# ON MULTIPLYING CURVES IN THE KAUFFMAN BRACKET SKEIN ALGEBRA OF THE THICKENED FOUR-HOLED SPHERE

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ABSTRACT. Based on the presentation of the Kauffman bracket skein module of the torus given by the third author in previous work, Charles D. Frohman and Răzvan Gelca established a complete description of the multiplicative operation leading to a famous product-to-sum formula. In this paper, we study the multiplicative structure of the Kauffman bracket skein algebra of the thickened four-holed sphere. We present an algorithm to compute the product of any two elements of the algebra, and give an explicit formula for some families of curves. We surmise that the algorithm has quasi-polynomial growth with respect to the number of crossings of a pair of curves. Further, we conjecture the existence of a positive basis for the algebra.

#### 1. INTRODUCTION

Skein modules were introduced in 1987 by the third author (see [\[Prz1,](#page-18-0) [Prz4\]](#page-18-1)) as an invariant of embeddings of codimension two, modulo ambient isotopy. In the case of 3-manifolds they generalize quantum invariants of classical links. In particular, the Kauffman bracket skein module and algebra are generalizations of the Kauffman bracket polynomial of links in the 3-sphere.

In 1997, Frohman and Gelca gave a famous product-to-sum formula for the Kauffman bracket skein algebra (KBSA) of the thickened torus [\[FG\]](#page-17-0). In this paper we give an algorithm to multiply any two curves in  $F_{0,4}\times I.^1$  $F_{0,4}\times I.^1$ We give several concrete applications of the algorithm. In particular, we give a simple formula for mutiplying unit fractions<sup>[2](#page-0-1)</sup> by  $\frac{1}{0}$ .

The paper is organized as follows. In the second section we recall background material: the Kauffman bracket skein module of 3-manifolds, the skein algebra structure for a thickened surface, and the exact structure (the generators and relations) of the algebras for the torus and the sphere with four holes,  $F_{0,4}$ . In Section [3,](#page-5-0) we give closed formulas for the product of some families of curves yielding relatively simple formulas. Finally, in the fourth section we recall the Farey diagram and its basic properties. We then describe an algorithm to compute a product-to-sum formula for every pair of fractions. In the last part we describe future directions and open problems.

### 2. Preliminaries

2.1. The Kauffman bracket skein module. The Kauffman bracket skein module (KBSM) of 3-manifolds is the most extensively studied object in the theory of skein modules. Let M be an oriented 3-manifold, k a commutative ring with unity and A a fixed invertible element in k. Let  $\mathcal{L}_{fr}$  be the set of unoriented framed links in M modulo topological equivalence, including the empty link, and  $k\mathcal{L}_{fr}$  the free  $k$ -module generated by  $\mathcal{L}_{fr}$ . Denote by  $S_{2,\infty}$ the submodule of  $k\mathcal{L}_{fr}$  generated by all the skein expressions of the form:

$$
\times - A \times - A^{-1} \times \text{and } \bigcirc \sqcup L + (A^2 + A^{-2})L,
$$

where  $\bigcirc$  denotes the trivial framed knot and the skein triple in the first relation represents framed links which can be isotoped to identical embeddings except within the neighborhood of a single crossing, where they differ as shown. The change of the crossing  $\times$  to  $\times$  and  $\times$  is called A-smoothing and B-smoothing, respectively.

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<span id="page-0-0"></span> $^1$ In [\[Deh2\]](#page-17-1), Max Dehn denotes  $F_{0,4}$  by  $L_4$  and calls it the four-holed sphere. In [\[FM\]](#page-17-2), Benson Farb and Dan Margalit call  $F_{0,4}$ , a lantern. Another name for  $F_{0,4}$  is T-shirt.

<span id="page-0-1"></span> $^{2}$ Of historical interest is the fact that unit fractions were the base for ancient Egyptian fraction calculus [\[Gol\]](#page-17-3).

The Kauffman bracket skein module of M is the quotient  $S_{2,\infty}(M; k, A) = k\mathcal{L}_{fr}/S_{2,\infty}$ . For brevity, let  $\mathcal{S}_{2,\infty}(M)$  denote  $\mathcal{S}_{2,\infty}(M;k,A),$  when  $k=\mathbb{Z}[A^{\pm 1}].$ 

The KBSM for a few 3-manifolds have been computed in [\[HP,](#page-17-4) [Le,](#page-18-2) [LT,](#page-18-3) [Mar,](#page-18-4) [Prz1\]](#page-18-0). If the 3-manifold is the product of a surface and the interval, one can simply project the links onto the surface and work with their link diagrams.

<span id="page-1-0"></span>**Theorem 2.1.** [\[Prz1,](#page-18-0) [Prz4\]](#page-18-1) Let F be an oriented surface and  $I = [0, 1]$ . Then,  $S_{2,\infty}(F \times I)$  is a free module generated by links (i.e. multicurves) including the empty link in  $F$  with no trivial components.

The generators in the statement of the above theorem are assumed to have blackboard framing. Let  $F_{a,b}$ denote a surface of genus g with b boundary components. Theorem [2.1](#page-1-0) applies, in particular, to handlebodies, since  $H_b = F_{0,b+1} \times [0,1]$ , where  $H_b$  is a handlebody of genus b.

We enrich the KBSM of  $F \times I$  with an algebra structure where the empty link is the identity element and the multiplication of two elements  $L_1\cdot L_2$  is defined by placing  $L_1$  above  $L_2$  that is,  $L_1\subset F\times(\frac{1}{2},1)$  and  $L_2\subset F\times (0,\frac12).$  When we switch the order of the factors in the product of two curves the roles of  $A$  and  $A^{-1}$ are reversed.

The Kauffman bracket skein algebra of a surface times an interval was extensively discussed in [\[BP,](#page-17-5) [Bul,](#page-17-6) [FKL,](#page-17-7) [PS1,](#page-18-5) PS2. If  $F$  is not a disc then the KBSA is an infinite dimensional module. As an algebra, however, it is finitely generated.

**Theorem 2.2.** [\[Bul\]](#page-17-6) The algebra associated to  $S_{2,\infty}(F \times I)$  is finitely generated, and the minimal number of generators is  $2^{rank(H_1(F))}-1.$ 

**Theorem 2.3.** [\[Prz3,](#page-18-7) [PS1,](#page-18-5) [PS2\]](#page-18-6) The center of the algebra  $S_{2,\infty}(F \times I)$  is a subalgebra generated by the boundary curves of  $F^3$  $F^3$ .

The exact structure of the KBSA has been computed for small surfaces  $F_{a,b}$ , namely, it is known for  $(g, b) \in$  $\{(0,0), (1,0), (1,1), (1,2), (0,1), (0,2), (0,3), (0,4)\}.$  In particular, the KBSA is commutative in the following cases:

Proposition 2.4. [\[BP\]](#page-17-5) As an algebra,

(1)  $S_{2,\infty}(F_{0,0} \times I; k, A) = S_{2,\infty}(F_{0,1} \times I; k, A) = k.$ 

(2)  $S_{2,\infty}(F_{0,2} \times I; k, A) = k[x]$ , where x is a curve parallel to a boundary component.

(3)  $S_{2,\infty}(F_{0,3} \times I; k, A) = k[x, y, z]$ , where  $x, y, z$  are curves parallel to the boundary components.

In all other cases the algebra  $S_{2,\infty}(F_{q,b} \times I)$  is noncommutative [\[BP\]](#page-17-5).

We continue with the study of the KBSA( $F_{0,4} \times I$ ), extending techniques developed by Frohman and Gelca in [\[FG\]](#page-17-0) for the torus. The remainder of this subsection is devoted to reviewing the description of the KBSA( $F_{1,0} \times I$ ) and KBSA $(F_{0,4}\times I)$ , as well as the relation between them based on the double branched cover  $F_{1,0}\longrightarrow S^2$ branched along four points (see Figure [2.1\)](#page-2-0).

The multicurves on the torus are parametrized by the pairs  $(d, n) \in \mathbb{Z}^+ \times \mathbb{Z}$ , often denoted by unreduced fractions  $\frac{n}{d}.$  The number of components of this multicurve is equal to  $r=gcd(d,n)$  and the reduced fraction  $\frac{n/r}{d/r}$ is called the slope of the multicurve. See Figure [2.2](#page-2-1) for a graphic representation of the product  $(1,0)*(0,1)*(1,1)$ in the torus. By convention, we denote the curve  $(-d, n)$  by  $(d, -n)$ ,  $d > 0$ . Further, let  $(0, 0)$  represent the empty link.

Similarly when  $F=F_{0,4}$ , we define the pair  $(d,n)\in\mathbb{Z}^+\times\mathbb{Z}$  (also denoted by  $\frac{n}{d}$ ) to be the multicurve whose (minimal) intersection numbers with the x-axis and the y-axis are  $2|n|$  and  $2d$ , respectively. The sign of n is given by the "direction" of the curve. This is a special case of Dehn coordinates of multicurves in any oriented surface with negative Euler characteristic (see [\[Deh1,](#page-17-8) [PH\]](#page-18-8) for a detailed description).

Remark 2.5. There are many ways of drawing curves of given slopes in  $F_{0,4}$ . Figure [2.3](#page-3-0) illustrates one such way of drawing the curve  $(d, n)$ ,  $n > 0$ , in  $F_{0,4} \times I$ . If  $n < 0$ , one reflects the curve  $(d, |n|)$  about the vertical axis.

<span id="page-1-1"></span> $^3$ In [\[FKL\]](#page-17-7) the center of the skein algebra is analyzed when  $A$  is a root of unity.



<span id="page-2-0"></span>FIGURE 2.1. Double branched cover of  $F_{1,0} \longrightarrow S^2$  branched along four points.



<span id="page-2-1"></span>FIGURE 2.2. The product  $(1,0) * (0,1) * (1,1)$  in the KBSA of  $F_{1,0} \times I$ .

A presentation of the algebras  $S_{2,\infty}(F_{1,0} \times I)$  and  $S_{2,\infty}(F_{0,4} \times I)$  is given in [\[BP\]](#page-17-5). We now recall this presentation for convenience, since it will be useful when comparing the multiplicative structure of  $S_{2,\infty}(F_{0,4}\times I)$ with that of  $S_{2,\infty}(F_{1,0}\times I)$ .

The KBSA of  $F_{1,0}\times I$  is generated by the simple curves  $(0,1),(1,0),(1,1)$ ; see [\[BP\]](#page-17-5). In order to get the set of relations, we start by computing the relations for the KBSA of  $F_{1,1}\times I$ . Figure [2.4](#page-3-1) illustrates the following products of curves:

<span id="page-2-3"></span><span id="page-2-2"></span>
$$
(1,0)*(0,1) = A(1,1) + A^{-1}(1,-1), \qquad (0,1)*(1,0) = A(1,-1) + A^{-1}(1,1).
$$

Therefore,

(2.1) 
$$
A(1,0) * (0,1) - A^{-1}(0,1) * (1,0) = (A^2 - A^{-2})(1,1).
$$

In a similar way one gets:

(2.2) 
$$
A(0,1) * (1,1) - A^{-1}(1,1) * (0,1) = (A^2 - A^{-2})(1,0),
$$

$$
3
$$



<span id="page-3-0"></span>FIGURE 2.3. The curve  $(3, 7) = \frac{7}{3}$  is drawn as follows. Start with a  $3 \times 7$  grid as shown in (a). The vertices of the rectangle represent the 4 punctures in  $S^2$ . We draw an arc starting at the lower left corner of the 'pillowcase' in (b). The arc in (c) is the core of the curve  $(3, 7)$  shown in (d).



<span id="page-3-1"></span>FIGURE 2.4. The product  $(1,0)*(0,1)$  in the KBSA of  $F_{1,1}\times I$ .

<span id="page-3-2"></span>(2.3) 
$$
A(1,1) * (1,0) - A^{-1}(1,0) * (1,1) = (A^2 - A^{-2})(0,1).
$$

Equations [2.1](#page-2-2)[-2.3](#page-3-2) constitute the relations of the KBSA of the punctured torus  $F_{1,1}$  (see Theorem [2.6\(](#page-3-3)1)). The curve  $\partial$ , parallel to the boundary, is generated by the former three generators, that is,

<span id="page-3-4"></span>
$$
(1,-1) * (1,1) = A2(1,0)2 + \partial + d + A-2(0,1)2,
$$

where  $d=-A^2-A^{-2}$  corresponds to the trivial contractible curve.

Equations [2.1-](#page-2-2)[2.3](#page-3-2) also hold for the KBSA of the torus. Moreover, since the boundary curve ∂ becomes contractible in  $F_{1,0}$ , we get the '*long relation*' for the KBSA of the torus (the  $\ell^{th}$ -power of a curve denotes  $\ell$ parallel copies of it):

$$
(2.4) \qquad A^2[(1,1)^2 + (1,0)^2] + A^{-2}(0,1)^2 - 2(A^2 + A^{-2}) - A((1,0)*(0,1)*(1,1)) = 0.
$$

<span id="page-3-3"></span>Theorem 2.6. [\[BP\]](#page-17-5) As an algebra,

- (1)  $S_{2,\infty}(F_{1,1} \times I) = k \{(1,0), (0,1), (1,1)\}/(2.1), (2.2), (2.3).$  $S_{2,\infty}(F_{1,1} \times I) = k \{(1,0), (0,1), (1,1)\}/(2.1), (2.2), (2.3).$  $S_{2,\infty}(F_{1,1} \times I) = k \{(1,0), (0,1), (1,1)\}/(2.1), (2.2), (2.3).$
- [\(2](#page-3-2))  $S_{2,\infty}(F_{1,0} \times I) = k \{(1,0), (0,1), (1,1)\}/(2.1), (2.2), (2.3), (2.4).$  $S_{2,\infty}(F_{1,0} \times I) = k \{(1,0), (0,1), (1,1)\}/(2.1), (2.2), (2.3), (2.4).$  $S_{2,\infty}(F_{1,0} \times I) = k \{(1,0), (0,1), (1,1)\}/(2.1), (2.2), (2.3), (2.4).$



<span id="page-4-0"></span>FIGURE 2.5. The product  $(1, 0) * (0, 1)$  in  $F_{0,4}$ .



<span id="page-4-5"></span>FIGURE 2.6. The product  $(-1, 1) * (1, 1)$  in  $F_{0,4}$ .

Now, we consider the KBSA of  $F_{0,4} \times I$ , which is generated (as an algebra) by the curves  $(1,0), (0,1), (1,1)$ together with the curves  $a_i$  where  $i = 1, 2, 3, 4$  and the curves  $a_i$  are parallel to  $p_i$  (see Figure [2.3](#page-3-0) (a)). Recall that the boundary curves generate the center of the algebra. We can consider a bigger commutative ring  $K =$  $k[a_1, a_2, a_3, a_4]$  and consider the KBSA over this ring. The computation of the product of the curves  $(1, 0)$  and  $(0, 1)$  is illustrated in Figure [2.5:](#page-4-0)

$$
(1,0)*(0,1) = A^2(1,1) + a_1a_3 + a_2a_4 + A^{-2}(1,-1).
$$

Therefore,

<span id="page-4-1"></span>
$$
(0,1) * (1,0) = A2(1,-1) + a1a3 + a2a4 + A-2(1,1).
$$

From these equations we obtain the following:

<span id="page-4-2"></span>
$$
(2.5) \qquad A^2(1,0) * (0,1) - A^{-2}(0,1) * (1,0) = (A^4 - A^{-4})(1,1) + (A^2 - A^{-2})(a_1a_3 + a_2a_4),
$$

<span id="page-4-3"></span>
$$
(2.6) \qquad A^2(0,1) * (1,1) - A^{-2}(1,1) * (0,1) = (A^4 - A^{-4})(1,0) + (A^2 - A^{-2})(a_1a_4 + a_2a_3),
$$

(2.7) 
$$
A^{2}(1,1) * (1,0) - A^{-2}(1,0) * (1,1) = (A^{4} - A^{-4})(0,1) + (A^{2} - A^{-2})(a_{1}a_{2} + a_{3}a_{4}).
$$

<span id="page-4-4"></span>The *long relation* for  $F_{0,4}$  is

$$
(2.8) \quad A^4 \left[ (1,1)^2 + A^2 (1,0)^2 \right] + A^{-4} (0,1)^2 + A^2 \left[ (a_1 a_4 + a_2 a_3)(1,0) + (a_1 a_3 + a_2 a_4)(1,1) \right] + A^{-2} (a_1 a_2 + a_3 a_4)(0,1) + \left[ a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_1 a_2 a_3 a_4 + (A^2 - A^{-2})^2 \right] - 2(A^4 + A^{-4}) - A^2 (1,0) * (0,1) * (1,1) = 0.
$$

<span id="page-5-1"></span>**Theorem 2.7.** [\[BP\]](#page-17-5) As an algebra,  $S_{2,\infty}(F_{0,4} \times I) = k[a_1, a_2, a_3, a_4] \{ (1,0), (0,1), (1, 1) \} / (2.5), (2.6), (2.7), (2.8).$  $S_{2,\infty}(F_{0,4} \times I) = k[a_1, a_2, a_3, a_4] \{ (1,0), (0,1), (1, 1) \} / (2.5), (2.6), (2.7), (2.8).$  $S_{2,\infty}(F_{0,4} \times I) = k[a_1, a_2, a_3, a_4] \{ (1,0), (0,1), (1, 1) \} / (2.5), (2.6), (2.7), (2.8).$  $S_{2,\infty}(F_{0,4} \times I) = k[a_1, a_2, a_3, a_4] \{ (1,0), (0,1), (1, 1) \} / (2.5), (2.6), (2.7), (2.8).$  $S_{2,\infty}(F_{0,4} \times I) = k[a_1, a_2, a_3, a_4] \{ (1,0), (0,1), (1, 1) \} / (2.5), (2.6), (2.7), (2.8).$ **Corollary 2.8.** [\[BP\]](#page-17-5) Let  $R_{0,1} = a_1a_2 + a_3a_4$ ,  $R_{1,0} = a_1a_4 + a_2a_3$ ,  $R_{1,1} = a_1a_3 + a_2a_4$  and  $y = a_1^2 + a_2^2 + a_3^2 + a_4^2$  $a_4^2+a_1a_2a_3a_4+(A^2-A^{-2})^2$ . Then, modulo  $R_{0,1},R_{1,0},R_{1,1}$  and  $y$ , the relations associated to the KBSA of  $F_{1,0} \times I$  and  $F_{0,4} \times I$  coincide upto switching  $A$  and  $A^2$  and keeping the slopes of the curves unchanged.

Corollary [2.8](#page-5-1) suggests that when looking for formulas for the multiplication of two curves in  $F_{0,4}$  based on the product-to-sum formula described in [\[FG\]](#page-17-0) (see Theorem [2.11\)](#page-5-2), one should work with a subring of the center of  $\mathcal{S}_{2,\infty}(F_{0,4}\times I)$  generated by the variables  $R_{0,1}, R_{1,0}, R_{1,1}$  and y. We denote this subring by  $K_0$ .

2.2. Chebyshev polynomials and the product-to-sum formula by Frohman and Gelca. In [\[FG\]](#page-17-0) Frohman and Gelca introduced a formula for the multiplication of two elements in the KBSA of the torus. Their formula is expressed in terms of Chebyshev polynomials of the first kind.

The Chebyshev polynomials of the first kind are defined by the recurrence relation

$$
T_n(x) = xT_{n-1}(x) - T_{n-2}(x)
$$

where  $T_0(x) = 2$ ,  $T_1(x) = x$  are the initial conditions.

Chebyshev polynomials of the second kind,  $S_n(x)$ , are defined by the same recurrence relation together with the initial conditions  $S_0(x) = 1$  and  $S_1(x) = x$ .

The following straighforward properties will be useful in the next sections:

<span id="page-5-3"></span>Proposition 2.9. [\[Lic,](#page-18-9) [MH\]](#page-18-10)

- (1) If we write  $x = a + a^{-1}$ , then  $T_n(x) = a^n + a^{-n}$  and  $S_n(x) = \frac{a^{n+1} a^{-n-1}}{a a^{-1}}$ .
- (2) The product-to-sum formula for Chebyshev polynomials is given by:

$$
T_m(x)T_n(x) = T_{m+n}(x) + T_{|m-n|}(x).
$$

(3) 
$$
T_n(x) = S_n(x) - S_{n-2}(x).
$$
  
(4) 
$$
S_n(x) = \left(\sum_{i=0}^{\lfloor (n-1)/2 \rfloor} T_{n-2i}\right) + \alpha(n), \text{ where } \alpha(n) = 0 \text{ if } n \text{ is odd and } 1 \text{ otherwise.}
$$

Given  $(d, n)$ , a multicurve in  $F_{1,0} \times I$  with  $r = gcd(d, n)$ , let

$$
(d,n)_T = T_r((d/r,n/r)) \quad \text{and} \quad (d,n)_S = S_r((d/r,n/r)).
$$

The above expressions are elements in  $S_{2,\infty}(F_{1,0} \times I)$  and denote formal linear combinations of muticurves on the torus where the variable of Chebyshev polynomials is substituted by the curves on the torus.

## Example 2.10.

(1)  $(8,2)_T = T_2((4,1)) = (4,1)^2 - 2 = (8,2) - 2$ , where 2 denotes  $2(0,0)$ . (2)  $(3, 6)_S = S_3((1, 2)) = (1, 2) * [(1, 2)^2 - 1] - (1, 2) = (3, 6) - 2(1, 2).$ 

<span id="page-5-2"></span>**Theorem 2.11.** [\[FG\]](#page-17-0) (Product-to-sum formula) Given two multicurves  $(d_1, n_1), (d_2, n_2) \in S_{2,\infty}(F_{1,0} \times I)$ ,

$$
(d_1, n_1)_T * (d_2, n_2)_T = A^{d_1 n_2 - n_1 d_2} (d_1 + d_2, n_1 + n_2)_T + A^{-(d_1 n_2 - n_1 d_2)} (d_1 - d_2, n_1 - n_2)_T.
$$

3. Product-to-sum formula for some families of curves in the thickened T-shirt

### <span id="page-5-0"></span>3.1. Basic formulas and Initial Data.

Given the product of two multicurves  $(d_1, n_1) * (d_2, n_2)$ , we associate to it the determinant  $(d_1n_2 - d_2n_1)$ . In this subsection we provide some basic computation results for the multiplication of two multicurves when the absolute value of their associated determinant is either 0, 1 or 2. These will serve as the initial data for the algorithm presented in Section [4.](#page-8-0)

As in the case of the torus, we denote by  $(d, n)_T$  the multicurve  $(d, n)$  decorated by the Chebyshev polynomial  $T_{gcd(d,n)}\left(\frac{d}{gcd(d,n)},\frac{n}{gcd(d,n)}\right)$ . Consider the product  $(d_1,n_1)*(d_2,n_2)$  with determinant  $d_1n_2-d_2n_1=0$ . We have  $(d_1, n_1) * (d_2, n_2) = (d_1 + d_2, n_1 + n_2)$ . From the product-to-sum formula for Chebyshev polynomials, Proposition [2.9](#page-5-3) (2), we have  $(d_1, n_1)_T$  \*  $(d_2, n_2)_T$  =  $(d_1 + d_2, n_1 + n_2)_T$  +  $(d_1 - d_2, n_1 - n_2)_T$ .

<span id="page-6-0"></span>**Lemma 3.1.** Consider two curves in  $F_{0,4}$ ,  $(d_1,n_1)=\frac{n_1}{d_1}$  and  $(d_2,n_2)=\frac{n_2}{d_2}$ , satisfying  $d_1n_2-d_2n_1=1$ . Then

$$
(d_1, n_1) * (d_2, n_2) = A^2(d_1 + d_2, n_1 + n_2) + A^{-2}(d_1 - d_2, n_1 - n_2) + R_{d_1 + d_2, n_1 + n_2},
$$

where the indices of  $R$  are taken modulo 2.

In particular we have:

- (i)  $(1,0)*(1,1) = A^2(2,1) + A^{-2}(0,1) + a_1a_2 + a_3a_4 = A^2(2,1) + A^{-2}(0,1) + R_{0,1}$
- (ii)  $(1, 0) * (0, 1) = A^2(1, 1) + A^{-2}(-1, 1) + a_1a_3 + a_2a_4 = A^2(1, 1) + A^{-2}(-1, 1) + R_{1,1}$
- (iii)  $(1, 1) * (0, 1) = A^2(1, 2) + A^{-2}(1, 0) + a_1a_4 + a_2a_3 = A^2(1, 2) + A^{-2}(1, 0) + R_{1,0}.$

Proof. It suffices to prove one of cases (i), (ii) or (iii). The lemma follows by applying the action of  $SL(2,\mathbb{Z})$ on the basic elements in the formulas (in Section [4](#page-8-0) we discuss the effects of this action; compare [\[FM\]](#page-17-2)). The calculation for case (ii) is presented in Figure [2.5.](#page-4-0)  $\Box$ 

<span id="page-6-1"></span>**Lemma 3.2.** Let  $(d_1, n_1), (d_2, n_2)$  be two curves in  $F_{0,4}$  so that  $gcd(d_1, n_1) = gcd(d_2, n_2) = 1$  and  $d_1n_2-d_2n_1 =$ 2. Then

$$
(d_1, n_1) * (d_2, n_2) = A^4(d_1 + d_2, n_1 + n_2) + A^{-4}(d_1 - d_2, n_1 - n_2) + y +
$$
\n
$$
((d_1 + d_1)/2(d_2 + n_1)/2) + A^{-2}B
$$
\n
$$
((d_1 + d_2)/2(d_2 + n_2)/2) + A^{-2}B
$$
\n
$$
((d_1 + d_2)/2(d_2 + n_1)/2) + y + z^2B
$$

$$
A^{2}R_{\frac{d_{1}+d_{2}}{2},\frac{n_{1}+n_{2}}{2}}((d_{1}+d_{2})/2,(n_{1}+n_{2})/2)+A^{-2}R_{\frac{d_{1}-d_{2}}{2},\frac{n_{1}-n_{2}}{2}}((d_{1}-d_{2})/2,(n_{1}-n_{2})/2).
$$

In particular, we have  $(1, -1) * (1, 1) = A^4 (2, 0)_T + A^{-4} (0, 2)_T + y + A^2 R_{1,0} (1, 0) + A^{-2} R_{0,1} (0, 1)$ .

*Proof.* Similarly, as in the previous lemma, it suffices to find the formula for  $(1, -1) * (1, 1)$ . Calculations for this case are shown in Figure [2.6.](#page-4-5)

<span id="page-6-2"></span>**Lemma 3.3.** Let  $(d_1, n_1), (d_2, n_2)$  be two multicurves in  $F_{0,4}$  so that  $d_1n_2 - d_2n_1 = 2$ .

(1) If 
$$
gcd(d_1, n_1) = 2
$$
, then  
\n
$$
(d_1, n_1)_T * (d_2, n_2) = A^4(d_1 + d_2, n_1 + n_2) + A^{-4}(d_1 - d_2, n_1 - n_2) +
$$
\n
$$
(d_1/2, n_1/2)R_{d_1/2 + d_2, n_1/2 + n_2} + (A^2 + A^{-2})R_{d_2, n_2}.
$$
\n(2) If  $gcd(d_2, n_2) = 2$ , then

$$
(d_1, n_1) * (d_2, n_2)_T = A^4(d_1 + d_2, n_1 + n_2) + A^{-4}(d_1 - d_2, n_1 - n_2) + R_{d_1 + d_2/2, n_1 + n_2/2}(d_2/2, n_2/2) + (A^2 + A^{-2})R_{d_1, n_1}.
$$

Proof. We prove case (1) by applying Lemma [3.1](#page-6-0) twice:

$$
(d_1, n_1)_T * (d_2, n_2) = {n_1 \choose d_1}_T * \frac{n_2}{d_2} = \frac{n_1/2}{d_1/2} * {n_1/2 \choose d_1/2} * \frac{n_2}{d_2} - 2\left(\frac{n_2}{d_2}\right) \frac{\text{Lemma 3.1}}{d_2}
$$

$$
\frac{n_1/2}{d_1/2} * \left(A^2 \frac{n_1/2 + n_2}{d_1/2 + d_2} + A^{-2} \frac{n_2 - n_1/2}{d_2 - d_1/2} + R_{d_1/2 + d_2, n_1/2 + n_2}\right) - 2\left(\frac{n_2}{d_2}\right) \frac{\text{Lemma 3.1}}{\text{Lemma 3.1}}
$$

$$
A^2 \left(A^2 \frac{n_1 + n_2}{d_1 + d_2} + A^{-2} \frac{n_2}{d_2} + R_{d_2, n_2}\right) +
$$

$$
A^{-2} \left(A^2 \frac{n_2}{d_2} + A^{-2} \frac{n_2 - n_1}{d_2 - d_1} + R_{d_2, n_2}\right) + \frac{n_1/2}{d_1/2} R_{d_1/2 + d_2, n_1/2 + n_2} - 2\left(\frac{n_2}{d_2}\right) =
$$

$$
A^4 \frac{n_1 + n_2}{d_1 + d_2} + A^{-4} \frac{n_2 - n_1}{d_2 - d_1} + \frac{n_1/2}{d_1/2} R_{d_1/2 + d_2, n_1/2 + n_2} + (A^2 + A^{-2}) R_{d_2, n_2}.
$$

The proof of case  $(2)$  is analogous.

**Remark 3.4.** After switching A by  $A^{-1}$  Lemmas [3.1](#page-6-0) holds when the determinant of the curves is equal to -1 and Lemmas [3.2](#page-6-1) and [3.3](#page-6-2) hold when the determinant of the curves is equal to  $-2$ .

3.2. Two closed formulas. In this subsection we present formulas for the product of two infinite families of curves and the curve  $(0, 1)$ .

<span id="page-7-2"></span>**Theorem 3.5.** Let  $m > 0$ . Then,

$$
(m,0)_T * (0,1) = A^{2m}(m,1) + A^{-2m}(m,-1) + (m-1,0)_S R_{1,1} + \sum_{i=1}^{m-1} (A^{2i} + A^{-2i})(m-1-i,0)_S R_{i+1,1}.
$$

*Proof.* For  $m = 1, 2$  the formula follows from Lemmas [3.1](#page-6-0) and [3.3.](#page-6-2) Now we assume that the formula holds up to  $m (m > 2)$  and proceed by induction on m by applying the recurrence relation of Chebyshev polynomials:

$$
(m+1,0)_T * (0,1) \xrightarrow{(2,2)} (0,1) * (m,0)_T * (0,1) - (m-1,0)_T * (0,1) \xrightarrow{(2,2)} (0,1)
$$
\n
$$
(1,0) * [A^{2m}(m,1) + A^{-2m}(m,-1)] - [A^{2m-2}(m-1,1) + A^{-2m+2}(m-1,-1)] + (1,0) * (m-1,0)_S R_{1,1} - (m-2,0)_S R_{1,1} + (1,0) * (m-1,-1)_S R_{1,1} - (m-2,0)_S R_{1,1} + (1,0) * \left[\sum_{i=1}^{m-1} (A^{2i} + A^{-2i})(m-1-i,0)_S R_{i+1,1}\right] - \sum_{i=1}^{m-2} (A^{2i} + A^{-2i})(m-2-i,0)_S R_{i+1,1} \xrightarrow{Chebyshev} B^{2m+2}(m+1,1) + A^{2m-2}(m-1,1) + A^{2m} R_{m+1,1} + A^{2m} R_{m+1,1} + A^{2-2m}(m-1,-1) + A^{-2m-2}(m+1,-1) + -A^{2m-2}(m-1,1) - A^{2-2m}(m-1,-1) + (m,0)_S R_{1,1} + \sum_{i=1}^{m-1} (A^{2i} + A^{-2i})(m-i,0)_S R_{i+1,1} = A^{2m+2}(m+1,1) + A^{-2m-2}(m+1,-1) + (m,0)_S R_{1,1} + \sum_{i=1}^{m} (A^{2i} + A^{-2i})(m-i,0)_S R_{i+1,1},
$$
\nas needed.<sup>4</sup>

Let  $[n]_q=1+q+q^2+...+q^{n-1}$  denote the  $n^{th}$  quantum integer. The following theorem, whose proof is given in the Appendix, gives the formula for the product of curves in the family  $(n, 1) * (0, 1)$ .

<span id="page-7-1"></span>**Theorem 3.6.** Let  $n$  be a non-negative integer. Then,

$$
(n,1)*(0,1) = A^{2n}(n,2)_T + A^{-2n}(n,0)_T + \sum_{i=0}^{n-1} \alpha_{n-1}\beta_i + A^2 \sum_{i=1}^{n-1} [n_i]_{A^4} \frac{1}{i} R_{i-1,1},
$$

where

$$
n_i = \begin{cases} [i]_{A^4} & \text{for } i \leq \frac{n}{2}, \\ [n-i]_{A^4} & \text{for } i \geq \frac{n}{2}, \end{cases}
$$

$$
\alpha_i = \left\{ \begin{array}{cc} R_{1,0} & i=1, \\ y & i=2, \\ \frac{(i-2)}{2}(R_{0,1}^2+R_{1,1}^2) & i>3 \text{ even}, \\ (i-2)R_{0,1}R_{1,1} & i>3 \text{ odd}, \end{array} \right. \quad \text{and} \quad \beta_i = \left\{ \begin{array}{c} \frac{(i-1)/2}{2}A^{-2i+8j}(i-2j,0)_S & i \text{ odd}, \\ \sum\limits_{j=0}^{i/2}A^{-2i+8j}(i-2j,0)_S & i \text{ even}. \\ \sum\limits_{j=0}^{i/2}A^{-2i+8j}(i-2j,0)_S & i \text{ even}. \end{array} \right.
$$

Notice that all the coefficients are positive and unimodal.

<span id="page-7-0"></span><sup>4</sup>Notice that we use the fact that the Chebyshev polynomials  $T_n$  and  $S_n$  have the same recursive relation.



FIGURE 4.1. The Farey diagram.

### <span id="page-8-1"></span>4. THE ALGORITHM

<span id="page-8-0"></span>In this section we describe how to compute the product of closed multicurves in  $F_{0,4} \times I$  algebraically.

Recall that the mapping class group of the torus is  $Mod(T^2) = SL(2, \mathbb{Z}) ([\text{Deh1}])$ , and that of  $F_{0,4}$  is the semidirect product:  $Mod(F_{0,4}) = PSL(2, \mathbb{Z}) \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$  (we allow permutations of the boundary components, compare [\[FM\]](#page-17-2)). The action of  $PSL(2, \mathbb{Z})$  on  $\mathbb{Q} \cup \frac{1}{0}$  is encoded by the Farey Diagram shown in Figure [4.1.](#page-8-1) Following is a short description (see [\[Hat\]](#page-17-9) for more detail).

The Farey diagram consists of vertices indexed by fractions  $\mathbb{Q}\cup\frac10,$  where vertices  $\frac{a}{b}$  and  $\frac{c}{d}$  are connected by an edge if the determinant  $(ad - bc)$  of the matrix  $\begin{bmatrix} a & c \ b & d \end{bmatrix}$  is  $\pm 1$ . The vertices of the Farey diagram are labeled according to the following scheme: if the labels at the ends of the long edge of a triangle are  $\frac{a}{b}$  and  $\frac{c}{d}$ , then the label on the third vertex of the triangle is the fraction  $\frac{a+b}{c+d}$ . This fraction is known as the *mediant* of  $\frac{a}{b}$  and  $\frac{c}{d}$ .

 $PSL(2, \mathbb{Z})$  acts on  $F_{0,4}$  as follows: the generators  $s_1=\left[\begin{array}{cc} 1 & 1\ 0 & 1 \end{array}\right]$  and  $s_2=\left[\begin{array}{cc} 1 & 0\ 1 & 1 \end{array}\right]$  act on the fractions  $\frac{n}{d}$ by matrix multiplication. The generators act on the center by switching  $a_3$  with  $a_4$  and  $a_1$  with  $a_4$  respectively. Therefore, under the action of  $s_1$ ,  $R_{1,1}$  goes to  $R_{1,0}$ ,  $R_{0,1}$  goes to  $R_{1,1}$  and  $R_{0,1}$  remains unchanged. The second generator,  $s_2$ , sends  $R_{1,1}$  to  $R_{0,1}$ ,  $R_{0,1}$  to  $R_{1,1}$  and keeps  $R_{1,0}$  fixed. The variable y remains unchanged by the action.

We now analyze the algorithm which, given an ordered pair of curves in  $F_{0,4} \times I$ , returns their product as an element of KBSA( $F_{0,4} \times I$ ) in terms of the generators in the basis. The first step in the algorithm consists of reducing the product  $(d_1, n_1) * (d_2, n_2)$  to the form  $(d, n) * (0, k)$ , with  $0 \le n < d$  and  $k > 0$ . This is achieved by using the action of  $PSL(2, \mathbb{Z})$  on  $F_{0,4}$  together with an adapted version of the Euclidean algorithm.

We now continue with the following lemmas in preparation for step 2 of the algorithm.

Lemma [4.1](#page-9-0) relates the determinant  $(d_1n_2-d_2n_1)$  of  $\left[\begin{array}{cc} d_1&n_1\ldots& d_n\end{array}\right]$  $d_2$   $n_2$ 1 with the geometric intersection number of

a pair of multicurves  $(d_1, n_1)$  and  $(d_2, n_2)$ . Lemma [4.2](#page-9-1) is used to show the correctness of the algorithm when  $k > 1$  or  $gcd(d, n) > 1$ . Lemmas [4.3](#page-9-2) and [4.4](#page-9-3) are used in Theorem [4.5](#page-10-0) to prove the correctness of the algorithm when  $k = 1$  and  $gcd(d, n) = 1$ .

<span id="page-9-0"></span>**Lemma 4.1.** Consider any two simple closed multicurves  $(c_1d_1, c_1n_1)$  and  $(c_2d_2, c_2n_2)$  with  $gcd(d_1, n_1) = 1$  $gcd(d_2, n_2)$  in  $F_{0,4} \times I$ , where  $c_i, d_i, n_i \in \mathbb{Z}$ , and  $i=1,2$ . Then, their geometric intersection number is equal to  $2|c_1d_1c_2n_2 - c_1n_1c_2d_2|$ . See section 2.2.5 in [\[FM\]](#page-17-2).

<span id="page-9-1"></span>**Lemma 4.2.** Consider  $(c_1d_1, c_1n_1) * (c_2d_2, c_2n_2)$  with  $gcd(d_1, n_1) = 1 = gcd(d_2, n_2)$ . Without loss of generality, assume  $c_1 > c_2 \ge 1$ . Suppose  $((c_1 - 1)d_1, (c_1 - 1)n_1) * (c_2d_2, c_2n_2) = \sum_i u_i(q_i, p_i)$ ,  $u_i \in Z(\mathcal{S}_{2,\infty}(F_{0,4}))$ , then we have  $|d_1p_i - n_1q_i| < |c_1d_1c_2n_2 - c_1n_1c_2d_2|$ .

*Proof.* The curves  $(q_i,p_i)$  are obtained after one smooths all the crossings of  $(c_1-1)$  curves  $(d_1,n_1)$  and  $c_2$  curves  $(d_2, n_2)$ . During this process, we observe that the number of crossings, say n, of the remaining copy of the curve  $(d_1,n_1)$  with  $(q_i,p_i)$  does not increase. Thus,  $n=2|d_1p_i-n_1q_i|\leq 2|d_1c_2n_2-n_1c_2d_2|\leq |c_1d_1c_2n_2-c_1n_1c_2d_2|<$  $2|c_1d_1c_2n_2 - c_1n_1c_2d_2|$ .

<span id="page-9-2"></span>**Lemma 4.3.** Consider a curve  $(d, n)$   $(d \ge 0)$  and let  $(d, n) * (0, 1) = \sum_i u_i(q_i, p_i)$ , with  $q_i \ge 0$ . Then,

- $(1)$   $|p_i| \leq n+1$  and  $q_i \leq d$ .
- (2)  $|p_i d q_i n| \leq d$ .
- (3) If  $n > 0$  then  $p_i \ge 0$ .

Proof.

- $(1)$  It follows from the observation that the coordinate lines cannot cut a  $\left(q_i,p_i\right)$  curve more than the number of times the curves  $(d, n)$  and  $(0, 1)$  together cut the  $(q_i, p_i)$  curve.
- (2) We notice that the curve  $(d,n)$  can intersect any curve of  $(d,n)*(0,1)$  (including  $(q_i,p_i))$  in no more than  $2d$  points. Thus, by Lemma [4.1,](#page-9-0) the statement follows.
- (3) This case follows from (2) since we have  $-d\leq p_id-q_in$  so  $\frac{n}{d}q_i-1\leq p_i,$  and therefore for  $n>0$  and  $q_i > 0$  we have  $p_i > -1$ . If  $q_i = 0$  then we can assume  $p_i \geq 0$  by convention.

 $\Box$ 

Lemma [4.3](#page-9-2) is generalized in Theorem [4.7.](#page-11-0)

#### <span id="page-9-3"></span>Lemma 4.4.

(1) Consider the fractions  $\frac{n_1}{d_1}, \frac{n_2}{d_2}$ , and  $\frac{p}{q}$  with positive denominators. Then

$$
|n_2q - d_2p| \le \frac{q}{d_1}|n_2d_1 - n_1d_2| + \frac{d_2}{d_1}|n_1q - d_1p|.
$$

(2) In particular, if  $|n_2d_1 - n_1d_2| = 1$ ,  $|n_1q - d_1p| \le d_1$  and  $q \le d_1$ , then

$$
|n_2q - d_2p| \le 1 + d_2.
$$

Proof. Inequality in (1) is essentially triangle inequality for fractions (rational numbers). We see this immediately when we divide two sides of the inequality in  $(1)$  by a positive number  $qd_2$ . We obtain the inequality:

$$
\frac{n_2}{d_2}-\frac{p}{q}|\leq |\frac{n_2}{d_2}-\frac{n_1}{d_1}|+|\frac{n_1}{d_1}-\frac{p}{q}|
$$

which is just a triangle inequality of rational numbers.  $\Box$ 



<span id="page-10-1"></span>FIGURE 4.2. Illustrations of Dehn coordinates.

<span id="page-10-0"></span>**Theorem 4.5.** Consider the product  $(d, n) * (0, 1)$  with  $1 < n < d \ge 3$  and  $gcd(d, n) = 1$ . Given products of pairs of curves with determinant less than d, we can compute  $(d, n) * (0, 1)$ .

*Proof.* Let  $n, d > 1$ ,  $gcd(n, d) = 1$ , and assume that the reduced fractions  $\frac{n_1}{d_1}$  and  $\frac{n_2}{d_2}$  "support"  $\frac{n}{d}$  in the Farey diagram, that is,  $n = n_1 + n_2$ ,  $d = d_1 + d_2$  and  $|n_2d_1 - n_1d_2| = 1$ .

With our assumptions,  $d_1, d_2$  and  $n_1, n_2$  satisfy  $0 < d_2 < d$ ,  $0 < d_1 < d-1$ ,  $0 < n_1, n_2$ . Then,  $(d, n) * (0, 1) =$  $A^{-2(d_1n_2-d_2n_1)}(d_1,n_1) * (d_2,n_2) * (0,1) - A^{-2(d_1n_2-d_2n_1)}R_{d,n}(0,1) - A^{-4(d_1z-d_2n_1)}(d_1-d_2,n_1-n_2) * (0,1).$ For  $(d_1 - d_2, n_1 - n_2) * (0, 1)$ , we know the product since the absolute value of determinant is  $|d_1 - d_2| < d$ . So let us focus on  $(d_1,n_1)*(d_2,n_2)*(0,1)$ . Assume  $(d_1,n_1)*(d_2,n_2)*(0,1)=(d_1,n_1)*\sum_i u_i(q_i,p_i)$  as in Lemma [4.3](#page-9-2) (1). By parts (1) and (2) of Lemma [4.3](#page-9-2) and part (2) of Lemma [4.4,](#page-9-3) we have  $|n_1p_i-d_1p_i|\leq 1+d_1 < d.$   $\Box$ 

We summarize the above results in the following theorem which proves the correctness of the algorithm.

**Theorem 4.6.** Consider the product  $(d, n) * (0, k)$  with  $0 \le n \le d > 0$ ,  $k \ge 0$ , one can compute the product algebraically with the knowledge of products of  $(d_1, n_1) * (d_2, n_2)$  with  $|d_1n_2 - d_2n_1| < dk$ .

Proof. If  $k = 0$ ,  $(d, n) * (0, 0) = (d, n)$ . If  $k > 1$ , by Lemma [4.2,](#page-9-1) the statement is true. Now, consider the case  $k = 1$ . We can assume that  $d = ms + r$ ,  $n = s$ , and we multiply  $(ms + r, s) * (0, 1)$  as before to get

$$
(ms + r, s) * (0, 1) = A^{2(ms + r)}(ms + r, s + 1) + A^{-2(ms + r)}(ms + r, s - 1) + \sum_{i} u_i(q_i, p_i).
$$

If  $n = 1$ , we have the closed formula in Theorem [3.6.](#page-7-1)

If  $n > 1$ , then

(1) when  $gcd(d, n) > 1$ , by Lemma [4.2,](#page-9-1) the statement is true.

(2) when  $gcd(d, n) = 1$  and  $d \geq 3$ , by Theorem [4.5,](#page-10-0) the statement is true.

The remaining finite cases for smaller determinants have been checked in the beginning of the previous section and in Lemmas [3.1,](#page-6-0) [3.2](#page-6-1) and [3.3.](#page-6-2)  $\square$ 

This completes the algorithm. The next theorem generalizes Lemma [4.3.](#page-9-2)

<span id="page-11-0"></span>**Theorem 4.7.** Consider the product  $(d, n) * (0, 1)$  with  $0 \le n \le d > 0$ . Then,

$$
(d,n) * (0,1) = A^{2d}(d,n+1) + A^{-2d}(d,n-1) + \sum_i u_i(q_i, p_i),
$$

with  $0 \le q_i \le d-1$  and  $0 \le p_i \le n$ .

*Proof.* Consider a curve (or multicurve) of slope type  $\frac{n}{d}$  with  $n, d > 0$ . In Dehn coordinates<sup>[5](#page-11-1)</sup>  $\frac{n}{d}$  has coordinates  $(2d, n)$ , that is, there are 4d points on Dehn annulus (that is, annulus along pants curve), say  $x_1, x_2, ..., x_{2d}$  on one boundary component and  $y_1,y_2,...,y_{2d}$  on the other. Consider now the diagram  $\frac{n}{d}\sqcup\frac{1}{0}$  (that is the product before skein relations). We can take as  $\frac{1}{0}$  curve the boundary of pants annulus on which there are points  $x_i$ . Every generic horizontal (slope  $\frac{0}{1})$ ) curve, say  $h$ , intersects  $\frac{n}{d}\sqcup\frac{1}{0}$  in  $2n+2$  points. We can always deform  $h$  to a curve, say  $h'$ , cutting the crossing, say  $x_i$ , and otherwise generic, and cutting  $\frac{n}{d}\sqcup\frac{1}{0}$  in  $2n$  additional points (there are always two choices, if  $x_i$  was the first then  $x_{2d+1-i}$  is another). If we smooth this crossing  $x_i$  in  $B-type\,$ than the diagram obtained by this smoothing is cut by  $h'$  in  $2n$  points. Furthermore, for any crossing  $x_i$  we can find its  $h$  type curve. Therefore, the first part of the lemma follows for any Kauffman state of  $\frac{n}{d}*\frac{1}{0}$  with at least one  $B$  smoothing. We find that its deformed curve  $h'$  cuts any Kauffman state of  $\frac{n}{d}*\frac{1}{0}$  at most  $2n$  times. Thus, any curves  $(q_i,p_i)=\frac{p_i}{q_i}$  satisfy  $p_i\leq n$  (in fact  $|p_i|\leq n).$  See Figures [4.3](#page-12-0) and [4.4.](#page-12-1)

Similar proof works for denominator: if a Kauffman state has at least one  $A$  smoothing and one  $B$  smoothing then always we can find them to be neighbors (along a circle) and draw  $\frac10$  curve which cuts a diagram of  $\frac n d \sqcup \frac10$ smoothed at those A and B crossings in no more than  $2(d-1)$  points. Compare Figures [4.4](#page-12-1) and [4.5.](#page-12-2)

This can be written more generally by first observing that  $A$  and  $B$  crossings do not have to be neighbors. Thus, we can pair A and B smoothings as far as some are left. Thus, consider the Kauffman state with  $n_A$  A-smoothings and  $n_B = 2d - n_A$  B-smoothings. Let  $n_c$  be the minimum of these numbers. Then the corresponding  $q_i$  satisfy  $q_i \leq d-n_c.$ 

<span id="page-11-1"></span> $^{5}$ In this proof we use notation from the book [\[PH\]](#page-18-8); compare Figure [4.2.](#page-10-1)



FIGURE 4.3. Passing via B-crossing.

<span id="page-12-0"></span>

<span id="page-12-1"></span>FIGURE 4.4. Passing via neighboring A and B crossings.



<span id="page-12-2"></span>FIGURE 4.5. Examples of A-B passing.

**Corollary 4.8.** Consider the product of (not necessarily reduced) positive curves  $\frac{n_1}{d_1} * \frac{n_2}{d_2}$  with  $d_1n_2 - d_2n_1 > 0$ , then in the skein module it can be written as

$$
(d_1, n_1) * (d_2, n_2) = A^{2(d_1n_2 - d_2n_1)}(d_1 + d_2, n_1 + n_2) + A^{-2(d_1n_2 - d_2n_1)}(d_1 - d_2, n_1 - n_2) + \sum_i u_i(q_i, p_i),
$$

where  $0 \le p_i < n_1 + n_2$  and  $0 \le q_i < d_1 + d_2$ .

*Proof.* Assume that the positive curves  $(d_1, n_1), (d_2, n_2)$  are obtained from the pair  $(d, n) * (0, 1)$  by the action of a nonnegative matrix of  $SL(2, Z)$  (upper half of the Farey diagram). In other words, there exist  $n_1$  and  $d_1$  so that

$$
\begin{bmatrix} n_2 & a_{1,2} \\ d_2 & a_{2,2} \end{bmatrix} \begin{bmatrix} n & 1 \\ d & 0 \end{bmatrix} = \begin{bmatrix} n_1 & n_2 \\ d_1 & d_2 \end{bmatrix}
$$

The condition (coming from Theorem [4.7](#page-11-0)<sup>[6](#page-13-0)</sup>) is now:  $0 \leq p_i \leq n_1 - a_{1,2}$ ,  $0 \leq q_i \leq d_1 - a_{2,2}$ .

The following result can be thought as a stronger version of Lemma [4.4](#page-9-3) (2).

 $\lceil$ 

**Corollary 4.9.** Consider the curves  $(d_1, n_1)$  and  $(d_2, n_2)$   $(d_2 \ge 0)$  and let  $(d_2, n_2) * (0, 1) = A^{2d_2}(d_2, n_2 + 1) +$  $A^{-2d_2}(d_2,n_2-1)\sum_i w_i(q_i,p_i).$  If  $|n_2d_1-n_1d_2|=1$ , then

$$
|n_1q - d_1p| \le d_1.
$$

*Proof.* By Theorem [4.7,](#page-11-0) we have  $0 \le q_i \le d_2 - 1$ . From Lemma [4.3](#page-9-2) (2) and Lemma [4.4](#page-9-3) (1), we have

$$
|n_1q_i - d_1p_i| \le \frac{q}{d_2}|n_2d_1 - n_1d_2| + \frac{d_1}{d_2}|n_2q - d_2p| < 1 + d_1 \le d_1.
$$

.

4.1. Motivation for Future Work. In the future we plan to compare our result with the study of positive basis for surface skein algebras (compare with [\[Thu\]](#page-18-11)). The third author heard about the positivity problem at the Aspen Physics Conference in March 2018 from Paul Wedrich. He heard about it from Dylan Thurston, who in turn had been motivated by a question of Edward Witten. We ask a similar question for the KBSA of  $F_{0.4} \times I$ .

### <span id="page-13-1"></span>Conjecture 4.10.

- $(1)$  If  $(d_2,n_2)_T*(d_1,n_1)_T=\sum_i w_i(q_i,p_i)_T,$  where  $w_i\in K_0,$  then  $w_i$  is a nonnegative polynomial in variables  $A, R_{0,1}, R_{1,1}, R_{1,0}, y.$
- (2)  $If (d_2, n_2)_T * (d_1, n_1)_T = A^{n_1d_2 n_2d_1} (d_1 + d_2, n_1 + n_2)_T + A^{-n_1d_2 + n_2d_1} (d_1 d_2, n_1 n_2)_T + \sum_i v_i (q_i, p_i)_S$ with  $v_i \in K_0$ , then  $v_i$  are nonnegative polynomials in variables  $A, R_{0,1}, R_{1,1}, R_{1,0}, y$ .

Notice that by Proposition [2.9](#page-5-3) (4), Conjecture [4.10,](#page-13-1) (2) is stronger than (1). All our computations and the formulas in Theorem [3.5](#page-7-2) and [3.6](#page-7-1) support the former conjecture.

### Question 4.11.

- (1) What is the complexity of multiplying fractions in the KBSA of  $F_{0,4} \times I$ ?
- (2) Is the complexity of the former algorithm quasi-polynomial in terms of the determinant of the curves being multiplied (that is, of complexity  $det^{log(det)}$  where det is the determinant of a pair of fractions)?

Appendix: Proof of Theorem [3.6](#page-7-1)

Before we prove Theorem [3.6,](#page-7-1) we provide one instructive example  $(7, 1) * (0, 1)$ .

$$
(7,1) * (0,1) = A^{14}(7,2)_T + A^{-14}(7,0)_T +
$$
  
\n
$$
y(A^{-10}(5,0)_S + A^{-2}(3,0)_S + A^6(1,0)) +
$$
  
\n
$$
A^2[R_{1,1}(6,1) + [2]_{A^4}R_{0,1}(5,1) + [3]_{A^4}R_{1,1}(4,1) + [3]_{A^4}R_{0,1}(3,1) + [2]_{A^4}R_{1,1}(2,1) + R_{0,1}(1,1)] +
$$

<span id="page-13-0"></span> $^6$ Let  $(d,n)*(0,1)=A^{2d}(d,n+1)+A^{-2d}(d,n-1)+\sum_i u_i(q_i',p_i').$  According to Theorem [4.7,](#page-11-0)  $0\leq p_i'\leq n$  and  $0\leq q_i'\leq d-1.$ Therefore,  $p_i = n_2 p'_i + a_{1,2} q'_i \le n_2 n + a_{1,2}(d-1) = n_1 - a_{1,2}$  and  $q_i = d_2 p'_i + a_{2,2} q'_i \le d_2 n + a_{2,2}(d-1) = d_1 - a_{2,2}$ .

$$
R_{1,0}[A^{12} + A^4(2,0)_S + A^{-4}(4,0)_S + A^{-12}(6,0)_S] +
$$
  
\n
$$
(R_{1,0} + R_{1,1}R_{0,1})[A^8 + (2,0)_S + A^{-8}(4,0)_S] +
$$
  
\n
$$
(R_{1,1}^2 + R_{0,1}^2)[A^2(1,0) + A^{-6}(3,0)_S + 2A^{-2}(1,0)] +
$$
  
\n
$$
+5R_{0,1}R_{1,1} + 3R_{0,1}R_{1,1}[A^{-4}(2,0)_S + A^4].
$$

**Theorem [3.6.](#page-7-1)** Let  $n$  be a non-negative integer. Then,

$$
(n,1)*(0,1) = A^{2n}(n,2)_T + A^{-2n}(n,0)_T + \sum_{i=0}^{n-1} \alpha_{n-1}\beta_i + A^2 \sum_{i=1}^{n-1} [n_i]_{A^4} \frac{1}{i} R_{i-1,1},
$$

where

$$
n_i = \begin{cases} [i]_{A^4} & \text{for } i \leq \frac{n}{2}, \\ [n-i]_{A^4} & \text{for } i \geq \frac{n}{2}, \end{cases}
$$

$$
\alpha_i = \left\{ \begin{array}{cc} R_{1,0} & i=1, \\ y & i=2, \\ R_{0,1}R_{1,1}+R_{1,0} & i=3, \\ \frac{(i-2)}{2}(R_{0,1}^2+R_{1,1}^2) & i>3 \text{ even}, \\ (i-2)R_{0,1}R_{1,1} & i>3 \text{ odd}, \end{array} \right. \quad \text{and} \quad \beta_i = \left\{ \begin{array}{c} \frac{(i-1)/2}{2}A^{-2i+8j}(i-2j,0)_S & i \text{ odd}, \\ \sum\limits_{j=0}^{i/2}A^{-2i+8j}(i-2j,0)_S & i \text{ even}. \\ \sum\limits_{j=0}^{i-1}A^{-2i+8j}(i-2j,0)_S & i \text{ even}. \end{array} \right.
$$

Notice that all the coefficients are positive and unimodal.

Proof. Clearly the formula is true for  $n = 0, 1, 2$  using initial data. Assume the formula is true for  $n < N$ . We prove the case  $n = N$  by induction. First, consider the case when N is even. Then,  $(n, 1) * (0, 1) =$  $A^{-2}(1,0)*(n-1,1)*(0,1)-A^{-2}R_{n,1}(0,1)-A^{-4}(n-2,1)*(0,1).$  Now, we consider each term separately.

The simpler one is the second product:

$$
(n-2,1)*(0,1) = A^{2(n-2)}(n-2,2)_T + A^{-2(n-2)}(n-2,0)_T + \sum_{i=1}^{(n-4)/2} A^2[i]_{A^4} R_{i,1}(i,1) + A^2[(n-2)/2]_{A^4} R_{(n-2)/2,1}((n-2)/2,1) + \sum_{i=n/2}^{n-3} A^2[n-2-i]_{A^4} R_{i,1}(i,1) + \sum_{i=0}^{n-3} \alpha_{n-2-i} \beta_i.
$$

Now we continue with the computation of the first product of the triple. By the induction hypothesis, we know

the product of 
$$
(n-1,1) * (0,1)
$$
. Thus,  $(1,0) * (n-1,1) * (0,1) = A^{2(n-1)}(1,0) * (n-1,2)_T + A^{-2(n-1)}(1,0) * (n-1,0)_T + \sum_{i=1}^{(n-2)/2} A^2[i]_{A^4} R_{i-1,1}(1,0) * (i,1) + \sum_{i=(n-2)/2+1}^{n-2} A^2[n-1-i]_{A^4} R_{i-1,1}(1,0) * (i,1) + \sum_{i=0}^{(n-2)} A^{2n-1-i}(1,0) * \beta_i = A^{2(n-1)}[A^4(n,2)_T + A^2R_{n/2,1}(n/2,1) + y + A^{-2}R_{(n-2)/2,1}((n-2)/2,1) + A^{-4}(n-2,2)_T] + A^{-2(n-1)}(1,0) * (n-1,0)_T + \sum_{i=1}^{(n-2)/2} A^2[i]_{A^4} R_{i-1,1}(1,0) * (i,1) + \sum_{i=(n-2)/2+1}^{n-2} A^2[n-1-i]_{A^4} R_{i-1,1}(1,0) * (i,1) + \sum_{i=0}^{n-2} \alpha_{n-2-i}(1,0) * \beta_i.$ 

Combining the two parts, we have

$$
(n,1) * (0,1) = A^{-2}(1,0) * (n-1,1) * (0,1) - A^{-2}R_{n,1}(0,1) - A^{-4}(n-2,1) * (0,1) = A^{2n}(n,2)_T + A^{2n-2}R_{n/2,1}(n/2,1) + A^{2n-4}y + A^{2n-6}R_{(n-2)/2,1}((n-2)/2,1) + A^{2n-8}(n-2,2)_T + A^{-2n}(1,0) * (n-1,0)_T + A^{2n-1}(n-2,1)_T
$$

$$
\sum_{i=1}^{(n-2)/2} [i]_{A^4} R_{i-1,1}[A^2(i+1,1) + R_{i+1,1} + A^{-2}(i-1,1)] +
$$
  
\n
$$
\sum_{i=0}^{(n-2)/2+1} [n-1-i]_{A^4} R_{i-1,1}[A^2(i+1,1) + R_{i+1,1} + A^{-2}(i-1,1)] +
$$
  
\n
$$
\sum_{i=0}^{n-2} A^{-2} \alpha_{n-1-i}(1,0) * \beta_i - A^{2(n-4)}(n-2,2)_T - A^{-2n}(n-2,0)_T - \sum_{i=1}^{(n-4)/2} A^{-2}[i]_{A^4} R_{i,1}(i,1) -
$$
  
\n
$$
A^{-2}[(n-2)/2]_{A^4} R_{(n-2)/2,1}((n-2)/2,1) - \sum_{i=n/2}^{n-3} A^{-2}[n-2-i]_{A^4} R_{i,1}(i,1) - \sum_{i=0}^{(n-3)/2} A^{-4} \alpha_{n-2-i} \beta_i - A^{-2} R_{n,1}(0,1)
$$
  
\n
$$
= A^{2n}(n,2)_T + A^{-2n}((1,0) * (n-1,0)_T - (n-2,0)_T) + A^{2n-2} R_{n/2,1}(n/2,1) + A^{2n-4} y +
$$
  
\n
$$
A^{2n-6} R_{(n-2)/2,1}((n-2)/2,1) + A^{2n-8}(n-2,2)_T + [\sum_{i=1}^{(n-4)/2} A^2[i]_{A^4} R_{i+1,1}(i+1,1) +
$$
  
\n
$$
A^2[(n-2)/2]_{A^4} R_{n/2,1}(n/2,1)] + \sum_{i=1}^{(n-2)/2} [i]_{A^4} R_{i-1,1} R_{i+1,1} + [\sum_{k=1}^{(n-4)/2} [k+1]_{A^4} R_{k,1} A^{-2}(k,1) +
$$
  
\n
$$
A^{-2} R_{0,1}(0,1)] + \sum_{k=(n+2)/2}^{n-1} [n-k]_{A^4} R_{k,1} A^2(k,1) + \sum_{i=n/2}^{(n-2
$$

By the recursive relation of Chebyshev polynomial, we have  $(n, 1) * (0, 1) =$ 

$$
A^{2n}(n,2)_T + A^{-2n}(n,0)_T + A^{2n-2}R_{n/2,1}(n/2,1) + A^{2n-4}y + A^{2n-6}R_{(n-2)/2,1}((n-2)/2,1) + A^{2n-8}(n-2,2)_T + \frac{(n-4)/2}{i-1}A^2[i]_{A^4}R_{i+1,1}(i+1,1) + A^2[(n-2)/2]_{A^4}R_{n/2,1}(n/2,1)] + \sum_{i=1}^{(n-2)/2} [i]_{A^4}R_{i-1,1}R_{i+1,1} + \frac{(n-4)/2}{i-1}[k+1]_{A^4}R_{k,1}A^{-2}(k,1) + A^{-2}R_{0,1}(0,1)] + \sum_{k=(n+2)/2}^{n-1} [n-k]_{A^4}R_{k,1}A^2(k,1) + \sum_{i=n/2}^{n-2} [n-1-i]_{A^4}R_{i-1,1}R_{i+1,1} + \sum_{k=(n-2)/2}^{n-3}A^{-2}[n-2-k]_{A^4}R_{k,1}(k,1) + \sum_{i=0}^{n-2}A^{-2}\alpha_{n-1-i}(1,0) * \beta_i - A^{2n-8}(n-2,2)_T - \sum_{i=1}^{(n-4)/2}A^{-2}[i]_{A^4}R_{i,1}(i,1) - A^{-2}[(n-2)/2]_{A^4}R_{(n-2)/2,1}((n-2)/2,1) - \sum_{i=n/2}^{n-3}A^{-2}[n-2-i]_{A^4}R_{i,1}(i,1) - \sum_{i=0}^{n-3}A^{-4}\alpha_{n-2-i}\beta_i - A^{-2}R_{n,1}(0,1).
$$

The blue terms cancel in pairs and we have  $(n, 1) * (0, 1) =$ 

$$
A^{2n}(n,2)_T + A^{-2n}(n,0)_T + A^2[A^{2n-4} + [(n-2)/2]_{A^4}]R_{n/2,1}(n/2,1) + A^{2n-4}y +
$$
  
\n
$$
A^{2n-6}R_{(n-2)/2,1}((n-2)/2,1) + \left[\sum_{k=2}^{(n-4)/2} A^2[k-1]_{A^4}R_{k,1}(k,1) + A^2[n/2-2]_{A^4}R_{(n-2)/2,1}((n-2)/2,1)\right] +
$$
  
\n
$$
\sum_{i=1}^{(n-2)/2} [i]_{A^4}R_{i-1,1}R_{i+1,1} + \sum_{i=n/2}^{n-2} [n-1-i]_{A^4}R_{i-1,1}R_{i+1,1} + \left[\sum_{k=1}^{(n-4)/2} A^{-2}[k+1]_{A^4}R_{k,1}(k,1) -
$$
  
\n
$$
\sum_{i=1}^{(n-4)/2} A^{-2}[i]_{A^4}R_{i,1}(i,1)] + \sum_{k=(n+2)/2}^{n-1} A^2[n-k]_{A^4}R_{k,1}(k,1) + \left[\sum_{k=(n-2)/2}^{n-3} A^{-2}[n-2-k]_{A^4}R_{k,1}(k,1) -
$$
  
\n
$$
\sum_{i=n/2}^{n-3} A^{-2}[n-2-i]_{A^4}R_{i,1}(i,1) - A^{-2}[n-2/2]_{A^4}R_{(n-2)/2,1}((n-2)/2,1)] +
$$

$$
\sum_{i=0}^{n-2} A^{-2} \alpha_{n-1-i}(1,0) * \beta_i - \sum_{i=0}^{n-3} A^{-4} \alpha_{n-2-i} \beta_i = A^{2n}(n,2)_T + A^{-2n}(n,0)_T + A^2 [n/2]_{A^4} R_{n/2,1}(n/2,1) + A^{2n-4} y + A^{2n-6} R_{(n-2)/2,1}((n-2)/2,1) + \left[ \sum_{k=2}^{(n-4)/2} A^2 [k-1]_{A^4} R_{k,1}(k,1) + A^2 [n/2-2]_{A^4} R_{(n-2)/2,1}((n-2)/2,1) \right] + A^{2n-4} y + A^{2n-6} R_{(n-2)/2,1}((n-2)/2,1) + \left[ \sum_{k=2}^{(n-4)/2} A^2 [k-1]_{A^4} R_{k,1}(k,1) + A^2 [n/2-2]_{A^4} R_{(n-2)/2,1}((n-2)/2,1) \right] + A^{2n-2} \sum_{i=1}^{(n-4)/2} A^2 [n]_{A^4} R_{i-1,1} R_{i+1,1} + \sum_{k=1}^{(n-4)/2} A^2 A^{4k-4} R_{k,1}(k,1) + A^{2n-1} \sum_{i=0}^{(n-4)/2} A^{-2} [n-1]_{A^4} R_{k,1}(k,1) + A^{2n-2} \alpha_{n-1-i}(1,0) * \beta_i - \sum_{i=0}^{n-3} A^{-4} \alpha_{n-2-i} \beta_i = A^{2n}(n,2)_T + A^{-2n}(n,0)_T + A^2 [n/2]_{A^4} R_{n/2,1}(n/2,1) + A^{2n-4} y + [A^2 R_{1,1}(1,1) + \sum_{k=2}^{(n-4)/2} A^2 A^{4k-4} R_{k,1}(k,1) + A^2 [n/2 - 2]_{A^4} R_{(n-2)/2,1}((n-2)/2,1) + A^{2n-6} R_{(n-2)/2,1}((n-2)/2,1)] + A^{2n-6} R_{(n-2)/2,1}((n-2)/2,1)] + A^{2n-6} R_{(n-2)/2,1}((n-2)/2,
$$

Thus, we have  $(n, 1) * (0, 1) =$ 

$$
A^{2n}(n,2)_T + A^{-2n}(n,0)_T + A^2[n/2]_{A^4}R_{n/2,1}(n/2,1) + A^{2n-4}y + [A^2R_{1,1}(1,1) + \sum_{k=2}^{(n-4)/2} A^2A^{4k-4}R_{k,1}(k,1) + \sum_{k=2}^{(n-4)/2} A^2[k-1]_{A^4}R_{k,1}(k,1) + A^2[n/2-2]_{A^4}R_{(n-2)/2,1}((n-2)/2,1) + A^{2n-6}R_{(n-2)/2,1}((n-2)/2,1)] + \sum_{i=1}^{(n-2)/2} [i]_{A^4}R_{i-1,1}R_{i+1,1} + \sum_{i=n/2}^{n-2} [n-1-i]_{A^4}R_{i-1,1}R_{i+1,1} + \sum_{k=(n+2)/2}^{n-1} A^2[n-k]_{A^4}R_{k,1}(k,1) + \sum_{i=0}^{n-2} A^{-2}\alpha_{n-1-i}(1,0) * \beta_i - \sum_{i=0}^{n-3} A^{-4}\alpha_{n-2-i}\beta_i = \sum_{i=0}^{(n-2)/2} A^{2n}(n,2)_T + A^{-2n}(n,0)_T + \sum_{i=1}^{(n-2)/2} A^2[i]_{A^4}R_{i,1}(i,1) + A^2[n/2]_{A^4}R_{n/2,1}(n/2,1) + \sum_{i=(n+2)/2}^{(n-1)} A^2[n-i]_{A^4}R_{i,1}(i,1) + A^{-2}[n/2]_{A^4}R_{n/2,1}(n/2,1) + \sum_{i=(n+2)/2}^{(n-1)} A^2[n-i]_{A^4}R_{i,1}(i,1) + A^{-2}[n/2]_{A^4}R_{n/2,1}(n/2,1) + A^{-2}[n/2]_{A^4}R_{i,1}(i,1) + A^{-2}[n/2]_{A^4}R_{n/2,1}(n/2,1) + A^{-2}[n/2]_{A^4}R_{i,1}(i,1) + A^{-2}[n/2]_{A^4}R_{n/2,1}(n/2,1) + A^{-2}[n/2]_{A^4}R_{i,1}(i,1) + A^{-2}[n/2]_{A
$$

$$
A^{2n-4}y + \sum_{i=1}^{(n-2)/2} [i]_{A^4} R_{i-1,1} R_{i+1,1} + \sum_{i=n/2}^{n-2} [n-1-i]_{A^4} R_{i-1,1} R_{i+1,1} + \sum_{i=0}^{n-2} A^{-2} \alpha_{n-1-i} (1,0) * \beta_i - \sum_{i=0}^{n-3} A^{-4} \alpha_{n-2-i} \beta_i.
$$
  
\nNow, let us consider 
$$
\sum_{i=0}^{n-2} A^{-2} \alpha_{n-1-i} (1,0) * \beta_i - \sum_{i=0}^{n-3} A^{-4} \alpha_{n-2-i} \beta_i =
$$
  
\n
$$
A^{-2} \alpha_{n-1} (1,0) + \sum_{i=1}^{n-2} \alpha_{n-1-i} [A^{-2} (1,0) * \beta_i - A^{-4} \beta_{i-1}].
$$
 Now,

$$
A^{-2}(1,0) * \beta_i - A^{-4}\beta_{i-1} = \begin{cases} \beta_{i+1} & i \text{ even,} \\ \beta_{i+1} - A^{2i+2} & i \text{ odd.} \end{cases}
$$

So, 
$$
\sum_{i=0}^{n-2} A^{-2} \alpha_{n-1-i}(1,0) * \beta_i - \sum_{i=0}^{n-3} A^{-4} \alpha_{n-2-i} \beta_i = A^{-2} \alpha_{n-1}(1,0) + \sum_{i=1}^{n-2} \alpha_{n-1-i} \beta_{i+1} - \sum_{k=1}^{(n-2)/2} \alpha_{n-1-(2i-1)} A^{2(2i-1)+2} =
$$
  
\n
$$
A^{-2} \alpha_{n-1}(1,0) + \sum_{i=2}^{n-1} \alpha_{n-i} \beta_i - \sum_{i=1}^{(n-2)/2} \alpha_{n-1-(2i-1)} A^{2(2i-1)+2} = A^{-2} \alpha_{n-1}(1,0) + \sum_{i=2}^{n-1} \alpha_{n-i} \beta_i - \sum_{i=1}^{(n-2)/2} \alpha_{n-2i} A^{4i} =
$$
  
\n
$$
\sum_{i=1}^{n-1} \alpha_{n-i} \beta_i - \sum_{i=1}^{(n-2)/2} \alpha_{n-2i} A^{4i}.
$$

Thus,

$$
A^{2n}(n,2) + A^{-2n}(n,0) + \sum_{i=1}^{(n-2)/2} A^{2}[i]_{A^{4}}R_{i,1}(i,1) + A^{2}[n/2]_{A^{4}}R_{n/2,1}(n/2,1) + \sum_{i=(n+2)/2}^{n-1} A^{2}[n-i]_{A^{4}}R_{i,1}(i,1) + A^{2n-4}y + \sum_{i=1}^{(n-2)/2} [i]_{A^{4}}R_{i-1,1}R_{i+1,1} + \sum_{i=1}^{n-2} [n-1-i]_{A^{4}}R_{i-1,1}R_{i+1,1} + \sum_{i=0}^{n-2} A^{-2}\alpha_{n-1-i}(1,0) * \beta_{i} - \sum_{i=0}^{n-3} A^{-4}\alpha_{n-2-i}\beta_{i} =
$$
  

$$
A^{2n}(n,2) + A^{-2n}(n,0) + \sum_{i=1}^{(n-2)/2} A^{2}[i]_{A^{4}}R_{i,1}(i,1) + A^{2}[n/2]_{A^{4}}R_{n/2,1}(n/2,1) + \sum_{i=(n+2)/2}^{n-1} A^{2}[n-i]_{A^{4}}R_{i,1}(i,1) + \sum_{i=1}^{(n-2)/2} A^{2}[n-i]_{A^{4}}R_{i,1}(i,1) + \sum_{i=1}^{(n-2)/2} A^{2}[n-i]_{A^{4}}R_{i,1}(i,1) + \sum_{i=1}^{(n-2)/2} \alpha_{n-2i}A^{4i} =
$$
  

$$
A^{2n}(n,2) + A^{-2n}(n,0) + \sum_{i=1}^{(n-2)/2} [i]_{A^{4}}R_{i-1,1}R_{i+1,1} + \sum_{i=n/2}^{(n-2)} [n-1-i]_{A^{4}}R_{i-1,1}R_{i+1,1} - \sum_{i=1}^{(n-2)/2} \alpha_{n-2i}A^{4i} =
$$
  

$$
A^{2n}(n,2) + A^{-2n}(n,0) + \sum_{i=1}^{(n-2)/2} A^{2}[i]_{A^{4}}R_{i,1}(i,1) + A^{2}[n/2]_{A^{4}}R_{n/2,1}(n/2,1) + \sum_{i=(n+2)/
$$

The case when  $n = N$  is odd can be proved in a similar way.

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#### **REFERENCES**

- <span id="page-17-6"></span>[Bul] D. Bullock, A finite set of generators for the Kauffman bracket skein algebra. Math. Z., 231, 1999, no. 1, 91-101.
- <span id="page-17-5"></span>[BP] D. Bullock, J. H. Przytycki, Multiplicative structure of Kauffman bracket skein module quantizations. Proc. Amer. Math. Soc., 128, 2000, no. 3, 923-931. eprint: [arXiv:math/9902117](http://arxiv.org/abs/math/9902117) [math.QA]
- <span id="page-17-8"></span>[Deh1] M. Dehn, Lecture notes from Breslau, The Archives of The University of Texas at Austin, 1922.
- <span id="page-17-1"></span>[Deh2] M. Dehn, Papers on group theory and topology. Translated from the German and with introductions and an appendix by John Stillwell. Springer-Verlag, New York, 1987. viii+396 pp. ISBN: 0-387-96416-9.
- [DLP] M. K. Dabkowski, C. Li, J. H. Przytycki, Catalan states of lattice crossing. Topology Appl., 182, 2015, 1-15. e-print: [arXiv:1409.4065](http://arxiv.org/abs/1409.4065) [math.GT]
- [DP] M. K. Dabkowski, J. H. Przytycki, Catalan states of lattice crossing: Application of Plucking polynomial. e-print: [arXiv:1711.05328](http://arxiv.org/abs/1711.05328) [math.GT]
- <span id="page-17-0"></span>[FG] C. Frohman, R. Gelca, Skein modules and the noncommutative torus, Trans. Amer. Math. Soc., 352, 2000, no. 10, 4877-4888. eprint: [arXiv:math/9806107](http://arxiv.org/abs/math/9806107) [math.QA]
- <span id="page-17-7"></span>[FKL] C. Frohman, J. Kania-Bartoszynska, T. Le, Unicity for Representations of the Kauffman bracket Skein Algebra. eprint: [arXiv:1707.09234](http://arxiv.org/abs/1707.09234) [math.GT]
- <span id="page-17-2"></span>[FM] B. Farb, D. Margalit, A primer on mapping class groups, Princeton University Press, Princeton, NJ, 2012. xiv+472 pp. ISBN: 978-0-691-14794-9.
- <span id="page-17-3"></span>[Gol] S. W. Golomb, Classroom Notes: An Algebraic Algorithm for the Representation Problems of the Ahmes Papyrus. Amer. Math. Monthly, 69, 1962, no. 8, 785-786.
- <span id="page-17-9"></span>[Hat] A. Hatcher, Topology of Numbers. 2018, https://www.math.cornell.edu/ hatcher/TN/TNpage.html
- <span id="page-17-4"></span>[HP] J. Hoste, J. H. Przytycki, The  $(2,\infty)$ -skein module of lens spaces; a generalization of the Jones polynomial. Journal of Knot Theory and Its Ramifications, 2, 1993, no. 3, 321-333.
- [Kau] L. H. Kauffman, State models and the Jones polynomial, Topology, 26, 1987, no. 3, 395-407.

- <span id="page-18-2"></span>[Le] T. T. Q. Lê, The colored Jones polynomial and the A-polynomial of knots. Adv. Math., 207, 2006, no. 2, 782-804.
- <span id="page-18-9"></span>[Lic] W. B. R. Lickorish, An introduction to knot theory. Graduate Texts in Mathematics, 175. Springer-Verlag, New York, 1997. x+201 pp. ISBN: 0-387-98254-X.
- <span id="page-18-3"></span>[LT] T. T. Q. Lê, A. T. Tran, The Kauffman bracket skein module of two-bridge links. Proc. Amer. Math. Soc., 142, 2014, no. 3, 1045-1056.
- <span id="page-18-4"></span>[Mar] J. Marché, The skein module of torus knots. Quantum Topol., 1, 2010, no. 4, 413-421.
- <span id="page-18-10"></span>[MH] J. C. Mason, D. C. Handscomb, Chebyshev Polynomials. Chapman & Hall / CRC, Boca Raton, FL, 2003, xiv+341 pp. ISBN: 0-8493-0355-9.
- <span id="page-18-8"></span>[PH] R. C. Penner, J. L. Harer, Combinatorics of train tracks. Annals of Mathematical Studies, 125, Princeton University Press, Princeton, NJ, 1992. xii+216 pp. ISBN: 0-691-08764-4; 0-691-02531-2.
- <span id="page-18-0"></span>[Prz1] J. H. Przytycki, Skein modules of 3-manifolds. Bull. Polish Acad. Sci. Math., 39, 1991, no. 1-2, 91-100. e-print: [arXiv:math/0611797](http://arxiv.org/abs/math/0611797) [math.GT]
- [Prz2] J. H. Przytycki, The Kauffman bracket skein algebra of a surface times the interval has no zero divisors. Proceedings of the 46th Japan Topology Symposium at Hokkaido University, July 26-29, 1999, 52-61.
- <span id="page-18-7"></span>[Prz3] J. H. Przytycki, Fundamentals of Kauffman bracket skein modules. Kobe J. Math. J., 16, 1999, no. 1, 45-66. e-print: [arXiv:math/9809113](http://arxiv.org/abs/math/9809113) [math.GT].
- <span id="page-18-1"></span>[Prz4] J. H. Przytycki, KNOTS: From combinatorics of knot diagrams to the combinatorial topology based on knots (chapter IX). Cambridge University Press (to appear), 2020, 950 pp. e-print:<http://arxiv.org/abs/math.GT/0602264>
- <span id="page-18-5"></span>[PS1] J. H. Przytycki, A. S. Sikora, On Skein Algebras and  $Sl_2(\mathbb{C})$ -Character Varieties. Topology, 39, 2000, no. 1, 115-148. e-print: <http://front.math.ucdavis.edu/q-alg/9705011>
- <span id="page-18-6"></span>[PS2] J. H. Przytycki, A. S. Sikora, Skein algebras of surfaces. Transactions of the American Mathematical Society (accepted for publication), DOI: https://doi.org/10.1090/tran/7298. e-print: <http://arxiv.org/abs/1602.07402>
- [QR] H. Queffelec, H. M. Russell, Chebyshev Polynomials and the Frohman-Gelca Formula. J. Knot Theory Ramifications, 24, 2015, no. 4, 1550023, 17 pp. e-print: [arXiv:1403.3716](http://arxiv.org/abs/1403.3716) [math.GT]
- [Sal1] P. Sallenave, Structure of the Kauffman bracket skein algebra of  $T^2 \times I$ . J. Knot Theory Ramifications, 8, 1999, no. 3, 367-372.
- [Sal2] P. Sallenave, On the Kauffman bracket skein algebra of parallelized surfaces. Ann. Sci. École Norm. Sup. (4), 33, 2000, no. 5, 593-610.
- <span id="page-18-11"></span>[Thu] D. P. Thurston, Positive basis for surface skein algebras. Proc. Natl. Acad. Sci. USA, 111, 2014, no. 27, 9725-9732. e-print: [arXiv:1310.1959v](http://arxiv.org/abs/1310.1959)2 [math.GT]

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