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ANALYTICAL REPRESENTATIONS FOR THE TRUNCATED SPECTRAL CHARACTERISTICS OF THE FOUR-POINT COHERENCE FUNCTION OF A LASER BEAM IN A TURBULENT MEDIUM

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New analytical representations for the truncated spectral characteristics of the four-point coherence function of a laser beam propagating in a turbulent medium are obtained. These representations are valid for any level of fluctuation of the refractive index in air. They turn into exact analytical representations previously derived for two particular cases by using an integral-functional equation for truncated spectral characteristics of the four-point coherence function. A constructive procedure for obtaining approximate analytical expressions for the four-point coherence function of a laser beam propagating in a turbulent medium is proposed.

Keywords: *integral-functional equation, turbulent medium, fluctuations, four-point coherence function, truncated spectral characteristics, virtual parameters of invariant embedding, bijective connections, analytical representations, laser beam.*

Introduction. A whole series of scientific and technical problems in information transfer in open optical communication systems, ranging, geophysics, astronomy, and acoustics and diagnostics of biological specimens can be solved by establishing the behavior of wave propagation (in particular, electromagnetic and acoustic) in stochastic media, the properties of which vary randomly on spatial scales much less than the overall propagation length of these waves in them. Such a situation occurs, e.g., upon propagation of laser radiation in a turbulent terrestrial atmosphere. A theoretical study of electromagnetic wave propagation in such an atmosphere can be reduced to solving the following three tasks. Modeling of stochastic processes describing the change of local, in particular optical, characteristics of the atmosphere itself. Derivation (in the framework of several physical and mathematical hypotheses and assumptions) of equations for those characteristics of wave fields in a turbulent atmosphere that can be experimentally found. Development of accurate, asymptotic, approximate analytical or numerical methods for solving these equations. These three tasks of the total problem of studying electromagnetic wave propagation in the turbulent terrestrial atmosphere are only partially solved because of their complexity or the inadequate methods used to solve them despite over a half century of research.

The seminal basic research to solve the first task was performed in the last century [1–3]. However, classical models describing local characteristics of a turbulent atmosphere on surface trajectories and in the ionosphere [4, 5, and references therein] are still being refined. Strictly speaking, the need to solve the boundary tasks for stochastic wave equations or their consequences, which the stochastic Helmholtz equations address, must be resolved first to solve the second task because the local properties of a turbulent atmosphere vary randomly. However, even formal rigorous solutions of scalar stochastic equations (wave, Helmholtz, and even much simpler ones) have not yet been obtained. Information about the general mathematical properties of stochastic differential equations and the complications with producing their explicit rigorous solutions has been published [6–10]. Various methods have been used for small-scale fluctuations in a turbulent atmosphere that take into account insignificant backscattering on converting from the corresponding stochastic differential equations to dynamic differential equations for various statistical moments of wave fields because of the appearance of such difficulties in solving the problem of propagation of electromagnetic (laser) radiation at rather long distances [10, 11]. These methods

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enabled closed dynamic differential equations for any order of statistical moments to be obtained. Attempts to solve the boundary problems for differential equations in partial derivatives that they satisfy have been made considering experimental methods for finding the statistical moments [12]. The finding of solutions to these tasks relates to the third component of the above problem. Several works [13–15 and references therein] produced solutions of the corresponding boundary problems for first- and second-order statistical moments. However, a fourth-order moment must be found to determine signal-to-noise ratios, i.e., a four-point coherence function $\Gamma_{22}(\ldots)$ is actually sought [3]. Unfortunately, adequately justified mathematical methods and algorithms have not been proposed for finding a function $\Gamma_{22}(\ldots)$ that in general is a function of nine scalar variables for delimited laser beams. In particular, only approximate and numerical methods were previously used to find this function [15–20]. The exception was work in which a new integral-functional equation for a four-dimensional Fourier transform (it had the sense of a truncated spectral characteristic) of the four-point coherence function $\Gamma_{22}(\ldots)$ of a beam of laser radiation in a turbulent atmosphere was derived [21]. This equation was obtained from several heuristic procedures from a reduction method for common invariance relations (GIRRM), which was a common and effective method for solving multi-dimensional problems in radiative transfer theory and mathematical physics [21–30 and references therein]. Accurate analytical representations for a family of integral characteristics of the function $\Gamma_{22}(\ldots)$ were first found explicitly by analyzing the structure of this equation [21].

The present work showed that the integral-functional Equation (13) [21] could also be effectively used to obtain various analytical representations for both truncated spectral characteristics of the function $\Gamma_{22}(\ldots)$ and the function itself.

Statement of the Problem. Let us examine a closed half-space $[V]$ on the boundary S of which lies the plane OXY in rectangular right-handed Cartesian coordinates $OXYZ$ with the Z axis directed into the half-space $[V]$. Let $[V]$ be filled randomly with an inhomogeneous medium, the properties of which are identical to those of a certain transparent part of a turbulent terrestrial atmosphere. Four points M_1, M_2, M_3, and M_4 are selected on any plane z = const (const ≥ 0). Their positions are determined by radius vectors $\mathbf{r}_1 = (\rho_{11}, \rho_{12}, z)$, $\mathbf{r}_2 = (\rho_{21}, \rho_{22}, z)$, $\mathbf{r}'_1 = \rho'_{11}, \rho'_{12}, z)$, and $\mathbf{r}'_2 = (\rho'_{21}, \rho'_{22}, z)$; henceforth let us use the notations $\boldsymbol{\rho}_1\ (\rho_{11}, \rho_{12})$, $\boldsymbol{\rho}_2 = (\rho_{21}, \rho_{22})$, $\boldsymbol{\rho}'_1 = (\rho'_{11}, \rho'_{12})$, and $\boldsymbol{\rho}'_2 = (\rho'_{21}, \rho'_{22})$. Let us suppose that a semi-infinite medium is irradiated with a monochromatic linearly polarized beam of radiation, the projection of the electric-field strength of which on the X and Y axes can be written as $e^{i(\omega t - kz)}U(\boldsymbol{\rho}; z)$, where i is the imaginary unit; $k = 2\pi/\lambda$, the wavenumber; λ, radiation wavelength; ω, its circular frequency; $U(\boldsymbol{\rho}; z)$, a complex amplitude that is a random function and varies insignificantly at distances of the order of the radiation wavelength; $\boldsymbol{\rho} = (\rho_1, \rho_2)$, a two-dimensional vector parallel to the OXY plane. Let us consider that the beam power is a finite quantity and that the beam is almost finite (in fact limited to any of its transverse cross sections). The ratio λ/a satisfies the inequality $\lambda/a << 1$ (a is the exact upper facet of a set of chord lengths connecting any two points of the boundary of the transverse cross section of this laser radiation beam and is equal to double the effective radius of this cross section). Let us also assume that the center of the transverse cross section of the initial laser radiation beam lies on the Z axis and that the beam complex amplitude for any finite $z \to [0, +\infty)$ allows for evaluation of $U(\boldsymbol{\rho}; z) = O[\exp(-w_0|\boldsymbol{\rho}|)]$ as $|\boldsymbol{\rho}| \to +\infty$ (w_0 is a certain positive finite number of dimension $[\mathrm{L}^{-1}]$ inverse to the dimension of length [L]). Also, let us consider that the volume of known information on the coherence properties of the radiation beam is sufficient to specify the four-point coherence function $\Gamma_{22}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \boldsymbol{\rho}'_1, \boldsymbol{\rho}'_2; z)$ [31, 32] on the $z = 0$ plane in the $OXYZ$ system:

$$\Gamma_{22}\left(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \boldsymbol{\rho}'_1, \boldsymbol{\rho}'_2; z\right) = \langle U\left(\boldsymbol{\rho}_1; z\right) U\left(\boldsymbol{\rho}_2; z\right) U^*\left(\boldsymbol{\rho}'_1; z\right) U^*\left(\boldsymbol{\rho}'_2; z\right)\rangle. \tag{1}$$

Here $\langle\ldots\rangle$ denotes the operation of averaging over an ensemble of instances; $*$, a symbol of complex conjugation; $U(\boldsymbol{\rho}_1; z)$, $U(\boldsymbol{\rho}_2; z)$, $U(\boldsymbol{\rho}'_1; z)$, and $U(\boldsymbol{\rho}'_2; z)$ signify the complex amplitudes of the wave field on the plane specified by the z-axis and parallel to the OXY plane at points M_1, M_2, M_3, and M_4, respectively.

Second partial derivatives over all variables ρ_{11}, ρ_{12}, ρ_{21}, ρ_{22}, ρ'_{11}, ρ'_{12}, ρ'_{21}, and ρ'_{22} and the first-order partial derivative over variable z are introduced into the initial differential equation in partial derivatives, the solution of which is the function $\Gamma_{22}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \boldsymbol{\rho}'_1, \boldsymbol{\rho}'_2; z)$ [21, 31–33]. Let function $\Gamma_{22}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \boldsymbol{\rho}'_1, \boldsymbol{\rho}'_2; 0)$ describing the properties of the initial laser radiation beam in the OXY plane and giving one of the boundary conditions for the sought function $\Gamma_{22}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \boldsymbol{\rho}'_1, \boldsymbol{\rho}'_2; z)$ have continuous partial derivatives up to the nth order ($n \geq 2$) inclusive over all variables ρ_{11}, ρ_{12}, ρ_{21}, ρ_{22}, ρ'_{11}, ρ'_{12}, ρ'_{21}, and ρ'_{22}. Then, it is natural to seek a solution of the boundary problem for the initial differential equation for $\Gamma_{22}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \boldsymbol{\rho}'_1, \boldsymbol{\rho}'_2; z)$ in the class of functions that have a continuous derivative for the variable z ($z \in [0, +\infty)$) and continuous partial derivatives to the nth order inclusive over all variables $\rho_{11}, \rho_{12}, \ldots, \rho'_{21}, \rho'_{22}$. Let us set as the second boundary condition the estimate $\Gamma_{22}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \boldsymbol{\rho}'_1, \boldsymbol{\rho}'_2; z) = O\{\exp[-w_0(|\boldsymbol{\rho}_1| + |\boldsymbol{\rho}_2| + |\boldsymbol{\rho}'_1| + |\boldsymbol{\rho}'_2|)]\}$ when just one of the quantities $|\boldsymbol{\rho}_1|$, $|\boldsymbol{\rho}_2|$, $|\boldsymbol{\rho}'_1|$, or $|\boldsymbol{\rho}'_2|$

approaches $+\infty$ (the above beams automatically satisfy this condition). Analytical representations for the truncated spectral characteristics of $\Gamma_{22}(\ldots)$ and the function itself are obtained in terms of the adopted assumptions.

Analytical Representations. The integral-functional Eq. (13) from before [21] is written:

$$\overline{\overline{\Gamma_{22}^{\times}}}\left(\widetilde{\boldsymbol{\omega}_1}-\tilde{z}\left(\boldsymbol{\gamma}+\boldsymbol{\zeta}\right),\widetilde{\boldsymbol{\omega}_2}-\tilde{z}\left(\boldsymbol{\gamma}-\boldsymbol{\zeta}\right),\boldsymbol{\zeta},\boldsymbol{\gamma};k\tilde{z}/2\right)=\exp\left(-f\left(\widetilde{\boldsymbol{\omega}_1},\widetilde{\boldsymbol{\omega}_2},\boldsymbol{\zeta},\boldsymbol{\gamma};\tilde{z};\xi,\boldsymbol{\alpha}\right)\right)$$

$$\times\left[\overline{\overline{\Gamma_{22}^{\times}}}\left(\widetilde{\boldsymbol{\omega}_1},\widetilde{\boldsymbol{\omega}_2},\boldsymbol{\zeta},\boldsymbol{\gamma};0\right)-2\pi k^3\int_0^{\tilde{z}}\exp\left(f\left(\widetilde{\boldsymbol{\omega}_1},\widetilde{\boldsymbol{\omega}_2},\boldsymbol{\zeta},\boldsymbol{\gamma};\tilde{z}';\xi,\boldsymbol{\alpha}\right)\right)g\left(\boldsymbol{\sigma},\boldsymbol{\theta},\boldsymbol{\zeta}\,\boldsymbol{\gamma};kz'/2;\xi,\boldsymbol{\alpha}\right)d\tilde{z}'\right] \tag{2}$$

$$=\exp\left(-f\left(\widetilde{\boldsymbol{\omega}_1},\widetilde{\boldsymbol{\omega}_2},\boldsymbol{\zeta},\boldsymbol{\gamma};\tilde{z};\xi,\boldsymbol{\alpha}\right)\right)\overline{\overline{\Gamma_{22}^{\times}}}\left(\widetilde{\boldsymbol{\omega}_1},\widetilde{\boldsymbol{\omega}_2},\boldsymbol{\zeta},\boldsymbol{\gamma};0\right)-B_0\left(\widetilde{\boldsymbol{\omega}_1},\widetilde{\boldsymbol{\omega}_2},\boldsymbol{\zeta},\boldsymbol{\gamma};\tilde{z};\xi,\boldsymbol{\alpha}\right),$$

where

$$\begin{cases}\tilde{z}=2k^{-1}z, \quad \widetilde{\boldsymbol{\omega}_1}=\boldsymbol{\omega}_1+2k^{-1}z\left(\boldsymbol{\gamma}+\boldsymbol{\zeta}\right), \quad \widetilde{\boldsymbol{\omega}_2}=\boldsymbol{\omega}_2+2k^{-1}z\left(\boldsymbol{\gamma}-\boldsymbol{\zeta}\right),\\ \boldsymbol{\omega}_s=\boldsymbol{\rho}_s-\boldsymbol{\rho}'_s, \quad \boldsymbol{\tau}_s=\boldsymbol{\rho}_s+\boldsymbol{\rho}'_s, \quad s\in\{1,2\}, \quad \mathbf{u}=\boldsymbol{\tau}_1-\boldsymbol{\tau}_2, \quad \mathbf{p}=\boldsymbol{\tau}_1+\boldsymbol{\tau}_2,\\ \boldsymbol{\rho}_1=2^{-1}\left(\boldsymbol{\omega}_1+2^{-1}\left(\mathbf{u}+\mathbf{p}\right)\right), \quad \boldsymbol{\rho}_2=2^{-1}\left(\boldsymbol{\omega}_2+2^{-1}\left(\mathbf{p}-\mathbf{u}\right)\right),\\ \boldsymbol{\rho}'_1=2^{-1}\left(2^{-1}\left(\mathbf{u}+\mathbf{p}\right)-\boldsymbol{\omega}_1\right), \quad \boldsymbol{\rho}'_2=2^{-1}\left(2^{-1}\left(\mathbf{p}-\mathbf{u}\right)-\boldsymbol{\omega}_2\right),\\ \boldsymbol{\sigma}=\widetilde{\boldsymbol{\omega}_1}-\tilde{z}'\left(\boldsymbol{\gamma}+\boldsymbol{\zeta}\right), \quad \boldsymbol{\theta}=\widetilde{\boldsymbol{\omega}_2}-\tilde{z}'\left(\boldsymbol{\gamma}-\boldsymbol{\zeta}\right);\end{cases} \tag{3}$$

$$\begin{cases}\Gamma_{22}^{\times}\left(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\mathbf{u},\mathbf{p};z\right)=\Gamma_{22}\left(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\boldsymbol{\rho}'_1,\boldsymbol{\rho}'_2;z\right),\\ \overline{\overline{\Gamma_{22}^{\times}}}\left(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\boldsymbol{\zeta},\boldsymbol{\gamma};z\right)=(2\pi)^{-2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\exp\left(i\left(\left(\boldsymbol{\zeta}\cdot\mathbf{u}\right)+\left(\boldsymbol{\gamma}\cdot\mathbf{p}\right)\right)\right)\\ \times\Gamma_{22}^{\times}\left(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\mathbf{u},\mathbf{p};z\right)du_1du_2\,dp_1dp_2;\end{cases} \tag{4}$$

$$g\left(\boldsymbol{\sigma},\boldsymbol{\theta},\boldsymbol{\zeta},\boldsymbol{\gamma};k\tilde{z}'/2;\xi,\boldsymbol{\alpha}\right)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Phi_\varepsilon^\circ\left(2\boldsymbol{\eta};k\tilde{z}'/2\right)\left[\cos\left(\left(\boldsymbol{\eta}\cdot\left(\boldsymbol{\sigma}-\boldsymbol{\theta}\right)\right)\right)-\cos\left(\left(\boldsymbol{\eta}\cdot\left(\boldsymbol{\sigma}+\boldsymbol{\theta}\right)\right)\right)\right]$$

$$\times\left[\overline{\overline{\Gamma_{22}^{\times}}}\left(\boldsymbol{\sigma},\boldsymbol{\theta},\boldsymbol{\zeta}-\boldsymbol{\eta},\boldsymbol{\gamma};k\tilde{z}'/2\right)-\xi\cos\left(\left(\boldsymbol{\eta}\cdot\boldsymbol{\alpha}\right)\right)\overline{\overline{\Gamma_{22}^{\times}}}\left(\boldsymbol{\sigma},\boldsymbol{\theta},\boldsymbol{\zeta},\boldsymbol{\gamma};k\tilde{z}'/2\right)\right]d\eta_1d\eta_2, \tag{5}$$

$$\boldsymbol{\eta}=\left(\eta_1,\eta_2\right),$$

$$f\left(\widetilde{\boldsymbol{\omega}_1},\widetilde{\boldsymbol{\omega}_2},\boldsymbol{\zeta},\boldsymbol{\gamma};\tilde{z};\xi,\boldsymbol{\alpha}\right)=\frac{k^3}{16}\int_0^{\tilde{z}}\varkappa_1\left(\widetilde{\boldsymbol{\omega}_1}-\tilde{z}''\left(\boldsymbol{\gamma}+\boldsymbol{\zeta}\right),\widetilde{\boldsymbol{\omega}_2}-\tilde{z}''\left(\boldsymbol{\gamma}-\boldsymbol{\zeta}\right);k\tilde{z}''/2;\xi,\boldsymbol{\alpha}\right)d\tilde{z}'', \tag{6}$$

$$\varkappa_1\left(\boldsymbol{\sigma}',\boldsymbol{\theta}',k\tilde{z}''/2;\xi,\boldsymbol{\alpha}\right)=8\pi\int_{-\infty}^{+\infty}\Phi_\varepsilon^\circ\left(\mathbf{q};k\tilde{z}''/2\right)\chi_1\left(\boldsymbol{\sigma}',\boldsymbol{\theta}';\mathbf{q};\xi,\boldsymbol{\alpha}\right)dq_1dq_2,$$

$$\boldsymbol{\sigma}'=\widetilde{\boldsymbol{\omega}_1}-\tilde{z}''\left(\boldsymbol{\gamma}+\boldsymbol{\zeta}\right), \quad \boldsymbol{\theta}'=\widetilde{\boldsymbol{\omega}_2}-\tilde{z}''\left(\boldsymbol{\gamma}-\boldsymbol{\zeta}\right), \mathbf{q}=\left(q_1,q_2\right),$$

$$\chi_1\left(\boldsymbol{\sigma}',\boldsymbol{\theta}';\mathbf{q};\xi,\boldsymbol{\alpha}\right)=1+\xi\cos\left(\frac{\left(\mathbf{q}\cdot\boldsymbol{\alpha}\right)}{2}\right)\cos\left(\frac{\left(\mathbf{q}\cdot\left(\boldsymbol{\sigma}'-\boldsymbol{\theta}'\right)\right)}{2}\right)$$

$$-\cos\left(\frac{\left(\mathbf{q}\cdot\left(\boldsymbol{\sigma}'+\boldsymbol{\theta}'\right)\right)}{2}\right)\left(\xi\cos\left(\frac{\left(\mathbf{q}\cdot\boldsymbol{\alpha}\right)}{2}\right)+\cos\left(\frac{\left(\mathbf{q}\cdot\left(\boldsymbol{\sigma}'-\boldsymbol{\theta}'\right)\right)}{2}\right)\right).$$

Functions $\boldsymbol{\Phi}_\varepsilon^\circ(\mathbf{q};z)=\text{const}\,\boldsymbol{\Phi}_\varepsilon(\mathbf{q};z)$, where function $\boldsymbol{\Phi}_\varepsilon(\mathbf{q};z)$ has the significance of the spectral density of fluctuations of dielectric permittivity ε in air that is directly related to the density of fluctuations of refractive index n (const if a positive number that is determined by the choice of the form for the forward and inverse Fourier transform) considering the relationship $n = \sqrt{\varepsilon}$. Expressions of the type $(\mathbf{a}\cdot\mathbf{l})$ in Eqs. (4)–(6) denote the scalar product of vectors $\mathbf{a}$ and $\mathbf{l}$. Function $\mathbf{\Phi}_\varepsilon^\circ(\mathbf{q}; z)$ satisfies the equality $\mathbf{\Phi}_\varepsilon^\circ(\mathbf{q}; z) = \mathbf{\Phi}_\varepsilon^\circ(-\mathbf{q}; z)$, which is automatically fulfilled when $\mathbf{\Phi}_\varepsilon^\circ(\mathbf{q}; z) = \mathbf{\Phi}_\varepsilon^\circ(|\mathbf{q}|; z)$. It is noteworthy that the quantity z and the components of two-dimensional real vectors $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, $\mathbf{u}$, $\mathbf{p}$, $\boldsymbol{\rho}_1$, $\boldsymbol{\rho}_2$, $\boldsymbol{\rho}_1'$, $\boldsymbol{\rho}_2'$, $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, $\tilde{\boldsymbol{\omega}}_1$, $\tilde{\boldsymbol{\omega}}_2$, $\boldsymbol{\sigma}$, $\boldsymbol{\theta}$, $\boldsymbol{\sigma}'$, $\boldsymbol{\theta}'$, and $\boldsymbol{\alpha}$ have the dimension of length [L] while components of vectors ξ, $\boldsymbol{\gamma}$, and $\mathbf{q}$, dimensions of $[L^{-1}]$.

The quantity ξ, components α_1 and α_2, and vector $\boldsymbol{\alpha}$ in Eqs. (2), (5), and (6) are arbitrary real numbers on which the solution of Eq. (2) does not depend. However, the values of terms included in the right part of integral-functional Eq. (2) depend on these virtual invariant embedding parameters. These parameters should be chosen so that the modulus of the complex-value function $g(\boldsymbol{\sigma}, \boldsymbol{\theta}, \boldsymbol{\zeta}, \boldsymbol{\gamma}; kz'/2; \xi, \boldsymbol{\alpha})$ adopts a value considerably less than the modulus of this function when parameters ξ and $\boldsymbol{\alpha}$ are not used, i.e., $\xi = 0$ and $\boldsymbol{\alpha} = \mathbf{0} = (0, 0)$ when seeking approximate or asymptotic analytical solutions of this equation.

Let us suppose that $\xi = 1$ and $\boldsymbol{\alpha} = \mathbf{0}$ in Eq. (2). Then, considering Eq. (3), the parity of the cosine and the equality $\mathbf{\Phi}_\varepsilon^\circ(\mathbf{q}; z) = \mathbf{\Phi}_\varepsilon^\circ(\mathbf{q}; z)$, and the existence of evenly limited and continuous partial first- and second-order derivatives of the complex-value function $\overline{\overline{\Gamma_{22}^\times}}(\boldsymbol{\sigma}, \boldsymbol{\theta}, \boldsymbol{\zeta} - \boldsymbol{\eta}, \boldsymbol{\gamma}; k\tilde{z}'/2)$ over real variables (η_1 and η_2), Eq. (2) can be transformed to the form

$$\overline{\overline{\Gamma_{22}^\times}}(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\boldsymbol{\zeta},\boldsymbol{\gamma};k\tilde{z}/2) = \exp\left(-f\left(\widetilde{\boldsymbol{\omega}_1},\widetilde{\boldsymbol{\omega}_2},\boldsymbol{\zeta},\boldsymbol{\gamma};\tilde{z};1,\mathbf{0}\right)\right)\overline{\overline{\Gamma_{22}^\times}}\left(\widetilde{\boldsymbol{\omega}_1},\widetilde{\boldsymbol{\omega}_2},\boldsymbol{\zeta},\boldsymbol{\gamma};0\right) - B_1\left(\widetilde{\boldsymbol{\omega}_1},\widetilde{\boldsymbol{\omega}_2},\boldsymbol{\zeta},\boldsymbol{\gamma};\tilde{z}\right), \tag{7}$$

$$\begin{aligned} B_1\left(\widetilde{\boldsymbol{\omega}_1},\widetilde{\boldsymbol{\omega}_2},\boldsymbol{\zeta},\boldsymbol{\gamma};\tilde{z}\right) &= 2\pi k^3\int_0^{\tilde{z}}\exp\left(-\left(f\left(\widetilde{\boldsymbol{\omega}_1},\widetilde{\boldsymbol{\omega}_2},\boldsymbol{\zeta},\boldsymbol{\gamma};\tilde{z};1,\mathbf{0}\right) - f\left(\widetilde{\boldsymbol{\omega}_1},\widetilde{\boldsymbol{\omega}_2},\boldsymbol{\zeta},\boldsymbol{\gamma};\tilde{z}';1,\mathbf{0}\right)\right)\right)d\tilde{z}' \\ &\times \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Phi_\varepsilon^\circ\left(2\boldsymbol{\eta};k\tilde{z}'/2\right)\left[\cos\left(\left(\boldsymbol{\eta}\cdot(\boldsymbol{\sigma}-\boldsymbol{\theta})\right)\right) - \cos\left(\left(\boldsymbol{\eta}\cdot(\boldsymbol{\sigma}+\boldsymbol{\theta})\right)\right)\right] \\ &\times\left[\frac{1}{2}d^2_{\boldsymbol{\eta}^\Delta}\overline{\overline{\Gamma_{22}^\times}}\left(\boldsymbol{\sigma},\boldsymbol{\theta},\boldsymbol{\zeta}-\boldsymbol{\eta}^\Delta,\boldsymbol{\gamma};k\tilde{z}'/2\right)\right]d\eta_1 d\eta_2 = B_0\left(\widetilde{\boldsymbol{\omega}_1},\widetilde{\boldsymbol{\omega}_2},\boldsymbol{\zeta},\boldsymbol{\gamma};\tilde{z};1,\mathbf{0}\right). \end{aligned} \tag{8}$$

Here $\boldsymbol{\sigma} = \boldsymbol{\omega}_1 - (\tilde{z} - \tilde{z}')(\boldsymbol{\gamma} + \boldsymbol{\zeta})$; $\boldsymbol{\theta} = \boldsymbol{\omega}_2 - (\tilde{z} - \tilde{z}')(\boldsymbol{\gamma} - \boldsymbol{\zeta})$; $\boldsymbol{\eta}^\Delta = (\eta_1^\Delta, \eta_2^\Delta) = c\boldsymbol{\eta}$ [dimensionless parameter $c \in (0, 1)$]; and $d^2_{\boldsymbol{\eta}^\Delta}(\ldots)$ is the second-order differential of the complex-value function $\overline{\overline{\Gamma_{22}^\times}}(\boldsymbol{\sigma}, \boldsymbol{\theta}, \boldsymbol{\zeta} - \boldsymbol{\eta}^\Delta, \boldsymbol{\gamma}; k\tilde{z}'/2)$ over variables $(\eta_1^\Delta, \eta_2^\Delta)$ at a certain point $\boldsymbol{\eta}^\Delta = (c\eta_1, c\eta_2)$. This differential, considering definitions of the functions $\Gamma_{22}^\times(\ldots)$ and $\overline{\overline{\Gamma_{22}}}(\ldots)$ and the bijective relationships between the various quantities [Eqs. (3) and (4)], can be expressed as the four-point coherence function $\Gamma_{22}(\ldots)$

$$\begin{aligned} &d^2_{\boldsymbol{\eta}^\Delta}\overline{\overline{\Gamma_{22}^\times}}\left(\boldsymbol{\sigma},\boldsymbol{\theta},\boldsymbol{\zeta}-\boldsymbol{\eta}^\Delta,\boldsymbol{\gamma};k\tilde{z}'/2\right) \\ &= -\left(\frac{4}{\pi^2}\right)\exp\left\{-2i\left(\boldsymbol{\gamma}\cdot\boldsymbol{\omega}_2\right)\right\}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(\boldsymbol{\eta}\cdot\mathbf{y}\right)^2\exp\left\{i\left(\left(\boldsymbol{\zeta}-c\boldsymbol{\eta}+\boldsymbol{\gamma}\right)\cdot\mathbf{y}\right)\right\}dy_1dy_2 \\ &\times\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\exp\left\{4i\left(\boldsymbol{\gamma}\cdot\mathbf{x}\right)\right\}\Gamma_{22}\left(\boldsymbol{\Lambda}_1,\boldsymbol{\Lambda}_2,\boldsymbol{\Lambda}_3,\boldsymbol{\Lambda}_4;k\tilde{z}'/2\right)dx_1dx_2, \end{aligned} \tag{9}$$

where $\boldsymbol{\Lambda}_1 = \mathbf{x} + 2^{-1}[\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2 + \mathbf{y} + (\tilde{z} - \tilde{z}')(\boldsymbol{\gamma} - \boldsymbol{\zeta})]$; $\boldsymbol{\Lambda}_2 = \mathbf{x} + 2^{-1}(\tilde{z} - \tilde{z}')(\boldsymbol{\gamma} - \boldsymbol{\zeta})$, $\boldsymbol{\Lambda}_3 = \mathbf{x} + 2^{-1}[\mathbf{y} - \boldsymbol{\omega}_1 - \boldsymbol{\omega}_2 - (\tilde{z} - \tilde{z}')(\boldsymbol{\gamma} - \boldsymbol{\zeta})]$, and $\boldsymbol{\Lambda}_4 = \mathbf{x} - \boldsymbol{\omega}_2 - 2^{-1}(\tilde{z} - \tilde{z}')(\boldsymbol{\gamma} - \boldsymbol{\zeta})$.

The modulus of function $g(\ldots)$ in Eq. (2) where $\xi = 1$ and $\boldsymbol{\alpha} = \mathbf{0}$ is equal to the modulus of the double integral over variables η_1 and η_2 in Eq. (8) in which the second-order differential is represented in the form of Eq. (9). Therefore, the value of this modulus depends essentially on the functional form of spectral density $\Phi_\varepsilon^\circ(2\boldsymbol{\eta}; k\tilde{z}'/2)$ and the quantities in the square brackets in Eq. (8). Let us briefly write a series of conditions for which the modulus of the function $g(\boldsymbol{\sigma}, \boldsymbol{\theta}, \boldsymbol{\zeta}, \boldsymbol{\gamma}; kz'/2; 1, \mathbf{0})$ reverts to zero or can take sufficiently small values as compared to the case where $\xi = 0$ and $\boldsymbol{\alpha} = \mathbf{0}$. First, the second terms in the right parts of Eqs. (2) and (7) revert to zero upon fulfilling any noncontradictory conditions,

$$\boldsymbol{\sigma} - \boldsymbol{\theta} = \pm(\boldsymbol{\sigma} + \boldsymbol{\theta}). \tag{10}$$

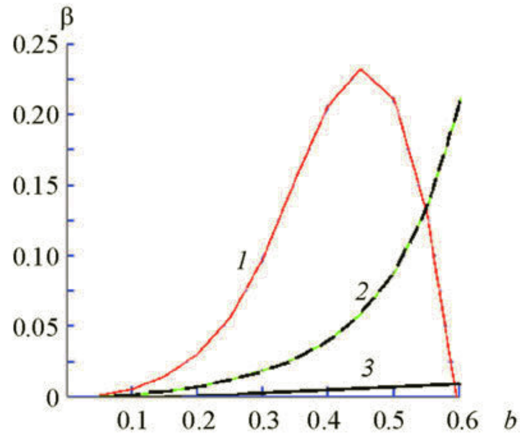

Fig. 1. Dependence of modulus β on effective beam radius b, c = 1 (1), 0.5 (2), and 0.2 (3).

With respect to Eq. (7), the conditions of Eq. (10) correspond to the situations $\boldsymbol{\omega}_2 = \mathbf{0}$, $\boldsymbol{\gamma} = \boldsymbol{\zeta}$, $\boldsymbol{\omega}_1$ is an arbitrary two-dimensional vector; $\boldsymbol{\omega}_1 = \mathbf{0}$, $\boldsymbol{\gamma} = -\boldsymbol{\zeta}$, and $\boldsymbol{\omega}_2$ is an arbitrary two-dimensional vector. If any of these conditions is fulfilled, function $\overline{\overline{\Gamma}}{}^{\times}_{22}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\zeta}, \boldsymbol{\gamma}; z)$ is equal to the first term in the right part of Eq. (7). Conditions of this type were used to obtain exact analytical representations for the integral characteristics of the four-point coherence function. Second, the first terms in the right parts of Eqs. (2) and (7) for finite values of $\tilde{z}$ are the main terms of the asymptote for function $\overline{\overline{\Gamma}}{}^{\times}_{22}(\ldots)$ [truncated spectral characteristics of the functions $\Gamma^{\times}_{22}(\ldots)$ and $\Gamma_{22}(\ldots)$ are expressed directly by it] for the cases 1) $\boldsymbol{\omega}_1$ is an arbitrary vector, $|\boldsymbol{\omega}_2| \to 0$, and $|\boldsymbol{\gamma} - \boldsymbol{\zeta}| \to 0$ and 2) $\boldsymbol{\omega}_2$ is an arbitrary vector, $|\boldsymbol{\omega}_1| \to 0$, and $|\boldsymbol{\gamma} - \boldsymbol{\zeta}| \to 0$. Third, the modulus of the second term in the right part of Eq. (7) can decrease significantly as compared to the situation where the virtual embedding parameters ξ and $\boldsymbol{\alpha}$ are not used at all in Eq. (2), i.e., when these parameters are replaced by zero and the zero vector in Eq. (2), because of the actual transverse limitation of the model and actual laser beams [34, 35], the presence of factor $2^{-1}(\boldsymbol{\eta}\cdot\mathbf{y})^2$ in the double integral over variables (y_1, y_2) in Eq. (9), and the rather rapid drop of the negative real function $\Phi^{\circ}_{\varepsilon}(2\boldsymbol{\eta}; k\tilde{z}'/2)$ with increasing $|\boldsymbol{\eta}|$ [4, 10, 32]. The integration in the right part of Eq. (9) for the above beams should essentially be performed only for the four-dimensional Euclidian space $\mathbb{R}_4$ if just one of the moduli $|\boldsymbol{\Lambda}_1|$, $|\boldsymbol{\Lambda}_2|$, $|\boldsymbol{\Lambda}_3|$, $|\boldsymbol{\Lambda}_4|$, $|\boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}_2|$, $|\boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}_3|$, $|\boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}_4|$, $|\boldsymbol{\Lambda}_2 - \boldsymbol{\Lambda}_3|$, $|\boldsymbol{\Lambda}_2 - \boldsymbol{\Lambda}_4|$, or $|\boldsymbol{\Lambda}_3 - \boldsymbol{\Lambda}_4|$ for the given set $[\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, (\tilde{z} - \tilde{z}'), \boldsymbol{\gamma}, \boldsymbol{\zeta})$ is greater than a. If just one of the inequalities $|\boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}_3| > a$ or $|\boldsymbol{\Lambda}_2 - \boldsymbol{\Lambda}_4| > a$ is valid, then the right part of Eq. (9) is practically equal to zero because the approximate equality $\Gamma_{22}(\ldots) \approx 0$ is true. For example, for the case where $\boldsymbol{\omega}_1 = \mathbf{0}$, $\boldsymbol{\omega}_2 = \mathbf{0}$, $\boldsymbol{\zeta} = \mathbf{0}$, and $\boldsymbol{\gamma}$ is an arbitrary two-dimensional vector or $\boldsymbol{\omega}_1 = \mathbf{0}$, $\boldsymbol{\omega}_2 = \mathbf{0}$, $\boldsymbol{\gamma} = \mathbf{0}$, and $\boldsymbol{\zeta}$ is an arbitrary two-dimensional vector, modulus $|\mathbf{y}|$ satisfies the inequality $|\mathbf{y}| \le 2a$. The quantity $B_1(\ldots)$ in the right part of Eq. (7) can adopt a modulus substantially less than the modulus of the second term in the right part of Eq. (2) when $\xi = 0$ and $\boldsymbol{\alpha} = \mathbf{0}$ with any type of limitation on modulus $|\mathbf{y}|$. This confirms that the quantity $B_1(\ldots)$ in Eq. (7) can be a correction to the first term in the right part and Eq. (7) itself can be used to obtain various analytical representations of the functions $\overline{\overline{\Gamma}}{}^{\times}_{22}(\ldots)$, $\Gamma^{\times}_{22}(\ldots)$, and $\Gamma_{22}(\ldots)$. Moduli of the second term in Eq. (2) for $\xi = 1$, $\boldsymbol{\alpha} = \mathbf{0}$ and $\xi = 0$, $\boldsymbol{\alpha} = \mathbf{0}$ were compared to validate this confirmation. A collimated Gaussian beam with a coherence radius equal to infinity, a normally distributed random field, and the given correlation function $\Gamma_{11}(\ldots)$ was used as the model beam. The function $\Gamma_{22}(\ldots)$ in the plane given by $z = 0$ was written [17, 19]:

$$\Gamma_{22}\left(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \boldsymbol{\rho}_1', \boldsymbol{\rho}_2'; 0\right) = \Gamma_{11}\left(\boldsymbol{\rho}_1, \boldsymbol{\rho}_1'; 0\right)\Gamma_{11}\left(\boldsymbol{\rho}_2, \boldsymbol{\rho}_2'; 0\right) + \Gamma_{11}\left(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2'; 0\right)\Gamma_{11}\left(\boldsymbol{\rho}_2, \boldsymbol{\rho}_1'; 0\right) \quad . \tag{11}$$

This model beam was chosen for several reasons: 1) the relative simplicity of the analytical representation of the function $\Gamma_{22}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \boldsymbol{\rho}_1', \boldsymbol{\rho}_2'; 0)$ for it; 2) the refractive index of $\Gamma_{22}(\ldots)$ in the framework of a quadratic approximation to the structural function retains its functional form for arbitrary values of z; and 3) the ability to perform most of the transformations and evaluate the multiple integrals analytically in Eqs. (2) and (7)–(9). The modulus of the ratio $\beta = \dfrac{B_0(\widetilde{\boldsymbol{\omega}_1}, \widetilde{\boldsymbol{\omega}_2}, \boldsymbol{\zeta}, \boldsymbol{\gamma}; \tilde{z}; 1, \mathbf{0})}{B_0(\widetilde{\boldsymbol{\omega}_1}, \widetilde{\boldsymbol{\omega}_2}, \boldsymbol{\zeta}, \boldsymbol{\gamma}; \tilde{z}; 0, \mathbf{0})}$ for a modified Karman center [4] for $\boldsymbol{\zeta} = \mathbf{0}$. Figure 1 shows dependences of moduli of ratios β on the effective cross-section radius of a Gaussian beam b for various values of c. It was assumed that the following conditions were fulfilled: $L_0 = 24l_k$; $|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2| = 2l_k$; $|\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 - 2\tilde{z}\boldsymbol{\gamma}| = 2.449l_k$; $b = a/2 = 0.05l_k$–$0.06l_k$; $l_0 = 0.04l_k$; l_0, L_0, and l_k are the internal, external, and transverse turbulence scales, respectively [12]; and l_k was assumed to be 0.08 m.

Figure 1 shows that a populated set of realistic quantities L_0, l_0, $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$, a, $\tilde{z}$, and $\boldsymbol{\alpha}$ exists for the chosen virtual parameters ξ and $\boldsymbol{\alpha}$ and the used limited laser beams [12, 34, 35], for which ratio β has a modulus almost or greater than an order of magnitude less than unity. Considering these facts and the above assumptions about the initial models of laser beams from Eqs. (2) and (7), we obtain analytical representations for the truncated spectral characteristic of function $\Gamma_{22}^{\times}(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \mathbf{u}, \mathbf{p}; z)$:

$$\overline{\overline{\Gamma_{22}^{\times}}}\left(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\zeta}, \boldsymbol{\gamma}; z\right) \simeq \overline{\overline{\Gamma_{22;0}^{\times}}}\left(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\zeta}, \boldsymbol{\gamma}; z\right) = \exp\left\{-f\left(\widetilde{\boldsymbol{\omega}_1}, \widetilde{\boldsymbol{\omega}_2}, \boldsymbol{\zeta}, \boldsymbol{\gamma}; \tilde{z}; 1, \mathbf{0}\right)\right\} \overline{\overline{\Gamma_{22}^{\times}}}\left(\widetilde{\boldsymbol{\omega}_1}, \widetilde{\boldsymbol{\omega}_2}, \boldsymbol{\zeta}, \boldsymbol{\gamma}; 0\right); \tag{12}$$

$$\begin{aligned}
&\Gamma_{22}^{\times}\left(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\zeta}, \boldsymbol{\gamma}; z\right) \simeq \overline{\overline{\Gamma_{22;1}^{\times}}}\left(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\zeta}, \boldsymbol{\gamma}; z\right) = \overline{\overline{\Gamma_{22;0}^{\times}}}\left(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\zeta}, \boldsymbol{\gamma}; z\right)\\
&- 2\pi k^3 \int_0^{\tilde{z}} \exp\left\{-\left(f\left(\widetilde{\boldsymbol{\omega}_1}, \widetilde{\boldsymbol{\omega}_2}, \boldsymbol{\zeta}, \boldsymbol{\gamma}; \tilde{z}; 1, \mathbf{0}\right) - f\left(\widetilde{\boldsymbol{\omega}_1}, \widetilde{\boldsymbol{\omega}_2}, \boldsymbol{\zeta}, \boldsymbol{\gamma}; \tilde{z}'; 0\right)\right)\right\} d\tilde{z}'\\
&\times \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Phi_{\varepsilon}^{\circ}\left(2\boldsymbol{\eta}; k\tilde{z}'/2\right)\left[\cos\left(\boldsymbol{\eta}\cdot(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2 + 2(\tilde{z} - \tilde{z}')\boldsymbol{\zeta})\right)\right.\\
&\left. - \cos\left(\boldsymbol{\eta}\cdot(\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + 2(\tilde{z} - \tilde{z}')\boldsymbol{\gamma})\right)\right] \times \left[\exp\left\{-f\left(\widetilde{\boldsymbol{\omega}_1}, \widetilde{\boldsymbol{\omega}_2}, \boldsymbol{\zeta} - \boldsymbol{\eta}, \boldsymbol{\gamma}; \tilde{z}'; 1, \mathbf{0}\right)\right\}\right.\\
&\times \left.\overline{\overline{\Gamma_{22}^{\times}}}\left(\widetilde{\boldsymbol{\omega}_1}, \widetilde{\boldsymbol{\omega}_2}, \boldsymbol{\zeta} - \boldsymbol{\eta}, \boldsymbol{\gamma}; 0\right) - \exp\left\{f\left(\widetilde{\boldsymbol{\omega}_1}, \widetilde{\boldsymbol{\omega}_2}, \boldsymbol{\zeta}, \boldsymbol{\gamma}; \tilde{z}'; 1, \mathbf{0}\right)\right\} \overline{\overline{\Gamma_{22}^{\times}}}\left(\widetilde{\boldsymbol{\omega}_1}, \widetilde{\boldsymbol{\omega}_2}, \boldsymbol{\zeta}, \boldsymbol{\gamma}; 0\right)\right] d\eta_1 d\eta_2 .
\end{aligned} \tag{13}$$

If any of the conditions of Eq. (10) are fulfilled, the sign $\simeq$ in Eqs. (11) and (12) can be replaced by an equal sign. It is noteworthy that the arguments of the exponents in Eqs. (11) and (12) are nonpositive because the equality occurs:

$$\begin{aligned}
&f\left(\widetilde{\boldsymbol{\omega}_1}, \widetilde{\boldsymbol{\omega}_2}, \boldsymbol{\zeta}, \boldsymbol{\gamma}; \tilde{z}; 1, \mathbf{0}\right) = \frac{\pi k^3}{2}\int_0^{\tilde{z}} d\widetilde{z}'' \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_{\varepsilon}^{\circ}\left(\mathbf{q}''; k\widetilde{z}''/2\right)\left[\sin\left(2^{-1}\left(\mathbf{q}''\cdot\boldsymbol{\omega}_1^{\times}\right)\right) + \sin\left(2^{-1}\left(\mathbf{q}''\cdot\boldsymbol{\omega}_2^{\times}\right)\right)\right]^2 dq_1'' dq_2'',\\
&f\left(\widetilde{\boldsymbol{\omega}_1}, \widetilde{\boldsymbol{\omega}_2}, \boldsymbol{\zeta}, \boldsymbol{\gamma}; \tilde{z}; 1, \mathbf{0}\right) - f\left(\widetilde{\boldsymbol{\omega}_1}, \widetilde{\boldsymbol{\omega}_2}, \boldsymbol{\zeta}, \boldsymbol{\gamma}; \tilde{z}'; 1, \mathbf{0}\right)\\
&= \frac{\pi k^3}{2}\int_{\tilde{z}'}^{\tilde{z}} d\widetilde{z}'' \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_{\varepsilon}^{\circ}\left(\mathbf{q}''; k\widetilde{z}''/2\right)\left[\sin\left(2^{-1}\left(\mathbf{q}''\cdot\boldsymbol{\omega}_1^{\times}\right)\right) + \sin\left(2^{-1}\left(\mathbf{q}''\cdot\boldsymbol{\omega}_2^{\times}\right)\right)\right]^2 dq_1'' dq_2'' ,
\end{aligned} \tag{14}$$

where $\boldsymbol{\omega}_1^{\times} = \boldsymbol{\omega}_1 + (\tilde{z} - \tilde{z}'')(\boldsymbol{\gamma} + \boldsymbol{\zeta})$ and $\boldsymbol{\omega}_2^{\times} = \boldsymbol{\omega}_2 + (\tilde{z} - \tilde{z}'')(\boldsymbol{\gamma} - \boldsymbol{\zeta})$. Equations (14) were obtained by elementary transformations of the function $f(\tilde{\boldsymbol{\omega}}_1, \tilde{\boldsymbol{\omega}}_2, \zeta, \gamma; h; 1, \mathbf{0})$ in which argument h was equal to $\tilde{z}$ or $\tilde{z}'$ ($\tilde{z} \in [0, \tilde{z}]$).

Equations (11)–(14) could be used to find a series of analytical representations for truncated spectral characteristics of functions $\Gamma_{22}(\ldots)$ and the function itself. Let us write the simplest analytical representations of this series to illustrate the confirmation. These were obtained based on Eq. (11) in the framework of the described assumptions. In particular, the following analytical representations were found:

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\{[((\boldsymbol{\gamma} - \boldsymbol{\zeta})\cdot\mathbf{x}) + ((\boldsymbol{\gamma} + \boldsymbol{\zeta})\cdot\mathbf{y})]i\} \overline{\overline{\Gamma_{22}}}\left(\mathbf{y}, \mathbf{x}, \mathbf{y} - \mathbf{a}, \mathbf{x} - \mathbf{b}; z\right) dx_1 dx_2 dy_1 dy_2\\
&\simeq \exp\left\{-2^{-1}\tilde{z}\left(|\boldsymbol{\gamma}|^2 - (\boldsymbol{\gamma}\cdot\boldsymbol{\zeta})\right)i\right\} \exp\{-A_0\} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\{[((\boldsymbol{\gamma} - \boldsymbol{\zeta})\cdot\mathbf{x}) + ((\boldsymbol{\gamma} + \boldsymbol{\zeta})\cdot\mathbf{y})]i\}\\
&\times \Gamma_{22}\left(\mathbf{y} + 2^{-1}\tilde{z}\boldsymbol{\zeta}, \mathbf{x}, \mathbf{y} - \mathbf{a} - 2^{-1}\tilde{z}\boldsymbol{\gamma}, \mathbf{x} - \mathbf{b} - 2^{-1}\tilde{z}\left(\boldsymbol{\gamma} - \boldsymbol{\zeta}\right); 0\right) dx_1 dx_2 dy_1 dy_2,
\end{aligned} \tag{15}$$

$$\Gamma_{22}\left(\rho_1, \rho_2, \rho_1 - \mathbf{a}, \rho_2 - \mathbf{b}; z\right) \approx \left(1/16\pi^4\right) \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \exp\left\{-\left(\left(\mathbf{m}\cdot\rho_2\right) + \left(\mathbf{n}\cdot\rho_1\right)\right) i\right\} \exp\left\{A_1\right\}$$

$$\times \exp\left\{\left(-4^{-1}\widetilde{z}\left((\mathbf{m}\cdot\mathbf{n}) + (\mathbf{m}\cdot\mathbf{m})\right)i\right\} dm_1 dm_2 dn_1 dn_2 \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \exp\left\{\left(\left(\mathbf{m}\cdot\mathbf{x}'\right) + \left(\mathbf{n}\cdot\mathbf{y}'\right)\right) i\right\} \mathfrak{H} dx_1' dx_2' dy_1' dy_2'.$$

Here, $\mathfrak{H}$ signifies the function $\Gamma_{22}(\mathbf{y}' - 4^{-1}\tilde{z}\,(\mathbf{n} - \mathbf{m}), \mathbf{x}', \mathbf{y}' - 4^{-1}\tilde{z}\,(\mathbf{n} + \mathbf{m}) - \mathbf{a}, \mathbf{x}' - 2^{-1}\tilde{z}\,\mathbf{m} - \mathbf{b}; 0)$ while the symbol A_0 means the expression

$$\frac{\pi k^3}{2}\int\limits_0^{\tilde{z}} d\widetilde{z}'' \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overset{\circ}{\Phi}_\varepsilon\left(\mathbf{q}''; k\widetilde{z}''/2\right)\left[\sin\left(2^{-1}\left(\mathbf{q}''(\mathbf{a} + 2^{-1}\left(\tilde{z} - \tilde{z}''\right)\left(\gamma + \zeta\right)\right)\right)\right.$$
$$\left. + \sin\left(2^{-1}\left(\mathbf{q}''(\mathbf{b} + 2^{-1}\left(\tilde{z} - \tilde{z}''\right)\left(\gamma - \zeta\right)\right)\right)\right]^2 dq_1'' dq_2'', \tag{16}$$

where **a** and **b** are arbitrary two-dimensional vectors, each component of which has the dimension [L]; $\mathbf{m} = (m_1, m_2)$, $\mathbf{n} = (n_1, n_2)$; and the quantity A_1 is given by the right part of Eq. (16) in which vectors $\gamma + \zeta$ and $\gamma - \zeta$ are replaced by vectors **n** and **m**.

Conclusions. Analytical representations [Eqs. (11)–(16)] generalized and confirmed previous results [21] and allowed important quantities describing statistical characteristics of limited beams of laser radiation propagated in a turbulent terrestrial atmosphere to be found. In particular, truncated spectral characteristics of a four-point coherence function, the function itself, and the twinkling effect [10] for actual types of model beams could be found using the obtained analytical representations. Heuristic procedures from a reduction method for common invariance relations [25, 27, 29] used before [21] to derive Eq. (2) allowed various analogs of this equation based on other partitions and representations of terms introduced into the second-order partial differential equation for the four-point coherence equation to be obtained. This expanded the capabilities of a search for more exact analytical representations for truncated spectral characteristics of a four-point coherence function and the function itself. The proposed approach allowed a generalization for statistical moments of any order and enabled exact and approximate analytical expressions for them to be obtained.

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