On Reconfiguration Graphs of Independent Sets under Token Sliding

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Reconfiguration problems arise when we want to study the relationship between *solutions* of some given computational problem (called a *source problem*) such as SATISFIABILITY, INDEPENDENT SET, DOMINATING SET, VERTEX-COLORING, and so on. The main goal of reconfiguration problems is to study the so-called *reconfiguration graph*—a graph whose nodes are solutions and their adjacency can be defined via some given *reconfiguration rule*. A typical example is the well-known classic Rubik's cube puzzle, where each configuration of the Rubik's cube corresponds to a solution, and two configurations (solutions) are adjacent if one can be obtained from the other by rotating a face of the cube by either 90, 180, or 270 degrees. The question is whether there exists a path from an arbitrary node to the one where each face has only one color. Reconfiguration graphs have been studied in the literature from three major viewpoints: *structural properties* (connectivity, Hamiltonicity, planarity, and so on), *realizability* (which graph can be realized as a certain type of reconfiguration graph), and *algorithmic properties* (whether certain question, such as finding a (shortest) path between two given nodes, can be answered efficiently, and if so, design an algorithm to do it). For an overview of this research area, readers are referred to the surveys [10,15,16].

One of the classic reconfiguration rules is the so-called *Token Sliding* (TS). Any vertex subset of a graph G can be seen as a set of tokens placed on some vertices of G. Under TS, a token on some vertex v can only be moved to one of v's unoccupied adjacent vertices. Two vertex subsets are *adjacent* under TS if one can be obtained from the other via a single TS-move. The *token graph* $F_k(G)$ [14] is a reconfiguration graph whose nodes are size-k vertex subsets of G and edges are defined under TS. The study of $F_k(G)$ dates back to the 1990s, when Alavi, Behzad, Erdős, and Lick [1] considered several basic structural properties (e.g., regularity, bipartitedness, Eulerianity, etc.) of $F_2(G)$ (which they called the *double vertex graph*). Other well-known reconfiguration rules are *Token Jumping* (TJ) and *Token Addition/Removal* (TAR), which respectively involve moving a token to any unoccupied vertex and adding/removing a single token to/from some unoccupied/occupied vertex.

In this paper, we take INDEPENDENT SET as the source problem and consider two types of reconfiguration graphs whose edges are defined under TS: the TS_k-reconfiguration graphs, for some given positive integer k, whose nodes are size-k *independent sets* of G (i.e., vertex subsets whose members are pairwise non-adjacent, also called *stable sets*) and the TS-reconfiguration graphs whose nodes are independent sets of arbitrary size. We denote these graphs by $\mathsf{TS}_k(G)$ and $\mathsf{TS}(G)$, respectively. (Similar definitions hold for TJ.) In particular, each $\mathsf{TS}_k(G)$ is an induced subgraph of $\mathsf{TS}(G)$ and also a subgraph of $F_k(G)$.

The focus of this paper is on purely graph theoretic properties of $\mathsf{TS}(G)$ and $\mathsf{TS}_k(G)$ as opposed to algorithmic properties, which have been well-studied in the literature. In particular, the tractability/intractability of whether there is a path between two given nodes and several subsequent questions (e.g., if yes, whether a shortest one can be found efficiently; whether the statement holds for any pair of nodes; and so on) have been well-investigated for several graphs G. Readers are referred to [16, Sections 4–5] for a quick summary of the recent results. In

particular, some structural properties regarding the connectivity and diameter of $\mathsf{TS}_k(G)$ can be derived from many of these algorithmic results [3,4,5,7,9,11,12]. (See Appendix A.) On the other hand, the realizability and structural properties of $\mathsf{TS}(G)/\mathsf{TS}_k(G)$ have not yet been systematically studied and we initiate this study here. More precisely, given a graph G, we provide some initial results regarding two main questions: (1) Whether the TS_k-reconfiguration graph of G belongs to some graph class $\mathcal{G}$ and (2) If G satisfies some property $\mathcal{P}$, whether $\mathsf{TS}(G)/\mathsf{TS}_k(G)$ also satisfies $\mathcal{P}$ too, and vice versa.

The outline of this paper is as follows. In Section 2, we give an example of TS_k-reconfiguration graphs used as a model for a problem in computational geometry. In Section 3, we define some terminology and notation that are used throughout this paper. In Section 4, for different graph classes $\mathcal{G}$ (including complete graphs, paths, cycles, complete bipartite graphs, connected split graphs, maximal outerplanar graphs, and complete graphs minus one edge), we study whether there is a graph G whose TS_k-reconfiguration graph belongs to $\mathcal{G}$. (See Table 1.) In Section 5, for different graph properties $\mathcal{P}$ (including s-partitedness, planarity, Eulerianity, girth, and the clique's size), we focus on answering the following question: if a graph G satisfies $\mathcal{P}$, whether $\mathsf{TS}(G)$ ($\mathsf{TS}_k(G)$) does too, and vice versa. (See Table 2.) In Section 6, we present a simple way of decomposing $\mathsf{TS}_k(G)$ into TS_k-reconfiguration graphs of G's subgraphs, provided that G contains certain structure. Finally, in Section 7, we summarize our results and propose some problems and directions for future study.

2   A Geometrical Example
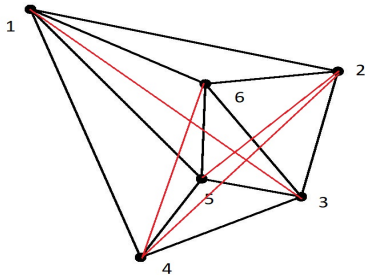
**Fig. 1.** Point set P with its line segments. A triangulation is shown by the black lines.
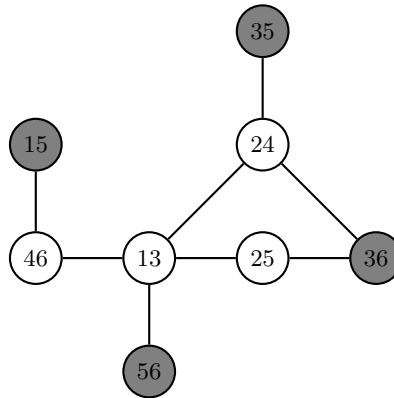
Fig. 2. Edge intersection graph G. Each number ab inside a node represents an intersecting line segment of P. The stable set corresponds to the triangulation in Fig. 1 is shaded grey.

In this section we present an example of a TS-reconfiguration graph arising from a well studied problem in computational geometry: triangulations on a set of planar points. For terminology and notation related to computational geometry which are not defined here, readers are referred to [17].

Given a set P of n points in the plane, no three collinear and no four co-circular. Two line segments are *intersecting* if they cross each other at an interior point of each segment, and *non-intersecting* otherwise. A *triangulation* of P is any maximal set on non-intersecting segments. It is well known that all triangulations have the same number of edges. The *edge intersection graph* G of P is the graph whose vertices V are the line segments with endpoints in P that intersect at least one other line segment. Let L be the remaining set of line segments defined by P. Note that any edge on the convex hull of P is in L and there may be other such line segments. In Fig. 1 we give a set of 6 points and show the 15 segments they define. A triangulation is shown by the solid black lines. The set L consists of the seven segments $\{12, 14, 16, 23, 26, 34, 45\}$. We see that the edges $\{16, 26, 45\}$ are internal to the convex hull. Two vertices in G are connected by an edge if their corresponding line segments intersect. In Fig. 2 we give the intersection graph G for the example, which has $\alpha(G) = 4$. The triangulation appears in G as the stable set $\{15, 35, 36, 56\}$.

Each vertex of $\mathsf{TS}_k(G)$ corresponds to a set of k pairwise non-intersecting line segments. Fig. 3 shows $\mathsf{TS}_4(G)$ for the example, where the independent sets are shown as their corresponding triangulations. The top left triangulation corresponds to the stable set $\{15, 35, 36, 56\}$. By token sliding we see that it is adjacent in $\mathsf{TS}_4(G)$ to stable sets $\{15, 35, 25, 56\}$, $\{46, 35, 36, 56\}$ and $\{15, 35, 36, 13\}$ corresponding to the triangulations T_2, T_6 and T_7. Geometrically we see that token sliding corresponds to *fliping* the diagonal of a convex quadrilateral in the triangulation. Lawson [13] showed that flipping well-chosen intersecting diagonals of convex quadrilaterals will lead to the unique Delaunay triangulation, which proves that $\mathsf{TS}_{\alpha(G)}(G)$ is connected. This result was
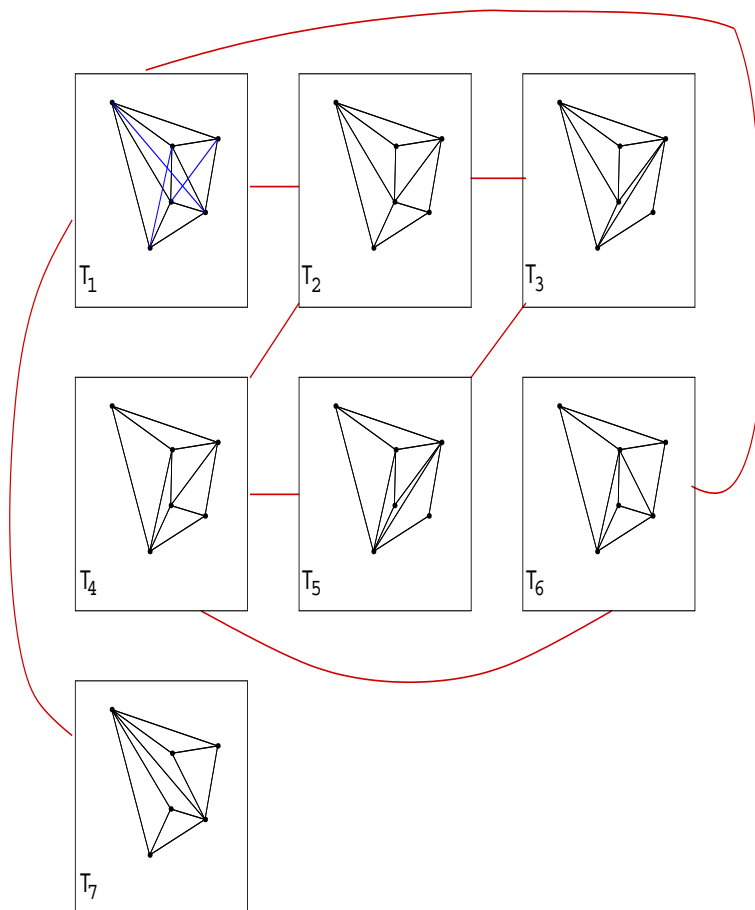
Fig. 3. Triangulations of P. The red edges correspond to edge flips. In T_1 (the triangulation in Fig. 1) the three possible edge flips, corresponding to token slides 15-46, 56-13 and 36-25 in G (Fig. 2), are shown in blue.

generalized by Bern and Eppstein [2] to the case where a non-intersecting set of the line segments can be specified and the triangulations are *constrained* to include these segments. In this case we add these constrained segments and any segments that intersect them to L. They were able to show that the corresponding constrained Delaunay triangulation could be obtained after at most $O(n^2)$ diagonal flips, giving a corresponding bound on the diameter of $\mathsf{TS}_{\alpha(G)}(G)$. Similar results on connectivity and diameter for some other graph classes are given in the Appendix A.

3 Preliminaries

For terminology and notation not defined here, readers are referred to [8]. Let G be a simple, undirected graph. We respectively use $V(G)$ and $E(G)$ to denote its vertex-set and edge-set. For two sets I, J, we sometimes use $I - J$ and $I + J$ to indicate $I \setminus J$ and $I \cup J$, respectively. Additionally, if $J = \{u\}$, we sometimes write $I - u$ and $I + u$ instead of $I - \{u\}$ and $I + \{u\}$, respectively. The *symmetric difference* of I and J, denoted by $I \triangle J$, is simply the set $(I - J) + (J - I)$. The *neighbors* of a vertex v in G, denoted by $N_G(v)$, is the set $\{w \in V(G) : vw \in E(G)\}$. The *closed neighbors* of v in G, denoted by $N_G[v]$, is simply the set $N_G(v) + v$. Similarly, for a vertex subset $I \subseteq V(G)$, its *neighbor* $N_G(I)$ and *closed neighbor* $N_G[I]$ are respectively $\bigcup_{v \in I} N_G(v)$ and $N_G(I) + I$. The *degree* of a vertex v in G, denoted by $\deg_G(v)$, is $|N_G(v)|$. For a vertex subset I, we denote by $G[I]$ the subgraph of G *induced* by vertices in I. An *independent set* (or *stable set*) of G is a vertex subset I such that for any $u, v \in I$, we have $uv \notin E(G)$. On the other hand, a *clique* of G is a vertex subset K such that for any $u, v \in K$, we have $uv \in E(G)$. We denote by $\alpha(G)$ and $\omega(G)$ the *maximum* size of an independent set and a clique of G, respectively. The *girth* of G, denoted by $\mathrm{girth}(G)$, is the smallest size of a cycle in G. In case G has no cycles, we define $\mathrm{girth}(G) = \infty$, and say that it has *infinite girth*, and otherwise G has *finite girth*. Two graphs G_1 and G_2 are *isomorphic* if there exists a bijection $f : V(G_1) \to V(G_2)$ such that $uv \in E(G_1)$ if and only if $f(u)f(v) \in E(G_2)$. For isomorphic graphs G_1 and G_2, we write $G_1 \simeq G_2$ or $G_1 \simeq_f G_2$ to emphasize that their isomorphism can be verified by the bijection f. A graph is *H-free* if it has no graph H as an induced subgraph.

We respectively denote by K_n, P_n, and C_n a *complete graph*, a *path*, and a *cycle* on n vertices. The graph $K_3 \simeq C_3$ is also called a *triangle*. $K_n - e$ is the graph obtained from K_n by removing exactly one edge. We

denote by $K_{m,n}$ a *complete bipartite graph* whose partite sets are of sizes m and n, for some positive integers $m \leq n$. The graph $K_{1,n}$ is also called a *star*. G is a *split graph* if $V(G)$ can be partitioned into two sets K and S, called a *KS-partition* of $V(G)$, such that K is a clique and S is a stable set of G. It is well-known that any split graph G has a unique *K-max KS*-partition of $V(G)$ such that $|K| = \omega(G)$ (e.g., see [6]) and we write $G = (K \cup S, E)_{K\text{-max}}$.

For a graph G and a positive integer k, the TS_k*-reconfiguration graph* of G, denoted by $\mathsf{TS}_k(G)$, takes all size-k independent sets of G as its nodes. Similarly, the TS*-reconfiguration graph* of G takes all independent sets of G as its nodes. Two nodes (independent sets) I and J are *adjacent* in either $\mathsf{TS}(G)$ or $\mathsf{TS}_k(G)$ if there exist $u, v \in V(G)$ such that $I - J = \{u\}$, $J - I = \{v\}$, and $uv \in E(G)$. Naturally, we call a graph F a TS_k*-reconfiguration graph* if there exists a graph G such that $F \simeq \mathsf{TS}_k(G)$ or more precisely $F \simeq_f \mathsf{TS}_k(G)$ where f is some bijection that can be used for verifying their isomorphism. One can think of f as a way to *label vertices of F by size-k independent sets of G*.

4   Graphs That Are (Not) TS_k-Reconfiguration Graphs

In this section, we study the realizability of TS_k-reconfiguration graphs. It is trivial that any graph G is also a TS_1-reconfiguration graph, since $G \simeq \mathsf{TS}_1(G)$. Therefore, in this section, we always assume $k \geq 2$.

For a graph G, let $L_k(G)$ be the graph whose nodes are size-k cliques of G and two nodes are adjacent if they have exactly $k-1$ vertices in common and its *complement* $\overline{G}$ is the graph whose vertices are $V(G)$ and two vertices are adjacent in $\overline{G}$ if they are not adjacent in G. In particular, $L_2(G)$ is also known as the *line graph* of G. The following lemma describes a relationship between $L_k(G)$ and the TS_k-reconfiguration graph of its complement $\overline{G}$. Fig. 4 illustrates this relationship for $k = 2$.

Lemma 1. *Given a graph G. Then, $\mathsf{TS}_k(\overline{G})$ is a subgraph of $L_k(G)$. Moreover, $L_k(G) \simeq \mathsf{TS}_k(\overline{G})$ if and only if G is K_{k+1}-free.*

Proof. By definition, for any $I = \{a_1, \ldots, a_k\} \in V(\mathsf{TS}_k(\overline{G}))$, we have $a_i a_j \notin E(\overline{G})$ and therefore $a_i a_j \in E(G)$ for $1 \leq i < j \leq k$, which implies that $a_1 \ldots a_k \in V(L_k(G))$. As a result, the mapping $f : V(\mathsf{TS}_k(\overline{G})) \to V(L_k(G))$ defined by $f(\{a_1, \ldots, a_k\}) = a_1 \ldots a_k$ is bijective. Moreover, if $I = \{a_1, \ldots, a_k\}$ and $J = \{a_1', \ldots, a_k'\}$ are adjacent in $\mathsf{TS}_k(\overline{G})$, we must have $|I \cap J| = k-1$, and therefore $a_1 \ldots a_k$ and $a_1' \ldots a_k'$ are adjacent in $L_k(G)$. Therefore, $\mathsf{TS}_k(\overline{G})$ is a subgraph of $L_k(G)$.

We now claim that $\mathsf{TS}_k(\overline{G}) \simeq_f L_k(G)$ if and only if G is K_{k+1}-free.

($\Rightarrow$) Suppose to the contrary that $\mathsf{TS}_k(\overline{G}) \simeq_f L_k(G)$ and G has a $K_{k+1} = a_1 \ldots a_k a_{k+1}$. Thus, $I = \{a_1, \ldots, a_{k-1}, a_k\}$ and $J = \{a_1, \ldots, a_{k-1}, a_{k+1}\}$ are vertices of $\mathsf{TS}_k(\overline{G})$. Since $a_1 \ldots a_{k-1} a_k$ and $a_1 \ldots a_{k-1} a_{k+1}$ are adjacent in $L_k(G)$, we have $IJ \in E(\mathsf{TS}_k(\overline{G}))$. It follows that $a_k a_{k+1} \in E(\overline{G})$, which means $a_k a_{k+1} \notin E(G)$, which is a contradiction. Therefore, G is K_{k+1}-free.

($\Leftarrow$) Suppose that G is K_{k+1}-free. It suffices to show that $IJ \in E(\mathsf{TS}_k(\overline{G}))$ if and only if $f(I)f(J) \in E(L_k(G))$. Since $\mathsf{TS}_k(\overline{G})$ is a subgraph of $L_k(G)$, the only-if direction is clear. We now show the if direction. Without loss of generality, let $a_1 \ldots a_{k-1} a_k$ and $a_1 \ldots a_{k-1} a_{k+1}$ be two adjacent vertices in $L_k(G)$. Since G is K_{k+1}-free, $a_k a_{k+1} \notin E(G)$, which means $a_k a_{k+1} \in E(\overline{G})$ and therefore $IJ \in E(\mathsf{TS}_k(\overline{G}))$, where $I = f^{-1}(a_1 \ldots a_{k-1} a_k)$ and $J = f^{-1}(a_1 \ldots a_{k-1} a_{k+1})$.

$\square$

In Fig. 4 we see that G contains a triangle and indeed $L_2(G)$ is not isomorphic to $\mathsf{TS}_2(\overline{G})$. If we break the triangle by deleting the edge $v_1 v_2$ this corresponds to deleting the vertex 12 in $L_2(G)$ and $\mathsf{TS}_2(\overline{G})$ along with adding the edge from 15 to 25. We now see that $L_2(G) \simeq \mathsf{TS}_2(\overline{G})$.

In the rest of this section, for some graph class $\mathcal{G}$, we answer the following question: Does there exist a graph G such that $\mathsf{TS}_k(G) \in \mathcal{G}$? Our results are described in Table 1.

We now prove some useful observations.

Proposition 2. *If H is an induced subgraph of G, then $\mathsf{TS}_k(H)$ is an induced subgraph of $\mathsf{TS}_k(G)$. The reverse does not hold for any $k \geq 2$.*

Proof. Suppose that there exist $I, J \in V(\mathsf{TS}_k(H))$ such that $IJ \in E(\mathsf{TS}_k(G))$. Since $IJ \in E(\mathsf{TS}_k(G))$, there exists $u, v \in V(G)$ such that $I - J = \{u\}$, $J - I = \{v\}$, and $uv \in E(G)$. Since $I, J \in V(\mathsf{TS}_k(H))$, it follows that $u, v \in V(H)$. As H is an induced subgraph of G and $uv \in E(G)$, we must have $uv \in E(H)$. Therefore, $IJ \in E(\mathsf{TS}_k(H))$.

Now, if $H = C_{2k}$ and $G = K_{1,k+1}$, one can readily verify that $\mathsf{TS}_k(H)$ is an induced subgraph of $\mathsf{TS}_k(G)$ but H is clearly not an induced subgraph of G.

$\square$

Proposition 3. *Given a graph H and let $G = \mathsf{TS}_{\alpha(H)}(H)$. Then, for every $k \geq \alpha(H)$, G is a TS_k-reconfiguration graph. More precisely, there exists a graph H' having $|V(H)| + k - \alpha(H)$ vertices and $|E(H)|$ edges such that $G \simeq \mathsf{TS}_k(H')$.*
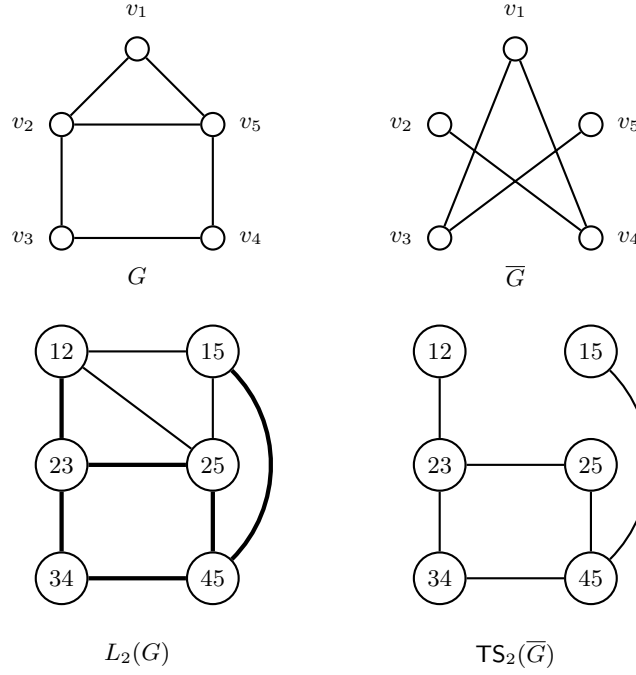
Fig. 4. A graph G, its complement $\overline{G}$, its line graph $L_2(G)$, and the graph $\mathsf{TS}_2(\overline{G})$. Each number of the form ab inside a node represents a vertex subset $\{v_a, v_b\}$ which forms both a node in $L_2(G)$ and an independent set of $\overline{G}$.

Table 1. Does G such that $\mathsf{TS}_k(G) \in \mathcal{G}$ exist $(k \geq 2)$?

$\mathcal{G}$	Does G exist?	Ref.
K_n	yes $\|V(G)\| = n + k - 1$ $\|E(G)\| = n(n-1)/2$	Cor. 4
P_n	yes $\|V(G)\| = (n+1) + k - 2 = n + k - 1$ $\|E(G)\| = n(n-1)/2$	Cor. 5
C_n	yes $\|V(G)\| = n + k - 2$ $\|E(G)\| = n(n-3)/2$	Cor. 5
$K_{m,n}$ $(m \leq n)$	yes, iff $m = 1$ and $n \leq k$ or $m = n = 2$ $\|V(G)\| = n + k$ or $\|V(G)\| = k + 2$ $\|E(G)\| = n(n+1)/2$ or $\|E(G)\| = 2$	Prop. 7
connected $F = (K \cup S, E)_{K\text{-max}}$	yes, iff $\|N_F(v) \cap S\| \leq k - 1$ and $\|N_F(w)\| = 1$ for every $v \in K$ and $w \in S$ $\|V(G)\| = \|K\| + \|S\| + k - 1$ $\|E(G)\| = \binom{\|K\|}{2} + \binom{\|S\|}{2} + \sum_{v \in K} \|N_F(v) \cap S\| +$ $+ \sum_{\substack{v \in K \\ \|N_F(v) \cap S\| \neq 0}} (\|K\| - 1)$	Prop. 9
maximal outerplanar $K_n - e$	yes, iff $n \leq 3$	Prop. 10

Proof. Let H' be the graph obtained by adding a set X of $(k - \alpha(H))$ new vertices to H. Any size-k independent set in H' is a disjoint union of X and a maximum independent set in H. Then, $G = \mathsf{TS}_{\alpha(H)}(H) \simeq \mathsf{TS}_k(H')$.

A direct consequence of Proposition 3 when $H = K_n$ is as follows.

Corollary 4. K_n *is a TS_k-reconfiguration graph, for any integers $k \geq 2$ and $n \geq 2$.*

A direct consequence of Lemma 1 and Proposition 3 is as follows.

Corollary 5. *(a) P_n is a TS_k-reconfiguration graph, for any integers $k \geq 2$ and $n \geq 1$.*
(b) C_n is a TS_k-reconfiguration graph, for any integers $k \geq 2$ and $n \geq 3$.

Proof. (a) Since P_n is triangle-free, from Lemma 1 we have

$$P_n \simeq L_2(P_{n+1}) \simeq \mathsf{TS}_2(\overline{P_{n+1}}).$$

This settles the case $k = 2$. For $k = 3$, note that $\alpha(\overline{P_{n+1}}) = \omega(P_{n+1}) = 2$, therefore it follows from Proposition 3 that P_n is a TS_k-reconfiguration graph for every $k \geq 2$.

(b) Since $C_3 \simeq K_3$, Corollary 4 settles the case $n = 3$. For $n \geq 4$, again, since C_n ($n \geq 4$) is triangle-free, from Lemma 1 we have

$$C_n \simeq L_2(C_n) \simeq \mathsf{TS}_2(\overline{C_n}).$$

This settles the case $k = 2$. Again, for $k = 3$, note that $\alpha(\overline{C_n}) = \omega(C_n) = 2$, therefore it follows from Proposition 3 that C_n ($n \geq 4$) is a TS_k-reconfiguration graph for every $k \geq 2$.

Under certain conditions, one can construct a new TS_k-reconfiguration graph from a known one, as in the following proposition.

Proposition 6. *Let I be an independent set of a given graph G. Let $k = |I| + 1$. Let G' be the graph obtained from G by adding a new vertex v_G and joining it to every vertex in $V(G) - I$. Then, $\mathsf{TS}_k(G')$ is obtained from $\mathsf{TS}_k(G)$ by adding a new node $I + v_G$ and all incident edges.*

Proof. By definition, G is an induced subgraph of G', and therefore by Proposition 2, $\mathsf{TS}_k(G)$ is an induced subgraph of $\mathsf{TS}_k(G')$. It suffices to show that $I + v_G$ is the unique node in $V(\mathsf{TS}_k(G')) - V(\mathsf{TS}_k(G))$. Observe that for any $J \in V(\mathsf{TS}_k(G')) - V(\mathsf{TS}_k(G))$, we must have $v_G \in J$. Now, if $J \neq I + v_G$, there must be some $w \in V(G) - I$ such that $w \in J$. However, by definition of G', we have $wv_G \in E(G')$, which contradicts $\{w, v_G\} \subseteq J \in V(\mathsf{TS}_k(G'))$. Therefore, $J = I + v_G$. Our proof is complete. $\square$

Proposition 7. *$K_{m,n}$ is a TS_k-reconfiguration graph for some integers $k \geq 2$ and $n \geq m \geq 1$ if and only if $m = 1$ and $n \leq k$ or $m = n = 2$.*

Proof.($\Leftarrow$) Corollary 5 settles the case $m = n = 2$, since $K_{2,2} \simeq C_4$. It remains to consider the case $m = 1$ and $n \leq k$. In this case, we claim that there exists a graph G such that $\mathsf{TS}_k(G) \simeq K_{1,n}$. Let $I_1 = \{a_1, \ldots, a_k\}$ be an independent set of size k, and let K_n be a clique whose vertices are $b_1, \ldots, b_n$. We construct G by joining each a_i with b_i, for $1 \leq i \leq n$.

It remains to show that $\mathsf{TS}_k(G) \simeq K_{1,n}$. Note that $V(\mathsf{TS}_k(G)) = I_1 \cup \bigcup_{i=1}^{n} J_i$, where $J_i = I_1 - a_i + b_i$ for $1 \leq i \leq n$. It follows that $\mathsf{TS}_k(G)$ has exactly $n + 1$ vertices. Additionally, from the construction of G, the set I_1 is adjacent to every J_i ($1 \leq i \leq n$) in $\mathsf{TS}_k(G)$, and for every $1 \leq i < j \leq n$, we always have $|J_j - J_i| = |J_i - J_j| = 2$, which implies that $J_i J_j \notin E(\mathsf{TS}_k(G))$. Therefore, $\mathsf{TS}_k(G) \simeq K_{1,n}$.

($\Rightarrow$) We first show that if $m = 1$ and $n \geq k + 1$, there does not exist any G such that $\mathsf{TS}_k(G) \simeq K_{m,n}$, where $k \geq 2$. Suppose to the contrary that $n \geq k + 1$ and G exists. Let $V(\mathsf{TS}_k(G)) = \{I_1, J_1, \ldots, J_n\}$ and assume without loss of generality that I_1 is adjacent to J_i in $\mathsf{TS}_k(G)$ for $1 \leq i \leq n$. Since $n \geq k + 1$, by the pigeonhole principle, there must be some vertex $u \in I_1$ such that sliding the token on u results at least two different size-k independent sets of G, say J_1 and J_2, that are both adjacent to I_1. Then, we can write $I_1 = I + x + u$, $J_1 = I + x + v$, and $J_2 = I + x + w$, for some size-$(k-2)$ independent set I of G such that none of the distinct vertices u, v, w, x is in I. By definitions of J_1 and J_2, both v and w are not in I_1. Now, let $J_3 = I + v + w$. One can verify that $J_3 \in V(\mathsf{TS}_k(G))$ and therefore must be adjacent to I_1 in $\mathsf{TS}_k(G)$, which implies $|J_3 - J_1| = 1$. However, note that $\{v, w\} \subseteq J_3 - I_1$, which is a contradiction.
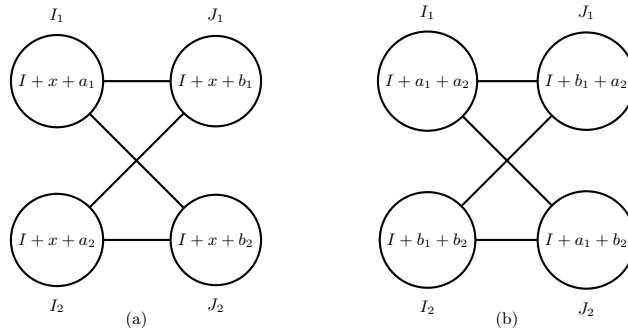
Fig. 5. Two possible forms of the 4-cycle $I_1 J_1 I_2 J_2$ in the proof of Proposition 7. Here I is a size-$(k-2)$ independent set of a graph G.

It remains to show that if $m \geq 2$ and $n > 2$, there does not exist any G such that $\mathsf{TS}_k(G) \simeq K_{m,n}$. Again, suppose to the contrary that $m \geq 2$, $n > 2$, and G exists. Suppose that $\mathsf{TS}_k(G)$ has the partite sets $X = \{I_1, \ldots, I_m\}$ and $Y = \{J_1, \ldots, J_n\}$, where I_i ($1 \leq i \leq m$) and J_j ($1 \leq j \leq n$) are size-k independent

sets of G. Let consider the length-4 cycle $I_1 J_1 I_2 J_2$, and suppose that we initially slide tokens in I_1. To form such a cycle, at most two tokens in I_1 can be moved from their original positions, otherwise we need to perform more than four token-slides to obtain a cycle. More formally, for some size-$(k-2)$ independent set I of G, the sets I_1, J_1, I_2, J_2 can only be in one of the following two forms (see Fig. 5):

(a) $I_i = I + x + a_i$ and $J_i = I + x + b_i$ ($1 \leq i \leq 2$). Intuitively, this corresponds to sliding a single token in I_1 along the cycle $a_1 b_1 a_2 b_2$ of G.

(b) $I_1 = I + a_1 + a_2$, $J_1 = I + b_1 + a_2$, $I_2 = I + b_1 + b_2$, and $J_2 = I + a_1 + b_2$. Intuitively, this corresponds to sliding tokens "back and forth" along the edges $a_1 b_1$ and $a_2 b_2$ of G.

In both cases, none of x, a_i and b_i is in I, for $1 \leq i \leq 2$. It remains to show that both cases lead to some contradiction. Now, if (a) happens, let $J = I + a_1 + a_2$. One can verify that $J \in \mathsf{TS}_k(G)$. By definition of J, it cannot be adjacent to I_1 in $\mathsf{TS}_k(G)$, otherwise $xa_2 \in E(G)$, which contradicts $I_2 = I + x + a_2 \in V(\mathsf{TS}_k(G))$. Additionally, J cannot be adjacent to J_1, because $\{a_1, a_2\} \subseteq J - J_1$. However, since $\mathsf{TS}_k(G) \simeq K_{m,n}$, it follows that J must be adjacent to either I_1 or J_1, which is a contradiction.

It remains to consider the case (b) happens. In this case, let $J \in \mathsf{TS}_k(G)$ be such that J is adjacent to both I_1 and I_2 and $J \notin \{J_1, J_2\}$. Since $n > 2$, such a set J exists. Now, if $\{a_1, a_2\} \subseteq J$, we also have $\{a_1, a_2\} \subseteq J - I_2 = J - (I + b_1 + b_2)$. (Both a_1 and a_2 are not in I.) This contradicts the adjacency of J and I_2. Therefore, $\{a_1, a_2\} \nsubseteq J$, and similarly, so does $\{b_1, b_2\}$. Since J is adjacent to both I_1 and I_2, it contains at least one member of $\{a_1, a_2\}$ and $\{b_1, b_2\}$, respectively. It follows that either $\{a_1, b_2\}$ or $\{a_2, b_1\}$ is in J. Now, if $\{a_1, b_2\} \subseteq J$, it follows that $J = I + a_1 + x = I + b_2 + y$ for some $x, y \in V(G)$ such that $xa_2, yb_1 \in E(G)$. Then, it follows that $x = b_2$ and $y = a_1$, that is, $J = I + a_1 + b_2 = J_1$, a contradiction. The case $\{a_2, b_1\} \subseteq J$ can be showed similarly. Our proof is complete.
□

Lemma 8. *Given a graph G. Then, if $\mathsf{TS}_k(G)$ has a K_n, so does G, for integers $k \geq 2$ and $n \geq 3$.*

Proof. We first consider the case $n = 3$, fix any $k \geq 2$ and suppose that $\mathsf{TS}_k(G)$ has a triangle labelled by independent sets I_1, I_2, I_3. Without loss of generality we may assume $I_1 = \{a_1, \ldots, a_{k-1}, w\}$ and $I_2 = \{a_1, \ldots, a_{k-1}, x\}$, where $wx \in E(G)$. Now, there are two possibilities for I_3: either (1) $I_3 = \{a_1, \ldots, a_{k-1}, y\}$ where $xy \in E(G)$ or (2) $I_3 = \{a_1, \ldots, a_{i-1}, y, a_{i+1}, \ldots, a_{k-1}, x\}$ where $a_i y \in E(G)$ for some $i \in \{1, \ldots, k-1\}$. Since I_1 and I_3 are adjacent in $\mathsf{TS}_k(G)$, only (1) can happen. Therefore, we must have $yw \in E(G)$ and so w, x, y define a triangle in G.
□

To conclude this section, we show that a connected split graph is a TS_k-reconfiguration graph if and only if it satisfies certain restricted conditions, as described in the following lemma.

Proposition 9. *Fix $k \geq 2$. A connected split graph $F = (K \cup S, E)_{K\text{-max}}$ is a TS_k-reconfiguration graph if and only if $|N_F(v) \cap S| \leq k - 1$ and $|N_F(w)| = 1$ for every $v \in K$ and $w \in S$.*

Proof. ($\Leftarrow$) For a connected split graph $F = (K \cup S, E)_{K\text{-max}}$ suppose that $|N_F(v) \cap S| \leq k - 1$ and $|N_F(w)| = 1$ for every $v \in K$, $w \in S$. Additionally, let $m = |K|$ and $n = |S|$. Suppose that $K = \{v_1, \ldots, v_m\}$ and let $n_i = |N_F(v_i) \cap S| = |\{w_1^i, \ldots, w_{n_i}^i\}|$ for $1 \leq i \leq m$. Observe that $n = \sum_{i=1}^{m} n_i$ and by assumption $n_i \leq k-1$ for every $1 \leq i \leq m$. We construct a graph G such that $\mathsf{TS}_k(G) \simeq F$ as follows. (See Fig. 6.)

- Let $I = \{a_1, \ldots, a_{k-1}\}$ be an independent set. Let K_m be a size-m complete graph with $V(K_m) = \{b_1, \ldots, b_m\}$. Let K_n be a size-n complete graph with $V(K_n) = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n_i} \{x_j^i\}$.
- The graph G has $V(G) = I \cup V(K_n) \cup V(K_m)$. The edges of G are defined by joining each $x_j^i \in V(K_n)$ to $a_j \in I$ ($1 \leq j \leq n_i$) and every vertex in $V(K_m) - b_i$.

Let $f : V(F) \to V(\mathsf{TS}_k(G))$ be such that $f(v_i) = K^i = I + b_i$ and $f(w_j^i) = K^i - a_j + x_j^i$, for $1 \leq i \leq m$ and $1 \leq j \leq n_i$. One can verify that f is well-defined and bijective. We claim that $F \simeq_f \mathsf{TS}_k(G)$, i.e., $uv \in E(F)$ if and only if $f(u)f(v) \in E(\mathsf{TS}_k(G))$.

($\Rightarrow$) Suppose that $uv \in E(F)$, i.e., either both u, v are in K or $u \in K$ and $v \in S$. If $u = v_i \in K$ and $v = v_j \in K$, since K_m is a complete subgraph of G, we have $f(u)f(v) = K^i K^j \in E(\mathsf{TS}_k(G))$. Otherwise, if $u = v_i \in K$ and $v = w_j^i \in S$, since a_j is the only vertex in $\{a_1, \ldots, a_{k-1}\}$ adjacent to x_j^i in G, we have $f(u)f(v) = K^i(K^i - a_j + x_j^i) \in E(\mathsf{TS}_k(G))$.

($\Leftarrow$) Suppose that $f(u)f(v) \in E(\mathsf{TS}_k(G))$, i.e., either both $f(u), f(v)$ are in $\bigcup_{i=1}^{m} K^i$ or $f(u) \in \bigcup_{i=1}^{m} K^i$ and $f(v) \in \bigcup_{i=1}^{m} \bigcup_{j=1}^{n_i} (K^i - a_j + x_j^i)$. If $f(u) = K^i$ and $f(v) = K^j$, by definition, we have $uv = v_i v_j \in E(F)$. Otherwise, if $f(u) = K^i$ and $f(v) \in \bigcup_{i=1}^{m} \bigcup_{j=1}^{n_i} (K^i - a_j + x_j^i)$, since $f(u)f(v) \in E(\mathsf{TS}_k(G))$, we must have $f(v) = K^i - a_j + x_j^i$ for some $j \in \{1, \ldots, n_i\}$, and therefore $uv = v_i w_j^i \in E(F)$.

($\Rightarrow$) Suppose that either (1) there exists $w \in S$ such that $|N_F(w)| \geq 2$ or (2) $|N_F(w)| = 1$ for every $w \in S$ and there exists $v \in K$ such that $|N_F(v) \cap S| \geq k$. We claim that F is not a TS_k-reconfiguration graph. Suppose to the contrary that there exists a graph G such that $F \simeq_f \mathsf{TS}_k(G)$. Since F is connected, so is $\mathsf{TS}_k(G)$. From Lemma 8, we must have $f(v_i) = I + b_i$ ($1 \leq i \leq m$) where $\bigcup_{i=1}^{m} \{b_i\}$ forms a complete subgraph K_m of G and $I = \{a_1, \ldots, a_{k-1}\}$ is an independent set of $G - N_G[V(K_m)]$.
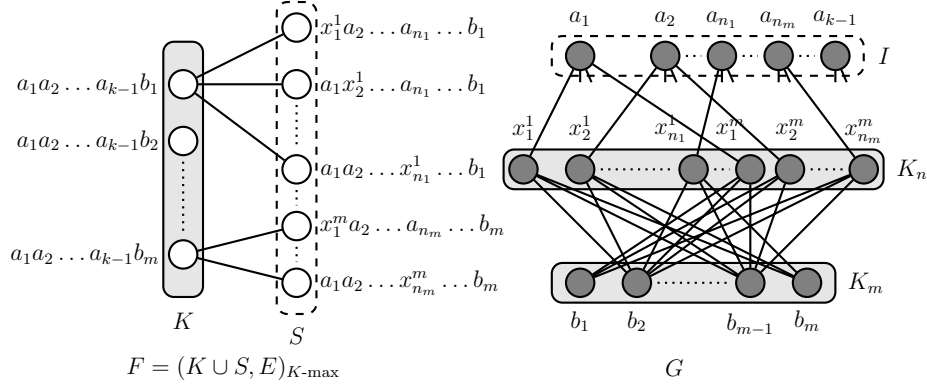
Fig. 6. Construction of a graph G such that $F \simeq \mathsf{TS}_k(G)$, where F is a connected split graph satisfying Proposition 9. Vertices of F are labeled by size-k stable sets of G. Vertices in a light gray (resp., dashed) box forms a clique (resp., stable set).

If (1) holds, since $|K| = \omega(F)$, there exists $i \in \{1, \ldots, m\}$ such that $v_i w \notin E(F)$, otherwise $K + w$ forms a clique in F of size $\omega(F) + 1$, which is a contradiction. Lemma 8 also implies that $f(w) = I + x$ for some $x \in V(G) - V(K_m) - I$. Since $v_i w \notin E(F)$, we also have $f(v_i)f(w) = (I + b_i)(I + x) \notin E(\mathsf{TS}_k(G))$, which implies $b_i x \notin E(G)$. Then, one can verify that $J = (I - a_1) + b_i + x \in V(\mathsf{TS}_k(G))$. Note that $b_i \notin f(v_j) = I + b_j$ for $j \neq i$ and $1 \leq j \leq m$. Since $b_i x \notin E(G)$, we have $x \neq b_j$. Additionally, since $x \notin I$, we must also have $x \notin f(v_j)$. Since $\mathsf{TS}_k(G)$ is connected, J must be adjacent to some $f(v_j)$ ($1 \leq j \leq m$), which is a contradiction since $\{b_i, x\} \subseteq J - f(v_j)$ for $j \neq i$ (i.e., J is not adjacent to any $f(v_j)$ for $j \neq i$) and $a_1 x \notin E(G)$ (i.e., J is not adjacent to $f(v_i) = I + b_i$).

If (2) holds, suppose that $v = v_i$ for some $i \in \{1, \ldots, m\}$. As before, one can show that if $f(w) = I + x$ for some $w \in N_F(v) \cap S$, there must be some contradiction. Therefore, for every $w \in N_F(v) \cap S$, we must have $f(w) = (I - a_j) + x + b_i$ for some $j \in \{1, \ldots, k-1\}$. Since $|N_F(v) \cap S| \geq k$, by the pigeonhole principle, there exists $j \in \{1, \ldots, k-1\}$ such that $f(v) = f(v_i) = I + b_i$ has two distinct non-adjacent neighbors $J_1 = (I - a_j) + x_1 + b_i$ and $J_2 = (I - a_j) + x_2 + b_i$. Again, both x_1 and x_2 are not in I, and since none of them are adjacent to b_i, they are also not in $\{b_1, \ldots, b_m\}$. This implies $\{x_1, x_2\} \cap f(v_i) = \emptyset$ for $1 \leq i \leq m$. One can verify that $J = (I - a_j) + x_1 + x_2 \in V(\mathsf{TS}_k(G))$ and it is not adjacent to any $f(v_i)$ ($1 \leq i \leq m$) since $\{x_1, x_2\} \subseteq J - f(v_i)$. This contradicts the connectivity of $\mathsf{TS}_k(G)$. $\qquad\square$

We conclude this section with a general result that applies to all graphs. $K_{1,3}$ is also known as the *claw* and adding an edge between any two of its degree-1 vertices results the *paw*. $K_4 - e$ is also known as the *diamond*. (See Fig. 7.)
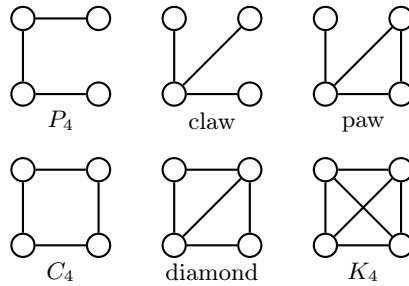
Fig. 7. Connected graphs on four vertices.

Proposition 10. *(a) Let G be a graph whose components are $G_1, G_2, \ldots, G_p$, for some $p \geq 1$. Suppose that all G_i ($1 \leq i \leq p$) are TS_k-reconfiguration graphs for some fixed integer $k \geq 2$. Then, so is G.*
(b) The diamond $K_4 - e$ is the unique smallest graph that is not a TS_k-reconfiguration graph for any $k \geq 2$.
(c) The following generalized graphs on $n \geq 4$ vertices of the diamond are not TS_k-reconfiguration graphs for any $k \geq 2$: Maximal outerplanar graphs and $K_n - e$.

(d) *There exists a disconnected* TS_k*-reconfiguration graph G such that one of its components is not a* TS_k*-reconfiguration graph for any $k \geq 2$.*

Proof. (a) Let H_i $(1 \leq i \leq p)$ be such that $G_i = \mathsf{TS}_k(H_i)$. Let H be the graph obtained by joining each vertex of H_i to all vertices of H_j, for every $1 \leq i < j \leq p$. From the above construction, any size-k independent set of H must be also an independent set of H_i, for some $i \in \{1, \ldots, p\}$. As a result, $\mathsf{TS}_k(H)$ is the disjoint union of $\mathsf{TS}_k(H_i)$ for all i, that is, $G = \mathsf{TS}_k(H)$.

(b) Graphs on at most three vertices are disjoint union of paths and cycles, and since any path or cycle is a TS_k-reconfiguration graph for $k \geq 2$ (Corollary 5), it follows from (a) that so are their disjoint unions. If a graph on four vertices is disconnected, each of its components has at most three vertices, and we are done. Thus, it suffices to consider six connected graphs on four vertices described in Fig. 7. Among these graphs, P_4, C_4, and K_4 are always TS_k-reconfiguration graphs for $k \geq 2$ and the claw $K_{1,3}$ is a TS_k-reconfiguration graph for $k \geq 3$. Let H be the *kite*—the graph obtained from a diamond by attaching a leaf to one of its degree-2 vertices. One can verify that $\mathsf{TS}_2(H)$ is indeed a paw, and since $\alpha(H) = 2$, it follows from Proposition 3 that $\mathsf{TS}_k(H + (k-2)K_1)$ is also a paw, i.e., the paw is always a TS_k-reconfiguration graph for $k \geq 2$.

It remains to show that the diamond is not a TS_k-reconfiguration graph for any $k \geq 2$. Suppose to the contrary that there exists a graph Q such that $K_4 - e = \mathsf{TS}_k(Q)$. It follows from Lemma 8 that vertices of $K_4 - e$ must be of the form $I_1 = I + a$, $I_2 = I + b$, $I_3 = I + c$, $I_4 = I + d$, where I is a size-$(k-1)$ independent set of Q and $\{a, b, c, d\} \subseteq V(Q) - I$ induce a diamond in which, say a and c are non-adjacent. However, for any $x \in I$, note that $J = (I - x) + a + c$ is a size-k independent set of Q, and therefore it must be a vertex of $K_4 - e$, which is a contradiction since $J \notin \{I_1, \ldots, I_4\}$.

(c) Use a similar argument as in the proof for the diamond.

(d) Take $G = \mathsf{TS}_k(\text{diamond} + (k-1)K_1)$. It is not hard to verify that the diamond is a component of G and from (b) it is not a TS_k-reconfiguration graph for any $k \geq 2$.

5   Properties of Reconfiguration Graphs

In this section we study various graph properties to see if they are inherited by their TS-reconfiguration graphs, and vice versa. More precisely, for a graph property $\mathcal{P}$ and a given graph G, we write $\mathcal{P}(G)$ to indicate G satisfies $\mathcal{P}$ and if $\mathcal{P}$ is inherited by $\mathsf{TS}(G)$ we write $\mathcal{P}(G) \Rightarrow \mathcal{P}(\mathsf{TS}(G))$. Conversely, $\mathcal{P}(\mathsf{TS}(G)) \Rightarrow \mathcal{P}(G)$ means that if $\mathcal{P}$ holds for $\mathsf{TS}(G)$ then it holds for G. For any fixed k we define similar notations $\mathcal{P}(G) \Rightarrow \mathcal{P}(\mathsf{TS}_k(G))$ and $\mathcal{P}(\mathsf{TS}_k(G)) \Rightarrow \mathcal{P}(G)$. Since $G \simeq \mathsf{TS}_1(G)$, the case $k = 1$ is uninteresting and therefore we only consider $k \geq 2$. Our results are summarized in Table 2.

Table 2. Some properties of (reconfiguration) graphs. Here $n = |V(G)|$. There are four cases: (a) $\mathcal{P}(G) \Rightarrow \mathcal{P}(\mathsf{TS}(G))$, (b) $\mathcal{P}(\mathsf{TS}(G)) \Rightarrow \mathcal{P}(G)$, (c) $\mathcal{P}(G) \Rightarrow \mathcal{P}(\mathsf{TS}_k(G))$, and (d) $\mathcal{P}(\mathsf{TS}_k(G)) \Rightarrow \mathcal{P}(G)$.

$\mathcal{P}$	G	(a)	(b)	(c)	(d)	Ref.
s-partite	general		yes		no	Prop. 11
planar	P_n	yes, iff $n \leq 8$		yes, iff $k = 2$, $n \geq 3$ or $k \geq 3$, $n \leq 8$		Prop. 12
	tree		yes, iff $n \leq 7$			Prop. 13
	C_n		yes, iff $n \leq 6$			Prop. 14
	connected					
Eulerian	C_n	no	yes	yes, iff $1 \leq k < n/2$		Prop. 16
	general	no	yes	no	no	Prop. 17
infinite girth	P_n	yes, iff $n \leq 4$		yes, iff $n \leq 2k$		Prop. 18
finite girth	C_n		yes		yes, iff $1 \leq k < n/2$	Prop. 19
having K_s	general		yes	no	yes	Prop. 20

5.1   *s*-Partitedness

A *proper s-coloring* of a graph G is a mapping $f : V(G) \to \{0, \ldots, s-1\}$ such that $f(u) \neq f(v)$ if $uv \in E(G)$, where $u, v \in V(G)$ and s is a positive integer. If G has a proper s-coloring, we call it a *s-partite* graph. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the smallest number s such that G has a proper s-coloring.

Proposition 11. *(a) G is s-partite if and only if $\mathsf{TS}(G)$ is s-partite. In other words, $\chi(G) = \chi(\mathsf{TS}(G))$.*
(b) For each $s \geq 2$, one can construct a graph G such that $\chi(\mathsf{TS}_k(G)) < \chi(G) = s$, for every $2 \leq k \leq \alpha(G)$.

Proof. (a) (by Masahiro Takahashi.) The if direction is directly followed from $G \simeq \mathsf{TS}_1(G)$. It remains to show that if G is s-partite, so is $\mathsf{TS}(G)$. Since G is s-partite, there exists a proper s-coloring $f : V(G) \to \{0, \ldots, s-1\}$ of vertices of G. For each independent set I of G, let $g(I) = \left(\sum_{v \in I} f(v) \right) \mod s$. Since G is s-partite, each TS-step only slides a token from one color class to another, and therefore if I and J are adjacent in $\mathsf{TS}(G)$, $g(I) \neq g(J)$. As a result, g is a s-coloring of $\mathsf{TS}(G)$, which means $\mathsf{TS}(G)$ is s-partite. To see that $\chi(G) = \chi(\mathsf{TS}(G))$, note that since $G \simeq \mathsf{TS}_1(G)$ is a subgraph of $\mathsf{TS}(G)$, we have $\chi(G) \leq \chi(\mathsf{TS}(G))$. On the other hand, if G has a $\chi(G)$-coloring, so does $\mathsf{TS}(G)$, i.e., $\chi(\mathsf{TS}(G)) \leq \chi(G)$.

(b) Fix $s \geq 2$. Let G' be any graph with $\chi(G') = s$ and construct G from G' by adding an additional vertex x adjacent to all vertices in $V(G')$. We have $\alpha(G) = \alpha(G')$ and since a new colour is required for x, $\chi(G) = \chi(G') + 1$. For each $2 \leq k \leq \alpha(G)$ the stable sets of size k in G and G' are identical and so is their adjacency, hence $\mathsf{TS}_k(G) \simeq \mathsf{TS}_k(G')$. Since $\mathsf{TS}_k(G')$ is a subgraph of $\mathsf{TS}(G')$ we may use part (a) to obtain: $\chi(\mathsf{TS}_k(G)) = \chi(\mathsf{TS}_k(G')) \leq \chi(\mathsf{TS}(G')) = \chi(G') = \chi(G) - 1$.

□

5.2   Planarity

A graph is *planar* if one can draw it in the plane such that its edges intersect only at their endpoints. The well-known Kuratowski's Theorem says that a graph G is planar if and only if it does not contain any subdivision of K_5 or $K_{3,3}$ as a subgraph.

Proposition 12. *(a) $\mathsf{TS}_2(P_n)$ is planar for every $n \geq 3$.*
(b) $\mathsf{TS}_3(P_n)$ is planar for every $n \leq 8$ and non-planar otherwise.
(c) $\mathsf{TS}(P_n)$ is planar for $n \leq 8$ and non-planar otherwise.

Proof. (a) Suppose that $P_n = v_1 \ldots v_n$ ($n \geq 3$). For each node $\{v_i, v_j\} \in V(\mathsf{TS}_2(P_n))$ ($1 \leq i, j \leq n$ and $|i-j| \geq 2$), its neighbors form a non-empty subset of $\left\{ \{v_{i-1}, v_j\}, \{v_{i+1}, v_j\}, \{v_i, v_{j-1}\}, \{v_i, v_{j+1}\} \right\}$. Therefore, $\mathsf{TS}_2(P_n)$ can be embedded into a $n \times n$ planar grid graph, which means it is also planar.

(b) In Fig. 8, we show a planar embedding of $\mathsf{TS}_3(P_8)$. In Fig. 9 we show a subdivision of $K_{3,3}$ that is contained in $\mathsf{TS}_3(P_9)$.

(c) $\mathsf{TS}_1(P_8) \simeq P_8$ and it can readily be verified that $\mathsf{TS}_4(P_8) \simeq P_5$. From (a), $\mathsf{TS}_2(P_8)$ is planar. From (b), $\mathsf{TS}_3(P_8)$ is planar and $\mathsf{TS}_3(P_9)$ is not.

□

Proposition 13. $\mathsf{TS}(T)$ *is planar for all trees T having at most seven vertices and non-planar otherwise.*

Proof. A computer search showed that:

- All eleven trees T on seven vertices satisfy $\mathsf{TS}(T)$ is planar.
- Among twenty-three trees T on eight vertices, seven of them satisfy $\mathsf{TS}(T)$ is non-planar.

See https://hoanganhduc.github.io/graphs/ for a list of the mentioned graphs and their corresponding TS-reconfiguration graphs.

□

Proposition 14. *(a) $\mathsf{TS}(C_n)$ is planar for $n \leq 6$ and non-planar otherwise.*
(b) With respect to the number of vertices, C_7 is the unique smallest graph G for which $\mathsf{TS}(G)$ is non-planar.
(c) With respect to the number of edges, including C_7, there are eight smallest graphs G for which $\mathsf{TS}(G)$ is non-planar.

Proof. A computer search showed that

(i) For each of the 99 connected planar graphs G on six vertices, $\mathsf{TS}(G)$ is planar.
(ii) There are seven trees T on eight vertices whose $\mathsf{TS}(T)$ is non-planar. (Proposition 13.)
(iii) $\mathsf{TS}_2(C_7)$ is non-planar. (Fig. 10)

See https://hoanganhduc.github.io/graphs/ for a list of the mentioned graphs and their corresponding TS-reconfiguration graphs.

□

Corollary 15. *(i) If $girth(G) \geq 7$, the graph $\mathsf{TS}(G)$ is non-planar.*
(ii) There exists a graph G whose girth is 3 and $\mathsf{TS}(G)$ is non-planar.

Proof. (i) Let C be a cycle in G whose length is $girth(G) \geq 7$. Note that C does not contain any chord; otherwise, we can find a cycle smaller than C, which contradicts our assumption. Now, consider the subgraph $\mathcal{H}$ of $\mathsf{TS}(G)$ induced by only independent sets in C. It follows from Proposition 14 that $\mathcal{H}$ is non-planar, and so is $\mathsf{TS}(G)$.

(ii) Take G as the graph obtained by connecting every vertex of P_n to some vertex v not in P_n. The subgraph of $\mathsf{TS}(G)$ induced by independent sets in P_n is non-planar when $n \geq 9$ (see Proposition 12).

□
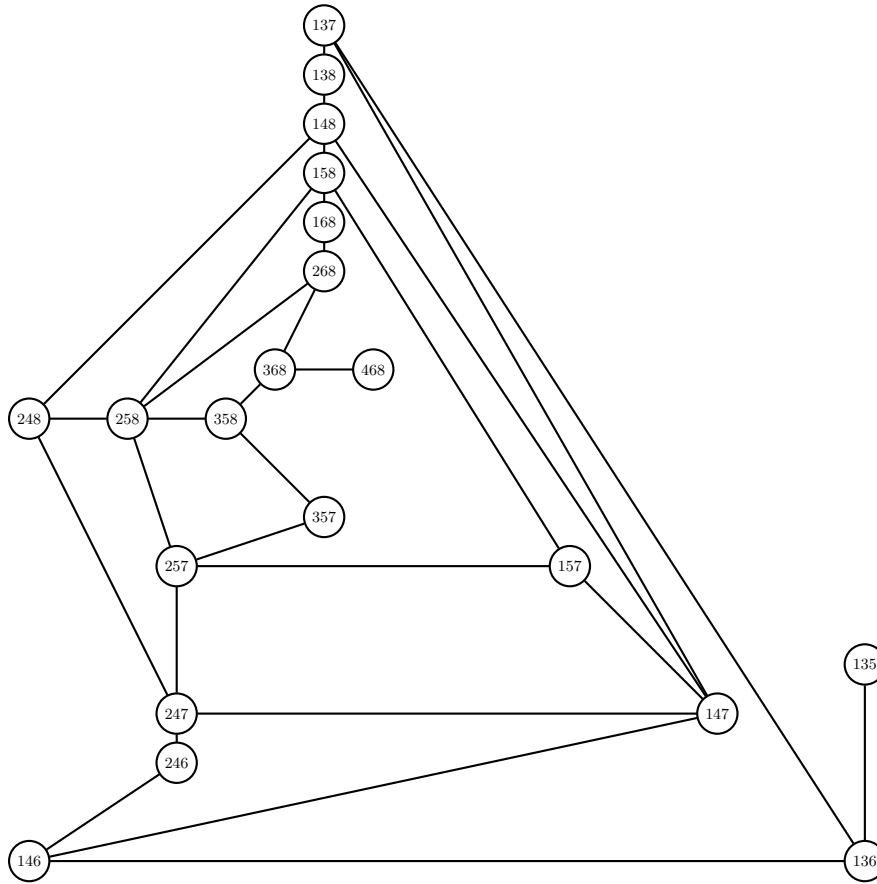
Fig. 8. A planar drawing of $\mathsf{TS}_3(P_8)$. Each number of the form abc inside a node represents an independent set $\{v_a, v_b, v_c\}$ of $P_8 = v_1 \ldots v_8$.
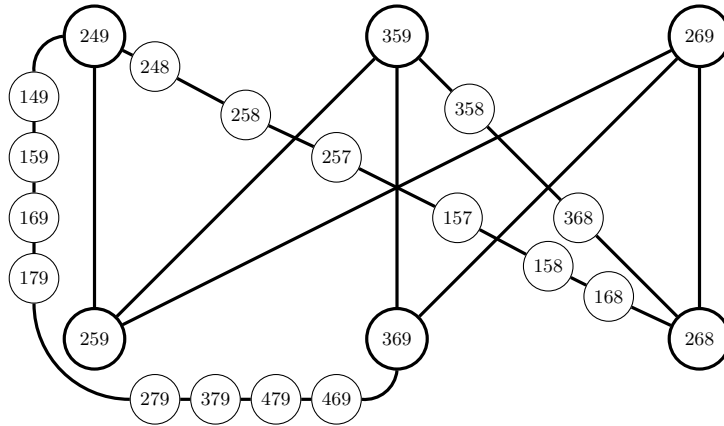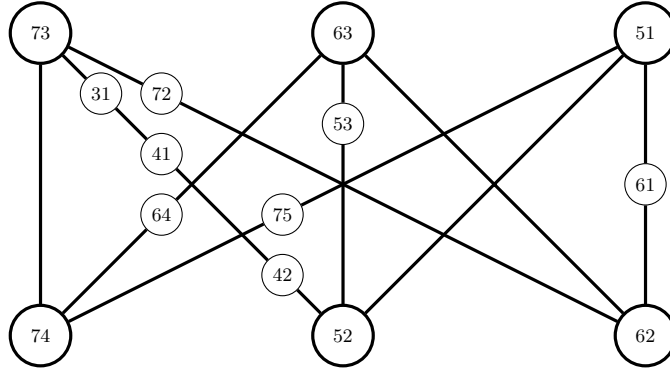

Fig. 9. A subdivision of $K_{3,3}$ that is contained in $\mathsf{TS}_3(P_9)$. Each number of the form abc inside a node represents an independent set $\{v_a, v_b, v_c\}$ of $P_9 = v_1 \ldots v_9$.

5.3   Eulerianity

A graph G is *Eulerian* if it has an *Eulerian cycle*—a cycle that visits each edge of G exactly once. It is well-known that a graph G is Eulerian if and only if it is connected and every vertex has even degree.

Proposition 16. *The graph $\mathsf{TS}_k(C_n)$ is Eulerian, for $1 \le k < n/2$*

Proof. Suppose that $C_n = w_1 w_2 \ldots w_n$. It is not hard to see that $\mathsf{TS}_k(G)$ is connected, since any independent set of size k, where $1 \le k < n/2$, can be reconfigured to the canonical independent set $\{w_1, w_3, \ldots, w_{2k+1}\}$. Let I be any independent set of C_n of size k. We show that $\deg_{\mathsf{TS}(C_n)}(I)$ is even. Note that only the maximal odd-length paths $P = v_1 v_2 \ldots v_{2i+1}$ in C_n satisfying $\{v_1, v_3, \ldots, v_{2i+1}\} \subseteq I$ affect $\deg_{\mathsf{TS}(C_n)}(I)$, and each of such path contributes either 0 or 2 to $\deg_{\mathsf{TS}(C_n)}(I)$. Thus, $\deg_{\mathsf{TS}(C_n)}(I)$ is even.   □

Fig. 10. A subdivision of $K_{3,3}$ that is contained in $\mathsf{TS}_2(C_7)$. Each number of the form ab inside a node represents an independent set $\{v_a, v_b\}$ of $C_7 = v_1 \ldots v_7 v_1$.

Proposition 17. *(a) For any Eulerian graph G on $n \geq 4$ vertices, every component of $\mathsf{TS}_2(G)$ is Eulerian.*
(b) There exists an Eulerian graph G where $\mathsf{TS}_k(G)$ is not Eulerian, for any $k \in \{3, \ldots, \alpha(G)\}$.
(c) For any graph G, if $\mathsf{TS}(G)$ is Eulerian, so is G. Moreover, for any $k \geq 2$, one can construct a graph G such that G is not Eulerian but $\mathsf{TS}_k(G)$ is.

Proof. (a) It suffices to show that all nodes of $\mathsf{TS}_2(G)$ have even degree. Take any independent set $I = \{v_1, v_2\}$. We have
$$\deg_{\mathsf{TS}_2(G)}(I) = \deg_G(v_1) + \deg_G(v_2) - 2|N_G(v_1) \cap N_G(v_2)|,$$
which is always even because G is Eulerian.

(b) For $k \geq 3$, take G as the graph containing exactly two cycles: C_3 and $C_{2\ell+1} = v_1 v_2 \ldots v_{2\ell+1}$. Note that $\alpha(G) = \ell + 1$. Suppose that vertices of $C_{2\ell+1}$ are in counter-clockwise order, $V(C_3) \cap V(C_{2\ell+1}) = \{v_1\}$. For $3 \leq k \leq \ell + 1$, let $I = \{w, v_{2\ell+1}, v_2, v_4, \ldots, v_{2(k-2)}\}$, where $w \in V(C_3) - v_1$. In these cases, we all have $\deg_{\mathsf{TS}_k(G)}(I) = 3$, and therefore $\mathsf{TS}_k(G)$ is not Eulerian.

(c) Clearly, if $\mathsf{TS}(G)$ is Eulerian, it must be connected, and therefore $\mathsf{TS}(G) \simeq \mathsf{TS}_1(G) \simeq G$, which implies G is also Eulerian. For $k = 2$, let G be a graph obtained by joining K_4 with a single vertex. Since G has a vertex of degree 1, it is not Eulerian. One can verify that $\mathsf{TS}_2(G) \simeq C_3$ and therefore it is Eulerian. For $k \geq 3$, let G be a graph obtained by joining a vertex of C_3 with an endpoint of P_3 and with $k - 2$ new vertices. Again, since G has a vertex of degree 1, it is not Eulerian. One can verify that $\mathsf{TS}_k(G) \simeq C_4$ and therefore it is Eulerian. $\qquad\square$

5.4   Girth

Recall that the *girth* of a graph G is the smallest size of a cycle in G, and is ∞ if G is a *forest*, i.e., it has no cycles.

Proposition 18. *For every $n \geq 2k + 1$, girth$(\mathsf{TS}_k(P_n))$ is 4 and ∞ otherwise. Consequently, girth$(\mathsf{TS}(P_n))$ is 4 for every $n \geq 5$ and ∞ otherwise.*

Proof. From Proposition 11, since $\mathsf{TS}(P_n)$ is bipartite, it does not contain any C_3, and therefore neither does $\mathsf{TS}_k(P_n)$. Let $P_{2k+1} = v_1 v_2 \ldots v_{2k+1}$. One can readily verify that $\mathsf{TS}_k(P_{2k+1})$ has a C_4 which contains the size-k independent set $I = \{v_1, v_4, v_7, v_9, \ldots, v_{2k+1}\}$. For $n \geq 2k + 1$, since P_{2k+1} is an induced subgraph of P_n, Proposition 2 implies that $\mathsf{TS}_k(P_{2k+1})$ is also an induced subgraph of $\mathsf{TS}_k(P_n)$, and therefore $\mathsf{TS}_k(P_n)$ also contains a C_4. Thus, girth$(\mathsf{TS}_k(P_n))$ is 4 for every $n \geq 2k + 1$. It remains to show that when $n \leq 2k$, the graph $\mathsf{TS}_k(P_n)$ has no cycles. To see this, note that we also have $k \leq \alpha(P_n) = \lceil n/2 \rceil$. Therefore, either $k = n/2$ or $k = (n+1)/2$ and in both cases one can verify that $\mathsf{TS}_k(P_n)$ has no cycles. Consequently, since $\mathsf{TS}_2(P_n)$ is an induced subgraph of $\mathsf{TS}(P_n)$, we have girth$(\mathsf{TS}(P_n))$ is 4 for every $n \geq 5 = 2 \times 2 + 1$ and ∞ otherwise. $\qquad\square$

Proposition 19. *For $1 \leq k < n/2$, girth$(\mathsf{TS}_k(C_n)) = n$. If $k = n/2$, we have girth$(\mathsf{TS}_k(C_n)) = \infty$.*
Consequently, girth$(\mathsf{TS}(C_n)) = n$.

Proof. Suppose that $I = \{v_1, v_2, \ldots, v_k\}$ is an independent set of C_n, where $1 \leq k < n/2$ and $\{v_i\}_{i=1,\ldots,k}$ are ordered in counter-clockwise direction. Since no token can "jump" above any other tokens, any cycle $\mathcal{C}$ in $\mathsf{TS}(C_n)$ containing I must be formed by moving tokens in counter-clockwise direction such that finally, the token originally placed on v_i is placed on v_{i+1} ($i = 1, 2, \ldots, k - 1$), and the token originally placed on v_k is placed on v_1. One can achieve this plan with exactly n token-slides, provided that $1 \leq k < n/2$. If $k = n/2$, the

graph $\mathsf{TS}_k(C_n)$ contains exactly two isolated vertices. Then, its girth is ∞. Consequently, since $\mathsf{TS}_k(C_n)$ is a subgraph of $\mathsf{TS}(C_n)$ for $1 \le k \le n/2$, it follows that $\mathrm{girth}(\mathsf{TS}(C_n)) = n$.

$\square$

5.5 Clique of given size

Proposition 20. *(a) G has a K_s if and only if $\mathsf{TS}(G)$ has a K_s ($s \ge 3$).*
(b) There exists a split graph G such that G has a K_s and $\mathsf{TS}_k(G)$ ($k \ge 2$) does not.

Proof. (a) $G \simeq \mathsf{TS}_1(G)$ implies the only-if direction. Lemma 8 implies the if direction.
(b) Take a split graph $G = (K \cup S, E)_{K\text{-max}}$ such that $|K| = s$ and there exists $v \in K$ with $\bigcup_{w \in S} N_G(w) = \{v\}$. Suppose to the contrary that $\mathsf{TS}_k(G)$ ($k \ge 2$) has a K_s. Thus, from Lemma 8, there must be a token t which traverses through all vertices in K. Since $k \ge 2$, there is always at least one token $t' \ne t$ in S, which implies that t can never be slid to v, a contradiction. Thus, $\mathsf{TS}_k(G)$ ($k \ge 2$) has no K_s.

$\square$

6   Decompositions and Joins

A major problem in constructing the TS_k-reconfiguration graph of a given graph G is that the number of independent sets in G may be exponentially large and hence so may be $\mathsf{TS}_k(G)$. However, the structure of G may be such that it can be decomposed into smaller subgraphs for which the TS_k-reconfiguration graphs are more easily obtained. This will be useful whenever $\mathsf{TS}_k(G)$ can be built from these smaller graphs. We present one such decomposition in this section.

Definition 21. *The H_1, H_2 join G of vertex disjoint simple graphs G_1 and G_2, where $H_i \subseteq G_i, i = 1, 2$, is formed by adding all edges between H_1 and H_2.*

Fig. 11(a) illustrates a join of the graphs G_1 and G_2. In cases where the graph G can be decomposed along a join we can decompose $\mathsf{TS}_k(G)$ into connected components. For the decomposition theorem we need a method of combining stable sets of possibly different sizes from disjoint graphs.

Definition 22. *Let G_1 and G_2 be simple connected graphs on different sets of vertices. For $1 \le s < k$ we define the product $\mathsf{TS}_s(G_1) \otimes \mathsf{TS}_{k-s}(G_2)$. The vertices are of the form $S_1 \cup S_2$ where $S_1 \in V(\mathsf{TS}_s(G_1))$ and $S_2 \in V(\mathsf{TS}_{k-s}(G_2))$. The edges in the product are of the form $(S_1 \cup S_2, S_1' \cup S_2)$ where $(S_1, S_1') \in E(\mathsf{TS}_s(G_1))$ and $(S_1 \cup S_2, S_1 \cup S_2')$ where $(S_2, S_2') \in E(\mathsf{TS}_{k-s}(G_2))$*

Fig. 11(b) shows some examples of this definition. In the following proposition we show that if graph G is the join of G_1 and G_2 then $\mathsf{TS}_k(G)$ is the union of $\mathsf{TS}_k(G_1)$, $\mathsf{TS}_k(G_2)$ and products of TS_k-reconfiguration graphs for smaller values of k. It is illustrated in Fig. 11.

(a) G is H_1, H_2 join of G_1, G_2.   (b) A decomposition of $\mathsf{TS}_k(G)$.
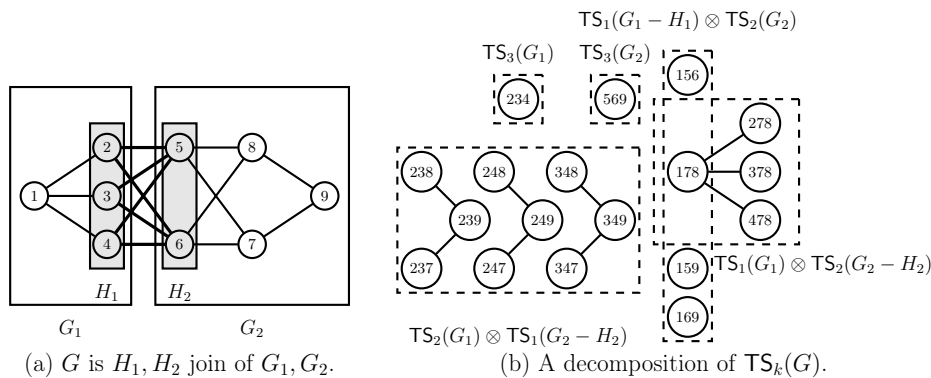
Fig. 11. A graph G and a decomposition of $\mathsf{TS}_k(G)$ using Proposition 23.

Proposition 23. *Let G be the H_1, H_2 join of graphs G_1, G_2. Choose an integer $k \geq 2$ so that G_1, G_2 each contain a stable set of size k.*
(a) $\mathsf{TS}_k(G)$ be decomposed into $k + 1$ (possibly empty) vertex disjoint subgraphs $\mathsf{TS}_k(G_1)$, $\mathsf{TS}_k(G_2)$, and for $1 \leq s < k$ the union of

$$\mathsf{TS}_s(G_1) \otimes \mathsf{TS}_{k-s}(G_2 - H_2) \quad and \quad \mathsf{TS}_s(G_1 - H_1) \otimes \mathsf{TS}_{k-s}(G_2). \tag{1}$$

(b) Fix $i = 1$ or 2. Every stable set S_i of size k in G_i satisfies

$$|S_i \cap V(H_i)| \neq 1 \tag{2}$$

if and only if $\mathsf{TS}_k(G_i)$ is disconnected from the remaining k subgraphs of $\mathsf{TS}_k(G)$.

Proof. Let S be an independent set of size k in G and, for $i = 1, 2$ let $S_i = S \cap V(G_i)$. Define $s = |S_1|$.

(a) We consider the three cases.
   (i) $s = k$.
      In this case S_1 is an independent set in G_1, so $S \in V(\mathsf{TS}_k(G_1))$.
   (ii) $s = 0$.
      The symmetric case, which yields $S \in V(\mathsf{TS}_k(G_2))$.
   (iii) $0 < s < k$.
      First suppose that $S \cap V(H_1) \neq \emptyset$ implying, by construction of the join, that $S \cap V(H_2) = \emptyset$. We have that $S_1 \in V(T_s(G_1)), S_2 \in V(T_{k-s}(G_2 - H_2))$ and therefore $(S_1, S_2) \in T_s(G_1) \otimes T_{k-s}(G_2 - H_2)$. Symmetrically, if $S \cap V(H_2) \neq \emptyset$ we conclude that $(S_1, S_2) \in T_s(G_1 - H_1) \otimes T_{k-s}(G_2)$. Note that S will appear in both sides of the union in (1) if and only if $S \cap V(H_i) = \emptyset, i = 1, 2$.
      The vertices of the $k + 1$ subgraphs of $\mathsf{TS}_k(G)$ in (a) correspond to stable sets in G with different values of s, so they are vertex disjoint.
(b) Suppose $i = 1$. A similar argument applies to the case $i = 2$.
   ($\Rightarrow$) We show that it is impossible to slide a vertex $v \in S_1$ to a vertex $w \in G_2$ to create a new stable set. Indeed, for such a slide to be possible we must have $v \in H_1$ and so by (1) $S_1 \cap H_1$ must contain another vertex u. Now $w \in H_2$ so it is adjacent to u and hence v cannot be slid to w. Hence there are no edges from $\mathsf{TS}_k(G_1)$ to any of the other k subgraphs in $\mathsf{TS}_k(G)$.
   ($\Leftarrow$) Suppose, by way of contradiction, that there is an edge between $\mathsf{TS}_k(G_1)$ and one of the other k subgraphs. Then there is a vertex $v \in H_1$ that can be slide to a vertex $w \in H_2$. This is only possible if $|S_1 \cap V(H_1)| = 1$ yielding the required contradiction.
                                                                                                  □

Observe that equation (2) in (b) is automatically satisfied if for either $i = 1$ or 2 we have $|V(G_i)| < |V(H_i)| + k - 1$. We remark that even when part (b) applies there may be additional edges between the $k - 1$ graphs described by the products in part (a).

7 Concluding Remarks

In this paper, we mainly studied the realizability and structural properties of reconfiguration graphs of independent sets under Token Sliding.

We presented in Section 4 necessary and sufficient conditions for a graph G to be a TS_k-reconfiguration graph ($k \geq 2$), where G belongs to certain restricted graph classes, namely complete graphs, paths, cycles, complete bipartite graphs, and connected split graphs. Even for $k = 2$, it remains unknown what the necessary and sufficient conditions for a forest to be a TS_k-reconfiguration graph are. We remark that "being a TS_k-reconfiguration graph" is *not* hereditary, even for trees. For example $K_{1,3}$ is not a TS_2-reconfiguration graph (Proposition 7) but if we replace one edge by a P_4 it is (Fig. 12). Proposition 18 provided a useful insight: given a forest F, if there exists G such that $F \simeq \mathsf{TS}_2(G)$, the graph G must be P_5-free (e.g., see the graph G in Fig. 12).

In Section 5, we shifted our focus to some graph properties, namely s-partitedness, planarity, Eulerianity, girth, and the clique's size, and provided examples and proofs showing that for some given graph G, certain $\mathsf{TS}(\mathsf{G})$ and $\mathsf{TS}_k(\mathsf{G})$ graphs do (not) inherit some properties from G, and vice versa. As the structural properties of $\mathsf{TS}(G)$ and $\mathsf{TS}_k(G)$ have not yet been systematically investigated, a large number of open questions in this direction can be obtained by strengthening our results or simply specifying either a graph class or a property which has not yet been considered.

In Section 6 we showed a way of decomposing a graph that induced a decomposition of its TS_k-reconfiguration graphs. By inverting the construction we have a way of building larger reconfiguration graphs from smaller pieces.
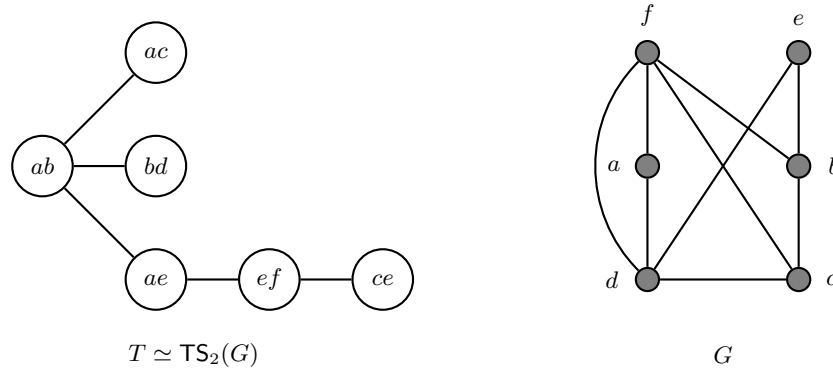
Fig. 12. Replacing an edge of $K_{1,3}$ by a P_4 results a tree T which is also a TS_2-reconfiguration graph. Each node ab in T represents a size-2 stable set of G.

Acknowledgements

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A   Connectivity and Diameter of $\mathsf{TS}_k(G)$ for Specific Graph Classes

For certain graph G, Table 3 includes some properties of the connectivity and diameter of $\mathsf{TS}_k(G)$ that can be derived from known results.

Table 3. Connectivity and diameter of $\mathsf{TS}_k(G)$ ($2 \leq k \leq \alpha(G)$). Here $n = |V(G)|$.

	G (simple, connected)	$\mathsf{TS}_k(G)(1 \leq k \leq \alpha(G))$		Ref.
		always connected?	diameter	
1	perfect	no	$O(2^n)$	
2	even-hole-free	yes, if $k = \alpha(G)$	$O(n)$, if $k = \alpha(G)$	[11,12]
3	P_4-free	no	$O(n^2)$	
4	claw-free	yes	$O(\text{poly}(n))$	[4]
5	tree	no	$O(n^2)$	[7]
6	path	yes		
7	bipartite permutation	no	$O(n^2)$	[9]
8	interval	no	$O(kn^2)$	[3,5]

1. In [12], Kamiński et al. showed the PSPACE-completeness of INDEPENDENT SET RECONFIGURATION (ISR) under any of TS, TJ, or TAR when the input graph is a perfect graph. Combining their reduction from SHORTEST PATH RECONFIGURATION (SPR) and an example of a reconfiguration graph of SPR having exponentially large diameter in the size of the input graph [11] gives us an example of $\mathsf{TS}_k(G)$ with exponentially large diameter in $n = |V(G)|$.
   Observe that one can construct a perfect graph G where $\mathsf{TS}_k(G)$ is not connected. For instance, take G as the star $K_{1,n}$. Then, for $n \geq k + 1$, $\mathsf{TS}_k(G)$ is not connected. This also holds for other graph classes such as P_4-free graphs, trees, bipartite permutation graphs, and interval graphs.
2. They also showed that $\mathsf{TJ}_k(G)$ is connected and its diameter is $O(n)$ when G is a connected even-hole-free graph. Observe that when $k = \alpha(G)$, we have $\mathsf{TJ}_k(G) \simeq \mathsf{TS}_k(G)$.
3. Kamiński et al. [12] designed a linear-time algorithm that decides whether there is a path between $I, J \in \mathsf{TS}_k(G)$, and if yes, outputs a shortest one, where G is P_4-free. One can verify that their algorithm indeed outputs a path in $\mathsf{TS}_k(G)$ of length $O(n^2)$.
4. Bonsma et al. [4] show that when G is a connected claw-free graph, $\mathsf{TS}_k(G)$ is always connected, and they provided a polynomial-time algorithm for outputting a path between any pair $I, J \in \mathsf{TS}_k(G)$.
5–6. Demaine et al. [7] designed a linear-time algorithm for deciding, whether there is a path between $I, J \in \mathsf{TS}_k(G)$, and if yes, output a path of length $O(n^2)$, provided that G is a tree. They also gave an example of an instance (G, I, J) where G is a path and the length of a shortest path between $I, J \in \mathsf{TS}_k(G)$ is $\Omega(n^2)$.
7. Fox-Epstein et al. [9] designed a cubic-time algorithm for deciding, whether there is a path between $I, J \in \mathsf{TS}_k(G)$, and if yes, output a path of length $O(n^2)$, provided that G is a bipartite permutation graph.
8. Bonamy and Bousquet [3] designed a polynomial-time algorithm for deciding whether $\mathsf{TS}_k(G)$ is connected when G is an interval graph. However, they did not provide any estimation on its diameter. Motivated by this question, Briański et al. [5] recently showed that the diameter of $\mathsf{TS}_k(G)$ is $O(kn^2)$.