



On metaphors in thinking about preparing mathematics for teaching

In memory of José (“Pepe”) Carrillo Yáñez (1959–2021)

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Abstract

This paper explores how different schools of thought in mathematics education think and speak about preparing mathematics for teaching by introducing and proposing certain metaphors. Among the metaphors under consideration here are the unpacking metaphor, which finds its origin in the Anglo-American school of thought of pedagogical reduction of mathematics; the elementarization metaphor, which has its origin in the German school of thought of didactic reconstruction of mathematics; and the recontextualization metaphor, which originates in the French school of thought of didactic transposition. The metaphorical language used in these schools of thought is based on different theoretical positions, orientations, and images of preparing mathematics for teaching. Although these metaphors are powerful and allow for different ways of thinking and speaking about preparing mathematics for teaching, they suggest that preparing mathematics for teaching is largely a one-sided process in the sense of an adaptation of the knowledge in question. To promote a more holistic understanding, an alternative metaphor is offered: preparing mathematics for teaching as ecological engineering. By using the ecological engineering metaphor, the preparation of mathematics for teaching is presented as a two-sided process that involves both the adaptation of knowledge and the modification of its environment.

Keywords Didactic transposition · Ecology of mathematical knowledge · Elementarization · Metaphor · Recontextualization · Unpacking mathematics

1 Introduction

As Wilhelm von Humboldt pointed out more than two centuries ago, language reveals something substantial about the worldview (“*Weltanschauung*”) of an individual or society who speaks that language (see Underhill, 2009). The diversity of languages, according to von Humboldt, reveals a variety of worldviews, which is reflected in the nature of

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the language being used. However, language is not only a fabric of thought. The language being used also affects perception and cognition (see Sapir, 1929; Whorf, 2012). Since language depicts reality differently, people of different languages perceive and think about reality differently.

Different languages use various metaphors that bring to light specific cultural ways of thinking (see Lakoff, 1987). For example, the English language uses conceptual metaphors such as “time is money” to illustrate that time can be spent, saved, or invested, while other languages may not speak about time in this way. Part of what is distinctive about metaphor is that in recourse to it, we speak of one thing (usually an abstraction, such as “time”) as in relation to another thing (usually more concrete, such as “money”). In this sense, metaphors are linguistic means to speak about one thing in terms that suggest another (Richards, 1936).

In addition to informing the description of certain concepts and phenomena within both scholarly and daily life contexts, metaphors also work to shape our fundamental understanding of such concepts. That is, metaphors influence how we think about and interact with the world (see Lakoff & Johnson, 1980). They are consequential for how we perceive, think about, and investigate phenomena of interest.

Metaphors can offer profound and imaginative ways of thinking; however, not all metaphors are equally suitable. Some metaphors can even project problematic positions and lead to misleading implications. Therefore, in any scientific discourse, metaphorical language should not be used uncritically. Instead, its theoretical foundations and implications should be made explicit and questioned, and alternative metaphors should be sought if necessary.

This paper examines the metaphorical language used to depict the *preparation of mathematics for teaching*, a critical practice seen in the work of mathematics teachers and other stakeholders of the educational system and society, including disciplinary experts, educational researchers, policy-makers, and curriculum developers. Although metaphors are prevalent and pervasive in thinking about preparing mathematics for teaching, they are rarely the subject of explicit consideration and critical examination. The present paper addresses this under-examined issue by investigating how different schools of thought impose various ways of thinking about preparing mathematics for teaching by introducing and proposing certain metaphors.

The paper is structured in three parts. The first part presents metaphors proposed by alternate schools of thought for thinking about the preparation of mathematics for teaching (Sect. 2). The second part discusses variances in the metaphorical language proposed by these schools of thought, which are conceptual as well as cultural and historical (Sect. 3). The third part aims at a more holistic understanding of the preparation of mathematics for teaching by proposing an alternative metaphor that suggests the preparation of mathematics for teaching as a two-sided process instead of a one-sided process as advocated by previous metaphors (Sect. 4).

2 Metaphors in thinking about the preparation of mathematics for teaching

Metaphors are not just a matter of words; they are not only linguistic ornaments. On the contrary, metaphors are fundamental linguistic and cognitive tools with which scholars can attempt to grasp, describe, or explain the object under consideration. Metaphors may

provide the linguistic context in which scientific models can be proposed and conceived and may introduce new concepts that enrich the scientific vocabulary. They may also provide scholars with new ways of thinking that stimulate theory building and hypothesis generation.

Mathematics education scholars have become increasingly interested in the study of metaphors, especially their role in the teaching and learning of mathematics (see e.g., Font, Bolite, et al., 2010; Lakoff & Nuñez, 2000; Olsen et al., 2020; Pimm, 1981; Presmeg, 1992; Sfard, 1998, 2001; Zandieh et al., 2017). In particular, the study of metaphors that may be taken for granted due to the cultural context in which they function can help scholars to understand previously overlooked conceptual systems.

Numerous metaphors have been used when speaking and thinking about preparing mathematics for teaching. This section focuses on the metaphorical language used in three different schools of thought with regard to the important work of mathematics teachers and those who mediate between the educational system and society.¹ These schools of thought include (a) the Anglo-American school of thought of pedagogical reduction that proposes metaphors such as unpacking, deconstructing, and decompressing mathematics; (b) the German school of thought of didactic reconstruction that proposes the metaphor of elementarization (“*Elementarisierung*”); and (c) the French school of thought of didactic transposition (“*transposition didactique*”) that proposes the metaphor of recontextualization of mathematics.²

It should be noted that none of these schools of thought is a homogeneous or unified entity; they exhibit major differences among their individual theorists. Thus, it is difficult to address these schools of thought in terms of an overarching analysis and comparison. Given the difficulty of establishing a solid foundation for what these schools of thought are in themselves, this paper will focus primarily on path-defining approaches to thinking and speaking about the preparation of mathematics for teaching, without claiming to be representative of the entire school of thought. For example, in discussing the French school of thought of didactic transposition, we refer primarily to Chevallard (1991), as he has been regarded as path-defining in the discussion of didactic transposition and the preparation of mathematics for teaching. Due to space limitations and for the purposes of this paper, other theorists will be referred to in a less comprehensive manner.

2.1 The unpacking, deconstruction, and decompression metaphors

There seems to be widespread recognition in the Anglo-American literature that mathematics teachers need to do a special kind of work on mathematics in order to prepare and make it accessible for students: a pedagogical reduction of mathematics into less abstract, less complex, and less compressed forms that are suitable for students in their learning. This

¹ The metaphors considered in this paper are illustrative of various schools of thought in thinking about preparing mathematics for teaching. The metaphors discussed here have been selected to highlight fundamental similarities and differences in the theoretical positions and entailments they contain.

² Note that there is not “the” Anglo-American school of thought or “the” German school of thought or “the” French school of thought. No claim is made here that the approaches considered in this paper are representative of the respective national research paradigms. Each national research paradigm is much more diverse than could be presented in this paper. The presentation of schools of thought focuses on particular viewpoints of a group of scholars who share a common discourse and set of ideas about the preparation of mathematics for teaching. For readability, we have named the schools of thought to which we refer according to their places of origin (hence the designation of Anglo-American, German, and French).

pedagogical reduction of mathematics has been seen as crucial to the work of teaching and has been described using various metaphors. For instance, Fennema and Franke (1992) presented the metaphor of *translating* abstract and complex subject matter into concrete and understandable representations, a practice that is said to distinguish a mathematics teacher from a mathematician. Since mathematics is “composed of a large set of highly related abstractions” (Fennema & Franke, 1992, p. 153), the teacher must translate it into “real-world situations and concrete or pictorial representations” (p. 154) that students can understand. Ma (1999), on the other hand, proposed the metaphor of *unpacking* mathematics to describe the process of prying apart and exposing complex mathematical ideas in order to open up their constitutive analogies, illustrations, and representations. Similarly, Ball and Bass (2000) suggested the metaphorical language of *deconstructing* and *decompressing* mathematics into less polished and final forms in order to speak and reflect on the practice of making accessible and visible elemental components. In this view, it is assumed that while mathematicians often convert their ideas into highly abstract and compressed forms to facilitate mathematical manipulation, teachers use the reverse process of decompressing ideas to work with mathematics in its more elemental and basic form. McCrory et al. (2012), on the other hand, made a case for *trimming* the mathematical content “in a way that matches students’ current level of sophistication while treating the mathematics with integrity” (p. 604).

Although they differ in emphasis and type, the common theme of these practices is based on the belief that mathematics in general is represented in highly abstract, complex, and compressed forms that are inaccessible to students. Therefore, mathematics must be reduced to simpler and more basic forms, suitable for students while they are learning. Let us take as an example the well-known statement by Ball and Bass (2000), that teachers must “decompress” their previously compressed knowledge:

... one needs to be able to *deconstruct* one’s own mathematical knowledge into less polished and final form, where elemental components are accessible and visible. ... Paradoxically, most personal knowledge of subject matter, which is desirably and usefully *compressed*, can be ironically inadequate for teaching. ... Indeed, its polished, compressed form can obscure one’s ability to discern how learners are thinking at the roots of that knowledge. Because teachers must be able to work with content for students in its growing, not finished, state, they must be able to do something perverse: work backward from mature and compressed understanding of the content to *unpack* its constituent elements. (Ball & Bass, 2000, p. 98; italics added)

The rationale of unpacking, deconstructing, or decompressing mathematics rests on the compression of knowledge into abstract and highly usable forms, which goes hand in hand with increasingly sophisticated or advanced mathematical work, on which mathematicians in particular rely. Since these compressed forms of knowledge can hinder students’ learning, teachers must unpack, deconstruct, or decompress the mathematics at hand.

In this respect, unpacking, deconstructing, or decompressing is not only a crucial practice that mathematics teachers must apply in their work; it stands apart from the mathematical work of mathematicians. While the development of mathematics and thus the work of mathematicians is characterized by an increasing abstraction and compression of mathematical ideas, the teaching of mathematics requires the opposite process: compressed forms must be unpacked, deconstructed, or decompressed.

Unpacking, deconstructing, or decompressing mathematics has been seen as an essential mathematical practice that mathematics teachers must enact as they teach, a practice that is “not needed—or even desirable—in settings other than teaching” (Ball et al., 2008,

p. 400). In particular, the metaphor of unpacking mathematics has been seen as “a compelling description of the distinctiveness of the mathematical work that teachers do” (Adler & Davis, 2006, p. 274). In this view, teachers work with mathematics in its decompressed or unpacked form:

Teaching involves the use of decompressed mathematical knowledge that might be taught directly to students as they develop understanding. ... Teachers ... must hold unpacked mathematical knowledge because teaching involves making features of particular content visible to and learnable by students. (Ball et al., 2008, p. 400)

2.2 The elementarization metaphor

In the German-speaking didactics of mathematics (“*Didaktik der Mathematik*”), a central issue has been how the subject matter content (“*der Stoff*”) could be prepared for pupils in a way that is appropriate to the abilities and personal experiences of the children (“*kindgerecht*”) and compatible with the logic of the subject matter (“*sachgerecht*”). A fundamental guiding philosophy has been to make mathematical concepts and procedures accessible to the learners without distorting the central mathematical content (Blum & Kirsch, 1979; Kirsch, 1977, 1987). The mathematical concepts and procedures made accessible to students should be “intellectually honest” and “upwardly compatible” (Kirsch, 1987). That is, mathematical concepts and procedures should be taught with sufficient mathematical rigour while at the same time connecting with and expanding students’ knowledge of the subject.

For Kirsch (1977, 1987), mathematical content should be prepared in such a way that natural, intuitive approaches, essential basic ideas, and typical working methods become visible and ideal learning sequences emerge. Such learning sequences do not necessarily have to reflect the actual developmental history of the mathematics under consideration. Nevertheless, they should be reconstructed in a didactic manner so that learners gain a deep insight into the respective mathematical topic, subject area, or working method (see Kattmann et al., 1997). An essential method of such a reconstruction of mathematics is the didactically oriented subject matter analysis, which forms a cornerstone of the German tradition of subject matter didactics (“*Stoffdidaktik*”). By means of a didactically oriented subject matter analysis, subject content is to be explored and questioned regarding its role, meaning, and scope for deeper insights into mathematical practices and as a contribution to general educational goals. The didactic reconstruction of subject matter is therefore based not only on mathematical and cognitive considerations, but also on normative considerations according to its educational value (“*Bildungswert*”; see Klafki, 1958).

The didactic reconstruction may involve a simplification of the subject matter content, but it represents, in particular, a concentration on the elementary (“*das Elementare*”)—a practice described with the metaphor of *elementarization* (“*Elementarisierung*”, see Kirsch, 1977; Klein, 2016).³ Elementarization does not merely mean a reduction of subject matter content to its basic components but rather a concretization, an embodiment of the

³ The focus on the elementary (“*das Elementare*”) has a long history in the German-speaking context regarding considerations for teaching and learning mathematics. In 1933, for example, Felix Klein used the term “elementary” in his work “*Elementarmathematik vom höheren Standpunkte aus*”, not in the everyday meaning of “simple” or “basic”, but as a consequence of the elementarization of mathematics, an essential conceptual exposition that involves a restructuring of the subject matter content (for a discussion, see Schüring, 2019).

essential meaning inherent in the subject matter content, and a magnification of its educational value. In the process of elementarization, the subject matter in question can become even more complex, as it is concentrated, intensified, or abstracted on what is fundamental for developing a deep understanding of, and insight into, the mathematical topic, subject area, or working method in question.

A didactic reconstruction of subject matter that focuses on the elementary in order to deepen understanding of the content and bring about its educational value is a complex undertaking. Such an undertaking is generally based on mathematical, epistemological, and historical considerations (Griesel, 1974; Kirsch, 1987; Klein, 2016; Schubring, 1978), as well as on normative considerations, including overarching educational goals and the specific contribution of the subject to those educational goals (Winter, 1975; Wittmann, 1975).

The practice of elementarization and the identification of the elementary have fostered the development of critical constructs in the German-speaking didactics of mathematics, in particular the constructs of “fundamental ideas” (“*fundamentale Ideen*”) and “basic ideas” (“*Grundvorstellungen*”). Fundamental ideas describe the underlying principles or essence of a subject area, such as the idea of approximation or the idea of symmetry (Schreiber, 1983; Schweiger, 1992). Basic ideas are more local than fundamental ideas: they are adequate interpretations of the use of mathematical concepts, such as the idea of “equal sharing” for dividing natural numbers (vom Hofe, 1995; vom Hofe & Blum, 2016).

2.3 The recontextualization metaphor

The French didactics of mathematics (“*didactique mathématique*”) has been particularly concerned with the study of systemic features, conditions, and constraints of preparing mathematics for teaching, which are predominant in the didactic transposition (“*transposition didactique*”) first proposed by Chevallard in the 1980s. Didactic transposition describes the process of change from scholarly knowledge (“*savoir savant*”), produced by the scientific community and legitimized by the academic institution, to taught knowledge (“*savoir enseigné*”), which is used in a particular educational institution (Chevallard, 1991). In this sense, this school of thought has a deep epistemic orientation. It emphasizes that the knowledge and practices taught in school have their origins in other institutions, including universities and other scientific organizations. It also makes explicit that there is an inevitable difference or gap between the knowledge and practices that originate in one institution (e.g., an academic institution) and those transposed to and used in another institution (e.g., an educational institution).

In general, people from different communities are involved in the didactic transposition, including disciplinary experts, curriculum developers, and teachers. These people belong to the “*noosphère*”, the sphere of those who think about education, an intermediary between the educational system and society (Bosch & Gascón, 2006; Chevallard & Bosch, 2014).

Through various manifestations and further developments of the theory of didactic transposition, in particular the anthropological theory of didactic (Chevallard, 1992) and the theory of didactic situations (Brousseau, 1997), scholars have realized that when scholarly knowledge and practices are transposed to schools, the knowledge in question must

be recontextualized.⁴ The metaphor of *recontextualization* describes a specific practice of organizing appropriate conditions in the environment to make the knowledge in question meaningful and useful in another context. The practice of recontextualization differs from the work of scholars or researchers who reorganize their personal knowledge and insights (“*connaissance*”) into decontextualized and depersonalized forms in order to produce communicable and socially distributable knowledge (“*savoir*”) (see Brousseau, 1997, p. 227). As Brousseau (1997) noted,

The teacher’s work is to some extent the opposite of the researcher’s; she must produce a *recontextualization* and *repersonalization* of the knowledge. It must become the student’s knowledge, that is to say, a fairly natural response to relatively particular conditions, conditions that are essential if she is to make sense of this knowledge. (p. 23, italics in original)

An important characteristic of the French school of thought has been to relate the analysis of local didactic situations to the more global levels of institutional, social, and political constraints that contribute to the organization of knowledge in the classroom and the concrete levels from which didactic phenomena and transformations can be considered and how they explain the changes in knowledge (Artigue, 1994; Gascón & Nicolás, 2019). In general, a distinction can be made between an external didactic transposition (from scholarly knowledge to knowledge to be taught), undertaken mainly by curriculum developers and policy-makers, and an internal didactic transposition (from knowledge to be taught to the knowledge actually taught in the classroom), undertaken by the teacher (see Chevallard, 1991, p. 35).

In short, the didactic transposition emphasizes the “institutional relativity of knowledge”, attempting to take into account the various constraints to which diverse actors in the transposition process are subject, and seeks to expose the “transparency illusion” of those who consider the transposition of knowledge as something deliberately chosen (see Bosch & Gascón, 2006).

3 On the theoretical positions and orientations of the different metaphors

It has been widely recognized that mathematics in its final and finished form is unsuitable for teaching and learning mathematics, as it often hides the thought processes involved in the development of the mathematical concepts at stake. The mathematical content in itself cannot be directly transferred into mathematical content for teaching. In the previous section, we outlined different metaphors and practices in thinking about preparing and making mathematics accessible for students. The Anglo-American school of thought of pedagogical reduction suggests that the mathematics at stake needs to be unpacked, deconstructed, or decompressed to be suitable for students’ learning. The German school of thought of didactic reconstruction, on the other hand, submits that the mathematics in question needs to be elementarized to make it worthwhile for teaching and accessible to students, and the French school of thought of didactic transposition proposes that the mathematics under

⁴ Although Chevallard (1991) referred to the notion of recontextualization (“recontextualisation”, p. 188) in his theory of didactic transposition, this notion has found greater appeal primarily through Brousseau’s (1997) theory of didactic situations.

consideration needs to be recontextualized to be meaningful and usable in other contexts or institutions.

However, these metaphors are not just different terms used to describe the preparation of mathematics for teaching. They provide the key concepts that respective scientific communities use to talk and think about the preparation of mathematics for teaching. The point here is not that different scientific communities use different metaphorical language, but rather that the metaphorical language is based on different theoretical positions and orientations of preparing mathematics for teaching (see Table 1).

The Anglo-American school of thought of pedagogical reduction suggests that the unpacking, deconstruction, and decompression of mathematics take place within the individual teacher—as cognitive-psychological processes in the teacher’s mind. In this view, the individual teacher and her cognitive processes are the subjects of analysis, while the sociocultural environment that frames the pedagogical reduction of mathematics is of marginal significance. The prefixes “un-” and “de-” in the terms unpacking, deconstructing, and decompressing are used to express a reversal of the processes packing, constructing, and compressing mathematics—processes that are considered to be crucial to the work of mathematicians. In this sense, unpacking, deconstructing, and decompressing can be understood as a kind of “undoing” of packing, constructing, and compressing. Mathematical knowledge can be unpacked and packed, deconstructed and constructed, and decompressed and compressed in this way—similar to physical objects of the material world, such as a chair or table, which can be taken apart and put back together.⁵ In this process, the mathematics to be taught is rarely questioned, and its interest and importance are justified within the discipline. Compared to the other two schools of thought, little or no emphasis is placed on epistemological considerations.

The German school of thought of didactic reconstruction of mathematics, on the other hand, is based on normative educational concepts and uses a humanistic approach to promote and cultivate the intellectual and moral abilities of the learners. The elementarization of mathematics is based on formal and normative criteria for unlocking the educational value of mathematics and promoting *Bildung*.⁶ Further cultural-historical and critical-didactic considerations on the relevance of learning and teaching mathematics frame the activity of elementarization and have been made the object of analysis. Although the elementarization of mathematics can be read as something like a decomposition of mathematics and thus could be mistakenly equated with the unpacking, deconstruction, and decompression of mathematics, it suggests instead an intensification of the elementary, a concretization and embodiment of the essential meaning of the mathematics in question. This normative orientation of the elementarization of

⁵ The assumption that knowledge is similar to physical objects in the material world is widespread in mathematics education discourses and underpins the common-sense understanding of learning as the acquisition or construction of knowledge (for a critical stance, see Radford, 2013; Towers & Davis, 2002). For an elaboration of the metaphor of mathematical entities as physical objects, see also Font, Godino, et al. (2010).

⁶ The German concept of *Bildung* is of particular importance in the German-speaking didactics (“Didaktik”). It refers to the personal and cultural formation and maturation of the learner as a whole person. There is no corresponding English translation of the concept of *Bildung*. The English term “education” seems to reflect the German word “Erziehung” rather than the German word “Bildung”, as it refers to the preparation of the individual for the demands of society. *Bildung*, however, has a holistic view, in which individuation (such as freedom, emancipation, and autonomy) is interwoven with cultivation (such as rationality, humanity, and morality) (for a theory of *Bildung*, see von Humboldt, 1960).

Table 1 Theoretical positions and orientations of different metaphors in thinking about preparing mathematics for teaching

Metaphor	Position	Orientation (or dimension)			
		Cognitive-psychological	Epistemic	Normative	Institutional-systemic
Unpacking, deconstruction, and decompression	Pedagogical reduction	•			
Elementarization	Didactic reconstruction	•	•	•	
Recontextualization	Didactic transposition		•		•

Note. The underlying orientations of the respective schools of thought often remain implicit and vague and are therefore subject to interpretation. A dot signifies the indication of that orientation in the respective school of thought, a qualification made based on our reading of the path-defining contributions in the respective schools of thought

mathematics, especially in its function of unlocking the educational value of the mathematics in question, is the distinctive characteristic of the German school of thought of didactic reconstruction—even more so than its cognitive-psychological and epistemic orientation.

The French school of thought of didactic transposition understands the transposition of mathematics as socially and culturally produced. In this view, mathematical knowledge is situated and contextualized, bound and constituted by its sociocultural environment. For mathematical knowledge to be meaningful and useful in other settings or institutions, it must be recontextualized. The practice of recontextualization goes far beyond the individual teacher. It involves people from different communities or institutions, including disciplinary experts, educational researchers, and curriculum developers, who are seen as mediators between the educational system and society. In the school of thought of didactic transposition, the object of analysis is the social and cultural system (and its systemic features) that enables or hinders the transposition and recontextualization of mathematics. Mathematical knowledge is understood as something that “lives” in institutions. To live in an educational institution, scholarly knowledge (like academic mathematics) must be transposed, like a piece of music:

Knowledge is not a substance which has to be transferred from one place to another; it is a world of experience which, through a creative process, has to be transposed, to be adapted to a different ‘key’—the child—and to a new ‘instrument’—the classroom. (Chevallard, 1999, p. 7)

4 Towards an ecological viewpoint in thinking about preparing mathematics for teaching

In the previous section, we have shown that different schools of thought involve different ways of thinking about preparing mathematics for teaching—by introducing different metaphors. In doing so, these different metaphors are powerful mediators of specific world-views and bring out different cultural ways of thinking about preparing mathematics for

teaching—a diversity that can be seen as one of the strengths of the field.⁷ In this way, they offer tangible progress towards a meaningful understanding of the preparation of mathematics for teaching in different historical and cultural contexts.

We should keep in mind, however, that metaphors that are powerful and helpful for one school of thought may be less important or even disruptive for another, given cultural-historical constraints and sociopolitical pressures. Therefore, we should consider metaphors in their cultural context, as they paint their own unique cognitive landscape (Bolton, 2010). Consequently, we must be careful with our eagerness to impose our brushstrokes onto the canvases of others. Language is embedded in a particular set of social circumstances, shaped by the context in which it occurs: “the word does not exist in a neutral and impersonal language ..., but rather it exists in other people’s mouths, in other people’s contexts, serving other people’s intentions” (Bakhtin, 1981, p. 294).

While metaphors can offer tangible progress towards a meaningful understanding of preparing mathematics for teaching, they can also lead to oversimplifying the complexity of the issues under consideration (Lakoff & Johnson, 1980). When aware of their limitations, the metaphors discussed above can, however, open up new perspectives and provide a more holistic understanding of preparing mathematics for teaching.

In order to understand the preparation of mathematics for teaching from a point of view that complements the viewpoints advocated by the three schools of thought (the Anglo-American, German, and French), here, we take an *ecological viewpoint* of mathematical knowledge, as proposed in Godino (1994) and Godino and Batanero (1998) and inspired by the work of Chevallard (1991), and use it as a driving idea for promoting a more holistic view.⁸ These authors base their ecological view of mathematical knowledge on the ecological paradigm in epistemology. Toulmin (1972) introduced the term “intellectual ecology”:

to show how, by comparing (i) the intellectual demands of the problem-situations which are the occasions for conceptual change with (ii) the ecological demands of the niches which are the loci of adaptation in the organic sphere, we can throw light in turn on the whole process of conceptual development in a collective rational enterprise. (p. 300)

Similarly, Morin (1992) dealt with the conditions and determinisms that affect the subject of knowledge, including its habitat, life, customs, and organization. Knowledge, according to Morin (1992), is not a mere reflection of society: “knowledge develops against social pressures but in a socially conditioned way” (p. 120; translated by the authors). Morin (1992) argued that although ideas (and therefore mathematical concepts) are not physical realities, they have an objective existence. That is, ideas form a noosphere, produced and dependent on human reality and interposed between the individual and the world. The locus of mathematical reality is, as White (1947) suggested, the common world of cultural life—the cultural tradition—a pre-existing organization of beliefs, tools, symbols, customs, and institutions that shapes, and is shaped by, the

⁷ Indeed, the field has become increasingly aware that, at times, different theoretical positions are needed to address the complexity and multifaceted nature of the phenomena under consideration and to enhance a deeper understanding of the issues of interest (Bikner-Ahsbabs & Prediger, 2014). Theoretical perspectives can sometimes be coordinated in such a way that even supposedly incommensurable positions turn out to be complementary, dialectical, or interdependent (Scheiner, 2020).

⁸ It is interesting to note that Godino (1994) and Godino and Batanero (1998) mentioned Chevallard (1989) as a source of inspiration for their elaboration of the ecological metaphor as an important tool for understanding the genesis, development, and function of mathematical knowledge in human institutions.

behaviour of each generation of human organisms. As White (1947) stated, “within the body of mathematical culture there is action and reaction among the various elements. Concept reacts upon concept; ideas mix, fuse, form new syntheses” (p. 298).

Chevallard (1991) has also offered some considerations on the ecology of knowledge:

A given knowledge is found in various types of institutions, which are to it, in terms of the ecology of knowledge, many different habitats. However, when considering these habitats, it is immediately apparent that the knowledge in question often occupies very distinct niches. (p. 210; translated by the authors)

Following Chevallard’s (1991) work where the ecological metaphor is introduced to describe didactic transposition, several scholars have expanded and made more explicit the ecology of knowledge within the anthropological theory of the didactic (e.g., Assude, 1996; Barquero et al., 2013; Chambris, 2010; Gascón, 2011). These contributions have made the ecological view more concrete by describing knowledge in terms of praxeologies and specifying the constraints that influence the development and evolution of praxeologies in terms of different levels of co-determination (Gascón & Nicolás, 2019).

According to Godino and Batanero (1998), an ecological viewpoint takes mathematical knowledge as a complex system of objects and practices, a knowledge system that “adopts different ‘ways of life and operation’ within different human groups” (p. 179). The ecological viewpoint, advocated here, views mathematics as somewhat like an “organism” that lives, grows, and develops in specific habitats or niches (Godino, 1994). In other words, mathematics is not so much like a physical object that can be unpacked or elementarized but rather like a “life form” that adapts and evolves. In this regard, Godino (1994) noted,

The analysis of the institutional ecology of a piece of knowledge leads us to study its habitat, i.e., the places where we can find the objects with which it is associated, the supporting structures and the functions of these interrelations, that is, the ecological niches of the different aspects of mathematical knowledge. (p. 150)

From the ecological viewpoint, institutions are habitats that form an integrated whole and a dynamically responsive system, having both “biotic” and “abiotic” complexes. The theory of didactic transposition focuses on the study of the “habitat, or rather, the different types of habitat best suited to the development [of the knowledge in question] and the exercise of its functions” (Chevallard, 1991, p. 231; translated by the authors)—as well as how mathematical knowledge (as a living form) dwells in an institution and what would be required to modify that knowledge (Gascón & Nicolás, 2019).

Different fields of knowledge, such as pure mathematics, applied mathematics, and mathematics education, often compete—like different species—for the same space or similar resources in a given environment. However, these knowledge fields can also coexist within the same institution by using the institutional environment differently, a process known in the field of ecology as “niche differentiation”.

Like organisms exposed to environmental pressures, mathematics is subject to societal, political, institutional, and economic constraints. For knowledge to survive, it must adapt or be adapted to its ecological niche. As Godino (1994) argued,

Mathematics should be contextualized, adapted to the conditions of particular habitats. ... Didacticians, the group of people who critically and systematically reflect on the production and communication of knowledge, play the role of ‘fertilizers’ for the knowledge to fully develop its potential. (pp. 154–155)

The metaphors of unpacking, elementarization, and recontextualization seem to share the assumption that changes in mathematical knowledge are responsible for creating the “knowledge-environment match”. This view of the adaptation of knowledge as a one-sided process is widespread. It is assumed that the knowledge in question adapts or is adapted to its environment. Yet, it is not assumed that the environment adapts or is adapted to the knowledge in question.

However, organisms do not merely adapt to existing conditions in their environment; they actively construct and modify their habitats in a way that influences their living conditions (see Levins & Lewontin, 1985; Lewontin, 1983). This process by which organisms change their habitat is called “niche construction” in ecology (Odling-Smee et al., 2003). It involves physical changes in the environment that can have lasting effects and can be experienced by the offspring of the organism or other descendants, a process known as “ecological inheritance” (Odling-Smee et al., 2003). For example, in many European and North American countries around the period from the 1960s to the 1970s, educational legislation in primary and secondary education was strongly influenced by Bourbaki’s perspective (see Kilpatrick, 2012). Bourbaki (1970) aimed to develop an academic mathematical language and propose to students the use of mathematical symbolism, formalism, and set theory. The emergence of a neoliberal agenda in education and a modernist ideology of curricula in the 1970s and 1980s fostered discussion about the direction of school mathematics curricula and, more recently, the inclusion of problem solving in curricula as a transversal way of approaching mathematics focused on the development of mathematical skills and competencies, replacing Bourbaki’s perspective. This is an exemplary case of how sociopolitical and economic pressures can shape not only knowledge but also the environment in which it functions.⁹

However, teachers, and those who mediate the educational system and society (the *noosphère*), also modify environmental conditions in specific ways—for instance, by imposing a systematic bias in selecting knowledge and allowing certain types of knowledge to exert some influence over others. Preparing mathematics for teaching is thus not only socially and culturally conditioned, but also socially and culturally constitutive (Scheiner, 2022). It shapes the situations, the objects of learning, and the social identities and relationships of people and groups of people. It raises critical issues of authority and power—by maintaining and reproducing the social status quo or by contributing to its transformation. Those involved in the didactic transposition (the *noosphère*) can then be considered “ecosystem engineers”. That is, they not only adapt knowledge to its environment, but also create and change the conditions in the environment.¹⁰

From this ecological viewpoint, preparing mathematics for teaching goes far beyond adapting knowledge to its habitat. The preparation of mathematics for teaching, we claim, is an ecological process—an ecological engineering that involves both the adaptation of

⁹ For example, Gispert and Schubring (2011) have shown that changes in epistemological conceptions of mathematics and pedagogical conceptions of mathematics teaching in both Germany and France were closely related to changing conceptions of social, economic, and cultural modernity and sociopolitical transformations in both countries.

¹⁰ In ecology, ecosystem engineers are those organisms that make, change, or maintain a habitat (see Wright et al., 2002). Dam-building beavers are the original model for ecosystem engineers. Dams built by beavers cause physical changes to the habitat and alter resources available to the other organisms present. Humans, on the other hand, are considered to be “the ultimate ecosystem engineer” (Smith, 2007). Through agricultural practices and urban development, humans have changed the way they interact with the environment, often with unforeseen consequences.

mathematical knowledge to its habitat and the modification of the environment through niche construction. This complementary match of knowledge and environment should not be considered the result of a one-sided process in which knowledge is merely adapted to fit its context. Instead, it should be understood as a two-sided process involving both the adaptation of knowledge and the modification of its environment.

The ecological metaphor highlights two aspects of knowledge engineering: first, engineering adapts knowledge (such as transformation or elementarization of knowledge), and second, engineering constructs the environment (niche construction) in which knowledge emerges and thrives. In this way, new forms of knowledge emerge (e.g., school mathematics) that have their own consistency and *raison d'être* in the ecological contexts in which they fulfil their functions. An example is the use of GeoGebra to explore some properties of geometric figures and to promote intuitive and visual understanding. New mathematical practices and objects, such as figure dragging, are created as a means of justification that do not correspond to existing forms of disciplinary knowledge. A new epistemological form emerges with its own ecological niche in which it fulfils its functions. Nevertheless, this creation of new concepts, procedures, and modes of justification in the school context may not be free from epistemological and cognitive problems, since the progression of learning in successive educational levels requires ensuring the coherent fit of different meanings. Thus, justification of geometric properties based on dynamic changes of figures may later cause difficulties with the logical-deductive demonstrations required in university teaching.

5 Conclusions

As Chevallard (1999) remarked, “Scientific communities are responsible for the tools they use, including their linguistic tools” (p. 7). This paper focused on the linguistic tools for speaking and thinking about preparing mathematics for teaching. Three schools of thought were examined in this paper: the Anglo-American school of thought of pedagogical reduction, which suggests that mathematics should be unpacked, deconstructed, or decompressed to be suitable for students in their learning; the German school of thought of didactic reconstruction, which submits that mathematics should be elementarized to make it worthwhile for teaching and accessible for students; and the French school of thought of didactic transposition, which proposes that mathematics should be recontextualized to be meaningful and usable in contexts other than its origin.

The exploration of, and reflection on, the metaphorical language used in these schools of thought reveal the many profound and creative ways of thinking about preparing mathematics for teaching. Such exploration and reflection are fraught with difficulties, as it is hard to determine the metaphors used; however, the differences in the metaphorical language used in these schools of thought suggest differences in their theoretical positions and orientations—differences that are conceptual, as well as cultural and historical.

The unpacking, deconstruction, and decompression metaphors used in the Anglo-American school of thought of pedagogical reduction of mathematics can be understood in such a way that the preparation of mathematics for teaching is a cognitive-psychological process in the mind of the individual teacher. On the other hand, the elementarization metaphor used in the German school of thought of didactic reconstruction of mathematics can be considered in such a way that the preparation of mathematics is a cultural-historical and normative practice aimed at unlocking the educational value of mathematics for the promotion of *Bildung*. The recontextualization metaphor used in the French school of thought of

didactic transposition of mathematics can be understood to mean that the preparation of mathematics for teaching is a social practice developed jointly by people from different communities, including disciplinary experts, educational researchers, curriculum developers, as well as teachers.

Although these metaphors are powerful and allow for different ways of thinking and speaking about preparing mathematics for teaching, they suggest that preparing mathematics for teaching is largely a one-sided process in the sense of an adaptation of the knowledge in question.¹¹ To promote a more holistic understanding of the preparation of mathematics for teaching, an ecological viewpoint has been proposed here that complements existing views of preparing mathematics for teaching. By offering an alternative metaphor—namely, the preparation of mathematics for teaching as ecological engineering—we not only rewrite what the other metaphors already embody but describe something that is still mostly unexplored. This is the portrayal of the preparation of mathematics for teaching as a two-sided process that includes both the adaptation of knowledge to its environment and the modification of its environment.

The ecological engineering metaphor, we hope, will enable scholars to articulate and develop a more holistic understanding of the complexity of preparing mathematics for teaching. However, the relevance and usefulness of the ecological metaphor rest not in re-expressing a previously described process (such as the adaptation of mathematical knowledge to its context) but in highlighting aspects of the preparation of mathematics for teaching that have gone unnoticed and are now accessible through the new metaphor.

The ecological engineering metaphor is thus not so much an embellishment of what is already known about preparing mathematics for teaching but rather a vehicle for a new insight or way of thinking, which is made available by introducing terms like “adaptation” and “niche construction” to refer to various plausibly postulated ecology-like aspects of the preparation of mathematics for teaching (e.g., the suggestion that when preparing mathematics for teaching, there are processes like adaptation and niche construction that are analogous to the processes of adaptation and niche construction of organisms).

These metaphorically constituted viewpoints then offer the possibility of viewing the preparation of mathematics for teaching more holistically—as a two-sided process that involves both the adaptation of the mathematical knowledge in question and the modification of its environment. These viewpoints can be capitalized upon, we assert, not only when mathematics is conceived of as a system of cultural objects that are transformed or adapted to be made useful for the classroom (a view under which the three schools of thought could be considered) but also when mathematics is conceived of as a human activity, that is, a practice that gives meaning to mathematical discourse and constitutes the *raison d'être* of the objects.

Not only do we believe that the metaphorically constituted viewpoints offered here can provide an organizing framework to further explore, examine, and explain the complexity of preparing mathematics for teaching. They can also reinforce Wittmann's (1995) view of mathematics education as a “systemic-evolutionary design science”. They make it possible to view mathematics education as a design science not only because it is concerned with the design and making of “artificial objects”, such as teaching units on particular

¹¹ We recognize that the French school of thought of didactic transposition takes a systemic view by addressing both the change of knowledge and the change of habitats or niches. Knowledge, in the theory of didactic transposition, is “a changing reality, which adapts to its institutional habitat where it occupies a more or less narrow niche” (Chevallard, 2007, p. 132).

mathematical topics (see Wittmann, 1995, p. 362), but also because it is concerned with the design and making of the environment in which the mathematical topics are brought to bear. This is indeed “systemic-evolutionary” in that it resembles the complexity of living systems, a characterization underscored by the ecological metaphor proposed here.

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