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# Small Noise in Signaling Selects Pooling on Minimum Signal

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**Abstract**

In this paper we study how the presence of a small amount of noise in signaling games impacts on the likelihood of separation and, hence, the likelihood of information transmission. We consider a variant of a standard signaling model where a source of exogenous noise affects the signals that agents observe. Noise, even if tiny, poses tight constraints on beliefs by making all signals possible along the equilibrium path. We show that separation cannot be obtained in equilibrium if the noise is small enough – but not nil. In particular, for any separating profile, if noise is sufficiently small then the sender has a profitable deviation consisting of a signal reduction. Instead, the pooling equilibrium where all sender's types pool on the minimum signal always exists, independently of the level of noise. These results provide a new source of interest in pooling equilibria.

**JEL classification code:** D82, D83.

**Keywords:** noise; separation; pooling; information transmission.

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# 1 Introduction

The setup that we are going to consider is a straightforward variant of a standard signaling model. The main novelty that we introduce concerns the presence of frictions in the form of exogenous noise affecting signals. Our analysis aims at establishing the relative prominence of pooling and separating equilibria when a tiny amount of noise affects the signal observed by the receiver. Since a small amount of noise can be considered as a realistic feature, it is interesting to check whether standard results in signaling games are robust to its introduction. In particular, our model is consistent with two different sources of noise: errors in the transmission of the signal, and mistakes in the observation made by the receiver. To fix ideas think of voiced messages (we will often use message instead of signal in the paper). In the presence of ambient noise (a source of transmission errors) the message that reaches the recipient can indeed differ from the one that has been sent by the sender. In addition, if the receiver does not pay full attention, another kind of errors arises, due to mistakes done by the receiver in the elaboration of the signal, so that the observed message differs from the received one.

Noise is important because it poses tight constraints on beliefs, since all signals become possible along the equilibrium path. Quite surprisingly, when the noise tends to become very small, such constraints deliver a powerful selection of equilibria, allowing the survival of only one equilibrium: the pooling equilibrium where all sender's types pool on the minimum signal.

To have an intuitive grasp of our main result, consider a signaling setting with noise where, as the noise goes to zero, the observed message becomes extremely close to the true message with arbitrarily high probability. Starting from a positive level of the signal, the sender has a profitable deviation consisting in a small enough reduction of the signal. By doing so, indeed, the sender does not significantly compromise separation, since the observed message is very likely to be sufficiently close to the original message as not to affect remarkably the receiver's beliefs (see the example in Section 2 for a better intuition). At the same time, such a reduction in signal entails a save in the cost of signaling which makes such a deviation profitable, against equilibrium. As a result, we are able to show that a unique equilibrium survives the progressive reduction in the amount of noise: the pooling equilibrium where all sender's types pool on the minimum signal.

These results suggest two important considerations. The first is that the presence of noise makes full separation less likely, meaning that revelation of types is harder to obtain in equilibrium even when noise is very small. In particular, less noise need not be better for information transmission. The second consideration is that pooling equilibria with non-minimal signals are also less likely in the presence of noise. This makes the pooling equilibrium with minimum signal rather focal in terms of prediction.

This paper is part of a broader project that studies the consequences of introducing frictions in signaling games. In particular, two general classes of frictions are considered. The first class comprises exogenous frictions: the signal sent by the sender is subject to a friction that reduces its informativeness for the receiver. We study this case in the present paper. The second class comprises instead endogenous frictions: the signal sent by the sender is subject to a friction whose intensity depends on the choices of either the sender, the receiver, or both. In Bilancini and Boncinelli (2014) we investigate the role of endogenous frictions by focusing on the case of costly acquisition of signals.

The paper is organized as follows. In Section 2 we give a rather simple example which is used to provide an intuitive understanding of the results that we obtain. In Section 3 we define the class of noisy signaling games. In Section 4 we present our main results on the non-existence of separating equilibra when noise is sufficiently small; we also show that there is a unique pooling equilibrium that survives at all (positive) levels of noise. In Section 5 we discuss our contribution in relationship with the relevant literature. Section 6 summarizes and provides some additional comments.

## 2 An example

Consider a sender $S$ who has to choose a message $m \in [0, 1]$. Sender's type is denoted with $t$, which can be equal to either 1 or 2, with prior probability $p$ and $1 - p$ respectively. The message sent is subject to an error $e$, which is normally distributed with density $f(e) = \frac{1}{\sigma\sqrt{2\pi}} \exp(\frac{-x^2}{2\sigma^2})$, where $\sigma$ is the standard deviation. A receiver $R$ observes $m + e$ and then has to choose an action $a \in \mathbb{R}$. The utility for $S$ is $a - m/t$. We note that the single crossing condition is satisfied, since $m$ is relatively cheaper for type-2 than for type-1. The utility for $R$ is $-(a - t)^2$.

It is straightforward to observe that the pooling equilibrium where both sender's types choose the minimum message 0 is an equilibrium for any level of noise $\sigma$. By means of the first order condition, it is easy to find that $R$'s best reply when the belief to meet a type-1 sender is $p$ is equal to $2 - p$. When both types pool on the same message, then every observed message yields a conditional belief that is equal to the prior belief, and hence $R$'s action is $2 - p$ for every observed message. Consequently, since messages are costly, both sender's types choose the zero message. This kind of reasoning is independent of the amount of noise $\sigma$, as long as it is positive. Remarkably, no refinement based on the restriction of out-of-equilibrium beliefs can be fruitfully applied to this setting with perturbed messages, because every observed message has a positive probability to be observed so that the Bayes rule can be always applied to find posterior probabilities and, hence, no out-of-equilibrium messages exist.

Perhaps more interestingly, any positive message level cannot be part of an equilibrium when the amount of noise becomes sufficiently small. This not only rules out every pooling equilibrium with a non-minimum message level, but also all separating equilibria. The reason behind this result is that, when $\sigma$ is small enough, there always exists a sufficiently small reduction in the message level that is profitable for the sender. Indeed, there are two effects arising from such a tiny reduction: one is the save in the cost of signaling, whose marginal change is equal to either 1 or 0.5 depending on the sender's type, while the other comes from the re-action by $R$ to the change in her posterior beliefs. Much of this paper is about showing that this latter effect is negligible. To give an intuitive understanding of this, we rely on figure 1, which represents conditional beliefs as a function of the observed message.

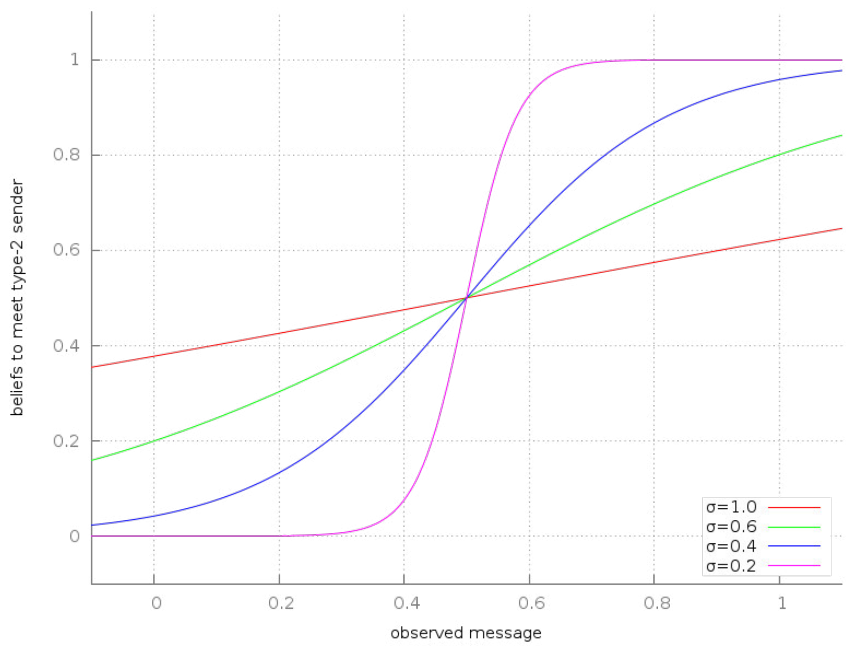


Figure 1: We consider a sender's strategy where type-1 sender and type 2 sender play 0 and 1, respectively. By considering a normal distribution of errors with mean zero and four different values of the standard deviation, we use the Bayes rule to compute the conditional probability that the observed message comes from a type-2 sender.

As we can see from the figure, the marginal effect of a reduction in the message level on the probability to be believed a type-2 sender, computed around the true message level, tends to zero when $\sigma$ approaches zero. In particular, as $\sigma$ gets lower and lower most of errors concentrate around the true message level, so that a type whose current message level is positive sees the negative effect of slightly reducing the message becoming vanishingly small. This should intuitively convince us that this type, for $\sigma$ small enough, can profitably exploit the save in the cost of signaling that is associated to a marginal reduction in the message.

All the rest of the paper is devoted to prove formally these results in a general setting of perturbed signaling.

# 3 The Model

We consider a setting with two players, a sender $S$ and a receiver $R$, who are engaged in a noisy signaling game.

*The sender.* Sender $S$ observes his own type $t \in T$ (with $T$ a finite set of cardinality $n$) and then chooses a signal (or message) $m \in [0, 1]$. Types are drawn according to probabilities $p = (p_1, p_2, \ldots, p_n)$, with $p_t > 0$ for every $t \in T$ and $\sum_{t \in T} p_t = 1$.

*Errors.* The signal is subject to an error $e \in \mathbb{R}$, so that the observed signal (or observed message) is $\hat{m} = m + e$. The error $e$ has a density function $f^\sigma$ which is always positive, differentiable – we denote its derivative with $f^\sigma_e$ – and single-peaked at zero, i.e., $f^\sigma(e') < f^\sigma(e)$ whenever $|e'| > |e|$. The parameter $\sigma \in (0, \bar{\sigma}]$ measures the amount of noise affecting signaling. In particular, as $\sigma$ decreases we have that most of errors will have a tiny effect on the observed signal. Formally, we assume that for every $\epsilon \in (0, 1)$ and every $\delta > 0$, a threshold level $\tilde{\sigma}$ can be found such that if $\sigma \leq \tilde{\sigma}$ then $\int_{-\delta}^{+\delta} f^\sigma(e) \mathrm{d}e \geq 1 - \epsilon$. Finally, as regularity condition we assume that $f^\sigma_e$ is uniformly convergent[1] as $\sigma$ tends to zero on every interval of the form $(-\infty, -a]$ or $[+a, +\infty)$ with $a > 0$.[2]

*The receiver.* Receiver $R$ observes the signal $\hat{m}$ and then chooses a reply $a \in A \subseteq \mathbb{R}$.

*Utilities.* Utility for $S$ is $U : T \times [0, 1] \times A \to \mathbb{R}$, and utility for $R$ is $V : T \times [0, 1] \times A \to \mathbb{R}$.

[1]A parameterized real-valued function $g^\sigma$ is said to be uniformly convergent on an interval $I$ if there exists a limit function $g$ such that, for every $\epsilon > 0$, there exists a threshold $\tilde{\sigma}$ such that for all $\sigma \leq \tilde{\sigma}$ we have that $|g^\sigma(x) - g(x)| < \epsilon$ for all $x \in I$.

[2]We exclude 0 from these intervals because, if we consider an interval including 0, then for every $\sigma > 0$ we might find a point $e$ in the interval where $f^\sigma_e(e)$ differs significantly from zero (intuitively, this happens in the proximity of 0 when the density function jumps over).

We assume that, for all $t \in T$, $U$ is differentiable with respect to both $m$ and $a$ with countinuous derivatives denoted, respectively, by $U_m$ and $U_a$; we also assume that $U_m < 0$.

*Strategies.* A strategy for $S$ is $\mu : T \to [0,1]$. A strategy for $R$ is $\alpha : \mathbb{R} \to A$.

*The receiver's beliefs.* Given $\mu$ and $\sigma$, $R$ observes $\hat{m}$ and then derives posterior beliefs on the type and the true signal, by means of the Bayes rule. In particular, the probability conditional upon the observation of $\hat{m}$ that $S$ is of type $t$ and, consequently, the true signal is $\mu(t)$ is:

$$\beta_t(\hat{m}|\mu, \sigma) = \frac{p_t f^\sigma(\hat{m} - \mu(t))}{\sum_{t' \in T} p_{t'} f^\sigma(\hat{m} - \mu(t'))}. \tag{1}$$

*The receiver's best reply.* Once $\hat{m}$ is observed, and beliefs $\beta(\hat{m}|\mu,\sigma) = (\beta_1(\hat{m}|\mu,\sigma), \ldots, \beta_n(\hat{m}|\mu,\sigma))$ are formed, $R$ replies with the action maximizing $R$'s utility. We assume that such best reply action is always unique, and we denote the resulting function with $\rho : \Delta T \to A$. Moreover, and importantly, we assume that $\rho$ is differentiable with continuous derivatives, and we denote its partial derivatives with $\rho_{\beta_t}$.

*Noisy signaling game and equilibrium.* We will refer to the setting that we have introduced as *noisy signaling game*, and we will denote it as $\Gamma(T, p, U, V, f^\sigma)$. An *equilibrium* of the noisy signaling game $\Gamma(T, p, U, V, f^\sigma)$ is a pair $(\mu, \alpha)$ such that:

- $\alpha(\hat{m}) = \rho(\beta(\hat{m}|\mu,\sigma))$, for all $\hat{m} \in \mathbb{R}$;
- $\mu(t) \in \arg\max_{m \in M} \int_{-\infty}^{+\infty} f^\sigma(e) U(t, m, \rho(\beta(m+e|\mu,\sigma)))\mathrm{d}e$, for all $t \in T$.

*Equilibrium robust to noise.* We are interested to understand which equilibria will persist in the presence of a tiny but positive amount of noise. Before doing that, we observe that standard refinements of signaling games that use restrictions of out-of-equilibrium beliefs[3] have no bite in a noisy signaling game. The reason is simply that, due to the presence of noise, every signal can be observed by the receiver, so that no out-of-equilibrium information set exists, and hence Bayesian update can always be applied to compute posterior beliefs.

Formally, we introduce the following equilibrium refinement to capture the idea of robustness to a vanishing amount of noise: given a family of noisy signaling games $\{\Gamma(T, p, U, V, f^\sigma)\}_{\sigma \in (0, \bar{\sigma}]}$, we say that a pair $(\mu, \alpha)$ is an *equilibrium robust to noise* if there exists $\tilde{\sigma}$ such that $(\mu, \alpha)$ is an equilibrium of the noisy signaling game $\Gamma(T, p, U, V, f^\sigma)$ for every $\sigma \leq \tilde{\sigma}$.

[3] We can remind the Intuitive Criterion, D1, and D2 (Cho and Kreps, 1987), Divinity and Universal Divinity (Banks and Sobel, 1987), the Undefeated Equilibrium (Mailath et al., 1993), and the Perfect Sequential Equilibrium (Grossman and Perry, 1986).

# 4 Results

Our take-home result – which we state in Propositions 1 and 2 – relies on an intermediate technical result that we provide in the following lemma. In a few words, small changes in the message level around $\mu(t')$ for some $t' \in T$ generate a negligible change in the beliefs held by $R$ that the sender she is facing is of (generic) type $t \in T$.

Lemma 1. *Given a noisy signaling game $\Gamma(T, p, U, V, f^\sigma)$, for every $t, t' \in T$, and for every $\ell > 0$, there exists $\tilde{\sigma} > 0$ such that, for every $\sigma < \tilde{\sigma}$, there exists $\tilde{\delta} > 0$ such that, for every $|\delta| < \tilde{\delta}$:*

$$\left.\frac{\mathrm{d}\beta_t(\hat{m}|\mu,\sigma)}{\mathrm{d}\hat{m}}\right|_{\hat{m}=\mu(t')+\delta} < \ell.$$

*Proof.* Consider any $t, t' \in T$. For every $\sigma > 0$, we have that:

$$\left.\frac{\mathrm{d}\beta_{\mathbf{t}}(\hat{m}|\mu,\sigma)}{\mathrm{d}\hat{m}}\right|_{\hat{m}=\mu(t')+\delta} = \lim_{\epsilon\to 0}\frac{\beta_t(\mu(t')+\delta+\epsilon|\mu,\sigma)-\beta_t(\mu(t')+\delta|\mu,\sigma)}{\epsilon},$$

which has an indeterminate form of the kind $0/0$. Since the involved functions are differentiable, we can apply L'Hôpital's rule and we obtain:

$$\lim_{\epsilon\to 0}\frac{\beta_t(\mu(t')+\delta+\epsilon|\mu,\sigma)-\beta_t(\mu(t')+\delta|\mu,\sigma)}{\epsilon} = \lim_{\epsilon\to 0}\frac{\mathrm{d}\beta_t(\mu(t')+\delta+\epsilon|\mu,\sigma)}{\mathrm{d}\epsilon}.$$

We elaborate on the last expression by using (1):

$$\lim_{\epsilon\to 0}\frac{\mathrm{d}\beta_t(\mu(t')+\delta+\epsilon|\mu,\sigma)}{\mathrm{d}\epsilon} = \tag{2}$$

$$= \lim_{\epsilon\to 0}\frac{p_t f^\sigma_e(\mu(t')+\delta+\epsilon-\mu(t))\left[\sum_{t''\in T} p_{t''} f^\sigma(\mu(t')+\delta+\epsilon-\mu(t''))\right]}{\left[\sum_{t''\in T} p_{t''} f^\sigma(\mu(t')+\delta+\epsilon-\mu(t''))\right]^2} +$$
$$-\lim_{\epsilon\to 0}\frac{p_t f^\sigma(\mu(t')+\delta+\epsilon-\mu(t))\left[\sum_{t''\in T} p_{t''} f^\sigma_e(\mu(t')+\delta+\epsilon-\mu(t''))\right]}{\left[\sum_{t''\in T} p_{t''} f^\sigma(\mu(t')+\delta+\epsilon-\mu(t''))\right]^2}. \tag{3}$$

We now show that, in the above two-line expression (3), if we take a sufficiently small $\sigma$, and then choose $\delta + \epsilon$ small enough, some terms in the fractions are bounded from above, while other terms grow unboundedly; this will allow us to show that expression (3) can be made smaller than an arbitrarily fixed $\ell$. We prove this by relying on two lemmas that are given in the appendix.

In particular, by applying Lemma 6 (with $\delta+\epsilon$ replacing $\delta$ in the lemma's statement), we know that there exist $\ell'$ and $\tilde{\sigma}_1$ such that, if $\sigma \leq \tilde{\sigma}_1$, then we can find $\tilde{\delta}_1$ such that, if $\delta+\epsilon \leq \tilde{\delta}_1$, then $f^\sigma(\mu(t') + \delta + \epsilon - \mu(t'')) < \ell'$ for every $\mu(t'') \neq \mu(t')$, and $f_e^\sigma(\mu(t') + \delta + \epsilon - \mu(t'')) < \ell'$ for every $\mu(t'')$.

Moreover, by applying Lemma 2 (with $\delta + \epsilon$ replacing $\delta$ in the lemma's statement), we obtain that for every $k > 0$, we can find $\tilde{\sigma}_2$ such that, if $\sigma \leq \tilde{\sigma}_2$, then we can find $\tilde{\delta}_2$ such that, if $\delta + \epsilon \leq \tilde{\delta}_2$ then $f^\sigma(\mu(t') + \delta + \epsilon - \mu(t'')) > k$ when $\mu(t') = \mu(t'')$.

We fix $\ell' \leq 1$, and we restrict ourselves to considering $\sigma \leq \tilde{\sigma}_1(\ell')$, and $\delta \leq \tilde{\delta}_1(\ell', \sigma)$. By virtue of the above results, we can affirm about the two-line expression (3) that: (i) each term of the sum at the numerator is bounded from above by $k$, so that the whole numerator is bounded from above by $2nk$; (ii) the denominator is bounded from below by $\tilde{p}_t^2 k^2$, with $\tilde{p}_t = \min_T p_t$ (we have also neglected all other terms of the denominator, all of which are positive). Therefore, expression (3) on the whole is bounded from above by $(2nk)/(\tilde{p}_t^2 k^2)$. Since such an upper bound converges to 0 as $k$ grows unboundedly, we can find $k'$ such that, for $k \geq k'$, expression (3) is lower than an arbitrarily fixed $\ell$. Then, we set $\tilde{\sigma} = \min\{\tilde{\sigma}_1(\ell'), \tilde{\sigma}_2(k')\}$, and $\tilde{\delta} = \min\{\tilde{\delta}_1(\ell', \sigma), \tilde{\delta}_2(k', \sigma)\}$. We complete the proof by simply noting that, for every $\delta < \tilde{\delta}$ we have that $\delta + \epsilon < \tilde{\delta}$ for $\epsilon$ that is sufficiently small (this is not a limitation since we are considering the limit for $\epsilon$ going to zero). □

Before proceeding, we define the *no-signal pooling equilibrium* as the pair $(\mu, \alpha)$ such that:

- $\mu(t) = 0$ for all $t \in T$;
- $\alpha(\hat{m}) = \rho(p)$ for all $\hat{m} \in \mathbb{R}$.

We are ready to state – in Proposition 1 – the first part of our main result: the equilibrium where all sender's types choose the minimum signal is robust to the introduction of a tiny amount of noise.

Proposition 1. *Given a family of noisy signaling games* $\{\Gamma(T, p, U, V, f^\sigma)\}_{\sigma \in (0, \bar{\sigma}]}$*, the no-signal pooling equilibrium is an equilibrium robust to noise.*

*Proof.* Given that all types of sender $S$ pool on the same signal, we have that $\beta_t(\hat{m}|\mu, \sigma) = p_t$ for every $\hat{m} \in \mathbb{R}$, every $\sigma \in (0, \bar{\sigma}]$, and every type $t \in T$, i.e., the posterior beliefs always coincide with the prior beliefs. Therefore, the best reply for $R$ is to choose $\rho(p_1, \ldots, p_n)$, irrespectively of the observed signal $\hat{m} \in \mathbb{R}$, and this is true for every $\sigma \in (0, \bar{\sigma}]$.

Since the reply by $R$ is not affected by the observed signal, signals are useless and sender $S$ is only motivated to save on signaling costs, and this is true for every $\sigma \in (0, \bar{\sigma}]$. The assumption that $U_m$ is always negative implies that the least costly signal is the minimum signal, i.e., 0, for all types $t \in T$.

The above arguments show that the no-signal pooling equilibrium is actually an equilibrium of the noisy signaling game $\Gamma(T, p, U, V, f^\sigma)$ for every $\sigma \in (0, \bar{\sigma}]$, and this implies that it is an equilibrium robust to noise. □

The following proposition complements the previous proposition and completes the illustration of our main result on equilibrium selection in noisy signaling games. In particular, Proposition 2 states a kind of converse result with respect to Proposition 1: there is no equilibrium other than the no-signal pooling equilibrium that is robust to noise.

PROPOSITION **2.** *Given a family of noisy signaling games* $\{\Gamma(T, p, U, V, f^\sigma)\}_{\sigma \in (0,\bar{\sigma}]}$*, if* $(\mu, \alpha)$ *is different from the no-signal pooling equilibrium then it is not an equilibrium robust to noise.*

*Proof.* If all sender's types choose the minimum signal, then either the profile coincides with the no-signal pooling equilibrium or $R$'s strategy is non-optimal.

Suppose $\mu(t) \neq 0$. Consider the derivative of the expected utility of type $t$ with respect to the chosen message:

$$\frac{\partial}{\partial m} \int_{-\infty}^{+\infty} f^\sigma(e) \cdot U(t, \mu(t), \rho(\beta(\mu(t) + e|\mu, \sigma))) \cdot \mathrm{d}e =$$
$$= \int_{-\infty}^{+\infty} f^\sigma(e) \cdot U_m(t, \mu(t), \rho(\beta(\mu(t) + e|\mu, \sigma))) \cdot \mathrm{d}e +$$
$$\int_{-\infty}^{+\infty} f^\sigma(e) \cdot U_a(t, \mu(t), \rho(\beta(\mu(t) + e|\mu, \sigma))) \cdot \left( \sum_{t \in T} \rho_{\beta_t}(\beta(\mu(t) + e|\mu, \sigma)) \cdot \frac{\mathrm{d}\beta_t(\mu(t) + e|\mu, \sigma)}{\mathrm{d}\hat{m}} \right) \cdot \mathrm{d}e. \tag{4}$$

We observe that $U_m(t, m, \cdot) \leq \overline{U}_m < 0$. This is true because $U_m(t, m, \cdot)$ takes a maximum value, denoted with $\overline{U}_m$, when beliefs vary by virtue of the extreme value theorem, since $U_m(t, m, \cdot)$ is continuous and beliefs are defined over a compact set; moreover, since $U_m$ is assumed to be always negative, we have that $\overline{U}_m < 0$. Therefore, the first integral of the two-line expression (4) is negative and always not greater than $\overline{U}_m \int_{-\infty}^{+\infty} f^\sigma(e)\mathrm{d}e = \overline{U}_m$, for every $\sigma \in (0, \bar{\sigma}]$.

We now argue that the second integral of expression (4) can be made as small as desired, and hence lower than $|\overline{U}_m|$, by choosing $\sigma$ sufficiently low, so that a profitable deviation for type $t$ exists, consisting of a tiny reduction of the message. Hence, the profile under consideration is not an equilibrium of the noisy signaling game when $\sigma$ is small enough, i.e., the profile is not an equilibrium robust to noise.

By choosing $\sigma$ sufficiently low, Lemma 1 then ensures that $\delta$ can be found such that, for every $e \in (-\delta, +\delta)$, the derivatives $\mathrm{d}\beta_t(\mu(t) + e|\mu, \sigma)/\mathrm{d}\hat{m}$ for all $t \in T$ are not greater than an arbitrary $\ell > 0$. We rewrite the integral under consideration dividing the range of integration in three intervals: from $-\infty$ to $-\delta$, from $-\delta$ to $+\delta$, and from $+\delta$ to $+\infty$.

We start focusing on the integral from $-\delta$ to $+\delta$. We observe that $U_a(t, m, \cdot)$ and all the derivatives $\rho_{\beta_t}$ are bounded when beliefs vary by virtue of the boundedness theorem, since the functions are continuous and beliefs are defined over a compact set. We denote the upper bounds with $\overline{U}_a$ and $\overline{\rho}_{\beta_t}$ for $t \in T$. The integral under consideration is hence not greater than $\overline{U}_a \ell \sum_{t \in T} \overline{\rho}_{\beta_t} \int_{-\delta}^{+\delta} \mathrm{d}e$, which in turn is not greater than $\overline{U}_a \ell \sum_{t \in T} \rho_{\beta_t}(\beta(\mu(t) + e|\mu, \sigma))$. Since $\ell > 0$ can be chosen freely, we conclude that the integral from $-\delta$ to $\delta$ can be made as small as desired when $\sigma$, and consequently $\delta$, are chosen small enough.

We now focus on the integral ranging from $-\infty$ to $-\delta$, and we rewrite it by using the formula of integration by parts:

$$[f^\sigma(e) \cdot U(t, \mu(t), \rho(\beta(\mu(t) + e|\mu, \sigma)))]_{-\infty}^{-\delta} - \int_{-\infty}^{-\delta} f_e^\sigma(e) \cdot U(t, \mu(t), \rho(\beta(\mu(t) + e|\mu, \sigma)))\mathrm{d}e. \quad (5)$$

We prove that (5) can be made as small as desired when $\sigma$ approaches zero, by showing that it is so for both of its terms. Preliminarily, we observe that $U(t, m, \cdot)$ is a bounded function when beliefs vary; this is obtained, analogously to what done for $U_a(t, m, \cdot)$, by applying the boundedness theorem once you note that beliefs are defined over a compact set, and functions $\rho$ and $U$ are both continuous (indeed, differentiability implies continuity). We denote with $\overline{U}$ and $\underline{U}$ the upper and lower bounds, respectively. We consider the first term of the sum, noting that:

$$[f^\sigma(e) \cdot U(t, \mu(t), \rho(\beta(\mu(t) + e|\mu, \sigma)))]_{-\infty}^{-\delta} \leq f^\sigma(-\delta) \cdot \overline{U} - \lim_{a \to -\infty} f^\sigma(a) \cdot \underline{U}.$$

By uniform convergence to zero established in Lemma 5, both $f^\sigma(-\delta)$ and $\lim_{a \to -\infty} f^\sigma(a)$ become arbitrarily close to zero when $\sigma$ approaches zero, and hence the same holds for the whole term. Then we focus on the second term of the sum, noting that:

$$\int_{-\infty}^{-\delta} f_e^\sigma(e) \cdot U(t, \mu(t), \rho(\beta(\mu(t) + e|\mu, \sigma)))\mathrm{d}e \leq \overline{U} \int_{-\infty}^{-\delta} f_e^\sigma(e) \cdot \mathrm{d}e = \overline{U} \, [f^\sigma(e)]_{-\infty}^{-\delta}.$$

which, by the same argument used above, can be made as small as desired when $\sigma$ approaches zero.

The integral from $+\delta$ to $+\infty$ can be dealt with similarly to what already done for the integral $-\infty$ to $-\delta$, and this completes the proof. □

Propositions 1 and 2 contrast with standard results in signaling games where pooling equilibria are usually found to be less robust than separating equilibria. In noisy signaling games, instead, the no-signal pooling equilibrium soars to be the prominent equilibrium that is likely to emerge.

# 5 Relation to the literature

Although noisy signaling has not attracted much attention so far, there exists a small literature on the topic.

Matthews and Mirman (1983) are perhaps the first to explicitly consider noise in signaling games. They apply this idea to the study of price signaling with stochastic demand (extending Milgrom and Roberts, 1982), assuming that the source of noise is the imperfect message transmission.

Hertzendorf (1993) studies multi-dimensional signaling in the presence of noise, extending the advertising model by Milgrom and Roberts (1986). His main finding is that the recipients of advertising signals will only rarely be informed about the exact advertising budget of a company, typically receiving a noisy signal.

Truyts (2012) studies the case of stochastic costly signaling in the presence of exogenous imperfect information, and provides condition for equilibrium signaling to decrease or increase in the accuracy of exogenous information.

Daley and Green (2013) consider a signaling environment where receivers observe, in addition to the signal chosen by the sender, the stochastic outcome of a test which is correlated with the sender's actual type. They show that if the noise affecting the test is not too strong, then equilibria where sender's types pool on the costly signal can become more plausible than separating equilibria.

The paper most closely related to ours is Carlsson and Dasgupta (1997). They analyze the noise-proofness of equilibria in signaling games where the receiver has only two actions available. A noise-proof equilibrium of an ordinary signaling game has the property that it can be approximated by equilibria of sequences of noisy signaling games with vanishing noise. Carlsson and Dasgupta (1997) show that a noise-proof equilibrium always exists and that it is in general "insufficiently revealing" in the sense that there is always a positive probability that the receiver misinterprets the observed signal and takes an action that is suboptimal against the actual type of the sender.

Besides the slightly different approach to the study of the robustness of signaling equilibria to noise, the main difference between our paper and Carlsson and Dasgupta (1997) is that the receiver has only two actions available. This leads to a discontinuity of the sender's expected payoff function in the signal sent, so that a marginal reduction in the signal might entail a large loss. This allows the survival of some separation in equilibrium, even when noise is very small. To put it differently, the main elements of our results are already contained in Carlsson and Dasgupta (1997), but their discrete specification of the receiver's action space leads to a much less strong selection in favor of pooling. This also shows that a continuum of actions for the receiver is an important ingredient of our result.

Another quite related paper is de Haan et al. (2011) where noisy signaling is explored both theoretically and experimentally. They consider the traditional setup by Spence (1973) where the receiver has only two actions as in (Carlsson and Dasgupta, 1997). They find, among other things, that if the amount of noise increases, then high types aiming for separation increase their signaling expenditures, and that for intermediate and high levels of noise, a separating and pooling equilibrium co-exist. More relevantly, they provide a result that is closely related to ours, namely that a separating equilibrium ceases to exist when two conditions hold: a low enough level of noise and a pessimistic enough prior. In a sense, our result confirms the theoretical analysis by de Haan et al. (2011) showing that pessimistic priors are not a crucial ingredient when the receiver has a continuum of actions.

Finally, noise in signaling games is considered also by Jeitschko and Normann (2012) who contrast a standard deterministic signaling game with one where the signal-generating mechanism is stochastic. In particular, they show that with stochastic signals a unique equilibrium emerges that involves separation where the degree of separation depends on the prior type distribution. They do not investigate what happens when the noise goes to zero.

# 6 Conclusions

In this paper we have considered a signaling framework and we have shown that the introduction of a tiny amount of noise in the transmission/observation of signals – so that the observed signal only slightly differs from the true one – has a big impact on the equilibrium profile that is likely to emerge. In particular, the unique equilibrium that survives in the presence of a vanishing amount of noise is the pooling equilibrium where all sender's types choose the minimum signal. This result is in stark contrast with the literature on the refinements of signalling equilibria in the absence of noise, where separating equilibria have been considered as the prominent outcome (see Riley, 2001).

We think that our research, despite its mainly theoretical nature, can have potentially relevant consequences for applied research, because of the relevance of signaling models in applied theory. In particular, consider the following sketched argument. The presence of noise in the transmission/observation of signals can be understood as an obstacle to the separation of types through signaling, which is considered a standard tool to overcome the adverse selection problem. In a sense, our results confirm this concern. In the light of this, a benevolent public authority may try to intervene and invest in order to reduce noise in relevant signaling setups (e.g., by providing a better transmission technology). A correct interpretation of our results, however, suggests that such an intervention could be a total waste of money, if not worsen the situation. The reason is that, for any positive level of noise, a separating equilibrium may well exist where different types choose different signals, so that observed signals allow a proper update of beliefs (see Carlsson and Dasgupta, 1997); however, as the amount of noise shrinks to zero, a slight reduction of the signal becomes a profitable deviation for all sender's types – as we have shown in this paper – making the original separating equilibrium unsustainable. A detailed investigation of the relationship between noise and welfare might be an interesting follow up of the present contribution.


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# Appendix

Here is a list of lemmas that are used to prove the main results in the paper. Lemmas from 2 to 5 show useful properties that hold in noisy signaling games, while Lemma 6 packages a result which is directly usable in the proof of Proposition 1.

LEMMA **2.** *For every $k > 0$, there exists $\tilde{\sigma}_1(k)$ such that, for every $\sigma \leq \tilde{\sigma}_1(k)$, there exists $\tilde{\delta}_1(\sigma)$ such that, if $\delta < \tilde{\delta}_1(\sigma)$, then $f^\sigma(\delta) > k$.*

*Proof.* Fix $k > 0$. Then fix $\epsilon \in (0, 1)$ and $\delta > 0$ such that $(1 - \epsilon)/(2\delta) > k$. By assumption there exists a threshold level $\tilde{\sigma}$ such that if $\sigma \leq \tilde{\sigma}$ then $\int_{-\delta}^{+\delta} f^\sigma(e)\mathrm{d}e \geq 1 - \epsilon$. Since $f^\sigma(0) \geq f^\sigma(e)$ for every $e \in \mathbb{R}$, because $f^\sigma$ is single-peaked with the peak at zero, it follows that $f^\sigma(0) > (1 - \epsilon)/(2\delta)$ if $\sigma \leq \tilde{\sigma}$. By continuity of $f^\sigma$, we can conclude that, if $\sigma \leq \tilde{\sigma}$, there exists $\tilde{\delta}_1(\sigma)$ such that, if $\delta < \tilde{\delta}_1(\sigma)$, then $f^\sigma(\delta) > k$. □

LEMMA **3.** $\lim_{\sigma \to 0} f^{\sigma}(e) = 0$, *for all* $e \neq 0$.

*Proof.* Suppose not. This means that it must exist $e' \neq 0$ and $k > 0$ such that we can always find a small enough $\sigma > 0$ such that $f^{\sigma}(e') > k$. Since for all $\sigma$ we have that $f^{\sigma}$ is single-peaked with peak at zero, then $f^{\sigma}(e) > k$ for all $e$ such that $|e| < |e'|$. But then $\int_{e'}^{\frac{e'}{2}} f^{\sigma}(e)\mathrm{d}e \geq ke'/2$, if $e' < 0$, or $\int_{\frac{e'}{2}}^{e'} f^{\sigma}(e)\mathrm{d}e \geq ke'/2$, if $e' > 0$, contradicting the assumption that if $\sigma$ is small enough then $\int_{-\delta}^{+\delta} f^{\sigma}(e)\mathrm{d}e$ becomes as close as desired to 1. □

LEMMA **4.** *For every* $\sigma > 0$, *and for every* $\ell > 0$, *there exists* $\tilde{\delta} > 0$ *such that, if* $|\delta| < \tilde{\delta}$, *then* $f_e^{\sigma}(\delta) < \ell$.

*Proof.* For every $\sigma > 0$, $f_e^{\sigma}(0)$ exists by assumption. If $f_e^{\sigma}(0) \neq 0$, then the assumption that $f^{\sigma}$ is single-peaked with peak at zero would be violated. Hence, $f_e^{\sigma}(0) = 0$ for every $\sigma > 0$. By continuity of $f_e^{\sigma}$, we conclude that, for every $\sigma > 0$ and every $\ell > 0$, there exists $\tilde{\delta} > 0$ such that, if $|\delta| < \tilde{\delta}$, then $f_e^{\sigma}(\delta) < \ell$. □

LEMMA **5.** *For every* $\delta > 0$, *we have that* $f_e^{\sigma}$ *converges uniformly to zero on* $(-\infty, -\delta] \cup [+\delta, +\infty)$ *when* $\sigma$ *tends to zero.*

*Proof.* Choose $\delta > 0$. By Lemma 3 we have that $f^{\sigma}$ converges pointwise to the zero constant function on $(-\infty, -\delta] \cup [+\delta, +\infty)$ when $\sigma$ tends to zero. Since $f_e^{\sigma}$ is assumed to be uniformly convergent on $(-\infty, -\delta]$ and $[+\delta, +\infty)$ when $\sigma$ tends to zero, and hence on $(-\infty, -\delta] \cup [+\delta, +\infty)$ as well, we now show that $f_e^{\sigma}$ converges uniformly on $(-\infty, -\delta] \cup [+\delta, +\infty)$ to the derivative of the limit function at which $f^{\sigma}$ converges, i.e., the derivative of the zero-constant function, which is the zero-constant function itself, proving the statement of the lemma. To be convinced of the above assertion, suppose that it is not true, i.e., there exists $e \in (-\infty, -\delta] \cup [+\delta, +\infty)$ such that $\lim_{\sigma \to 0} f_e^{\sigma}(e) \neq \lim_{\sigma \to 0} f^{\sigma}(e)$. Then we consider an interval $[a, b]$ such that $e \in [a, b]$. Due to the assumption that $f^{\sigma}$ converges pointwise to the zero constant function on $(-\infty, -\delta] \cup [+\delta, +\infty)$, and hence on $[a, b]$ as well, we can find $e_0 \in [a, b]$ where $\lim_{\sigma \to 0} f^{\sigma}(e_0)$ exists (and is equal to 0), and so we obtain a contradiction with, e.g., Theorem 7.17 of Rudin (1976). □

LEMMA **6.** *There exist* $\ell$ *and* $\tilde{\sigma}$ *such that, if* $\sigma < \tilde{\sigma}$, *then we can find* $\tilde{\delta}$ *such that, if* $\delta < \tilde{\delta}$ *then:*

- $f^{\sigma}(\mu(t') + \delta - \mu(t'')) < \ell$ *for every* $\mu(t'') \neq \mu(t')$;
- $f_e^{\sigma}(\mu(t') + \delta - \mu(t'')) < \ell$ *for every* $\mu(t'')$.

*Proof.* Fix $\ell > 0$. Choose $\tilde{\delta}_1 > 0$ such that $\tilde{\delta}_1 < \min_{t'' \in T} |\mu(t') - \mu(t'')|$. Consider $e_1 = -\min_{t'' \in T} |\mu(t') - \mu(t'')| + \tilde{\delta}_1$ and $e_2 = +\min_{t'' \in T} |\mu(t') - \mu(t'')| - \tilde{\delta}_1$. By applying Lemma 3 to both $e_1$ and $e_2$ and choosing the minimum threshold we are able to find $\tilde{\sigma}_1$ such that, if $\sigma < \tilde{\sigma}_1$, then both $f^\sigma(e_1)$ and $f^\sigma(e_2)$ are not greater than $\ell$. By the fact that $f^\sigma$ is single-peaked with peak at zero, we conclude that $f^\sigma(\mu(t') + \delta - \mu(t''))$ for every $\mu(t'') \neq \mu(t')$ and every $|\delta| < \tilde{\delta}_1$.

By Lemma 4, we know that for every $\sigma > 0$, there exists $\tilde{\delta}_2 > 0$ such that, if $|\delta| < \tilde{\delta}_2$, then $f_e^\sigma(\mu(t') + \delta + \epsilon - \mu(t'')) < \ell$ when $\mu(t') = \mu(t'')$.

Now we use again $e_1$ and $e_2$ as previously defined. By Lemma 5, we know that $f_e^\sigma$ is uniformly convergent on $(-\infty, -e] \cup [+e, +\infty)$, i.e., there exists $\tilde{\sigma}_2 > 0$ such that, if $\sigma < \tilde{\sigma}_2$ and $|\delta| < \tilde{\delta}_1$, then $f_e^\sigma(\mu(t') + \delta - \mu(t'')) < \ell$ for every $\mu(t'') \neq \mu(t')$. □

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