. Math. Soc., Providence, RI (1984) 327–362 MR Zbl

[21] **C Livingston**, *Knot* 4*–genus and the rank of classes in* $W(\mathbb{Q}(t))$, Pacific J. Math. 252 (2011) 113–126 MR Zbl

[22] **C Livingston**, **P Melvin**, *Abelian invariants of satellite knots*, from "Geometry and topology" (J Alexander, J Harer, editors), Lecture Notes in Math. 1167, Springer (1985) 217–227 MR Zbl

[23] **D McCoy**, *Null homologous twisting and the algebraic genus*, from "2019–10 MATRIX annals" (D R Wood, J de Grier, C E Praeger, T Tao, editors), MATRIX Book Ser. 4, Springer (2021) 147–165

[24] **A N Miller**, *Winding number m and $-m$ patterns acting on concordance*, Proc. Amer. Math. Soc. 147 (2019) 2723–2731 MR Zbl

[25] **L Rudolph**, *Some topologically locally-flat surfaces in the complex projective plane*, Comment. Math. Helv. 59 (1984) 592–599 MR Zbl

[26] **L Rudolph**, *Quasipositivity as an obstruction to sliceness*, Bull. Amer. Math. Soc. 29 (1993) 51–59 MR Zbl

[27] **L Rudolph**, *Quasipositive plumbing (constructions of quasipositive knots and links, V)*, Proc. Amer. Math. Soc. 126 (1998) 257–267 MR Zbl

[28] **H Schubert**, *Knoten und Vollringe*, Acta Math. 90 (1953) 131–286 MR Zbl

[29] **L R Taylor**, *On the genera of knots*, from "Topology of low-dimensional manifolds" (R A Fenn, editor), Lecture Notes in Math. 722, Springer (1979) 144–154 MR Zbl

[30] **A G Tristram**, *Some cobordism invariants for links*, Proc. Cambridge Philos. Soc. 66 (1969) 251–264 MR Zbl

Department of Mathematics, ETH Zurich
Zurich, Switzerland

Department of Mathematics and Statistics, Swarthmore College
Swarthmore, PA, United States

Department of Mathematics, University of Notre Dame
Notre Dame, IN, United States

peter.feller@math.ch, amille11@swarthmore.edu, jpinzonc@nd.edu

Received: 18 December 2019 Revised: 6 October 2020