



UCL

**A New Class of Quantile Processes with
Applications in Risk Analysis and
Valuation**

by

Holly Brannelly

Department of Mathematics

University College London

Gower Street, London WC1E 6BT, United Kingdom

Submitted to the University of London

for the degree of

Doctor of Philosophy

4 May 2022

Abstract

This thesis presents a novel approach for the construction of quantile processes, governing the stochastic dynamics of quantiles in continuous time. Two constructions are proposed, one producing a function-valued quantile process and the second, a process with random quantile levels. The latter method employs a distortion map composed of a distribution function and a quantile function, similar to a transmutation map, applied to each marginal of a ‘driving’ process with càdlàg paths. A multidimensional extension that utilises a copula is also presented. As a result, we obtain a one-step approach to constructing widely flexible classes of stochastic models, accommodating extensive ranges of higher-order moment behaviours (e.g., tail behaviours in the finite dimensional distributions, and asymmetry). Such features are parameterised in the composite map and are thus interpretable with respect to the driving process. Subclasses of quantile processes are explored, with emphasis placed on the Tukey family of models whereby skewness and kurtosis are directly parameterised and thus the composite map is explicable with regard to such statistical behaviours. It is also shown that the quantile processes induce a distorted probability measure that is interpretable in its properties (which may be intentionally constructed), leading to the central application developed in this thesis. We propose a general, time-consistent, and dynamic risk valuation principle under the induced measures of quantile processes, allowing for pricing in incomplete markets and thus having application in insurance pricing. Here, the distorted measures are considered ‘subjective’ and are constructed in such a way to account for external market characteristics and investor risk attitudes, leading to a parametric system of risk-sensitive probability measures, indexed by such factors. The properties of the valuation principle based on the quantile process distortion measures are discussed with regard to stochastic ordering and risk-loadings, and a case study is presented where insurance instruments linked to greenhouse gas emissions are considered.

Impact Statement

The main research contribution presented in this thesis is the development of a novel class of stochastic processes that models the dynamic evolution of quantiles in continuous time. The properties of this class of stochastic models are designed within a systematic and intuitive framework, consisting of a composite map applied marginally to some driving base process. The resulting class of quantile processes is highly flexible and statistically rich in its higher-order, finite-dimensional properties and tail behaviours. Interpretation is provided as to how each component of the parameterisation (composite map) determines the properties of the output quantile process, motivating the use of such processes in various applications across multiple areas within industry where a precise model specification is required, which we discuss as follows.

The use of quantiles is critical in the assessment of risk. For example, the specification of quantile-based risk measures, such as VaR or expected shortfall, are crucial in financial risk management, regulatory control or policy making, and capital requirements, e.g., those imposed under Solvency II regulations in EU insurance markets and those arising in Basel III for banking firms. In this work we are able to demonstrate how to consistently and rigorously transform a process to a quantile process in continuous time that serves the purpose of producing a measure flow of distorted quantiles, which will consequently be interpretable in its properties. As base processes, we consider diffusions such as geometric Brownian motion and Ornstein–Uhlenbeck processes which are widely adopted in finance, physics, and many other fields. The distortion that is applied allows one to ‘update’ the model in such a way to better capture the properties and features of data sets, e.g., stylised characteristics of financial asset returns, such as leptokurtosis and skewness. We focus on the Tukey family of models, as the g -and- h family spans a much larger area of the skew–kurtosis plane than many other skewed and heavy-tailed distributions. Control of such distributional features is direct.

The generality of the proposed framework presented means that its scope of ap-

plication is by no means limited to financial mathematics, however. We emphasize that such a methodology can be adopted for a variety of applications including dynamic risk measures in econometrics, behavioural economics and dynamic consumer preference models, operations research and dynamic decision making, signal processing in information theory, as well as to more general applications requiring dynamic risk measures such as disaster monitoring (earthquake hazard, flood hazard, increasing frequency of extreme temperatures and other natural disasters). We also draw attention to dynamic risk analysis in climate and environmental science where the risk measure is not a monetary quantity, necessarily, and the complexities of many modelling challenges within this realm motivates the use of highly parameterised models.

The key application in this thesis considers the pricing of insurance instruments linked to greenhouse gas emissions. The quantile process construction induces a parametric system of risk-sensitive distorted measures characterising the pricing principle, which leads to a well-defined mechanism for producing relativised prices. For example, beyond the stylised higher-order features of risk processes, one may also characterise investor risk preferences in the choice of the composite map. It is not novel to consider distributional distortions in the quantification of preferences, but we present the framework in this thesis as a flexible and unrestricted alternative to existing distortion-based pricing techniques, that are widely adopted for many types of risks, both in practice and in academic studies.

In summary, this thesis provides a mathematically rigorous theoretical framework for the construction of a class of quantile processes that, by the nature and generality of the framework, is widely applicable across various disciplines. Upon the completion of this PhD, the developed framework shows great potential and range of applications, and it is exciting to envisage the mathematical modelling challenges it may enhance going forward.

Acknowledgements

Firstly, I would like to thank my primary supervisor, Dr. Andrea Macrina, for his supervision and support, and for being understanding throughout tougher parts of the last four years. I am extremely appreciative of his constant enthusiasm toward our joint research as part of my PhD study. I am also very grateful for the fantastic co-supervision of my secondary supervisor, Prof. Gareth W. Peters, and his invaluable expertise that he has shared with me over the course of my PhD. I am greatly appreciative of the insightful academic discussions we have had, the time he has given for such discussions, as well as for his constant encouragement towards our research. There's no doubt it has been an unusual couple of years to work through the second half of a PhD, and without the momentum and continual contribution of exciting ideas from, and conversations with, both my supervisors, it would not have been possible.

Besides my supervisors, I would like to thank Prof. William Shaw for his input into making this exciting research project possible, as well as EPSRC and Winton Capital for their financial generosity that has provided me with the opportunity to undertake this work. In addition, I am grateful to the administrative staff of the Department of Mathematics at UCL for all their support, in particular, Helen Higgins for her kindness and always being available to lend a helping hand.

As part of my training as a PhD student, I was extremely lucky to have the opportunity to attend interesting workshops in, and beyond, London. I would like to thank Prof. David Taylor, as well as my team members and mentors, for making it possible for me to participate in the FMTC at the University of Cape Town, for two consecutive years. I thoroughly enjoyed my time here, and learnt a lot that contributed to my, both academic and non-academic, development as a PhD student. In addition, these trips lead me to develop a love for the city of Cape Town, that has brought me to return for some very enjoyable visits. I am also grateful to Dr. Tomoko Matsui for inviting me to visit the Institute of Statistical Mathematics, Tokyo, and the funding support received

that allowed me to work on my research amongst such a knowledgeable community, in such an incredible city. In addition, I am deeply appreciative of the opportunity to attend, and in some cases, present my work at, the UCL–Osaka conference on the Mathematics for Risk and Decisions, the London–Oxford–Warwick Financial Mathematics workshop, the Junior Female Researchers in Probability workshop in Berlin, and the 17th Winter School on Mathematical Finance in Lunteren.

Finally, I am deeply grateful for my supportive group of childhood friends, that have always been there when in need, or otherwise; to Talya for her unconditional friendship, and the coffees we have had together on campus; to Yupeng for always offering a helping hand in the first half of my PhD; and to my amazing parents for their unwavering support. Without all of you, this PhD journey would not have been possible.

I, Holly Brannelly, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the work.

Contents

1	Introduction and summary	15
2	Quantile transforms and modelling	23
2.1	Quantile transformation maps	24
2.2	Characterisation of quantile processes	34
3	Construction of Stochastic Quantile Processes	39
3.1	Random-level quantile processes	40
3.1.1	Canonical random-level quantile processes	49
3.1.2	Pivotal random-level quantile processes	53
3.1.3	SDEs satisfied by random-level quantile diffusions	57
3.2	Function-valued quantile processes	60
3.3	Flexible families of Tukey quantile processes	66
3.4	Multidimensional and multivariate extension	82
4	Measure distortions induced by Quantile Processes	85
4.1	Connection with the Radon-Nikodym derivative	88
4.2	The multidimensional case	90
5	Stochastic ordering	92
5.1	First-order stochastic dominance of univariate quantile processes	94
5.2	Second-order stochastic dominance of univariate quantile processes	99
5.3	First-order stochastic dominance of multidimensional quantile processes	107
6	Distortion-based pricing and valuation	110
6.1	Existing frameworks for decision-based valuation of risks	110

6.1.1	Distortion operators	114
6.2	A general stochastic valuation principle	118
6.3	A valuation principle based on measure distortions induced by quantile processes	123
6.3.1	Properties of the quantile process-based valuation principle . . .	125
6.4	Connection to insurance premium calculation principles	131
7	Application	138
7.1	Description of the data	142
7.2	Model calibration	145
7.3	Forecasting	149
7.4	Construction of the empirical quantile process	150
7.5	Pricing results and sensitivities	152
7.6	Multidimensional extension	155
8	Conclusions	162
	Bibliography	164

List of Figures

2.1	Cumulative distribution functions (CDFs) of the transmuted standard uniform, standard normal and exponential CDFs under a quadratic RTM for a range of parameter values $\lambda \in \{-1, -0.6, -0.2, 0, 0.2, 0.6, 1\}$.	28
2.2	Quantile functions of the Tukey- gh distribution for a range of g -and- h parameters, relative to the standard normal quantile function.	34
3.1	Three random-level quantile process paths for $\omega_1, \omega_2, \omega_3 \in \Omega$ and the corresponding realisations of the intermediate quantile level process (U_t) at times $0 < t_0 < t_1 < t_2 < \infty$. Here, $Z(t, \omega) = Q_\zeta(U(t, \omega))$	48
3.2	Three sample paths of the process (U_t) for $\omega_1, \omega_2, \omega_3 \in \Omega$, and the corresponding values of the quantile process (Z_t) at times $t_0 < t_1 < t_2 < \infty$.	49
3.3	One sample path of each parameter process in the function-valued quantile process construction, and the corresponding quantile function at times $t_0 < t_1 < t_2 < \infty$	62
3.4	Quantile functions of the Tukey- g distribution for $g \in \{0.3, 0.8, 1.5, 3\}$ and $g \in \{-0.3, -0.8, -1.5, -3\}$, relative to the standard normal quantile function.	71
3.5	Sample paths of the GBM driving process with parameters $\mu = 0.1$, $\sigma = 0.05$, the uniformly distributed process (U_t) , and the Tukey- g quantile process with parameter values $g = \{0.6, 2, -0.8\}$	73
3.6	Quantile functions of the Tukey- h distribution, relative to the standard normal quantile function, for $h \in \{-0.05, -0.1, 0.05, 0.1, 0.6, 1\}$, and h -quantile transforms for $h \in \{-0.6, -1\}$	77

3.7	Sample paths of the GBM driving process with parameters $\mu = 0.1$, $\sigma = 0.05$, and the Tukey- h quantile process with parameter values $h = \{0.02, 0.2\}$	79
4.1	Relation between the marginal distribution functions of the driving process under the \mathbb{P} - and \mathbb{P}^Z -measures at $t = 0.5$ for $\mu = 0$ and a range of g and h parameters.	87
4.2	Relation between the marginal distribution functions of the driving process under the \mathbb{P} - and \mathbb{P}^Z -measures at $t = 0.5$ for $\mu = 0.8$ and a range of g and h parameters.	88
4.3	Relation between the marginal distribution functions of the driving process under the \mathbb{P} - and \mathbb{P}^Z -measures at $t = 0.5$ for $\mu = 0$ and a range of g , h and λ parameters.	88
5.1	Values of u^* as $g_1 - g_2$ is increased for $g_2 = 0.1$ fixed, and $h_1 - h_2 \in \{0.2, 0.35, 0.5\}$ with $h_2 = 0.05$ fixed.	99
6.1	The one- and two-factor Wang transforms, $\nu_{\lambda,1}(u) = \Phi(\Phi^-(u) + \lambda)$ and $\nu_{\lambda,2}(u) = T_k(\Phi^-(u) + \lambda)$, respectively, applied to some base CDF $F(x)$ for risk aversion parameter values $\lambda \in \{0.1, 0.5, 1, 0, -0.1, -0.5, -1\}$ and degrees of freedom parameter (in the two-factor transform) values $k = \{1, 3, 7\}$	116
6.2	Premiums by layer under the Tukey- g QPVP, with $B = 0.01$, $\kappa = -100$ and $g \in \{0.05, 0.1, 0.3, 0.8, 1, 2\}$	136
7.1	Time series plot of the EUA December 2022 futures daily closing price.	143
7.2	Time series plot of the de-trended (once differenced) EUA December 2022 futures daily closing price.	144
7.3	Plots of the ACF of the once differenced EUA December 2022 futures daily close time series.	145
7.4	Time series for the EUA December 2022 (black line) against the fitted ARIMA(2,1,3) with drift $c = 0.0889$ model (red dashed line).	147
7.5	Plots of the ACF of the residuals of the fitted ARIMA(2,1,3) with drift $c = 0.0889$ model for the EUA December 2022 futures daily close time series.	148

7.6	Forecasts from the fitted ARIMA(2,1,3), with drift $c = 0.0889$, model for a forecast horizon $h = 249$ days.	149
7.7	Empirical CDFs of the forecasted quantile process at time $T = 31$ December 2022 for a range of parameters in the Tukey- gh quantile function. The grey dashed lines lie at the considered threshold values $a \in \{93.6476, 104.064, 121.408\}$	151
7.8	Forecasts from the fitted ARIMA(2,1,3) model for a forecast horizon $h = 249$ days.	157
7.9	Contour plots of the Bernstein copula based on the empirical EUA December 2022 and EUA March 2023 futures data at time windows $i \in \{1, 94, 188, 282, 376, 466\}$, where $i = 466$ corresponds to the last window, i.e., that at time T	159
7.10	Empirical CDFs of the forecasted quantile process at time $T = 31$ December 2022, for a range of parameters in the Tukey- gh quantile function. The grey dashed lines lie at the considered threshold values $a \in \{93.6476, 104, 064, 121, 408\}$	160

List of Tables

5.1	FOSD results for Tukey- gh quantile processes for different values of the skewness and kurtosis parameters.	99
6.1	Risk-adjusted premiums by layer under the Wang transform and Tukey- g distorted measure. The figures in bold correspond to the premiums matched across the different distortions, through the choice of κ	135
7.1	Output of the <code>auto.arima()</code> , function applied to the EUA December 2022 futures time series data	147
7.2	p -values for the Ljung-Box test statistic for the residuals of the fitted model at lags 1-5 and 10.	148
7.3	Forecast accuracy measures of the ARIMA(2,1,3) with drift $c = 0.0889$ model fitted to the EUA December 2022 data.	149
7.4	Premiums obtained under the QPVP for $A = 92.9430$, $B = 10$, and range of distortion parameters $g \in \{0.1, 0.5, 0.8, 1, 2, 4, 6\}$ and $h \in \{0.01, 0.1, 0.2, 0.5\}$. The thresholds of the stop-loss contract that are considered are $a \in \{93.6576, 104, 064, 121.408\}$	154
7.5	Premiums obtained under the QPVP for $A = 92.9430$, $B = 30$, and range of distortion parameters $g \in \{0.1, 0.5, 0.8, 1, 2, 4, 6\}$ and $h \in \{0.01, 0.1, 0.2, 0.5\}$. The thresholds of the stop-loss contract that are considered are $a \in \{93.6576, 104, 064, 121.408\}$	155
7.6	Output of the <code>auto.arima()</code> , function applied to the EUA March 2023 futures time series data	156
7.7	Forecast accuracy measures of the ARIMA(2,1,3) model fitted to the EUA March 2023 data.	157

-
- 7.8 Premiums obtained under the QPVP for $A = 106.8844$, $B = 10$, and range of distortion parameters $g \in \{0.1, 0.5, 0.8, 1, 2, 4, 6\}$ and $h \in \{0.01, 0.2, 0.5\}$. The thresholds of the stop-loss contract that are considered are $a \in \{93.6576, 104, 064\}$ 161
- 7.9 Premiums obtained under the QPVP for $A = 126.9485$, $B = 30$, and range of distortion parameters $g \in \{0.1, 0.5, 0.8, 1, 2, 4, 6\}$ and $h \in \{0.01, 0.1, 0.2, 0.5\}$. The thresholds of the stop-loss contract that are considered are $a \in \{93.6576, 104, 064\}$ 161

Chapter 1

Introduction and summary

When posed with mathematical challenges in the realm of statistical modelling, exploratory data analysis and various applications therein, literature that tackles such problems through the description of probability distribution functions is dense. The research addresses both static and dynamic problems, namely the distributional specification of random variables and stochastic processes. Alternatively, what if one chooses to describe statistical distributions through their quantile function? Although both functions convey largely the same information about the underlying random variable, can one exploit the properties of quantile functions—particularly those not shared by distribution functions—to the advantage of the mathematical problem at hand? For example, the sum of two quantile functions, or an increasing transformation of a quantile function, remains a quantile function, and so algebraic manipulations become ‘easier’, yet models remain tractable in light of the alteration of statistical properties and, perhaps, introduction of model complexity. This is not yet to mention the many distributions for which a distribution function does not (analytically) exist, and so the quantile function becomes the descriptive attribute of interest. In what follows, we exploit useful properties of quantile functions to build families of highly flexible, statistically rich and curiosity-evoking stochastic models, termed ‘quantile processes’. We start with something familiar, namely a general, continuous-time stochastic process with càdlàg paths, as an input, and construct a mechanism to produce an output that lies under a new class of continuous-time processes. The mechanism employed allows the output model to accommodate a wide range of higher-order moment and tail behaviours in its finite-dimensional distributions, all of which are directly interpretable

with regard to the constructive parameterisation. In other words, we address questions of the following type: “Can we add fluidity to the properties exhibited by existing models, regardless of how descriptive they may be, to alter or enhance, their modelling capability?”, “Is there a one-step, constructive and interpretable approach so that the magnitude to which the process is altered is at the discretion of the modeller?” and, accordingly, “What applications does the statistical versatility and richness of the approach inspire the study of in the context of such quantile processes?”.

The principal approach pursued in this thesis is based on a composite map, consisting of a distribution function and a quantile function, that maps each marginal of an input process to those of an output process that determines the evolution of quantiles in continuous time. We describe the effect of the map on the input process as a ‘distortion’. The construction allows for the base process to be distorted in such a way to better satisfy some target model or objective. As such, this work pertains to the study of how the finite-dimensional laws of stochastic processes behave, and a characterisation of their dynamics, under a distortion of the described type, *relative* to the law of some base process. Additionally, the class of distortion maps we employ belong to a class of ‘quantile-preserving maps’, so that quantiles of the input process are mapped to quantiles of the output process at any given quantile level $u \in [0, 1]$. An outline of such maps is given in Section 2.1, and the quantile process construction, characterised by a composite map transformation, is defined in Section 3.1, and adapted from the research paper by Brannelly et al. (2021b). The univariate case is predominantly focused on, however in Section 3.4 an extension to the random-level construction, in which a multivariate driving process is considered, is provided. This requires the introduction of a copula function in the composite map for the construction of quantile processes. The focus with respect to the class of quantile functions considered in the composite map is the Tukey family of distributions, as introduced by Tukey (1977) and discussed in Section 3.3, but this is not to say that one should not consider distributions beyond this family when exploring the class of models presented herein. We consider this family of models for the reason that they are expressed in terms of their quantile functions, and that skewness and kurtosis have direct parameterisation, thus providing a useful starting point for interpretability of the statistical behaviours of the resulting quantile process models.

The use of distortion maps as a means of altering probability distributions is ex-

explored extensively in works by Azzalini (2005, 2008), Azzalini and Capitanio (2003), Fischer and Klein (2004), Genton (2005), Haynes et al. (1997), Hoaglin (1985), Klein and Fischer (2002), MacGillivray (1981, 1992), Shaw and Buckley (2009, 2007), Tukey (1977), Wang (1995, 2000), Yan and Genton (2019), and many others. Additionally, distortions applied on the density space are ubiquitous in statistical science under the topics of measure and density approximation. In such settings, density distortions are used to expand, modify or tilt a base distribution such that the distortion alters the moments or cumulants of the distribution relative to the base distribution in such a fashion that the resulting distorted distribution may better satisfy a target objective, which is often expressed in terms of moments or cumulants, thereby producing an improved approximation. Common examples include the saddle point, Edgeworth and generalised Esscher transformations. Such approaches have enjoyed widespread use to produce transformations applied to a family of base densities in order to distort the moment or tail behaviour characteristics of the resulting distribution, relative to the base distribution, for some analytic purpose. For an overview of this large family of methods see, e.g., Barndorff-Nielsen and Cox (1979), Bickel (1974), Daniels (1954) and Wand et al. (1991). In this thesis we characterise a class of distortions directly in the quantile space, one reason being (other than the aforementioned useful properties of quantile functions) that many risk management problems are expressed in terms of quantile functions rather than a density. Additionally, we remark that the consideration of distributional distortions in the literature is significantly less present in the context of continuous-time stochastic models. In this regard, we see significant novelty in the proposed approach.

Whilst the focus of this thesis is the class of quantile process models discussed above, which we call ‘random-level’ quantile processes, a second approach is also considered whereby the class of quantile processes constructed are ‘function-valued’. Here, we choose to model the vector of parameters of a quantile function by a multivariate stochastic process, to produce a continuous-time, function-valued process in the space of quantile functions. In other words, we map from realisations of this parameter process to function-valued realisations of the quantile process, allowing one to dynamically model the entire quantile function at any instance in time. Again, we may employ quantile-preserving maps to enable the construction of statistically richer quantile functions from simpler quantile functions, as the starting point for these mod-

els. Function-valued quantile processes are defined in Section 3.2, and a comparison is made between these models and the random-level quantile processes defined in Section 3.1. We emphasise, however, that the focal point of this thesis is the construction of random-level quantile processes, as the discussion that can be had on its theoretical underpinnings and application is plentiful. Nevertheless, this is not to say that the function-valued quantile process construction does not accommodate interesting mathematical investigations going forward.

The notion of addressing statistical studies through the exploration of quantiles dates back to work by polymath Sir Francis Galton, see Galton (1875, 1882, 1883), in which the idea of dividing ranges of values into groups of equal size—‘equi-postiles’, ‘quartiles’, ‘octiles’, ‘deciles’ and ‘percentiles’—was established. This was followed by the introduction of the more general term ‘quantile’ by Kendall (1940), and subsequently the introduction of the quantile function as a tool for statistical modelling and data analysis by Tukey (1977) and Parzen (1979a,b)—see also Parzen (1993, 2004). More recently, quantile processes and dynamical quantile functions have been explored in the stochastic process literature, the statistical regression and time series econometrics literature, the risk management and insurance literature, as well as the mathematical statistical literature within the study of empirical processes. As a result, there are numerous meanings attributed to the terminology ‘quantile process’ or ‘quantile dynamics’ based on the definitions developed in earlier works. A comprehensive discussion on the use of quantile functions in discrete, time series based statistical modelling and data analysis is given by Gilchrist (2000), Koenker and Bassett Jr (1978), Koenker and Hallock (2001), Koenker and Xiao (2006), a review by Koenker (2017), and a tutorial review by Peters (2018). This tutorial review also draws a connection between quantile processes and quantile-preserving maps, following which a discrete-time, function-valued quantile process model is introduced by Chen et al. (2022). It is with this work that one may directly relate the class of function-valued quantile processes introduced in this thesis as the continuous-time analogue of such models. Distinct from these statistical time series modelling frameworks, there have also been developments of what are termed ‘quantile processes’ for empirical processes in mathematical statistics and probability literature—see for instance the sequence of works by Csörgő (1983), Csörgő and Révész (1978) and Csörgő et al. (1986). The literature on quantile processes in continuous time is less dense, however one may consider

the works of Akahori (1995), Dassios (1995), Embrechts et al. (1995) and Yor (1995) whereby—building on ideas in Miura (1992)—a Brownian process is considered and, at each instance in time, the distribution of a random variable, defined by the α -quantile of the diffusion at this time, is studied. From a financial mathematics perspective, Miura (1992) motivated such a consideration by introducing the ‘ α -percentile option’ whereby the underlying is given by the α -percentile of the price process over the life of the option, e.g., the median if $\alpha = 0.5$. Comparatively to work where the focus is the distributional behaviour of the quantiles of a continuous time process, our random-level approach focuses on constructing a model for the dynamic quantiles, directly. It is less transpicuous as to where the random-level quantile processes presented in this thesis overlaps with existing quantile process models, and thus we see the utility of quantile dynamics in this regard as under-explored and, to our knowledge, not yet proposed.

As with any new framework comes breadth to explore its potential in regard to statistical behaviours, sensitivities to model inputs, and scope of application. It is natural as a financial mathematician, particularly when considering quantiles, for ones mind to wonder to the realm of risk in its many facets: quantification, management, and modelling of it, as well as risk profiles, preferences and how to characterise them. As such, the key application explored in this thesis is the development of a general, time-consistent and dynamic approach for the valuation of risks: the ‘quantile process-based valuation principle’. In this setting, we see it natural to consider random-level quantile processes from the perspective of their induced probability measures and how such measures inherit the (intentionally constructed, highly flexible and directly parameterised) statistical properties of the quantile process. The manner in which we construct probability distortions is a flexible and unrestricted alternative to commonly used distortion-based pricing techniques such as those produced using distortion operators, see, e.g., Godin et al. (2012, 2019), Wang (1996, 2000, 2002), or a weighting/exponential tilting, see, e.g., Cruz et al. (2015), Esscher (1932), Furman and Zitikis (2008) and references therein. Unlike these existing methods, the approach developed in this work facilitates the direct parameterisation of higher-order features in the choice of composite map used in the quantile process construction, which thereby allows one to incorporate knowledge of asymmetries or tail risk, for example, and to directly quantify their impact (through the distorted probability measure) on the resultant valuations. Of key importance in this work is the explication of how the choice of the composite

map affects the behaviour of prices obtained under the stochastic valuation principle. We consider markets where arbitrage pricing may fall short due to the (possible) incompleteness of markets, such as insurance markets. The quantile process-induced ‘distorted’ probability measures are considered subjective, and to incorporate investor risk preferences, as well as external factors such as financial market risk, and characteristics associated with the underlying risk source that may not be accounted for when considering historical losses (e.g., technological advancements, changes in climate, geopolitical turmoil). The first of these factors is emphasised throughout, and as such, when the Radon–Nikodym derivative between the real–world and the distorted probability measure is derived and the connection with the pricing kernel is made, this is equivalent to the risk preferences of each market participant corresponding to a different, and not necessarily unique, pricing kernel. This is of course consistent with the assumption of market incompleteness. In the insurance setting, the framework presents a fairly general class of axiomatically justified premium principles with high levels of flexibility. These premium principles allow one to capture skewness and leptokurtosis that is commonly observed in markets, and they can be used to explicitly incorporate more structure into subjective assignments of elicitable information regarding an investor’s risk preferences. This is captured through the induced characteristics of the quantile process. When multidimensional quantile processes are considered to induce the probability measure that characterises the valuation principle, the copula used in the composite map for their construction presents another new element in the risk quantification and modelling framework based on probability measure distortions induced by quantile processes. In the context of the stochastic valuation principle, this allows for (auxiliary) risk factors, e.g., other highly correlated risks or external macroeconomic/ systemic risks, to also be considered. The idea is that the incorporation of such external risks in the construction of the distorted probability measure brings additional sources of risk drivers into the valuation principle. Here, one or more marginals of the multivariate driving process are the underlying risks on which the financial or insurance contract is written, and the remaining marginals of the driving process model the external risks that are accounted for in the valuation problem. The model is presented in the context of pricing insurance layer and stop–loss contracts. We envisage applications of this model to include, e.g., the pricing of insurance contracts written on some loss process where external risks related to climate change are likely

to impact. While the external risk sources may trigger the payout of an insurance contract, such risks are not regarded as the directly insurable loss, nor is their impact prevalent in the historical loss data of the risk being insured. As the realised impact of climate change is on the rise, it becomes more important for insurers to factor such risks into their pricing models. Here, the climate risk processes are taken as marginals of the multivariate driver used in the construction of the quantile process that induces the distorted probability measure. The composite map (consisting of a quantile function and a copula) can also be utilised to produce excess skewness or kurtosis that the insurer deems reasonable to factor into the model, but is not captured in historical loss data, or the risk preferences of the agent buying the insurance contract. It may be at the discretion of the insurer to determine the appropriate composite map used in the construction of the quantile process. A theoretically similar idea is formalised by Zhu et al. (2019), where a premium principle for agricultural losses is constructed by incorporating a re-weighting of the historical losses with systemic risk factors—here, the insurer may select the appropriate weighting function.

We build up to the valuation principle systematically in this thesis—the content of Chapters 4, 5 and 6 is adapted from the research paper by Brannelly et al. (2021a)—with an introduction to the distorted measures induced by random-level quantile processes given in Chapter 4. A connection with the Radon–Nikodym derivative is made. Following this, in Chapter 5, we derive necessary and sufficient conditions under which the quantile processes satisfy first- and second-order pathwise stochastic dominance, with respect to the choice of composite maps used in their construction. These results allow one to derive useful properties of the valuation framework, for instance the setting in which the valuation principle induces a consistent risk-loading and thus is axiomatically sound in the context of insurance prices. The stochastic ordering results also lead to ordered parametric families of prices or premiums related to some given base risk.

In Chapter 6, the general and time-consistent stochastic valuation principle is defined and the connection with dynamic, convex risk measures, see e.g., Acciaio and Penner (2011), Bion-Nadal (2006, 2008, 2009) and Detlefsen and Scandolo (2005), is made. The developed general framework is shown to incorporate well-known pricing frameworks built by concave distortion operators, see above references, and a large number of premium calculation principles (PCPs), see Laeven et al. (2008) for a de-

tailed survey on PCPs. A special case of the stochastic valuation principle is then specified, employing a conditional expectation under the distorted probability measure induced by the quantile process, thus automatically ensuring time-consistency. We emphasise that the choice of the probability measure, determined by the choice of the quantile distortion map, determines the behaviour of the output price process for any given input risk. For example, if the input risk process becomes more heavy-tailed or skewed, or both, under the distortion map, then the valuation principle induces a risk-loading; this may coincide with risk aversion of the considered market participant. In the context of risk profiles and preferences, connections are also made with the decision theory literature, where it is well-established to consider a market agent's preferences through a utility function. Again, we emphasise the advantage in the flexibility of the quantile process construction and thus ability to capture a somewhat unrestricted compass of risk profiles through the precise model specification.

The work in this thesis is presented as the origin of a theoretical framework in which a new class of continuous-time stochastic processes, modelling the dynamics of quantiles with statistically rich and purposefully constructed properties, is defined. We present this work as a steppingstone to the wider picture of what, under further mathematical study, such processes may achieve in applications across a variety of mathematical and statistical subdisciplines.

Chapter 2

Quantile transforms and modelling

The use of quantile functions in characterising distributions to describe data dates back to work by Galton (1902) and, in subsequent years, various aspects of quantile methods appeared in both theoretical and practical works, see e.g., Parzen (1979a,b) and Tukey (1977). In his book *Statistical Modelling with Quantile Functions*, Gilchrist (2000) both unified and developed such ideas to provide an extensive discussion on the use of quantile methods in statistical modelling procedures. The focus was to provide an additional perspective on existing ideas by re-expressing them in terms of quantile functions, and to emphasise the advantages of doing so. The purpose of this chapter is to present some background material in, and motivate, the theory of quantile transformation maps and quantile processes; the understanding of both concepts is principal to the construction of the quantile processes developed in this thesis.

We first provide the definition of a quantile function for a random variable, that shall be referred to throughout this thesis. This definition utilises the generalised inverse discussed by Embrechts and Hofert (2013), allowing one to consider instances where the corresponding distribution function may not be real-valued, continuous and strictly monotone, and hence the ordinary definition of the inverse it possesses on its range does not apply.

Definition 2.0.1. *For an increasing function $F : \mathbb{R} \rightarrow \mathbb{R}$ with $F(-\infty) = \lim_{x \downarrow -\infty} F(x)$ and $F(\infty) = \lim_{x \uparrow \infty} F(x)$, the generalised inverse $Q := F^- : \mathbb{R} \rightarrow \overline{\mathbb{R}} := [-\infty, \infty]$ of F is defined by $Q(y) = \inf\{x \in \mathbb{R} : F(x) \geq y\}$ for $y \in \mathbb{R}$ and with the convention that $\inf \emptyset = \infty$.*

We denote the distribution, quantile, and density functions of a random variable

by F , Q , and f , respectively, where an argument in the subscript of such functions will denote the random variable to which they correspond. When included in notation, parameters will follow a semicolon in the arguments of these functions.

Definition 2.0.2. *Let X be a real-valued random variable with distribution function $F_X : \mathbb{R} \rightarrow [0, 1]$. The quantile function of X is $Q_X = F_X^- : [0, 1] \rightarrow \overline{\mathbb{R}}$.*

2.1 Quantile transformation maps

It is common in statistical modelling practices to express quantile functions as the inverse of cumulative distribution functions (CDFs), as illustrated by Definition 2.0.2. In the case of more complex CDFs however, these inverses may be intractable, leading to the need for numerical methods when working with quantile functions directly. Alternatively, what if one were to consider transformations of simple quantile functions to build complex distributional, and more specifically, tractable, models? This idea was formalised by Gilchrist (2000) where, in Section 3.2 of his book, a set of construction rules obeyed by quantile functions is presented, and in Section 6 a discussion is given on transformation methods for building new models. Here, the requirement for model complexity is motivated by the properties of the data that the models seek to describe.

In this section, we focus on two classes of the given transformations, p - and Q -transformations, and their development into the notion of rank and sample transmutation maps, respectively, as introduced by Shaw and Buckley (2007, 2009). We draw comparisons with the elongation and reshaping maps of Tukey (1977) to further emphasise the power of simple transformations in the construction of statistically rich quantile functions.

First, imagine the plot of a quantile function $Q(u)$, where on the x -axis we have the quantile level $u \in [0, 1]$. The purpose of p - and Q -transformations is to transform the x - and the y -axis, respectively, whilst ensuring that the resulting plot remains that of a valid quantile function. The transformations are simple yet powerful, and are defined as follows.

Definition 2.1.1. *Let $T_p(u) : [0, 1] \rightarrow [0, 1]$ be a non-decreasing function such that $T_p(0) = 0$ and $T_p(1) = 1$. Then $T_p(u)$ is a p -transformation and for any quantile function $Q(u)$, the function $Q(T_p(u))$ is also a quantile function with the same range as $Q(u)$.*

Whilst the requirement that p -transformations be non-decreasing functions on $[0, 1]$ limits the breadth of available transformations, a wide range of distributions may still be produced. For example, the Power p -transformation, $T_p^P(u) := u^\alpha$ for $\alpha \in \mathbb{R}^+$, transforms the quantile level of the Weibull distribution, with quantile function $Q(u) = (-\log(1 - u))^\beta$ for $\beta \in \mathbb{R}^+$, and produces the quantile function of the generalised exponentiated Weibull distribution, $Q(u) = (-\log(1 - u^\alpha))^\beta$.

Before defining Q -transformations, we first give the definition of a general quantile-preserving map.

Definition 2.1.2. *A map $T : \mathbb{R} \rightarrow \mathbb{R}$ is a quantile-preserving map if for any quantile function $Q(u)$, the function $Q^T(u) := T(Q(u))$ is a quantile function for $u \in [0, 1]$.*

To illustrate the above definition, let Y be a real-valued random variable with distribution function F_Y , and $\eta \in \mathbb{R}$ be such that $F_Y''(y)$ exists and is bounded in the neighbourhood of η , and that $F_Y'(\eta) = f_Y(\eta) > 0$. Let $F_Y(\eta) = u$ for $u \in (0, 1)$, then η is the unique u -quantile of F_Y , see Bahadur (1966). A direct consequence of Definition 2.1.2 is that for a quantile-preserving map T and the real-valued random variable $Z := T(Y)$ with distribution function F_Z , η is such that $\eta_Z := T(\eta)$ is the unique u -quantile of F_Z . Another general example of quantile-preserving maps is the class of Q -transformations, defined as follows.

Definition 2.1.3. *Let $T_Q(x) : \mathbb{R} \rightarrow \mathbb{R}$ be a non-decreasing function. Then $T_Q(x)$ is a Q -transformation and is a quantile-preserving map.*

The practical value of many quantile functions obtained under Q -transformations has led to the naming of such distributions. For example, the lognormal quantile function is obtained by applying $T_Q^e(x) := \exp(x)$ to the normal quantile function $Q_N(u) = \mu + \sigma\sqrt{2}\operatorname{erf}^{-1}(2u - 1)$ for $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$. Considering the uniform quantile function $Q_U(u) = u$, one can construct the quantile function of power law using $T_Q^P(Q_U(u)) := Q_U(u)^\alpha$, the quantile function of the reflected exponential distribution using $T_Q^L(Q_U(u)) := \log(Q_U(u))$ and that of the reciprocal uniform distribution using $T_Q^R(Q_U(u)) := 1/Q_U(1 - u)$. Since the composition of two non-decreasing functions is also non-decreasing, the Pareto distribution can be constructed using $T_Q^{PR}(x) := T_Q^P(T_Q^R(x))$ and the exponential distribution by using $T_Q^{LR}(x) := T_Q^L(T_Q^R(x))$, again applied to the uniform quantile function $x = Q_U(u) = u$. Further examples of

well-known distributions constructed in such a manner are given by Gilchrist (2000) in Section 6.5, and by Klugman et al. (2012).

Analogous to p - and Q -transformations are rank and sample transmutation maps (RTMs and STMs), respectively, as introduced by Shaw and Buckley (2007, 2009), where the rather expository term ‘distributional alchemy’ is coined in the introduction of transmutation maps, capturing the transformational power of such mechanisms. The focus of these maps is the discovery of distributional families by modifying some base distribution in such a way that higher order moments, e.g., skewness and kurtosis, are introduced in a universal and relative manner to the resulting transformed quantile function. The maps are expressed as the composite of the CDF of one distribution, and the quantile function of another. Tractability of the base distribution is preserved in the production of richer parametric families of valid quantile functions, an example being skew-kurtotic variations of the base distribution. We note that the asymptotic analogue to transmutation maps, i.e., approximating one distribution in terms of another, is long-established and well documented under the realm of Edgeworth, or Gram-Charlier expansions, see Bickel (1974), Charlier (1905) and Edgeworth (1907), and Cornish-Fisher expansions, see Cornish and Fisher (1938). Here, skewness and kurtosis are implemented to a base distribution, notably normal or lognormal, by multiplying the base density, or sample (at a given quantile level), respectively, by an asymptotic series consisting of special functions based on the base distribution multiplied by coefficients, up to various orders, involving the output cumulants. The limitations of the practicality of such methods is outlined by Shaw and Buckley (2009), and one may refer to, e.g., Abramowitz and Stegun (1964) and Wallace (1958) for further documentation of these ideas. Henceforth, this thesis will focus on analytical classes of quantile transformation maps, that is, outside of the asymptotic domain. We refer to Shaw and Buckley (2007, 2009) for the following material.

Definition 2.1.4. *Consider two distribution functions F_1 and F_2 with a common sample space. A pair of general RTMs is given by*

$$G_{R_{12}}(u) = F_2(F_1^-(u)), \quad G_{R_{21}}(u) = F_1(F_2^-(u)), \quad (2.1.1)$$

where $G_{R_{ij}}(u) : [0, 1] \rightarrow [0, 1]$, and $G_{R_{ij}}(0) = 0$ and $G_{R_{ij}}(1) = 1$, for $i, j = 1, 2$ and $i \neq j$. Under suitable assumptions, $G_{R_{12}}(u)$ and $G_{R_{21}}(u)$ are mutual inverses.

An additional assumption that the RTMs be continuously differentiable is made in order to ensure that the densities of the mapped random variables are continuous, and one may also assume that they be monotone so that the distribution and quantile functions involved are well-defined. Intuitively, the map $G_{R_{ij}}(u)$ maps the ranks of the distribution F_i to those of the distribution F_j , thus introducing skewness or kurtosis in a universal way. We see that RTMs fall under the class of p -transformations given in Definition 2.1.1 in that they map a rank, or quantile level $u \in [0, 1]$, to a transformed rank, or quantile level, so that for $Q(u)$ any quantile function, $Q(G_{R_{ij}}(u))$ is a quantile function for $i, j = 1, 2$ and $i \neq j$.

Of particular interest is the quadratic RTM (QRTM), where the effect of the map is an introduction of skewness to the base distribution. The QRTM has the form

$$G_{R_{12}}(u) = u + \lambda u(1 - u) \quad (2.1.2)$$

for $|\lambda| < 1$. It follows that $F_2(x) = (1 + \lambda)F_1(x) - \lambda F_1(x)^2$, and the transmuted CDF, F_2 , will represent some skewed version of the base CDF, F_1 . As an example, consider base distributions belonging to the standard uniform, standard normal and exponential (with parameter $\beta > 0$) distributions, given by $F_U(x) = x$, $F_N(x) = \Phi(x) = 0.5(1 + \operatorname{erf}(x/\sqrt{2}))$ and $F_E(x) = 1 - \exp(-\beta x)$, respectively, on the relevant supports. Figure 2.1 shows the transmuted standard uniform, standard normal and exponential distributions under the QRTM for a range of values $\lambda \in [-1, 1]$, including $\lambda = 0$, i.e., the case of no transmutation to the base distribution. We observe the introduction of greater levels of positive (resp. negative) skewness, relative to the base distribution, the larger the absolute value of the positive (resp. negative) parameter λ .

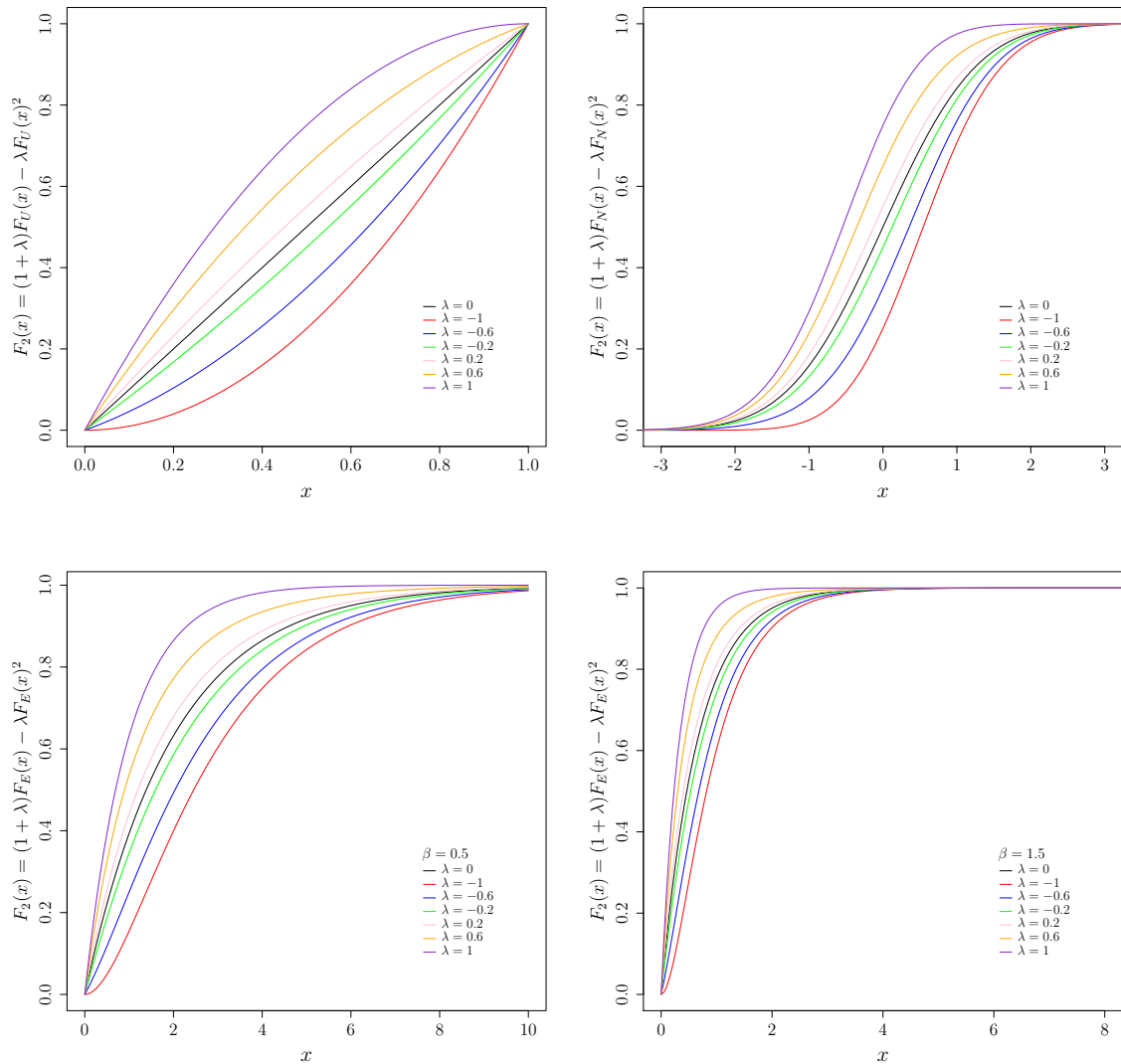


Figure 2.1: Cumulative distribution functions (CDFs) of the transmuted standard uniform, standard normal and exponential CDFs under a quadratic RTM for a range of parameter values $\lambda \in \{-1, -0.6, -0.2, 0, 0.2, 0.6, 1\}$.

We also note alternative existing literature in the context of modulating distributions to introduce skewness, see, e.g., Arellano-Valle et al. (2006), Azzalini (2005, 2008), Azzalini and Capitanio (2003), Genton (2005) and Vicari and Kotz (2005), and the references cited therein. An advantage of quantile transformation maps over models of the types given in these papers lies in the ability to introduce *relative* skewness to some base distribution, as opposed to an absolute amount of skewness, thus providing substantial practical flexibility and interpretation in the context of model assessment

and comparison. Additionally, RTMs may be generalised to treat the introduction of kurtosis by defining the general RTM

$$G_{R_{12}}(u) = u + u(1 - u)P(u) \quad (2.1.3)$$

where P is a polynomial with various parameters. The simplest type of such RTM is the symmetric cubic RTM, obtained by choosing $P(u) = \gamma(u - 0.5)$ for $\gamma \in \mathbb{R}$. This transmutation preserves symmetry and allows for the introduction of kurtosis.

We now draw our attention to STMs, which fall under the class of Q -transformations, given by Definition 2.1.3, in that they transform the entire distribution by converting samples from the inner distribution F_1 to those from F_2 .

Definition 2.1.5. *Consider two distribution functions F_1 and F_2 with sample spaces $D_1 \subseteq \mathbb{R}$ and $D_2 \subseteq \mathbb{R}$, respectively. A general STM $G_{S_{12}}(z) : D_1 \rightarrow D_2$ is given by $G_{S_{12}}(z) = F_2^{-1}(F_1(z))$ and is a quantile-preserving map.*

Applications of STMs include the recycling of samples, e.g., Monte Carlo samples, from one distribution to those from another. We highlight this aspect of the STM mechanism, rather than the fact that for many choices of distribution function F_2 , its inverse might be intractable and *awkward*. Instead, in this thesis, we advocate for the use of quantile transformation maps to produce complex quantile functions, rather than attempting to approximate inverses of CDFs. In case of interest, one may refer to Shaw et al. (2011) and Steinbrecher and Shaw (2008) for a detailed approach on obtaining a direct route to $G_{S_{12}}(u)$, given a choice of F_1 and F_2 , using differential equations, where the only requirement is that one can calculate the logarithmic derivatives of the two corresponding densities f_1 and f_2 .

We now introduce the classes of elongation and reshaping transformations, which allow for the introduction of *relative* skewness and kurtosis to a transformed quantile function. We refer to Hoaglin (1985) and Klein and Fischer (2002) for the following two definitions. First, we note that, when applied to some random variable, the notion of elongation pertains to the tail properties of the distribution of the output random variable being *relative* to those of the input random variable under such transformations. We consider transformations around the mode of the base distribution, under the assumption that this distribution is unimodal, which in the symmetric case coincides with the distribution median and mean. Considering a transformation around the median

of the base distribution in the case where such distribution is non-symmetric provides a relative measure of central tendency in regard to a fixed quantile level $u = 0.5$. This transformation, however, does not pertain to the standard interpretation of relative skew—this is further discussed in Section 3.3.

Definition 2.1.6. *Let X be a symmetric real-valued random variable, and $T_E(x) : \mathbb{R} \rightarrow \mathbb{R}$ be such that $T_E(x) \approx x + \mathcal{O}(x^2)$ for x around the mode of X . The function $T_E(x)$ is called an elongation transformation if, for $x > 0$, it is a strictly monotonically increasing, convex function satisfying $T'_E(x) > 0$, $T''_E(x) > 0$, and $T_E(x) = -T_E(-x)$.*

The properties of the elongation transformation $T_E(x)$ given in Definition 2.1.6 ensure that, when the transformation map is applied to a random variable X , symmetry of the random variable is preserved, deformation of the random variable around the mode is controlled, and increased heaviness of the tails of the transformed random variable, relative to X , is ensured. When applied to a quantile function, the elongation map must satisfy the modal property in Definition 2.1.6 around the mode of the random variable of which the quantile function is transformed. Since $T_E(x)$ is strictly monotonically increasing for $x > 0$ and $T_E(x) = -T_E(-x)$, the elongation transformation map is a quantile-preserving map and it follows that the quantile function of the random variable $\zeta := T_E(X)$ is given by $Q_\zeta(u) = T_E(Q_X(u))$ for all $u \in [0, 1]$.

There exist several families of elongation transformations to produce flexible, relative tail distortions in a controlled manner through the choice of parameters. We emphasise that the maps may be applied to some base random variable directly, to produce an output random variable with the desired properties, or to a quantile function to produce an output quantile function of some random variable with the desired properties. In the case of the introduction of relative skewness instead, a reshaping transformation is employed, and defined so to affect positive values of the input differently to negative ones.

Definition 2.1.7. *Let X be real-valued random variable and $T_R(x) : \mathbb{R} \rightarrow \mathbb{R}$ be such that $T_R(x) \approx x + \mathcal{O}(x^2)$ for x around the mode of X . The function $T_R(x)$ is called a reshaping transformation if it is a strictly increasing function, with continuous second derivative, satisfying $T_R(x) \neq -T_R(-x)$ for all $x \neq 0$.*

If $T_R(x)$ is convex, i.e., $T''_R(x) > 0$, the base distribution (of X) is skewed to the right (increased positive skewness), and if $T_R(x)$ is concave, i.e., $T''_R(x) < 0$, the base distri-

bution is skewed to the left (increased negative skewness). Since $T_R(x)$ is strictly monotonically increasing for $x \in \mathbb{R}$, the reshaping transformation is a quantile-preserving map and it follows that the quantile function of the random variable $\zeta := T_R(X)$ is given by $Q_\zeta(u) = T_R(Q_X(u))$ for all $u \in [0, 1]$.

In this thesis, we focus on the class of Tukey transformations, proposed by Tukey (1977) and further explored by Hoaglin (1985), Jorge and Boris (1984) and MacGillivray (1981, 1992), among others in more recent years. A multivariate extension is detailed by Field and Genton (2006). The family of models allows for the construction of skewed and leptokurtic distributions, covering very flexible ranges of these properties, through the transformation of some base distribution which is often taken to be Gaussian, however can be generalised to a non-Gaussian symmetric random variable, see e.g., Jiménez et al. (2015), Jiménez and Arunachalam (2016) and Klein and Fischer (2002). In the case of introducing heavy-tailedness, the Tukey transformation is an elongation map, as given by Definition 2.1.6, and in the case of introducing skewness, the Tukey transformation satisfies the properties of the reshaping transformation given in Definition 2.1.7. Each subfamily of the Tukey transformation is characterised by a parameterisation T . The subfamilies that have received the most attention are the g -and- h and the g -and- k , which have been explored in various practical contexts by Cruz et al. (2015), Degen et al. (2007), Jiménez and Arunachalam (2011), Peters and Sisson (2006), Peters et al. (2016) and by Haynes et al. (1997) and Hossain and Hossain (2009), respectively. The g -and- h family has gained significant attention due to its ability to adequately approximate a large range of distributions, including the exponential, Student- t , Cauchy and Weibull distributions. In this thesis we focus on the g -and- h family, however replacing the transformation of the type h with that of the type k allows one to obtain the g -and- k family, see Rayner and MacGillivray (2002). Similarly, the g -and- j family is given by Fischer and Klein (2004) where the h transformation is replaced with that of the type j . A detailed overview of the class of models and their extensions, in the context of loss distributions for non-life insurance modelling, is given by Peters et al. (2016), along with a robust procedure for estimating the model parameters. The generic specification of the Tukey transformation is given as follows.

Definition 2.1.8. *Consider a real-valued random variable X with quantile function*

$Q_X(u)$ for $u \in [0, 1]$. The function

$$r(x) := A + BxT(x)^\Theta \quad (2.1.4)$$

is a Tukey transformation with parameter $\Theta \in \mathbb{R}$, location and scale parameters $A \in \mathbb{R}$, $B \in \mathbb{R}^+$, and where $T(x) : \mathbb{R} \rightarrow \mathbb{R}$ characterises the type of Tukey transformation. The transformed random variable $\zeta := r(X)$ is Tukey-distributed with quantile function $Q_\zeta(u) = r(Q_X(u))$ for $u \in [0, 1]$.

The choice of function T and parameter Θ in Eq. (2.1.4) determines the statistical properties of the random variable ζ and thus of the quantile function $Q_\zeta(u)$. For the introduction of relative kurtosis or skewness to the base random variable X , the parameterisation T must ensure the Tukey transformation map given by Eq. (2.1.4) is an elongation map, or a reshaping transformation, respectively, when $A = 0$, $B = 1$. These location and scale parameters may then be applied to shift or re-scale the transformed random variable. As such, restrictions on the distribution of X are required to ensure the maps are well-defined around the distributional mode. Most generally, X is considered a standardised, symmetric random variable, i.e., with probability density function that is symmetric around the origin. Then the mean, median and mode of X lie at zero.

A series of kurtosis transformations that have been proposed in the literature are the h , k and j types, where the parameterisation T in Eq. (2.1.4) is given by

$$T_h(x) := \exp\left(\frac{hx^2}{2}\right), \quad (2.1.5)$$

$$T_k(x) := (1 + x^2)^k, \quad (2.1.6)$$

$$T_j(x) := \left[\frac{\exp(w) + \exp(-w)}{2}\right]^j, \quad (2.1.7)$$

respectively, for $h, j \in \mathbb{R}$, $k > 0.5$ and $\Theta = 1$. If the base distribution is Gaussian, setting $\Theta \geq 0$ allows for the tails to be made heavier under the transformation. We note that special treatment of the elongation map for the Tukey- h transform is required for $h < 0$ since $r_h(x) := A + BxT_h(x)$ is no longer monotonically increasing for $x^2 > -1/h$, see Hoaglin (1985). To introduce skewness, one may consider a transformation of the

g type, where the parameterisation T is given by

$$T_g(x) := \frac{\exp(gx) - 1}{gx} \quad (2.1.8)$$

for $g \in \mathbb{R} \setminus 0$ and $\Theta = 1$. We remark that $T_g(x) = T_{-g}(-x)$ and so the sign of g determines the direction of skewness. Additionally, considering the series expansion of $T_g(x)$, that is,

$$T_g(x) \approx 1 + \frac{gx}{2!} + \frac{g^2x^2}{3!} + \mathcal{O}(x^3), \quad (2.1.9)$$

it holds that $T_g(x) \approx 1$ for $x \approx 0$, and so the transformation scales a standardised, symmetric base distribution differently on either side of its mode (at the origin) via the parameter g , thus producing skewness.

One may introduce both skewness and kurtosis by considering the third transformation in the g -and- h family, the Tukey- gh distributional family, where the parameterisation T is given by the product of the g -and- h type transformations, that is $T_{gh}(x) := T_g(x)T_h(x)$, and $\Theta = 1$. We have

$$r_{gh}(x) = A + Bx \left(\frac{\exp(gx) - 1}{gx} \right) \exp \left(\frac{hx^2}{2} \right) \quad (2.1.10)$$

for $x \in \mathbb{R}$, $g \in \mathbb{R} \setminus 0$ and $h \in \mathbb{R}_0^+$. As $g \rightarrow 0$ the Tukey- gh family coincides with the Tukey- h family, and for $h = 0$ with the Tukey- g family. The random variable X is usually (and thus henceforth in this thesis) assumed to be a standard Gaussian random variable so that the quantile function of the Tukey- gh distribution function is given by

$$Q_{T_{gh}}(u; A, B, g, h) = A + \frac{B}{g} \left[\exp \left(g\sqrt{2}\text{erf}^{-1}(2u - 1) \right) - 1 \right] \exp \left(h \left(\text{erf}^{-1}(2u - 1) \right)^2 \right) \quad (2.1.11)$$

for $u \in [0, 1]$. Figure 2.2 shows the Tukey- gh quantile function for a range of skewness and kurtosis parameters, relative to the standard normal quantile function.

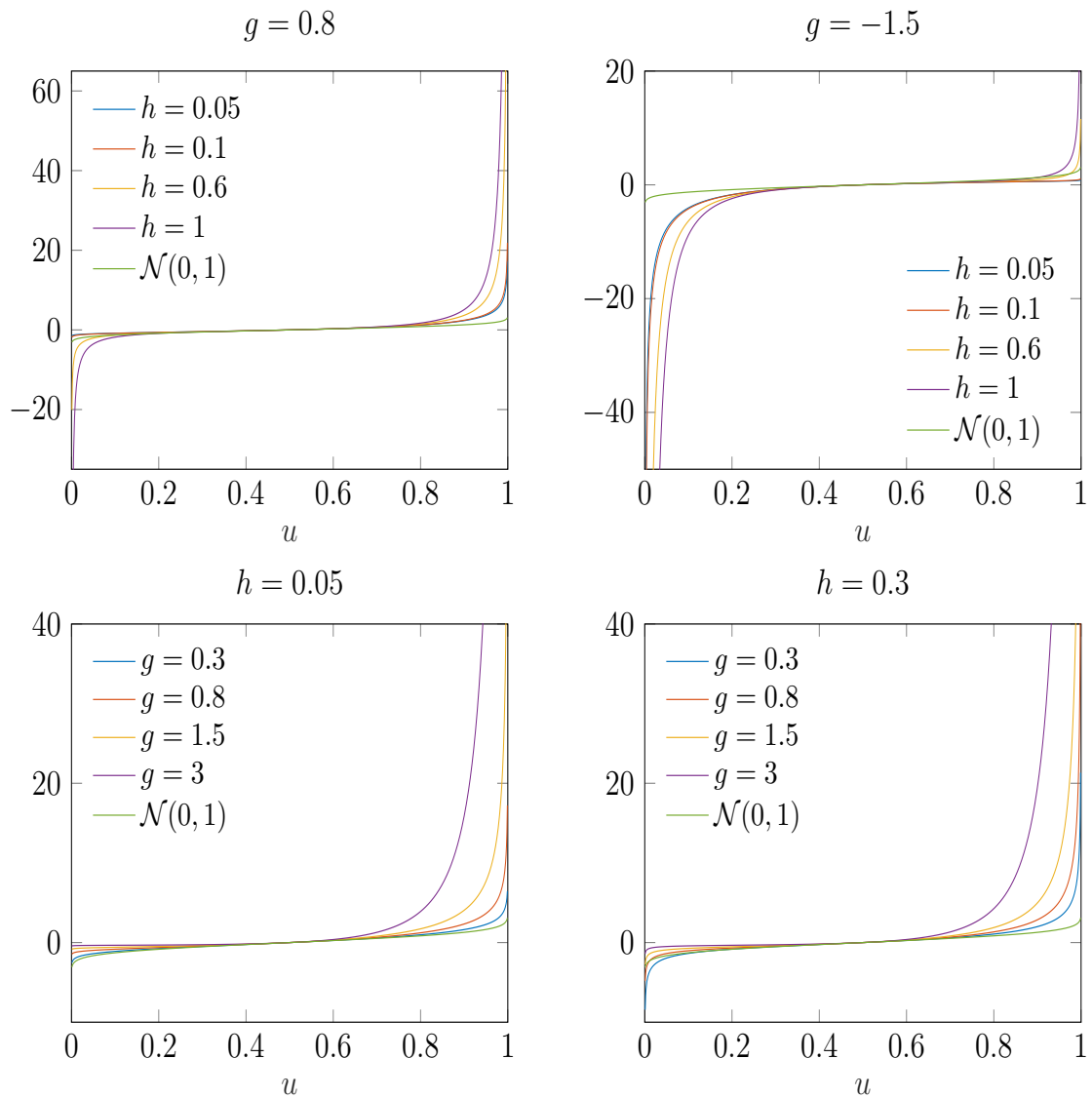


Figure 2.2: Quantile functions of the Tukey- gh distribution for a range of g -and- h parameters, relative to the standard normal quantile function.

2.2 Characterisation of quantile processes

We now introduce some widely adopted definitions of quantile processes with the intention of differentiating these definitions from the use of this terminology in our constructions. We emphasise the connection between quantile processes and quantile transformation maps, illustrated in the context of constructing parametric quantile time series models by Peters (2018).

In much of the literature, the formulation adopted by Csörgő and Révész (1978) is invoked when one refers to a quantile process, and it is based on the univariate, empirical quantile process, defined as follows.

Definition 2.2.1. *Let Y_1, Y_2, \dots, Y_n be a sequence of i.i.d. random variables with a continuous distribution function F_Y , and let $Y_{(1,n)} \leq Y_{(2,n)} \leq \dots \leq Y_{(n,n)}$ denote the order statistics of the random sample Y_1, Y_2, \dots, Y_n . Define the empirical distribution function $F_n(y)$ and the quantile function $Q_n(u)$ as follows:*

$$F_n(y) = \begin{cases} 0 & \text{if } Y_{(1,n)} > y \\ \frac{k}{n} & \text{if } Y_{(k,n)} \leq y < Y_{(k+1,n)}, \quad k = 1, 2, \dots, n-1 \\ 1 & \text{if } Y_{(n,n)} \leq y, \end{cases}$$

$$Q_n(u) = Y_{(k,n)} \quad \text{if } \frac{k-1}{n} < u \leq \frac{k}{n}, \quad k = 1, 2, \dots, n.$$

For $u \in (0, 1)$, the empirical quantile process is defined by $q_n(u) = n^{1/2} (Q_n(u) - F_Y^-(u))$.

This definition relates to the convergence of the law of the order statistics of an empirical process, which is observed as a sequence of independent and identically distributed (i.i.d.) random variables from a fixed distribution F_Y . The new class of quantile processes presented in this thesis is a constructive, parametric approach to modelling quantiles and so we highlight the distinction from Definition 2.2.1, and provide the definition to emphasise this. Similarly, we note another important definition associated to the term ‘quantile process’, being the α -quantile of a continuous-time stochastic process, introduced for a Brownian motion with drift by Dassios (1995), Embrechts et al. (1995) and Yor (1995), and in the context of a process with stationary and independent increments by Dassios (1996).

Definition 2.2.2. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \infty)}, \mathbb{P})$ be a filtered probability space and $(W_t)_{t \in [0, \infty)}$ and (\mathcal{F}_t) -adapted, one-dimensional, standard Brownian motion on the probability space. Define the process $(Y_t)_{t \in [0, \infty)}$ by $Y_t := \mu t + W_t$ for $\mu \in \mathbb{R}$ and let $\mathcal{F}_t = \sigma((Y_s)_{0 \leq s \leq t})$ for all $t \in [0, \infty)$. For $\alpha \in [0, 1]$, and $t \in [0, \infty)$, the α -quantile process of $(Y_s)_{0 \leq s \leq t}$ is given by the process $(M_t^\alpha)_{t \in [0, \infty)}$, defined by*

$$M_t^\alpha(\omega) := \inf \left\{ y : \int_0^t \mathbb{1}_{\{Y_s(\omega) \leq y\}} ds > \alpha t \right\} \quad (2.2.1)$$

for all $\omega \in \Omega$. For fixed $\omega \in \Omega$, $M_t^\alpha(\omega)$ is the α -quantile of the function $s \mapsto Y_s(\omega)$ for $s \leq t$.

The development of this definition was motivated by the problem in financial mathematics involving the pricing of ‘ α -percentile options’, see Akahori (1995) and Miura (1992). For each sample path of the underlying process, the definition produces the quantile at level $\alpha \in [0, 1]$ of the path (i.e., a function of time) up to some time $t \in [0, \infty)$. Whilst the quantile process models constructed in this thesis are also in the context of continuous-time stochastic processes, with a focus on diffusions, the approach is motivated by the ability to develop stochastic processes that model quantiles with specific statistical properties, rather than as a study of the quantiles of the paths of some given process through time. Henceforth, we denote the quantile level by $u \in [0, 1]$.

We now consider a somewhat *discrete-time analogue* to the quantile processes developed in this thesis. The notion of sample quantiles was generalised to quantile regression models by Buchinsky (1998) and Koenker and Bassett Jr (1978), and further developed by Koenker and Hallock (2001) and Koenker (2004), allowing one to estimate conditional quantile functions at any quantile level $u \in [0, 1]$. Quantile time series models were then developed by Koenker and Xiao (2006) for the class of quantile autoregressive (QAR) models, allowing for a representation of the model in which the autoregressive (AR) parameters vary with the quantile level, defined as follows.

Definition 2.2.3. Consider a univariate time series $\{Y_1, \dots, Y_t, \dots\}$ for $t \in \mathbb{N}$ and let $\mathcal{F}_t = \sigma(Y_0, Y_1, \dots, Y_t)$ denote the natural σ -algebra of the observed time series. Let $\{U_t\}$ be a sequence of i.i.d. standard uniform random variables for all $t \in \mathbb{N}$, and $\alpha_i(u) : [0, 1] \rightarrow \mathbb{R}$ be monotone increasing functions, for $i = 0, \dots, p$. A scalar (vector) on function QAR(p) model with random coefficients is defined by

$$Y_t = \alpha_0(U_t) + \sum_{i=1}^p \alpha_i(U_t)Y_{t-1}. \quad (2.2.2)$$

The corresponding QAR(p) model for the conditional quantile function of the random variable Y_t , conditioned on the observations of the time series until time $t - 1$ is characterised by

$$Q_{Y_t}(u|\mathcal{F}_{t-1}) = \alpha_0(u) + \sum_{i=1}^p \alpha_i(u)Y_{t-i}, \quad (2.2.3)$$

for all $u \in [0, 1]$.

The transition from Eq. (2.2.2) to Eq. (2.2.3) follows from the fact that for a standard uniform random variable U with quantile function $Q_U(u) = u$, and $g(u)$ any monotone increasing function, we have $Q_{g(U)}(u) = g(Q_U(u)) = g(u)$ for all $u \in [0, 1]$. Properties of the QAR models, such as quantile correlation (QACF) and quantile partial correlation (QPACF) are defined by Li et al. (2015). Further quantile time series regressions are outlined by Peters (2018) and the references cited therein. There, one finds a presentation of popular classes of quantile time series models that have been developed in the literature, followed by an extensive tutorial on the construction of such models from the perspective of the different model components. The general class of linear or nonlinear conditional quantile time series models is defined by the relation

$$Q_{Y_t}(u|\mathcal{F}_{t-1}, \mathcal{G}_t; \boldsymbol{\theta}) = T(\mathcal{F}_t, \mathcal{G}_t, Q_\epsilon(u; \boldsymbol{\gamma})) \quad (2.2.4)$$

where T is a quantile-preserving map given by Definition 2.1.2. The filtration $\mathcal{G}_t = \sigma(\mathbf{X}_0, \dots, \mathbf{X}_t)$ is generated by the series $\{\mathbf{X}_0, \dots, \mathbf{X}_t\}$, $\mathbf{X}_t \in \mathbb{R}^d$, of observed exogenous covariates, $\boldsymbol{\theta} \in \mathbb{R}^{d'}$ is a static vector of model parameters, $Q_\epsilon(u; \boldsymbol{\gamma})$ is a quantile error function of some white noise sequence ϵ_t with static vector of parameters $\boldsymbol{\gamma} \in \mathbb{R}^{d''}$, and the remaining notation is as per Definition 2.2.3. The focus of parametric models of the type given in Eq. (2.2.4) lies in the modelling choice of $Q_\epsilon(u; \boldsymbol{\gamma})$ and the map T , which may either be applied to the quantile error function to obtain more flexible families (as discussed in Section 2.1) or to the quantile time series relationship to produce nonlinear quantile time series models. The most common choice of quantile-preserving maps are the classes of linear additive maps, nonlinear multiplicative maps, or Q -transformations, as given by Definition 2.1.3. As such, here lies the connection between quantile transformation maps and the construction of flexible discrete-time quantile time series models.

As an example, an extension of the linear QAR model in Eq. (2.2.2), with quantile error function $Q_\epsilon(u; \boldsymbol{\gamma})$ and some choice of map T , is given by

$$Q_{Y_t}(u|\mathcal{F}_{t-1}, \mathcal{G}_t; \boldsymbol{\theta}) = \sum_{i=1}^p \alpha_i(u) Y_{t-1} + \sum_{j=1}^d \sum_{i=1}^k \beta_{j,i} X_{j,t-i} + T(Q_\epsilon(u; \boldsymbol{\gamma})) \quad (2.2.5)$$

for $\boldsymbol{\theta} = (\boldsymbol{\beta}, \boldsymbol{\gamma})$ and $\boldsymbol{\beta} = (\boldsymbol{\beta}_0, \dots, \boldsymbol{\beta}_d)$ for $\boldsymbol{\beta}_i \in \mathbb{R}^k$ and $i = 1, \dots, d$. Here, we assume the

map T is applied to the quantile error function directly, allowing for greater flexibility in its properties whilst preserving the linearity of the model. An example of a nonlinear quantile time series model of the class given by Eq. (2.2.4) is the dynamic quantile function model (DQM) introduced by Chen et al. (2022). Here, a parametric example is given in which T is considered to be a Tukey– gh transformation map, outlined as follows.

Definition 2.2.4. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X(u, \omega) : [0, 1] \times \Omega \rightarrow \mathbb{R}$ be such that if $u \in [0, 1]$ is fixed, $X(u, \cdot)$ is a real-valued random variable on the probability space, and if $\omega \in \Omega$ is fixed, $X(\cdot, \omega)$ is a real-valued quantile function. Here, X is called a quantile function-valued (QF-valued) random variable. Let $\{X_t\}$ denote the set of QF-valued random variables indexed by $t \in \mathbb{Z}$ and define the p -dimensional vector $\boldsymbol{\xi}_t = \mathcal{M}(X_t)$ for a mapping \mathcal{M} such that $\mathcal{M} : X(\cdot, \omega) \rightarrow \mathbb{R}^p$ for fixed $\omega \in \Omega$. Define the conditional distribution $F_{\boldsymbol{\xi}_t}$ on \mathbb{R}^p such that $\boldsymbol{\xi}_t | \mathcal{F}_{t-1} \sim F_{\boldsymbol{\xi}_t}$ where $\mathcal{F}_{t-1} = \sigma(\{\boldsymbol{\xi}_s : s \leq t-1\})$. The QF-valued one-step-ahead forecast is defined by*

$$\tilde{X}_t(u) := \mathcal{M}^{-1}(\mathbb{E}[\boldsymbol{\xi}_t | \mathcal{F}_{t-1}]) \quad (2.2.6)$$

for all $t \in \mathbb{Z}$.

Under the assumption that $X_t(u)$ is the quantile function of a Tukey– gh distribution, so that

$$X_t(u) = \begin{cases} A_t + B_t \frac{\exp(g_t \sqrt{2} \operatorname{erf}^{-1}(2u-1)) - 1}{g_t} \exp\left(h_t (\operatorname{erf}^{-1}(2u-1))^2\right), & g_t \neq 0 \\ A_t + B_t \sqrt{2} \operatorname{erf}^{-1}(2u-1) \exp\left(h_t (\operatorname{erf}^{-1}(2u-1))^2\right), & g_t = 0, \end{cases} \quad (2.2.7)$$

for $A_t \in \mathbb{R}$, $B_t \in \mathbb{R}^+$, $g_t \in \mathbb{R}$ and $h_t \in \mathbb{R}_0^+$, the parameterisation \mathcal{M} is chosen to be $\boldsymbol{\xi}_t = (A_t, \log(B_t), g_t, h_t)$. The quantile process in Eq. (2.2.7) is a nonlinear quantile function-valued time series model based on the Tukey– gh transformation map. In Section 3.2 we define a class of ‘function-valued’ quantile processes that could be viewed as a continuous-time analogue to those given by Chen et al. (2022), above.

Chapter 3

Construction of Stochastic Quantile Processes

Numerous classes of discrete- and continuous-time quantile processes have been explored in existing literature, as discussed in Section 2.2. The purpose of this chapter is to introduce the novel class of continuous-time stochastic quantile processes that constitutes the backbone of the work presented in this thesis. We develop two constructions of such processes, one based on a ‘process-driven’ (or ‘random-level’) construction, which produces a scalar-valued process, and the second based on a ‘parameter-driven’ (or ‘function-valued’) construction, which produces a function-valued process in the space of quantile functions. In this thesis, we focus largely on the first of these constructions. Here, quantile processes are developed by distorting each (time) marginal of a given univariate stochastic process under a composite map consisting of a distribution function and a quantile function, which in turn produces the marginals of the resulting quantile process. The family of quantile (distortion) processes will be characterised by the parametric form of the selected transformation composite map and the underlying base process; connections to the quantile transformation maps given in Section 2.1 are made. As such, we emphasise the wide class of models with directly interpretable statistical characteristics that arise from such a transformation.

In the function-valued construction, we model the parameters of a well-defined quantile function by a multivariate stochastic process, and hence map from realisations of each of these parameter processes to function-valued realisations of the quantile process. Each sample path of the multivariate parameter process will drive the resulting

function–valued quantile process, allowing one to dynamically model the entire quantile function at any instance in time, much like the discrete–time QAR process given by Eq. (2.2.3). The choice of parametric quantile function determines the statistical properties, e.g., asymmetry and tail–heaviness, that can be accommodated in the quantile function–valued output process.

We lay the mathematical foundations for these frameworks to be rigorously defined in the following two sections. The notation and definitions used in this thesis include the formulation of the relevant probability spaces. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \infty)}, \mathbb{P})$ denote a filtered probability space with filtration (\mathcal{F}_t) and $(W_t)_{t \in [0, \infty)}$ an (\mathcal{F}_t) –adapted, one–dimensional, standard Brownian motion. Unless stated otherwise, when we refer to the probability space on which a process is defined, we mean $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \infty)}, \mathbb{P})$.

3.1 Random–level quantile processes

The first type of quantile processes is constructed by a composite map (which we refer to as the ‘composite map’ or ‘distortion map’ throughout) applied marginally to a continuous–time stochastic process, which in principle could be multivariate, but the univariate case is considered prior to Section 3.4. This auxiliary process (which we refer to as the ‘driving process’ or ‘base process’ throughout) produces the stochasticity of the output quantile process, which characterises a law with relative statistical properties determined by the functions in the map. By the choice of its functional form, the map is a quantile preserving map, as given by Definition 2.1.2, which ensures that for every quantile level $u \in [0, 1]$, well–defined quantiles of the driving process are mapped to well–defined quantiles of the output process at all times $t \in (0, \infty)$ —hence, ‘quantile process’. The result is a wide class of models with appropriately chosen attributes, from a modelling perspective, both marginally and serially. Since the composite map is a quantile preserving map, if the serial dependence is captured by a copula, see Nelsen (2007), on the transition distribution of each (continuous) process, the serial dependence structure of the driving process is the same as that of the output quantile process. That is, the copula of the process is invariant under the proposed procedure. This follows from the fact that a quantile preserving map is increasing, and by Sklar’s theorem, see Sklar (1959), the copula of continuous random variables is invariant under a monotonic transformation of those random variables.

The class of base processes with càdlàg paths is considered, thus including Lévy processes as stochastic drivers, for example. As such, the composite map can be used to obtain quantile processes with either continuous or discrete (discontinuous) paths. We present the continuous random–level quantile process construction as follows. Here, the notation ‘ $\stackrel{d}{=}$ ’ means equal in distribution.

Definition 3.1.1. *Let $Q_\zeta(u; \boldsymbol{\xi})$ be the quantile function of some real–valued, continuous random variable ζ , where $u \in [0, 1]$ is the quantile level and $\boldsymbol{\xi} \in \mathbb{R}^d$ is a d –dimensional vector of parameters, for $d \in \mathbb{N}$. For $t \in (0, \infty)$, let $F(t, y; \boldsymbol{\theta}) : \mathbb{R}^+ \times \mathbb{R} \rightarrow [0, 1]$ be a continuous, time–dependent distribution function, where $\boldsymbol{\theta} \in \mathbb{R}^{d'}$ is a d' –dimensional vector of parameters, $d' \in \mathbb{N}$. Consider a real–valued process $(Y_t)_{t \in [0, \infty)}$ with continuous paths and $Y_0 = y_0 \in \mathbb{R}$. At each time $t \in [t_0, \infty)$, $t_0 > 0$, the continuous random–level quantile process is defined by*

$$Z_t \stackrel{d}{=} Q_\zeta(F(t, Y_t; \boldsymbol{\theta}); \boldsymbol{\xi}), \quad (3.1.1)$$

that is, $Z(t, \omega) = Q_\zeta(F(t, Y(t, \omega); \boldsymbol{\theta}); \boldsymbol{\xi}) : [t_0, \infty) \times ([t_0, \infty) \times \Omega) \rightarrow [-\infty, \infty]$. The map $t \mapsto Z(t, \omega)$ for each $\omega \in \Omega$ and all $t \in [t_0, \infty)$ is \mathcal{F}_t –measurable. Here, the random variable ζ characterises the family of quantile processes to which (Z_t) belongs.

The process (Z_t) is well–defined for any choice of initial time $t_0 := 0 + \epsilon$, $\epsilon > 0$, by continuity of the marginal distribution $F(t, y; \boldsymbol{\theta})$ for any $t > 0$. In principle, one could also set $Z_0 = z_0 \in \mathbb{R}$ and extend the time interval, on which (Z_t) is defined, to $t \in [0, \infty)$. Throughout this thesis, we consider the time domain on which a quantile process is defined to be $[t_0, \infty)$ where $t_0 = 0 + \epsilon$ for $\epsilon > 0$ arbitrarily small, and so we write $t_0 > 0$. Additionally, we may consider the ranges of the random variables ζ and Y_t , for each $t \in (0, \infty)$, as well as the domain of the distribution function F , to be some subset of the real line. Denote the range of Y_t for each $t \in (0, \infty)$ by $\text{ran}(Y_t)$ and the domain of the distribution function F by $\text{dom}(F)$. Here, we restrict the quantile process construction, given by Definition 3.1.1, to the case that $\max_{t \in (0, \infty)} \text{ran}(Y_t) = \text{dom}(F)$, so that each stage of the composite map is well–defined. It follows that $\text{ran}(Z_t) = \text{ran}(\zeta)$ for all $t \in [t_0, \infty)$.

Now, consider Definition 3.1.1 and assume the process (Y_t) is governed by the finite–dimensional distribution function $F_Y(t, y; \boldsymbol{\vartheta}) : \mathbb{R}^+ \times \mathbb{R} \rightarrow [0, 1]$ for $t \in (0, \infty)$ and $\boldsymbol{\vartheta} \in \mathbb{R}^k$ a k –dimensional vector of parameters, for $k \in \mathbb{N}$. Then the marginals of the

process given by $U_t \stackrel{d}{=} F_Y(t, Y_t; \boldsymbol{\vartheta})$ at any time $t \in [t_0, \infty)$ are uniformly distributed on $[0, 1]$. If $F = F_Y$ in Eq. (3.1.1), we may associate the value of the process (U_t) , at any time, as the quantile level that each marginal of the quantile process (Z_t) corresponds to at that given time. This concept motivates the name ‘random–level’ in the above quantile process construction. Additionally, when (Y_t) is a diffusion, the process (U_t) is referred to as a ‘uniformized diffusion process’ by Bibbona et al. (2016), and its dynamics are derived. It is also stated that the same (U_t) may be constructed from different driving processes and their marginal distributions, however this is equivalent to the driving processes having the same serial dependence as characterised by a unique copula, see Nelsen (2007), on the Fokker–Planck transition distribution. Also, we note that at $t = 0$, the probability mass of F_Y is concentrated on $Y_0 = y_0 \in \mathbb{R}$, i.e., $F_Y(t, y; \boldsymbol{\vartheta})$ depends on $y_0 \in \mathbb{R}$ (as a parameter), so for each y_0 we obtain a different quantile process (Z_t) .

Otherwise, in the general case that $F \neq F_Y$, the process given by $\tilde{U}_t \stackrel{d}{=} F(t, Y_t; \boldsymbol{\theta})$ at any time $t \in [t_0, \infty)$ is non–uniformly distributed on $[0, 1]$. We may also write $\tilde{U}_t \stackrel{d}{=} F(t, Y_t; \boldsymbol{\theta}) \stackrel{d}{=} F(t, Q_Y(t, U_t; \boldsymbol{\vartheta}); \boldsymbol{\theta})$ for (U_t) a process that is uniformly distributed at each $t \in [t_0, \infty)$ and where $Q_Y = F_Y^-$. It then follows that the value of the process (U_t) , at any time, is considered to be the quantile level that the process (Z_t) corresponds to at that given time. In either case, the quantile process (Z_t) models well–defined quantiles for all quantile levels in $[0, 1]$.

As previously discussed, of particular interest and importance in the random–level quantile process construction, given by Definition 3.1.1 is how (Z_t) behaves *relative* to (Y_t) , in regard to the statistical properties of its finite–dimensional distributions. Since the composite map employed in the construction of the quantile process, i.e., that in Eq. (3.1.1), is a quantile–preserving map, as given by Definition 2.1.2, at each $t \in [t_0, \infty)$, the quantile function of the output process is given by

$$Q_Z(t, u; \tilde{\boldsymbol{\xi}}) = Q_\zeta(F(t, Q_Y(t, u; \boldsymbol{\vartheta}); \boldsymbol{\theta}); \boldsymbol{\xi}) \quad (3.1.2)$$

for $u \in [0, 1]$ and where $\tilde{\boldsymbol{\xi}} := (\boldsymbol{\xi}, \boldsymbol{\theta}, \boldsymbol{\vartheta})$. We note that throughout this thesis, we consider unimodal distributions with finite moments of the considered order, unless stated otherwise. This simplifies the idea of considering the distortions produced by the composite map to be relative to some measure of centrality, which is often assumed

to be the mode of the base driving process. In the case of multimodal distributions, one may rederive the following results.

To determine whether the quantile process is more (positively or negatively) skewed, or has heavier tails, than the driving process, marginally, we may consider the skewness and kurtosis orderings, respectively, introduced by van Zwet (1964). Such results are reformulated in the context of random–level quantile processes in the following propositions. We note that by ‘more positively skewed’ we mean a random variable with density function that has longer right tails and shorter left tails—its mode is less than its median which is less than its mean—than the density function of some other random variable.

Proposition 3.1.1. *Consider Definition 3.1.1 such that at each time $t \in [t_0, \infty)$, the quantile function of the output quantile process is given by Eq. (3.1.2). Let $F_Y(t, y; \boldsymbol{\vartheta})$ be the marginal distribution function of the driving process, and define $D_Y \subseteq \mathbb{R}$, such that $D_Y := \{y : 0 < F_Y(t, y; \boldsymbol{\vartheta}) < 1\}$. Then for each $t \in [t_0, \infty)$, the marginal of the driving process is no more positively skewed than that of the quantile process if, and only if, the map $Q_\zeta(F(t, y; \boldsymbol{\theta}); \boldsymbol{\xi})$ is convex on D_Y for all $\boldsymbol{\theta} \in \mathbb{R}^d$ and $\boldsymbol{\xi} \in \mathbb{R}^d$. If $Q_\zeta(F(t, y; \boldsymbol{\theta}); \boldsymbol{\xi})$ is convex and nonlinear on D_Y , we say that the marginal of the quantile process is more positively skewed than that of the driving process for each $t \in [t_0, \infty)$.*

The proof is a direct consequence of the result proved by van Zwet (1964), given that, considering Eq. (3.1.2) and the composite map in Eq. (3.1.1), we have

$$Q_Z\left(t, F_Y(t, y; \boldsymbol{\vartheta}); \tilde{\boldsymbol{\xi}}\right) = Q_\zeta(F(t, y; \boldsymbol{\theta}); \boldsymbol{\xi}) \quad (3.1.3)$$

for all $t \in [t_0, \infty)$ and $y \in D_Y$.

To compare kurtosis (in the sense of heaviness of the tails) between distributions, it is common that symmetric distributions are considered. The interplay between kurtosis measures and orderings, and skewness are discussed in detail by Balanda and MacGillivray (1990) and MacGillivray and Balanda (1988). We consider, however, the kurtosis ordering of van Zwet (1964), for symmetric distributions, reformulated in the context of random–level quantile processes in the following proposition. We note that by ‘more kurtosis’ we mean a random variable with density function that has heavier tails (and thus peakedness) than the density function of some other random variable.

Proposition 3.1.2. *Consider the setting in Proposition 3.1.1 and assume $F_Y(t, y; \boldsymbol{\vartheta})$ and $Q_Z(t, u; \tilde{\boldsymbol{\xi}})$ are symmetric for each $t \in [t_0, \infty)$ and all $\boldsymbol{\vartheta} \in \mathbb{R}^k$ and $\tilde{\boldsymbol{\xi}} \in \mathbb{R}^{k'}$. For each $t \in [t_0, \infty)$, denote the median of Y_t by $m_t^Y \in D_Y$. Then for each $t \in [t_0, \infty)$, the marginal of the driving process has less kurtosis than that of the quantile process if, and only if, the map $Q_\zeta(F(t, y; \boldsymbol{\theta}); \boldsymbol{\xi})$ is convex for all $y > m_t^Y$ and all $\boldsymbol{\theta} \in \mathbb{R}^d$ and $\boldsymbol{\xi} \in \mathbb{R}^d$.*

The proof is analogous to that of Proposition 3.1.1, based on the result proved by van Zwet (1964). We remark that, by the assumed symmetry of the distribution functions, the composite map $Q_\zeta(F(t, y; \boldsymbol{\theta}); \boldsymbol{\xi})$ will be concave for $y < m_t^Y$. Since the marginal distribution of the driving process is assumed to be symmetric, its median, mean and mode will be equal, and kurtosis is introduced in the transformation relative to these measures of centrality. Various alternative measures, or orderings, of skewness and kurtosis that are largely based on those of van Zwet (1964) are explored by MacGillivray (1986), MacGillivray and Balanda (1988), Oja (1981) and the references cited therein. If one wishes to consider the non–symmetric case, instead of the studying the convexity of the composite map, as given in Proposition 3.1.2, one may consider the convexity of the composition of the ‘spread’ of the quantile process with the inverse of that of the driving process, where for some random variable Y , its spread is defined as $S_Y(u) := Q_Y(0.5 + u) - Q_Y(0.5 - u)$ for $u \in [0, 0.5)$. In other words, we replace $Q_\zeta(F(t, y; \boldsymbol{\theta}); \boldsymbol{\xi})$ with $S_Z(S_Y^-(y))$ in Proposition 3.1.2, and require convexity for $y \geq 0$ at each time. This is described in detail by Balanda and MacGillivray (1990), and applied to the quantile process setting analogously to the above Propositions.

In Section 3.3, we observe how the two classes of reshaping and elongation maps, given generally in Definitions 2.1.7 and 2.1.6, respectively, are defined with regard to Propositions 3.1.1 and 3.1.2, respectively. We note that both types of such maps also consider the distributional distortions relative to the mode of the base distribution. We discuss further the case in which one may wish to define the deformation relative to some measure of central tendency, next.

Depending on the modelling objective, one should consider each stage of the composite map carefully in regard to how the quantiles of the driving process are mapped to the quantiles of the output process at some given level; in other words, how each input value is mapped to each output value. For example, first assume $F = F_Y$ in the composite map and the driving process has median 100 for all $t \in (0, \infty)$. If the ran-

dom variable ζ is chosen such that its median is, e.g., zero (however the same argument holds for any value significantly less than, or greater than, the median of the driving process, marginally), then the composite map takes quantiles at level 0.5 that lie at 100 to those that lie at zero. Here, one may wish to adjust the parameters of $Q_\zeta(u; \boldsymbol{\xi})$, i.e., the components of the vector $\boldsymbol{\xi}$, so that the median of ζ , that is, $Q_\zeta(0.5, \boldsymbol{\xi})$, is approximately the marginal median of Y_t for $t \in (0, \infty)$. A similar argument can be made in the case where $F \neq F_Y$, and motivates the notion of defining the quantile transformation in Eq. (3.1.1) relative to some measure of central tendency, e.g., the mode, mean or median of each marginal of the driving process may be preserved under the composite map. This is discussed further in Section 3.3 for the flexible class of Tukey models, and illustrated in the following example.

Example 3.1.1. *Let (Y_t) be a driving process with marginal distribution function $F_Y(t, y; \boldsymbol{\vartheta})$ for all $t \in (0, \infty)$, and define $m_t^Y := Q_Y(t, 0.5; \boldsymbol{\vartheta})$, that is the marginal median of the process. In the quantile process construction, choose F such that $F(t, m_t^Y) = 0.5$ for all $t \in (0, \infty)$ and Q_ζ such that $Q_\zeta(0.5; \boldsymbol{\xi}(t)) = m_t^Y$ for all $t \in [t_0, \infty)$. The vector of parameters in the quantile function Q_ζ may be time–dependent, as shown here, (or, in some cases, stochastic), to ensure this holds for all times. Then the composite map is median–preserving.*

Now, for the random–level quantile process (Z_t) to have discrete paths, it suffices that either the driving process (Y_t) has discontinuous, càdlàg paths; the quantile function $Q_\zeta(u; \boldsymbol{\xi})$ is that of a discrete random variable ζ ; or that the distribution function $F(t, y; \boldsymbol{\theta}) : \mathbb{R}^+ \times \mathcal{D} \rightarrow \mathcal{U}$ (resp. $F_Y(t, y; \boldsymbol{\vartheta}) : \mathbb{R}^+ \times \mathcal{D} \rightarrow \mathcal{U}$) is a discrete distribution function, where \mathcal{D} and \mathcal{U} denote the collections of all countable subsets of \mathbb{R} and $[0, 1]$, respectively. Here, $Z_t(\omega) = Z(t, \omega) = Q_\zeta(F(t, Y_t(\omega); \boldsymbol{\theta}); \boldsymbol{\xi}) : \mathbb{R}^+ \times (\mathbb{R}^+ \times \Omega) \rightarrow \mathcal{D}$ (resp. $Z_t(\omega) = Z(t, \omega) = Q_\zeta(F_Y(t, Y_t(\omega); \boldsymbol{\vartheta}); \boldsymbol{\xi}) : \mathbb{R}^+ \times (\mathbb{R}^+ \times \Omega) \rightarrow \mathcal{D}$) at any time $t \in [t_0, \infty)$. In what follows, we wish to consider quantile processes with càdàg paths, characterised by the following proposition.

Proposition 3.1.3. *The quantile process (Z_t) will have càdlàg paths if, and only if, the map $t \mapsto F(t, y; \boldsymbol{\theta})$ is càdlàg for all $y \in \mathbb{R}$ and $\boldsymbol{\theta} \in \mathbb{R}^d$, and $Q_\zeta(u; \boldsymbol{\xi})$ is a continuous quantile function for all $\boldsymbol{\xi} \in \mathbb{R}^d$, that is, ζ is a continuous random variable.*

Proof. In the following proof we drop notational dependence of distribution and quantile functions on the parameters and assume the given statements hold for all values

in each parameter space. Since (Y_t) has càdlàg paths, for $s \in (0, \infty)$ and all $\omega \in \Omega$, $\lim_{t \downarrow s} Y(t, \omega) = Y(s, \omega)$ and the left limit, $\lim_{t \uparrow s} Y(t, \omega)$ exists. By the definition of a distribution function, the map $y \mapsto F(t, y)$ will be càdlàg for all $t \in (0, \infty)$. By Definition 2.0.1 of a generalised inverse, the quantile function $Q_\zeta(u)$ will be càglàd when $F_\zeta := Q_\zeta^-$ is càdlàg, and continuous when F_ζ is continuous. If $F(t, y)$ is a càdlàg function in both arguments, and $Q_\zeta(u)$ is a continuous function for $u \in [0, 1]$, it follows that

$$\begin{aligned} \lim_{t \downarrow s} Z(t, \omega) &= \lim_{t \downarrow s} Q_\zeta(F(t, Y(t, \omega))) = Q_\zeta(F(s, \lim_{t \downarrow s} Y(t, \omega))) \\ &= Q_\zeta(F(s, Y(s, \omega))) = Z(s, \omega), \end{aligned} \quad (3.1.4)$$

and the limit

$$\lim_{t \uparrow s} Z(t, \omega) = \lim_{t \uparrow s} Q_\zeta(F(t, Y(t, \omega))) = Q_\zeta(F(s, \lim_{t \uparrow s} Y(t, \omega))) \quad (3.1.5)$$

exists by the càdlàg property of the paths of (Y_t) , so $Z_t = Z(t, \omega)$ has càdlàg paths for all $\omega \in \Omega$. We loosen the restrictions of continuity and show, in turn, that if either $t \mapsto F(t, y)$ is not càdlàg for all $y \in \mathbb{R}$, or $Q_\zeta(u)$ is not continuous, (Z_t) will not have càdlàg paths and thus the proposition holds by contradiction.

First, assume $Q_\zeta(u)$ is continuous and $t \mapsto F(t, y)$ is not a càdlàg map. Then for each $\omega \in \Omega$,

$$\lim_{t \downarrow s} Z(t, \omega) = \lim_{t \downarrow s} Q_\zeta(F(t, Y(t, \omega))) = Q_\zeta\left(\lim_{t \downarrow s} F(t, Y(t, \omega))\right) \neq Z(s, \omega) \quad (3.1.6)$$

as $F(t, y)$ is not right–continuous and so

$$\lim_{t \downarrow s} F(t, Y(t, \omega)) \neq F(s, \lim_{t \downarrow s} Y(t, \omega)) = F(s, Y(s, \omega)). \quad (3.1.7)$$

It follows that $Z_t = Z(t, \omega)$ will not have right–continuous paths. Now, assume $t \mapsto F(t, y)$ is càdlàg and $Q_\zeta(u)$ is càglàd but not continuous. We have

$$\lim_{t \downarrow s} Q_\zeta(F(t, Y(t, \omega))) \neq Q_\zeta\left(\lim_{t \downarrow s} F(t, Y(t, \omega))\right) = Q_\zeta\left(F\left(s, \lim_{t \downarrow s} Y(t, \omega)\right)\right) \quad (3.1.8)$$

and so $Z_t = Z(t, \omega)$ can not have right–continuous paths. It follows that the limit in

Eq. (3.1.4) holds true, and that in Eq. (3.1.5) exists if, and only if, $F(t, y)$ is càdlàg in both arguments, and $Q_\zeta(u)$ is a continuous quantile function. We only specify that $t \mapsto F(t, y)$ for all $y \in \mathbb{R}$ is càdlàg as $y \mapsto F(t, y)$ for all $t \in (0, \infty)$ will be càdlàg by the definition of a distribution function. \square

We conclude this section by emphasising that the composite map takes the form of a general STM, as given by Definition 2.1.5. In a sampling setting, samples of the driving process (Y_t) , at some quantile level, are transformed via the map to samples of the quantile process (Z_t) , at the same quantile level, from some target distribution which, when $F = F_Y$ in Eq. (3.1.1), corresponds to that of the random variable ζ . Whilst the outer part of the map involves a quantile function which, as discussed in Section 2.1, is often intractable in the case of more statistically complex distributions, we will employ quantile–preserving maps (e.g., Q –transformations, elongation maps and reshaping transformations), applied to simple quantile functions, to produce richer and flexible families for the quantile function $Q_\zeta(u; \boldsymbol{\xi})$. As such, the construction in Definition 3.1.1 gives rise to a dynamic, continuous–time family of distortion processes characterised by some parametric form of the selected transformation map and some underlying base process. The utility of quantile dynamics in this context, to our knowledge, has not yet been proposed or studied in the setting of continuous–time càdlàg processes.

Further interpretation of random–level quantile processes is given by Figure 3.1, where the plots of three sample paths of some quantile process (Z_t) are shown. Considering three time points, $0 < t_0 < t_1 < t_2 < \infty$, we observe the following:

- (i) For each $\omega_i \in \Omega$, the path $Z(t, \omega_i)$ of the quantile process corresponds to a sequence of quantiles at different levels $U(t, \omega_i)$ for all $t \in [t_0, \infty)$.
- (ii) If $n \in \mathbb{N}$ paths of the quantile process are observed, at any time $t \in [t_0, \infty)$ the entire quantile function, that is over all quantile levels, is observed as $n \rightarrow \infty$.

Similarly, Figure 3.2 shows the plots of three sample paths of the intermediate process, $U_t \stackrel{d}{=} F(t, Y_t)$, defined on $[0, 1]$. If $F = F_Y$, when $n \in \mathbb{N}$ paths of the process are observed then, at any time $t \in [t_0, \infty)$, as $n \rightarrow \infty$ all values $u \in [0, 1]$ will be observed. This follows from the fact that, here, U_t is uniformly distributed on $[0, 1]$.

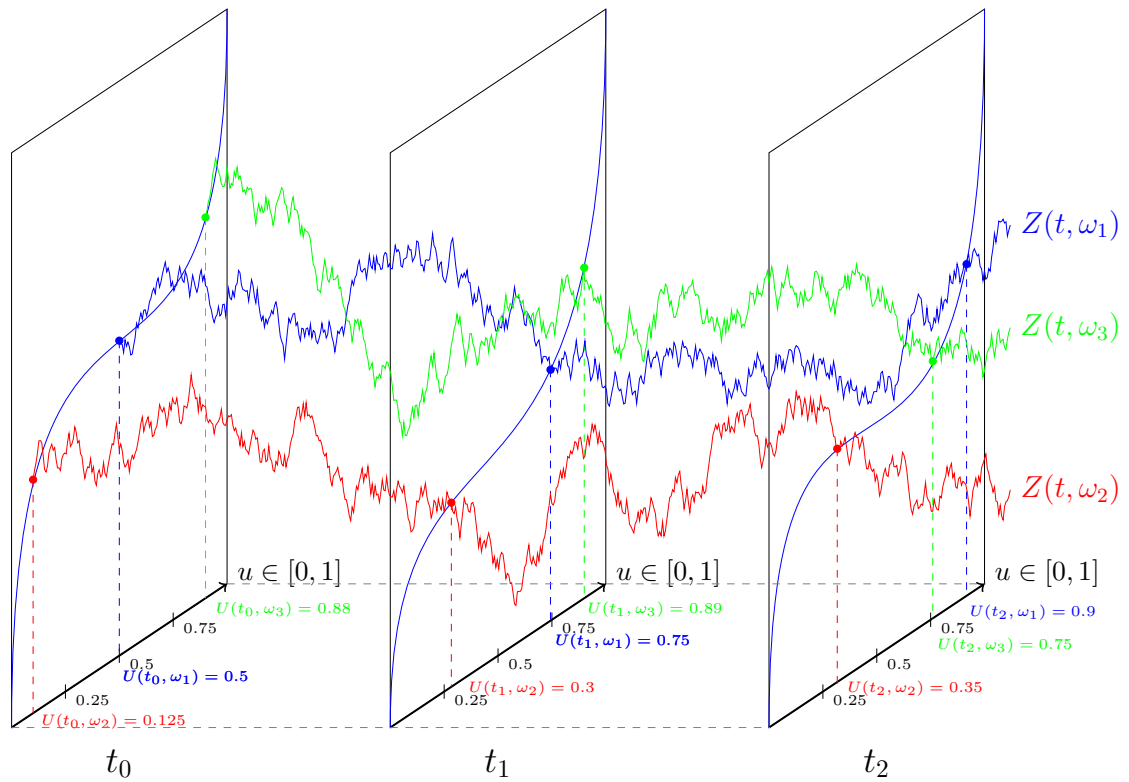


Figure 3.1: Three random-level quantile process paths for $\omega_1, \omega_2, \omega_3 \in \Omega$ and the corresponding realisations of the intermediate quantile level process (U_t) at times $0 < t_0 < t_1 < t_2 < \infty$. Here, $Z(t, \omega) = Q_\zeta(U(t, \omega))$.

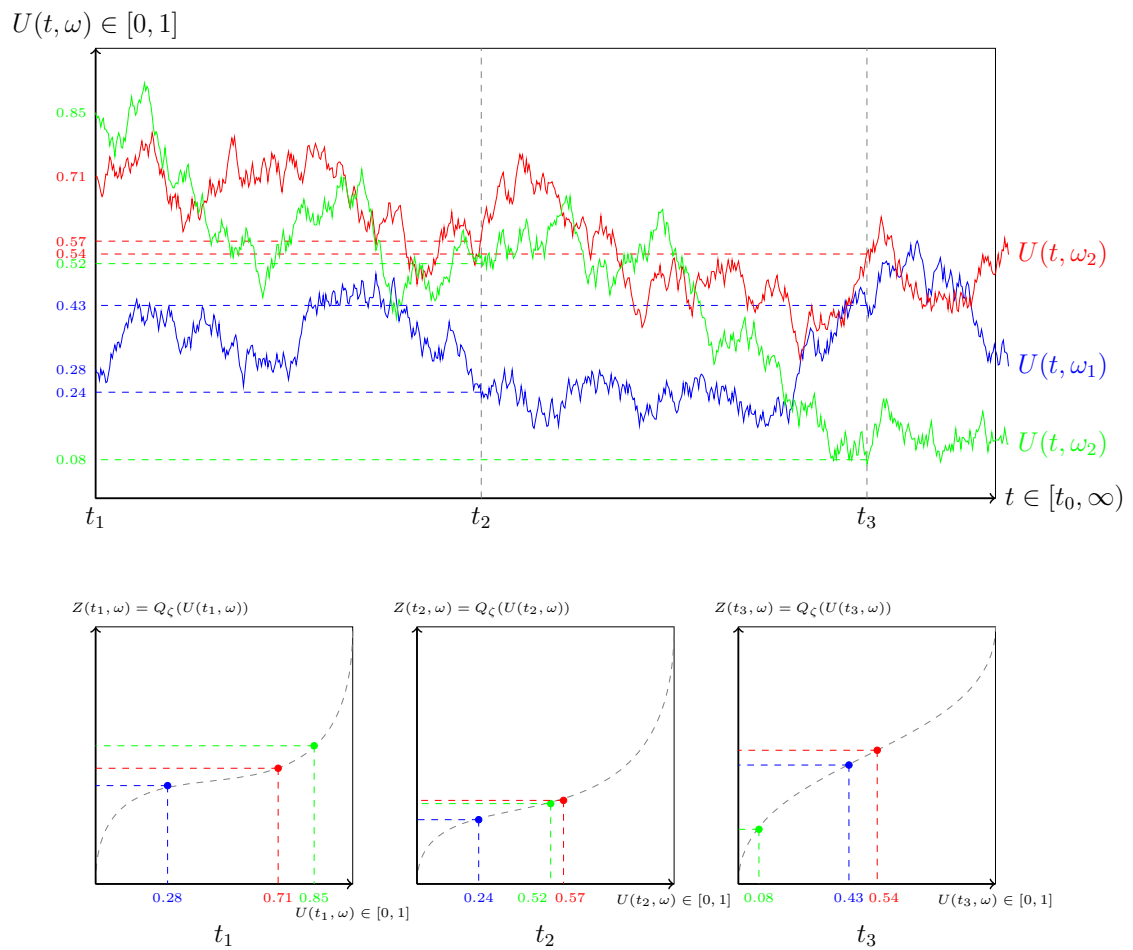


Figure 3.2: Three sample paths of the process (U_t) for $\omega_1, \omega_2, \omega_3 \in \Omega$, and the corresponding values of the quantile process (Z_t) at times $t_0 < t_1 < t_2 < \infty$.

3.1.1 Canonical random–level quantile processes

The construction–based Definition 3.1.1 of quantile processes gives rise to a very wide class of dynamic, continuous–time distortion processes, characterised by the parametric form of the composite map and the base process. As such, it is useful to distinguish certain sub–classes of random–level quantile processes where the motivations for such constructions, from a model–based perspective, are clear. In this section, we introduce the canonical random–level quantile processes with continuous and discrete paths, respectively. We present the following definitions as the *simplest* form of random–level quantile processes for which one may take to be a baseline when building complexity into the random–level models. From a model selection perspective, it may be useful to

explore the following canonical quantile processes as a *starting point* for some modelling objective. The continuous canonical quantile process is defined as follows, and utilises a standard, univariate Brownian motion driver.

Definition 3.1.2. *Let (W_t) denote a one–dimensional, (\mathcal{F}_t) –adapted standard Brownian motion, and set $Y_t = W_t$ in Definition 3.1.1. For all $t \in [t_0, \infty)$, the canonical quantile process is given by $Z_t \stackrel{d}{=} Q_\zeta(F_W(t, W_t); \boldsymbol{\xi})$, where $F_W(t, w) = [1 + \operatorname{erf}(w/\sqrt{2t})]/2$. In case that the range of the quantile function be restricted to some $D_\zeta \subset \mathbb{R}$, then $Z(t, \omega) = Q_\zeta(F_W(t, W_t(\omega)); \boldsymbol{\xi}) : [t_0, \infty) \times \Omega \rightarrow D_\zeta$. The canonical quantile process is then defined on the state space $(D_\zeta, \mathcal{B}(D_\zeta))$.*

We consider the following example of a canonical random–level quantile process, where the quantile function considered is that of the logistic distribution. The shape of the logistic distribution is similar to that of the normal distribution, however it is leptokurtic (has positive excess kurtosis and thus heavier tails than the normal distribution). As such, the following quantile process construction may be considered when one wishes to transform a Brownian motion to a heavier tailed process with finite–dimensional distributions that still resemble (in shape) the normal distribution.

Example 3.1.2. *Consider Definition 3.1.2 and let ζ be a logistic–distributed random variable with location and scale parameters $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$, respectively. The quantile function of ζ is given by $Q_\zeta(u; \mu, \sigma) := Q_{\text{Lg}}(u; \mu, \sigma) = \mu + \sigma \log(u/(1 - u))$ for all $u \in [0, 1]$. It follows that the canonical–logistic quantile process is given by*

$$Z_t \stackrel{d}{=} \mu + \sigma \log \left(\frac{1 + \operatorname{erf}(W_t/\sqrt{2t})}{1 - \operatorname{erf}(W_t/\sqrt{2t})} \right) \quad (3.1.9)$$

for each $t \in [t_0, \infty)$, $t_0 > 0$, and where $Z_{t_0} = z_{t_0} \in \mathbb{R}$. We may show, by differentiation, that the function $m(t, w) := \mu + \sigma \log((1 + \operatorname{erf}(w/\sqrt{2t})/(1 - \operatorname{erf}(w/\sqrt{2t})))$ is convex for $w > 0$ and so, referring to Proposition 3.1.2, the quantile process will have heavier tails (in its finite–dimensional distributions) than the driving Brownian motion.

Whilst the logistic distribution is leptokurtic, the level of excess kurtosis in each marginal distribution of the quantile process is fixed (and equal to 1.2), and not determined by the choice of parameters. As such, we also construct a canonical–Tukey– h quantile process, in the following example. Here, one may control the excess kurtosis through the parameter h .

Example 3.1.3. Consider Definition 3.1.2 and let ζ be a Tukey– h distributed random variable—see Definition 2.1.8 and Eq. (2.1.5)—with quantile function given by

$$Q_{T_h}(u; A, B, h) = \sqrt{2} \operatorname{erf}^{-1}(2u - 1) \exp\left(h \left(\operatorname{erf}^{-1}(2u - 1)\right)^2\right) \quad (3.1.10)$$

for all $u \in [0, 1]$ and $A \in \mathbb{R}$, $B \in \mathbb{R}^+$, $h \in \mathbb{R}_0^+$. Here, the parameter h controls the level of excess kurtosis, i.e., kurtosis exceeding that of the normal distribution. The canonical–Tukey– h quantile process is given by

$$Z_t \stackrel{d}{=} A + B \frac{W_t}{\sqrt{t}} \exp\left(h \frac{W_t^2}{2t}\right) \quad (3.1.11)$$

for all $t \in [t_0, \infty)$, $t_0 > 0$ and where $Z_{t_0} = z_{t_0} \in \mathbb{R}$. Again, we may show, by differentiation, that the function $m(t, y) := A + B(w/\sqrt{2t})\exp(hw^2/2t)$ is convex for $w > 0$ and so, referring to Proposition 3.1.2, the quantile process will have heavier tails, marginally, than the driving Brownian motion. If we consider kurtosis in regard to the fourth centralised, scaled moment, the kurtosis in the finite–dimensional distribution of the (stationary) quantile process in Eq. (3.1.11), at $t \in [t_0, \infty)$, is given by $\tilde{\mu}_4 = 3(1 - 2h^3)/(1 - 4h)^{5/2}$ —see, e.g., Jorge and Boris (1984). It holds that $\partial\tilde{\mu}_4/\partial h = (12h^3 - 18h + 30)/(1 - 4h)^{7/2} > 0$ for all $h > -1$ and so the level of excess kurtosis is increasing in the parameter h for $h > -1$.

In the discrete case, the driving process is considered to be a homogeneous Poisson process, i.e., a counting, or pure birth Markov process. We define $S := [0, \infty)$ to be a metric space with metric $d : S \times S \rightarrow [0, \infty)$, \mathcal{B} the Borel σ –algebra $\mathcal{B}(S)$ and \mathcal{S} the collection of all countable subsets of S . We refer to Kingman (1993) and define a one–dimensional, homogeneous Poisson process with state space $S = [0, \infty)$ and rate $\lambda \in \mathbb{R}^+$ to be the map $\mathcal{N} : \Omega \rightarrow \mathcal{S}$ satisfying:

- (i) for each $B \in \mathcal{B}$, $N(B) = \#\{\mathcal{N} \cap B\}$ is a Poisson random variable with parameter (or mean measure) $\mu(B) = \lambda B$, that is $\mathbb{P}(N(B) = n) = (\lambda B)^n e^{-\lambda B} / n!$ for any $n \in \mathcal{S}$ with $N(0) = 0$,
- (ii) for disjoint sets $B_1, B_2, \dots, B_n \in \mathcal{B}$, $N(B_1), N(B_2), \dots, N(B_n)$ are independent.

In this thesis, we refer to the homogeneous Poisson process \mathcal{N} on the positive real line with intensity parameter $\lambda \in \mathbb{R}^+$ by its associated Poisson random measure, $N_t :=$

$N([0, t]) = \#\{\mathcal{N} \cap [0, t]\}$ for any $t \in [0, \infty)$ and where $N_0 = 0$. In the discrete case, it is less clear (than in the continuous case) what statistical properties the quantile process will possess, due to the nature of the transformation. We leave the detailed study of this class of processes as further work. Nonetheless, we present the following example of a discrete canonical quantile process as follows.

Example 3.1.4. Let $(N_t)_{t \in [0, \infty)}$ denote a one–dimensional, (\mathcal{F}_t) –adapted homogeneous Poisson process on $[0, \infty)$ with intensity parameter $\lambda \in \mathbb{R}^+$. Consider Definition 3.1.1 however where we allow the time–dependent distribution function $F(t, y; \boldsymbol{\vartheta})$ to be a discrete distribution function. Set $Y_t = N_t$ so that for all $t \in [t_0, \infty)$, the discrete canonical quantile process is given by $Z_t \stackrel{d}{=} Q_\zeta(F_N(t, N_t; \lambda); \boldsymbol{\xi})$ where $F_N(t, n; \lambda) = \sum_{k=0}^n [\exp(-\lambda t)(\lambda t)^k / k!]$, and $\tilde{U}_t := F_N(t, N_t; \lambda)$ is non–uniformly distributed on $[0, 1]$. In the case that the range of the quantile function be restricted to some $D_\zeta \subset \mathbb{R}$, then $Z(t, \omega) = Q_\zeta(F_N(t, N_t(\omega); \lambda); \boldsymbol{\xi}) : [t_0, \infty) \times \Omega \rightarrow \tilde{D}_Z$ where

$$\tilde{D}_Z := \left\{ Q_\zeta(0; \boldsymbol{\xi}), Q_\zeta(e^{-\lambda t}; \boldsymbol{\xi}), Q_\zeta((\lambda t)e^{-\lambda t}; \boldsymbol{\xi}), Q_\zeta\left(\sum_{k=1}^2 \frac{(\lambda t)^k e^{-\lambda t}}{k!}; \boldsymbol{\xi}\right), \right. \\ \left. Q_\zeta\left(\sum_{k=1}^3 \frac{(\lambda t)^k e^{-\lambda t}}{k!}; \boldsymbol{\xi}\right), \dots, Q_\zeta\left(\sum_{k=1}^n \frac{(\lambda t)^k e^{-\lambda t}}{k!}; \boldsymbol{\xi}\right), \dots, Q_\zeta(1; \boldsymbol{\xi}) \right\} \quad (3.1.12)$$

is a discrete, countably infinite subset of D_ζ for all $t \in [t_0, \infty)$. For any $i \in \mathbb{Z}^+$, define $z_i := Q_\zeta\left(\sum_{k=0}^i (\lambda t)^k e^{-\lambda t} / k!; \boldsymbol{\xi}\right)$ and we have $\mathbb{P}(Z_t = z_i) = \mathbb{P}(U_t = u_i) = \mathbb{P}(N_t = i)$.

In the above example, we consider the (discrete) distribution function of the Poisson process as the inner function in the composite map, so that $\tilde{U}_t := F_N(t, N_t; \lambda)$ is the non–uniformly distributed input into the quantile function, Q_ζ . Alternatively, as a further study of this class of discrete quantile processes, one may consider the transformation introduced by Rüschendorf and de Valk (1993) to obtain a uniform random variable (by transforming the Poisson driving process) that will be input into the quantile function, Q_ζ , in the construction of the discrete, canonical quantile process.

We consider Example 3.1.4, and present the following discrete canonical random–level quantile process, where the quantile function considered is that of the Tukey– g distribution, as given by Eq. (2.1.11) with kurtosis parameter $h = 0$. The parameter $g \in \mathbb{R} \setminus 0$ allows for the introduction of skewness to the finite–dimensional distributions

of the process.

Example 3.1.5. Consider Example 3.1.4 and let ζ be a Tukey– g distributed random variable—see Definition 2.1.8 and Eq. (2.1.8)—with time–dependent parameters, and quantile function given by

$$Q_{T_g}(u; A(t), B(t), g(t)) = A(t) + \frac{B(t)}{g(t)} \exp\left(g(t)\sqrt{2}\operatorname{erf}^{-1}(2u - 1)\right) \quad (3.1.13)$$

for all $u \in [0, 1]$, and $A(t) \in \mathbb{R}$, $B(t) \in \mathbb{R}^+$ and $g(t) \in \mathbb{R} \setminus 0$ for all $t \in [0, \infty)$. The discrete canonical–Tukey– g quantile process with time–inhomogeneous parameters is given by

$$Z_t \stackrel{d}{=} A(t) + \frac{B(t)}{g(t)} \left[\exp\left(g(t)\sqrt{2}\operatorname{erf}^{-1}\left(2 \sum_{k=0}^{N_t} \frac{(\lambda t)^k e^{-\lambda t}}{k!} - 1\right)\right) - 1 \right] \quad (3.1.14)$$

for all $t \in [t_0, \infty)$, $t_0 > 0$.

3.1.2 Pivotal random–level quantile processes

The second formulation of random–level quantile processes we present is the pivotal subclass of models. The motivation behind this construction lies in increasing the interpretability of the statistical properties of the quantile process, relative to the base process, by *standardising* the driver. Whilst in the construction of quantile processes given in Definition 3.1.1 the driving process influences the behaviour and properties of the output quantile process, in many cases the relativity between the driving and quantile process properties (e.g., higher order moments) may not be explicit in terms of the model parameters. Hence, we introduce the pivotal quantile process construction. First, we refer to Shao (2006) for a definition of a pivotal quantity.

Definition 3.1.3. Let $X = (X_1, \dots, X_n)$ be a sample from a population $P \in \mathcal{P}$ and $\theta = \theta(P)$ denote a function from \mathcal{P} to $\Theta \in \mathbb{R}^k$, $k \in \mathbb{Z}^+$. A known Borel function \mathcal{R} of (X, θ) is called a pivotal quantity for θ if, and only if, the distribution of $\mathcal{R}(X, \theta)$ does not depend on P .

Pivotal quantities are most commonly used in statistics for inference procedures and for finding confidence intervals for one or more of the unknown population parameters.

Here, we instead wish to produce pivotal quantities for the parameters of the marginal distributions of the driving process in order to have a *standardised*, or *normalised* input to the composite map used in producing the quantile process. The continuous pivotal quantile process is defined as follows.

Definition 3.1.4. *Consider Definition 3.1.1 where we allow the parameter vectors in the distribution and quantile functions be time–dependent, i.e., $Y_t \sim F_Y(t, y; \boldsymbol{\theta}_Y(t))$ where the parameter vector $\boldsymbol{\theta}_Y(t) \in \mathbb{R}^d$ for $t \in (0, \infty)$. Consider another parameter vector $\tilde{\boldsymbol{\theta}}(t) \in \mathbb{R}^{d'}$, $d' \leq d$, and define the pivot process by $\tilde{Y}_t := \mathcal{R}(Y_t; \boldsymbol{\theta}_Y(t))$ where \mathcal{R} is a Borel function such that $\tilde{Y}_t \sim F_{\tilde{Y}}(t, \tilde{y}; \tilde{\boldsymbol{\theta}}(t))$, and the ‘reference law’ $F_{\tilde{Y}}$ does not depend on $\boldsymbol{\theta}_Y(t)$ for all $t \in (0, \infty)$. At any time $t \in [t_0, \infty)$, the continuous pivotal quantile process formulation is given by*

$$Z_t \stackrel{d}{=} Q_\zeta \left(F \left(t, \tilde{Y}_t; \boldsymbol{\theta}(t) \right); \boldsymbol{\xi}(t) \right), \quad (3.1.15)$$

where $F(t, y; \boldsymbol{\theta}(t))$ is a distribution belonging to the same family of distributions as $F_{\tilde{Y}}(t, y; \tilde{\boldsymbol{\theta}}(t))$, with parameters $\boldsymbol{\theta}(t) \in \mathbb{R}^{d'}$.

The pivot process (\tilde{Y}_t) serves as a base or reference process with respect to which the resulting quantile process is anchored, and relative properties between the two processes will be made explicit in terms of the model parameters. We illustrate this in the following example.

Example 3.1.6. *Consider some process $(Y_t)_{t \in [0, \infty)}$ such that at each $t \in (0, \infty)$, the process is normally distributed with some time–dependent mean and standard deviation parameters $\mu_Y(t) \in \mathbb{R}$ and $\sigma_Y(t) \in \mathbb{R}^+$, respectively. The process defined at each time by $\tilde{Y}_t := (Y_t - \mu_Y(t))/\sigma_Y(t)$ will be distributed according to the standard normal distribution, i.e., $\tilde{\boldsymbol{\theta}} = (0, 1)$, and so is a pivotal quantity for $\boldsymbol{\theta}_Y(t) = (\mu_Y(t), \sigma_Y(t))$. If we consider a normal distribution function F_N with $m(t), v(t)$ the mean and variance parameters, respectively, then the Gaussian pivotal random–level quantile process is given by*

$$Z_t \stackrel{d}{=} Q_\zeta \left(F_N \left(t, \tilde{Y}_t \right); \boldsymbol{\xi} \right) \stackrel{d}{=} Q_\zeta \left(\frac{1}{2} \left[1 + \operatorname{erf} \left(\frac{\tilde{Y}_t - m(t)}{\sqrt{2v(t)}} \right) \right]; \boldsymbol{\xi} \right) \quad (3.1.16)$$

for all $t \in [t_0, \infty)$. Consider the quantile function of the Tukey–g distribution (see Section 2.1 or Section 3.3) with time–dependent parameters $\boldsymbol{\xi}(t) = (A(t), B(t), g(t))$,

given by

$$Q_{T_g}(u; A(t), B(t), g(t)) = A(t) + \frac{B(t)}{g(t)} \left[\exp \left(g(t) \sqrt{2} \operatorname{erf}^{-1}(2u - 1) \right) - 1 \right], \quad (3.1.17)$$

for all $u \in [0, 1]$ and $t \in (0, \infty)$. Here, we have

$$\begin{aligned} Z_t &\stackrel{d}{=} A(t) + \frac{B(t)}{g(t)} \left[\exp \left(g(t) \frac{\tilde{Y}_t - m(t)}{\sqrt{v(t)}} \right) - 1 \right] \\ &\stackrel{d}{=} A^*(t) + \frac{B^*(t)}{g^*(t)} [\exp(g^*(t)Y_t) - 1] \end{aligned} \quad (3.1.18)$$

for all $t \in [t_0, \infty)$, $t_0 > 0$, where

$$A^*(t) = A(t) + \frac{B(t)}{g(t)} \left[\exp \left(-\frac{g(t)(\mu_Y(t) + \sigma_Y(t)m(t))}{\sigma_Y(t)\sqrt{v(t)}} \right) - 1 \right] \quad (3.1.19)$$

$$B^*(t) = \frac{B(t)}{\sigma_Y(t)\sqrt{v(t)}} \exp \left(-\frac{g(t)(\mu_Y(t) + \sigma_Y(t)m(t))}{\sigma_Y(t)\sqrt{v(t)}} \right) \quad (3.1.20)$$

$$g^*(t) = \frac{g(t)}{\sigma_Y(t)\sqrt{v(t)}}. \quad (3.1.21)$$

The quantile process in the second line of Eq. (3.1.18) has the form of a standard Tukey- g transform with time-dependent parameters given by Eqs (3.1.19)–(3.1.21), relative to the base (Y_t) . That is $Z_t \stackrel{d}{=} A^*(t) + B^*(t)Y_t T_g(Y_t)$ for all $t \in [t_0, \infty)$ and where $T_g(x)$ is given by Eq. (2.1.8). Thus, by definition of the Tukey- g transform, all skewness introduced by the transform will be relative to the base process (Y_t) , giving the parameters direct interpretability.

The first four standardised moments of Z_t at each $t \in [t_0, \infty)$ are given, in terms of the parameters of the composite map, by

$$\begin{aligned} \mu_Z &= A(t) + \frac{B(t)}{g(t)} \left[\exp \left(-\frac{m(t)g(t)}{\sqrt{v(t)}} + \frac{g^2(t)}{2v(t)} \right) - 1 \right], \\ \sigma_Z^2 &= \frac{B^2(t)}{g^2(t)} \left(\exp \left(\frac{g^2(t)}{v(t)} \right) - 1 \right) \exp \left(-\frac{2g(t)m(t)}{\sqrt{v(t)}} + \frac{g^2(t)}{v(t)} \right) \\ \tilde{\mu}_{3,Z} &= \left(\exp \left(\frac{g^2(t)}{v(t)} \right) + 2 \right) \sqrt{\exp \left(\frac{g^2(t)}{v(t)} \right) - 1} \end{aligned}$$

$$\tilde{\mu}_{4,Z} = \exp\left(4\frac{g^2(t)}{v(t)}\right) + 2\exp\left(3\frac{g^2(t)}{v(t)}\right) + 3\exp\left(2\frac{g^2(t)}{v(t)}\right) - 6.$$

where $\tilde{\mu}_{3,Z}$ and $\tilde{\mu}_{4,Z}$ are the third and fourth central moments, respectively. The skewness and kurtosis of Z_t are given by $\tilde{\mu}_{3,Z}/\sigma_Z^3$ and $\tilde{\mu}_{4,Z}/\sigma_Z^4$, respectively.

In the discrete case, we define the discrete pivotal quantile process by use of the Poisson mapping theorem, see Kingman (1993), which allows one to construct a homogeneous Poisson process pivotal quantity from an inhomogeneous Poisson process. By an inhomogeneous Poisson process, we mean a Poisson process \mathcal{N} on $S = [0, \infty)$ with intensity function $\lambda(t) : S \rightarrow [0, \infty)$ such that its mean measure is defined by $\mu(B) = \int_B \lambda(x)dx$ for any $B \in \mathcal{B}$ where \mathcal{B} is the Borel σ -algebra $\mathcal{B}(S)$. The Poisson mapping theorem is given as follows, and outlines the preservation of the properties of the process under a measurable mapping.

Definition 3.1.5. *Let \mathcal{N} be a Poisson process with state space S and σ -finite mean measure μ , and $\psi : S \rightarrow S'$ be a measurable map where S' is some other locally compact, separable metric space with Borel σ -algebra \mathcal{B}' . If the measure $\mu'(\cdot) := \mu(\psi^{-}(\cdot))$ is non-atomic, then $\psi(\mathcal{N}) := \{\psi(\zeta) : \zeta \in \mathcal{N}\}$ is a Poisson process with state space S' and mean measure μ' .*

The discrete pivotal quantile process is defined as follows.

Definition 3.1.6. *Let $Q_\zeta(u; \boldsymbol{\xi})$ be a quantile function for $u \in [0, 1]$ and where $\boldsymbol{\xi} \in \mathbb{R}^d$ is a vector of parameters. Let \mathcal{N} be an inhomogeneous Poisson process on $S = [0, \infty)$ with intensity function $\lambda(t) : S \rightarrow [0, \infty)$, so that its mean measure is given by $\mu(B) = \int_B \lambda(x)dx$ for any $B \in \mathcal{B}$. Let S' be a locally compact, separable metric space with Borel σ -algebra \mathcal{B}' , and $\psi : S \rightarrow S'$ be a measurable map such that $\psi^{-}(x) = \mu^{-}(\tilde{\lambda}x)$ for some $\tilde{\lambda} \in \mathbb{R}^+$. Define by $\tilde{\mathcal{N}} := \psi(\mathcal{N})$ the Poisson process on state space S' with mean measure $\mu'(B) = \mu(\psi^{-}(B))$ for any $B \in \mathcal{B}'$. The process $\tilde{\mathcal{N}}$ will be a homogeneous Poisson process with intensity parameter $\tilde{\lambda}$, and so is a pivotal quantity for $\lambda(t)$. Consider a homogeneous Poisson distribution F_N with intensity parameter $\theta \in \mathbb{R}^+$, $\theta \neq \tilde{\lambda}$, then the discrete pivotal quantile process formulation is given by*

$$Z_t \stackrel{d}{=} Q_\zeta\left(F_N\left(\tilde{N}_t; \theta\right); \boldsymbol{\xi}\right) = Q_\zeta\left(\sum_{k=1}^{\tilde{N}_t} \frac{(\theta t)^k e^{-\theta t}}{k!}; \boldsymbol{\xi}\right) \quad (3.1.22)$$

for all $t \in [t_0, \infty)$ and where $\tilde{N}_t := \tilde{N}([0, t]) = \#\{\tilde{\mathcal{N}} \cap [0, t]\}$.

Remark 1. When $\theta = \tilde{\lambda}$, we refer to the quantile process in Eq. (3.1.22) as the discrete canonical quantile process, as given by Example 3.1.4.

3.1.3 SDEs satisfied by random–level quantile diffusions

The purpose of this section is to present the stochastic differential equations (SDEs) satisfied by quantile diffusions, that is, quantile processes constructed as per Definition 3.1.1 with the driving process being a diffusion. The SDEs are useful for the simulation of quantile processes, as well as to provide insight into the effect the distortion map has on the drift and volatility coefficients of the quantile process, relative to the driving diffusion. We introduce a generic diffusion process in continuous time as the stochastic process (Y_t) , on the filtered probability space, satisfying

$$dY_t = \mu(t, Y_t) dt + \sigma(t, Y_t) dW_t \quad (3.1.23)$$

for all $t \in [0, \infty)$ and where $Y_0 = y_0 \in \mathbb{R}$ is a specified initial condition, $\mu(t, y) : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ is the drift function and $\sigma(t, y) : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}^+$ is the volatility function. Under conditions on the drift and volatility functions, see Ikeda and Watanabe (2014), Karatzas and Shreve (2012) and Øksendal (2013), the SDE 3.1.23 admits a unique solution $Y_t(\omega) = Y(t, \omega) : [0, \infty) \times \Omega \rightarrow \mathbb{R}$.

The following proposition and corollary guarantee that the quantile process (Z_t) , given by Definition 3.1.1, is a diffusion whenever the driving process (Y_t) is a diffusion. The corresponding infinitesimal drift and volatility coefficients are derived herein. We omit the dependence on the vectors of parameters in the notation for the following distribution, quantile and density functions.

Proposition 3.1.4. *Let $(Z_t)_{t \in [t_0, \infty)}$ be a quantile process given by Definition 3.1.1 with driving process $(Y_t)_{t \in [0, \infty)}$ that satisfies the SDE 3.1.23. Assume the following derivatives exist so that $f(t, y) := \partial_y F(t, y)$ and $f_\zeta(z) := \partial_z F_\zeta(z)$ are density functions. The dynamics of (Z_t) satisfy*

$$dZ_t = \alpha(t, Z_t) dt + \tilde{\sigma}(t, Z_t) dW_t \quad (3.1.24)$$

where

$$\begin{aligned} \alpha(t, Z_t) &= \frac{\partial_t F(t, x)|_{x=Q(t, F_\zeta(Z_t))}}{f_\zeta(Z_t)} + \mu(t, Q(t, F_\zeta(Z_t))) \frac{f(t, Q(t, F_\zeta(Z_t)))}{f_\zeta(Z_t)} \\ &\quad + \frac{1}{2} \sigma^2(t, Q(t, F_\zeta(Z_t))) \frac{f'(t, Q(t, F_\zeta(Z_t))) f_\zeta(Z_t)^2 - f(t, Q(t, F_\zeta(Z_t)))^2 f'_\zeta(Z_t)}{f_\zeta(Z_t)^3}, \end{aligned} \quad (3.1.25)$$

$$\tilde{\sigma}(t, Z_t) = \sigma(t, Q(t, F_\zeta(Z_t))) \frac{f(t, Q(t, F_\zeta(Z_t)))}{f_\zeta(Z_t)}, \quad (3.1.26)$$

for $t \in [t_0, \infty)$ and $Z_{t_0} = z_{t_0} \in \mathbb{R}$. The short-hand notation f' denotes differentiation with respect to the spatial variable.

Proof. The result follows from a straightforward application of Ito's formula and since

$$\frac{\partial Q(t, u)}{\partial u} = \frac{1}{f(t, Q(t, u))}, \quad (3.1.27)$$

as given by Eq. (64) in the paper by Steinbrecher and Shaw (2008). \square

Corollary 3.1.1. *Consider Proposition 3.1.4 and assume the distribution function in the composite map of Eq. (3.1.1) is the distribution function that governs the driving process, i.e., $F = F_Y$. The dynamics of (Z_t) satisfy the SDE 3.1.24 where*

$$\begin{aligned} \alpha(t, Z_t) &= \frac{\sigma^2(t, Q_Y(t, F_\zeta(Z_t))) f'_Y(t, Q_Y(t, F_\zeta(Z_t)))}{f_\zeta(Z_t)} \\ &\quad + \frac{f_Y(t, Q_Y(t, F_\zeta(Z_t))) (\sigma^2(t, Q_Y(t, F_\zeta(Z_t))))'}{2 f_\zeta(Z_t)} \\ &\quad - \frac{1}{2} \sigma^2(t, Q_Y(t, F_\zeta(Z_t))) \frac{f_Y(t, Q_Y(t, F_\zeta(Z_t)))^2 f'_\zeta(Z_t)}{f_\zeta(Z_t)^3}, \end{aligned} \quad (3.1.28)$$

and $\tilde{\sigma}(t, Z_t)$ is given by Eq. (3.1.26) for $t \in [t_0, \infty)$ and $Z_{t_0} = z_{t_0} \in \mathbb{R}$. Here, $f_Y(t, y)$ is the marginal density of the driving process (Y_t) starting with $y_0 \in \mathbb{R}$.

Proof. Similarly to the proof of Proposition (3.1.4), we apply Ito's formula to $Z_t = Q_\zeta(F_Y(t, Y_t))$. Since $F_Y(t, x)$ is the law of the process (Y_t) , we can use the Fokker–Plank equation to describe how the density of (Y_t) , that is $f_Y(t, y)$, evolves with time. The chain rule yields $\partial_t Q_\zeta(F_Y(t, y)) = \partial_t F_Y(t, y) / f_\zeta(Q_\zeta(F_Y(t, y)))$ and by the fundamental

theorem of calculus, we obtain

$$\partial_t \left(\int_{-\infty}^{\phi(t)} f_Y(t, x) dx \right) = f_Y(t, \phi(t)) \partial_t \phi(t) + \int_{-\infty}^{\phi(t)} \partial_t f_Y(t, x) dx.$$

Now, using the Fokker-Planck equation for the marginal density of (Y_t) , we have $\partial_t F_Y(t, y) = \int_{-\infty}^y \partial_t f_Y(t, x) dx = -\mu(t, y) f_Y(t, y) + \frac{1}{2} (\sigma^2(t, y) f_Y'(t, y) + f_Y(t, y) \partial_y \sigma^2(t, y))$, and therefore

$$\partial_t (Q_\zeta(F_Y(t, y))) = \frac{-\mu(t, y) f_Y(t, y) + \frac{1}{2} (\sigma^2(t, y) f_Y'(t, y) + f_Y(t, y) \partial_y \sigma^2(t, y))}{f_\zeta(t, Q_\zeta(t, F_Y(t, y)))}.$$

Noting that $Y_t = Q_Y(t, F_\zeta(Z_t))$, the result stated in the corollary follows. \square

To compare the effect that different distortion or transformation maps have on the drift and volatility coefficients of the quantile diffusion, we consider the following example involving different skewness–inducing transformations.

Example 3.1.7. *Let $(Y_t)_{t \in [0, \infty)}$ be a scaled Brownian motion, that is $Y_t = \sigma W_t$ for $\sigma \in \mathbb{R}^+$ and $Y_0 = y_0 = 0$. Consider the following quantile preserving transformation maps that, when applied to each marginal of the driving diffusion, produce an output process with increased relative skewness in its (lognormal) finite–dimensional distributions:*

- (i) *The exponential Q –transformation, $T_Q^e(y) := \exp(y)$ —see Definition 2.1.3.*
- (ii) *The composite map $Q_{T_g}(F(t, y; \boldsymbol{\theta}); g)$ where $Q_{T_g}(u; g)$ is the quantile function of the Tukey– g distribution, given by Eq. (2.1.11) with $A, h = 0$ and $B = 1$, and $F(t, y; \boldsymbol{\theta})$ is the distribution function of a real–valued random variable—see Definition 3.1.1.*

In case (i), for all $t \in [0, \infty)$ we have $Z_t \stackrel{d}{=} \exp(Y_t) \stackrel{d}{=} \exp(\sigma W_t)$, satisfying the SDE

$$dZ_t = \frac{1}{2} \sigma^2 Z_t dt + \sigma Z_t dW_t \quad (3.1.29)$$

with $Z_0 = z_0 = 1$. In case (ii), if we consider F to be the standard normal distribution function, for all $t \in [t_0, \infty)$, for $t_0 > 0$, we have $Z_t \stackrel{d}{=} [\exp(g\sigma W_t) - 1]/g$, satisfying the SDE

$$dZ_t = \frac{1}{2} \sigma^2 g (gZ_t + 1) dt + \sigma (gZ_t + 1) dW_t \quad (3.1.30)$$

with $Z_{t_0} = z_{t_0} \in [-1/g, \infty)$ when $g > 0$ and $z_{t_0} \in (-\infty, 1/g]$ when $g < 0$. If alternatively we consider $F = F_Y$ to be the distribution function associated to the driving process, for all $t \in [t_0, \infty)$, we have $Z_t \stackrel{d}{=} [\exp(gW_t/\sqrt{t}) - 1]/g$, satisfying the SDE

$$dZ_t = \left(\frac{g}{2t} - \frac{\log(gZ_t + 1)}{2gt} \right) (gZ_t + 1) dt + \frac{gZ_t + 1}{\sqrt{t}} dW_t \quad (3.1.31)$$

with $Z_{t_0} = z_{t_0} \in [-1/g, \infty)$ when $g > 0$ and $z_{t_0} \in (-\infty, 1/g]$ when $g < 0$. We remark that the drift and volatility functions in Eq. (3.1.30) are similar to those in Eq. (3.1.29), however the Z_t component of each function is shifted and scaled. The scaling parameter g allows for direct control of the amount of introduced skewness to the base process $Y_t \stackrel{d}{=} \sigma W_t$ in Eq. (3.1.30) and relative to the standardised Brownian motion $Y_t \stackrel{d}{=} W_t/\sqrt{t}$ in Eq. (3.1.31), thus providing greater model flexibility than the exponential Q -transformation. We also note that the Tukey- g transformation allows for negative skewness by setting $g < 0$, whereas the exponential Q -transformation (i.e., the lognormal distribution) does not. When $F = F_Y$ is considered in case (ii) to give the SDE 3.1.31, there is no dependence on driving process volatility parameter σ , there is a time scaling of each coefficient, and an extra term involving a logarithm is introduced.

3.2 Function-valued quantile processes

In this section we present the second type of quantile process whereby the construction produces a process that, at each time, models realisations of a quantile function over all levels $u \in [0, 1]$. Here, the stochasticity is produced by considering continuous-time, dynamic parameters of the chosen quantile function, and as such leads to quantile processes with dynamic statistical properties. Such a construction is a continuous-time generalisation of the class of dynamic quantile function models presented by Chen et al. (2022) and given in Definition 2.2.4. The advantages of this model lie in the fact that an entire parametric quantile function is considered as an observation of the output process. Consequently, this allows one to dynamically model extreme quantiles, or capture dynamic tail behaviour, more readily than by considering alternative types of models where there may not be sufficient data available to model such extremes, e.g., in histogram-valued time series models. Additionally, one may consider very flexible families of quantile functions with directly parameterised features, by considering the

class of quantile preserving maps given in Definition 2.1.2, e.g., those in Definitions 2.1.3 and 2.1.8. We present the function-valued quantile process construction as follows.

Definition 3.2.1. *Let ζ be a real-valued random variable with quantile function $Q_\zeta(u; \boldsymbol{\xi})$ for $u \in [0, 1]$ and $\boldsymbol{\xi} \in \mathbb{R}^d$ a d -dimensional vector of parameters, for $d \in \mathbb{N}$. Consider a d -dimensional (\mathcal{F}_t) -adapted process $(\boldsymbol{\xi}_t)_{t \in [0, \infty)}$ on the filtered probability space, with càdlàg paths. Let $Z(t, u, \omega) : [0, \infty) \times [0, 1] \times \Omega \rightarrow [-\infty, \infty]$ be a function of three variables such that if $t \in [0, \infty)$ and $\omega \in \Omega$ are fixed, $u \mapsto Z(t, u, \omega)$ is a well-defined quantile function. Then for all $t \in [0, \infty)$, the function-valued quantile process $(Z_t(u))_{t \in [0, \infty)}$ is defined by*

$$Z_t(u) = Q_\zeta(u; \boldsymbol{\xi}_t) \quad (3.2.1)$$

where $u \in [0, 1]$ is the quantile level and such that, by the equality in Eq. (3.2.1), we mean

$$\mathbb{P}(\{\omega \in \Omega : Z_t(u)(\omega) = Q_\zeta(u; \boldsymbol{\xi}_t(\omega)), \forall u \in [0, 1] \text{ \& } t \in [0, \infty)\}) = 1. \quad (3.2.2)$$

We have, $Z_t(u)(\omega) = Q_\zeta(u; \boldsymbol{\xi}(t, \omega)) = Z(t, \omega, u) : [0, \infty) \times \Omega \times [0, 1] \rightarrow [-\infty, \infty]$. We emphasise that the quantile process given by Definition 3.2.1 is *function-valued*, that is for each time $t \in [0, \infty)$, the process is a well-defined quantile function, instead of a random variable (representing some single quantile at a given level) as per the random-level quantile processes given by Definition 3.1.1. We illustrate this in Figure 3.3. Here, a function-valued quantile process, given by Definition 3.2.1, with vector of parameters $\boldsymbol{\xi}_t = (\xi_t^{(1)}, \xi_t^{(2)}) \in \mathbb{R}^2$ is considered. A single sample path of each parameter process is shown, and the values taken at times $t_0 < t_1 < t_2$ determine the shape of the quantile function at that time. For $\omega \in \Omega$, we have $\xi^{(1)}(t_i, \omega) \in \mathbb{R}$, $\xi^{(2)}(t_i, \omega) \in \mathbb{R}$ for $i = 0, 1, 2$, and the quantile function is given by $Z_{t_i}(u) = Q_\zeta(u; \xi^{(1)}(t_i, \omega), \xi^{(2)}(t_i, \omega))$ for all $u \in [0, 1]$. The function-valued quantile process models the time-evolution of this quantile function. Figure 3.3 illustrates that, with the function-valued construction, we observe the time evolution of the entire quantile function, with the observed quantile functions at times t_1, t_2, t_3 shown, based on the values of the sample paths of the parameter process at those times.

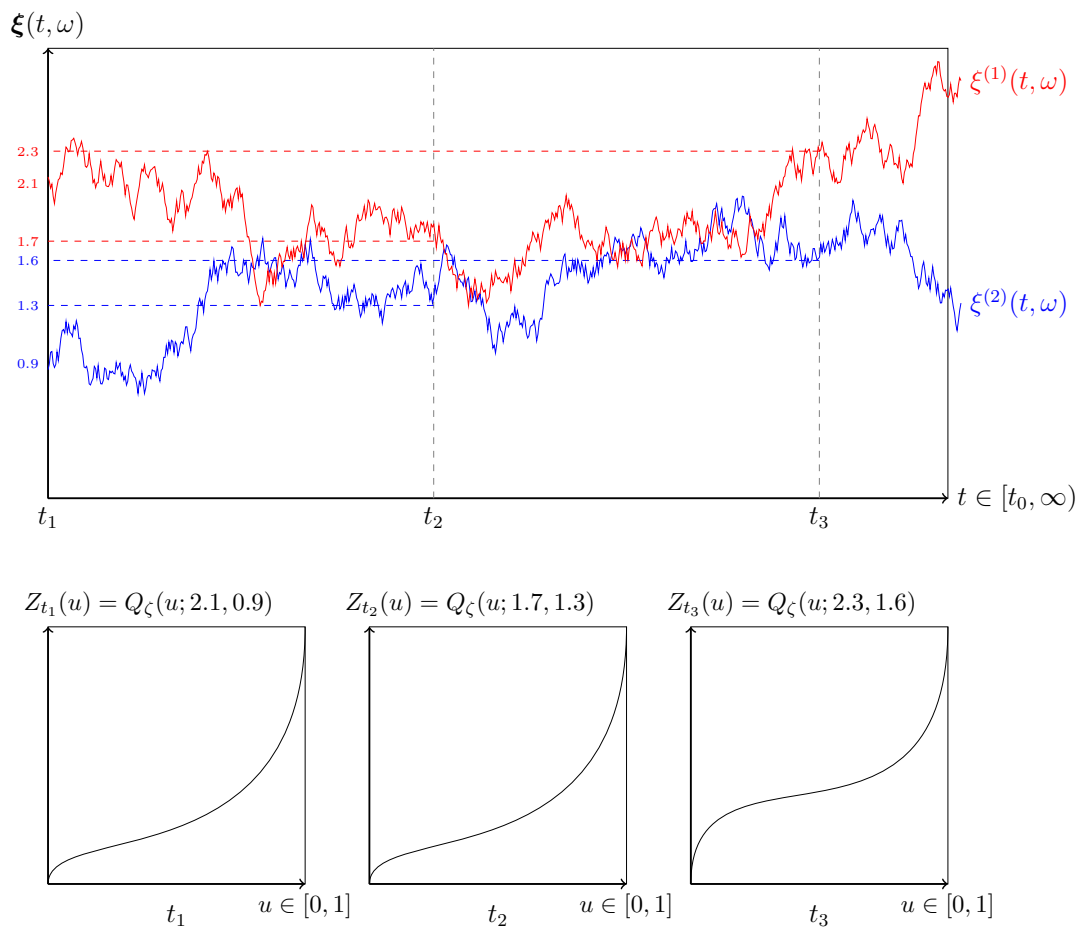


Figure 3.3: One sample path of each parameter process in the function-valued quantile process construction, and the corresponding quantile function at times $t_0 < t_1 < t_2 < \infty$

We remark that, since the quantile process given by Definition 3.1.1 is function-valued, a connection may be made between such a class of models and distributional regression methods and symbolic data analysis (SDA). We leave this for future work, and one may refer to, e.g., Beranger et al. (2018), Billard and Diday (2003), Billard (2011) and the references cited therein.

Since the process takes support on a function space corresponding to quantile functions, the sufficient statistical characterisation of such a process naturally points to working with probability weighted moments which, when viewed in a quantile setting, arise as L-moments. In the static case, L-moments, as introduced by Hosking (1990), provide a unique characterisation of a distribution with finite mean, and are often viewed as advantageous over conventional moments due to the fact that they will al-

ways exist whenever the mean of the distribution of interest does. Additionally, they may be expressed as a projection of the quantile function onto a sequence of orthogonal polynomials that form a basis of L^2 , as an alternative to their characterisation as linear combinations of order statistics. Let ζ be a random variable and L_k be the k^{th} shifted Legendre polynomial, see Legendre (1785). Then the k^{th} L-moment of ζ is defined as

$$l_k := \int_0^1 Q_\zeta(u) L_{k-1}(u) du. \quad (3.2.3)$$

As such, we may construct a time-dependent sequence of L-moments for the function-valued quantile process. Fix $t \in [0, \infty)$ and $\omega \in \Omega$ so that $\bar{\boldsymbol{\xi}}_t := \boldsymbol{\xi}(t, \omega) \in \mathbb{R}^d$ is fixed. Then the k^{th} instantaneous L-moment of the function-valued quantile process $(Z_t(u))$, characterised by ζ , is given by

$$l_{k,t} := \int_0^1 Q_\zeta(u; \bar{\boldsymbol{\xi}}_t) L_{k-1}(u) du = \int_0^1 Z_t(u) L_{k-1}(u) du \quad (3.2.4)$$

for all $t \in [0, \infty)$. The following proposition characterises when Eq. (3.2.4) is finite.

Proposition 3.2.1. *Consider a function-valued quantile process $(Z_t(u))_{t \in [0, \infty)}$, given by Definition 3.2.1. The k^{th} instantaneous L-moment, given by Eq. (3.2.4), for the parameter-driven quantile process is finite for all $t \in [0, \infty)$ and any $k \in \mathbb{N}$, $k > 0$ if the first moment of the random variable ζ with distribution function is finite.*

Proof. Consider Eq. (3.2.4). Make the change of variable $u := F_\zeta(z; \bar{\boldsymbol{\xi}}_t)$, where $F_\zeta(z; \bar{\boldsymbol{\xi}}_t) = Q_\zeta^-(z; \bar{\boldsymbol{\xi}}_t)$ for $z \in \mathbb{R}$, so we may write

$$\begin{aligned} l_{k,t} &= \int_{-\infty}^{\infty} z f_\zeta(z; \bar{\boldsymbol{\xi}}_t) L_{k-1}(F_\zeta(z; \bar{\boldsymbol{\xi}}_t)) dz \\ &= \sum_{j=0}^{k-1} \frac{(-1)^{k-j} (k+j)!}{(j!(k-j)!)} \int_{-\infty}^{\infty} z f_\zeta(z; \bar{\boldsymbol{\xi}}_t) F_\zeta^j(z; \bar{\boldsymbol{\xi}}_t) dz \\ &= \sum_{j=0}^{k-1} \frac{(-1)^{k-j} (k+j)!}{(j!(k-j)!)} \mathbb{E} [\zeta F_\zeta^j(\zeta; \bar{\boldsymbol{\xi}}_t)] \end{aligned} \quad (3.2.5)$$

where $f_\zeta(z; \bar{\boldsymbol{\xi}}_t) = \partial_z F_\zeta(z; \bar{\boldsymbol{\xi}}_t)$ and since

$$L_{k-1}(u) = \sum_{j=0}^{k-1} \frac{(-1)^{k-j} (k+j)!}{(j)! (k-j)!} u^j \quad (3.2.6)$$

for $u \in [0, 1]$. Define the random variable $A := \zeta F_\zeta^k(\zeta; \bar{\boldsymbol{\xi}}_t)$, where $A(\omega) \leq \zeta(\omega)$ for all $\omega \in \Omega$, almost surely, since $0 \leq F_\zeta^k(\zeta; \bar{\boldsymbol{\xi}}_t) \leq 1$, $k > 0$. It follows that

$$\mathbb{E} [\zeta F_\zeta^j(\zeta; \bar{\boldsymbol{\xi}}_t)] = \mathbb{E} [A] \leq \mathbb{E} [\zeta] \quad (3.2.7)$$

and so if $\mathbb{E}[\zeta]$ is finite, $l_{k,t}$ will be finite, by Eq. (3.2.5), as required. \square

The first four instantaneous L-moments of the Tukey-*gh* family, with quantile function given by Eq. (2.1.11), are given by Peters et al. (2016). We conclude this section by drawing a comparison between random-level and function-valued quantile processes. To obtain a quantile process given by Definition 3.2.1, that is similar to that in Definition 3.1.1, one could construct a quantile process of this type and fix the quantile level at some $\bar{u} \in [0, 1]$. We treat this special case, next. We consider the following case in which we can construct a version of random-level quantile processes, given by Definition 3.1.1, however now where the underlying driver is a stochastic vector of parameters $(\boldsymbol{\xi}_t)$, as in Definition 3.2.1, and we have control over the fixed quantile level corresponding to the quantiles modelled by the output process.

Definition 3.2.2. *Consider Definition 3.2.1 where $(\boldsymbol{\xi}_t)$ is the stochastic vector of parameters and $Q_\zeta(u; \boldsymbol{\xi}_t)$ is given in Eq. (3.2.1). Consider the special case of the function-valued construction in Definition 3.2.1, whereby we fix the quantile level $u = \bar{u} \in [0, 1]$, and so the quantile process given by Eq. (3.2.1) becomes*

$$Z_t^{\bar{u}} \stackrel{d}{=} Z_t(\bar{u}) \stackrel{d}{=} Q_\zeta(\bar{u}; \boldsymbol{\xi}_t), \quad (3.2.8)$$

where $Z_t^{\bar{u}}(t, \omega) = Q_\zeta(\bar{u}; \boldsymbol{\xi}(t, \omega)) : [0, \infty) \times \Omega \rightarrow [-\infty, \infty]$. This is distinct from the usual case whereby the function-valued construction models the dynamics of the entire quantile curve, and is equivalent to taking some fixed point on the quantile curve in the function-valued construction.

Now, let the functions $Q_{\tilde{\zeta}}(u; \tilde{\boldsymbol{\xi}})$ and $F(t, y; \boldsymbol{\vartheta})$ be the quantile function and distribution function, respectively, given in Definition 3.1.1. The process analogous to the

random-level quantile process, however now at fixed level \bar{u} and with the stochastic driver (ξ_t) , is defined by

$$Z_t \stackrel{d}{=} Q_{\tilde{\zeta}} \left(F(t, Z_t^{\bar{u}}; \vartheta); \tilde{\xi} \right) \quad (3.2.9)$$

for all $t \in [t_0, \infty)$, $t_0 > 0$, where $Z_t^{\bar{u}}$ is given by Eq. (3.2.8). Since Eq. (3.2.9) is implicitly dependent on \bar{u} , this process takes well-defined quantile values at the level \bar{u} .

If one considers Definition 3.1.1 and chooses the functions Q_{ζ} and $Q_Y = F_Y^-$, where F_Y is the law governing the driving process (Y_t) , such that $Q_Y(t, U_t) \stackrel{d}{=} Z_t^{\bar{u}}$ for each $t \in [t_0, \infty)$, one can ensure this quantile process matches that obtained by the usual random-level construction given in Eq. (3.1.1).

Example 3.2.1. Consider a uniformly distributed random variable $\zeta \sim U[a, b]$ where $-\infty < a < b < \infty$. Take $a = 0$ and consider a process $(b_t)_{t \in [0, \infty)}$ such that $b_t > 0$ for all $t \in [0, \infty)$. Using Definition 3.2.1, we construct a uniformly-distributed function-valued quantile diffusion by $Z_t(u) = Q_{\zeta}(u; b_t) = ub_t$ for $u \in [0, 1]$.

Fix $u = \bar{u} \in [0, 1]$ and define the process $Z_t^{\bar{u}} \stackrel{d}{=} \bar{u}b_t$. One obtains a special case of a random-level quantile process with stochasticity driven by the process (b_t) , producing output quantiles at level $\bar{u} \in [0, 1]$, by considering Definition 3.1.1 with $Y_t \stackrel{d}{=} Z_t^{\bar{u}}$ for all $t \in [0, \infty)$.

Example 3.2.2. Consider some random variable $\zeta_2 \sim F_{\zeta_2}$ that belongs to the location-scale family with location parameter $A \in \mathbb{R}$ and scale parameter $B \in \mathbb{R}^+$, that is $\zeta_2 \stackrel{d}{=} A + B\zeta_1$ for any random variable $\zeta_1 \sim F_{\zeta_1}$. Take $B = 1$ and consider the process $(A_t)_{t \in [0, \infty)}$ with associated law $F_A(t, a)_{0 < t < \infty}$. Using Definition 3.2.1, we construct a location-scale, function-valued quantile process by $Z_t(u) = Q_{\zeta_1}(u) + A_t$ for all $t \in [0, \infty)$ and $u \in [0, 1]$. Fix $u = \bar{u} \in [0, 1]$ and define the distribution function $F_Y(t, y) = F_A(t, y - Q_{\zeta_2}(\bar{u}))$ for all $t \in (0, \infty)$. For some choice of the functions Q_{ζ} and F in Definition 3.1.1, one can produce equivalent quantile processes in the two following ways:

1. Using the function-valued construction, taking Eq. (3.1.1) with Q_Y the quantile function corresponding to the distribution function $F_Y(t, y) = F_A(t, y - Q_{\zeta_2}(\bar{u}))$.
2. By $Z_t \stackrel{d}{=} Q_{\zeta}(F(t, Z_t^{\bar{u}}))$ where $Z_t^{\bar{u}} \stackrel{d}{=} Q_{\zeta_1}(\bar{u}) + A_t$ for each $t \in (0, \infty)$. This is a special case of a random-level quantile diffusion, where the driving process is the location process (A_t) and (Z_t) models quantiles at the chosen level \bar{u} .

We emphasise that the above examples illustrate the special case of quantile processes, that is not a new construction but instead an overlap between the random–level and function–valued constructions. In general, however, these two constructions are structurally distinct and will not be related.

3.3 Flexible families of Tukey quantile processes

In this section we employ Tukey transformation maps, as given by Definition 2.1.8, to construct families of quantile processes in which skewness and kurtosis are parameterised directly. We focus on the g –and– h family, which is comprised of the one–parameter Tukey– g and $-h$ transformations, and the two–parameter Tukey– gh transformation. Since these distributions are characterised by their quantile functions, we consider each Tukey transformation applied to a standard normal quantile function to produce the quantile function $Q_\zeta(u; \boldsymbol{\xi})$ used in each quantile process construction.

Consider Definitions 2.1.8 and 3.1.1. In the most general form, a Tukey random–level quantile process is given by

$$Z_t \stackrel{d}{=} A + B\sqrt{2}\operatorname{erf}^{-1}(2F(t, Y_t; \boldsymbol{\theta}) - 1) T \left(\sqrt{2}\operatorname{erf}^{-1}(2F(t, Y_t; \boldsymbol{\theta}) - 1) \right)^\Theta \quad (3.3.1)$$

for each $t \in [t_0, \infty)$ and $\Theta \in \mathbb{R}$. We are interested in the conditions under which the choice of parameterisation T ensures Eq. (3.3.1) is a well–defined reshaping function or elongation map applied to each marginal of the driving process. Inherently, this ensures that skewness and kurtosis, respectively, are introduced *relative* to each marginal of the driving process under the quantile process construction. Additionally, by Definitions 2.1.6 and 2.1.7, the distortions are defined with respect to the mode of the driving process.

We first consider the Tukey– g family, and set $A = 0$, $B = 1$ in the following proposition, however the location and scale of the transformed random variable may be considered by changing these parameters, respectively, after the reshaping transformation. In what follows, we may drop any notational dependence on the vector of parameters of the distribution and density functions. Additionally, we assume $F \neq F_Y$, where $Y_t \sim F_Y(t, y)$ for all $t \in [t_0, \infty)$, else the random variable defined by $\Phi_t \stackrel{d}{=} \sqrt{2}\operatorname{erf}^{-1}(2F_Y(t, Y_t) - 1)$ is standard normally distributed for all $t \in [t_0, \infty)$ and

we recover the usual Tukey- g transformation of a standard normal random variable. In this case, if the mode of Y_t lies at the origin for each $t \in [t_0, \infty)$, the skewness transformation occurs around the mode of the driving process and the composite map is a well-defined reshaping transformation, as given by Definition 2.1.7. If the mode of Y_t does not lie at the origin for each $t \in [t_0, \infty)$, we remark that the finite-dimensional distributions of the quantile process may still have greater positive (resp. negative) skewness than those of the driving process for $g > 0$ (resp. $g < 0$) by the convexity (resp. concavity) of the composite map—see Proposition 3.1.1.

Next, we consider the Tukey- g quantile process, in the case where $F \neq F_Y$ in the composite map from which it is constructed.

Proposition 3.3.1. *Consider Definition 3.1.1 and let $Q_\zeta(u; \boldsymbol{\xi}) = Q_{T_g}(u; A, B, g)$ be the quantile function of a Tukey- g distributed random variable, given by Eq. (2.1.11) with $h = 0$, and for $A \in \mathbb{R}$, $B \in \mathbb{R}^+$ and $g \in \mathbb{R} \setminus 0$. Set $A = 0$, $B = 1$. Consider the Tukey- g random-level quantile process, given by*

$$Z_t \stackrel{d}{=} \frac{1}{g} \left[\exp \left(g\sqrt{2} \operatorname{erf}^{-1} (2F(t, Y_t; \boldsymbol{\theta}) - 1) \right) - 1 \right] \quad (3.3.2)$$

for each $t \in [t_0, \infty)$ and where the vector of parameters $\boldsymbol{\theta}$ does not include g . The quantile process has the representation $Z_t \stackrel{d}{=} T_R(Y_t)$ for all $g \in \mathbb{R} \setminus 0$, where T_R is a reshaping transformation, given by Definition 2.1.7, if, and only if, for all $t \in [t_0, \infty)$, $F(t, y; \boldsymbol{\theta})$ has a continuous second derivative, the median of the random variable with distribution function $F(t, y; \boldsymbol{\theta})$ is equal to the mode of Y_t , where both lie at the origin, and $\partial_y F(t, y; \boldsymbol{\theta}) =: f(t, y; \boldsymbol{\theta}) \approx 1/\sqrt{2\pi}$ for $y \approx 0$. Here, ∂_y denotes first-order differentiation with respect to the spatial variable y .

Proof. First, assume for each $t \in [t_0, \infty)$, the mode of Y_t lies at some point $y_t^* \in \mathbb{R}$. Then there exists some $u_t^* \in [0, 1]$ such that $F(t, y_t^*) = u_t^*$ for each $t \in [t_0, \infty)$. Consider the function

$$m(t, y) := \frac{1}{g} \left[\exp \left(g\sqrt{2} \operatorname{erf}^{-1} (2F(t, y) - 1) \right) - 1 \right] \quad (3.3.3)$$

for all $t \in [t_0, \infty)$ and $y \in \mathbb{R}$. In order for it to hold that $m(t, y) \approx y$ for $y \approx y_t^*$, we

first require that $m(t, y_t^*) = y_t^*$ for all $t \in [t_0, \infty)$, i.e.,

$$\frac{1}{g} \left[\exp \left(g\sqrt{2} \operatorname{erf}^{-1} (2F(t, y_t^*) - 1) \right) - 1 \right] = \frac{1}{g} \left[\exp \left(g\sqrt{2} \operatorname{erf}^{-1} (2u_t^* - 1) \right) - 1 \right] = y_t^* \quad (3.3.4)$$

for all g . The left hand side of Eq. (3.3.4) does not depend on g if, and only if, $u_t^* = 0.5$, in which case we must have $y_t^* = 0$. Thus, for each $t \in [t_0, \infty)$, the mode of Y_t lies at the origin and is equal to $F(t, 0.5)$, that is, the median of the random variable with distribution function $F(t, y)$.

We now wish to study the behaviour of $m(t, y)$ when $y \approx 0$, at each $t \in [t_0, \infty)$. Considering the Taylor series expansion of the function $\exp(g\sqrt{2}\operatorname{erf}^{-1}(2F(t, y) - 1))$ around $y = 0$, we have

$$\begin{aligned} \exp \left(g\sqrt{2} \operatorname{erf}^{-1} (2F(t, y) - 1) \right) &\approx \exp \left(g\sqrt{2} \operatorname{erf}^{-1} (2F(t, 0) - 1) \right) \\ &+ g\sqrt{2\pi} f(t, 0) \exp \left((\operatorname{erf}^{-1} (2F(t, 0) - 1))^2 + g\sqrt{2} \operatorname{erf}^{-1} (2F(t, 0) - 1) \right) y \\ &+ \left\{ \frac{g\pi}{\sqrt{2}} f'(t, 0) \exp \left((\operatorname{erf}^{-1} (2F(t, 0) - 1))^2 \right) \right. \\ &\quad \left. + g\pi\sqrt{2} f(t, 0)^2 \operatorname{erf}^{-1} (2F(t, 0) - 1) \exp \left(3 (\operatorname{erf}^{-1} (2F(t, 0) - 1))^2 \right) \right. \\ &\quad \left. + g^2 \pi f(t, 0)^2 \exp \left(2 (\operatorname{erf}^{-1} (2F(t, 0) - 1))^2 \right) \right\} \\ &\times \exp \left(g\sqrt{2} \operatorname{erf}^{-1} (2F(t, 0) - 1) \right) y^2 + \dots \end{aligned} \quad (3.3.5)$$

and so

$$\begin{aligned} m(t, y) &\approx \frac{1}{g} \left[\exp \left(g\sqrt{2} \operatorname{erf}^{-1} (2F(t, 0) - 1) \right) - 1 \right] \\ &+ \sqrt{2\pi} f(t, 0) \exp \left((\operatorname{erf}^{-1} (2F(t, 0) - 1))^2 + g\sqrt{2} \operatorname{erf}^{-1} (2F(t, 0) - 1) \right) y + \mathcal{O}(y^2) \end{aligned} \quad (3.3.6)$$

for each $t \in [t_0, \infty)$. We consider each term of the expansion in turn. First, we have

$$\frac{1}{g} \left[\exp \left(g\sqrt{2} \operatorname{erf}^{-1} (2F(t, 0) - 1) \right) - 1 \right] = 0 \quad (3.3.7)$$

if, and only if, $F(t, 0) = 0.5$, and so the median of the random variable with distribution

function $F(t, y)$ must equal 0. It follows that

$$\sqrt{2\pi}f(t, 0)\exp\left(\left(\operatorname{erf}^{-1}(2F(t, 0) - 1)\right)^2 + g\sqrt{2}\operatorname{erf}^{-1}(2F(t, 0) - 1)\right) = \sqrt{2\pi}f(t, 0) \approx 1 \quad (3.3.8)$$

if, and only if, $f(t, 0) \approx 1/\sqrt{2\pi}$, as required. The continuity of the second derivative of $F(t, y)$ ensures the continuity of the second derivative of $m(t, y)$. \square

Considering Eq. (3.3.2), if F is the distribution function of the standard normal distribution at all $t \in [t_0, \infty)$, then we may write $Z_t \stackrel{d}{=} T_R(Y_t) \stackrel{d}{=} r_g(Y_t)$ when Y_t has mode equal to zero for all $t \in [t_0, \infty)$. This agrees with the result given in Proposition 3.3.1. Further examples of distribution functions one may consider are:

1. The Student-t distribution function with degrees of freedom parameter $\nu \rightarrow \infty$. If we consider, e.g., $\nu = 15$, we have $f(t, 0) \approx 1/\sqrt{2\pi}$ with a 1.65% error margin.
2. The Laplace distribution with location parameter $\mu = 0$ and scale parameter $b = \sqrt{\pi/2}$.
3. The Logistic distribution with location parameter $\mu = 0$ and scale parameter $s = \sqrt{\pi}/(2\sqrt{2})$.
4. The Gumbel distribution with location parameter $\mu = 0$ and scale parameter $\beta = \sqrt{2\pi}/e$.

The following corollary considers the case where the parameter g accounts for all relative skewness introduced to the driving process, in the construction of the quantile process. This allows one to control the skewness introduced to the driving process, marginally, via the composite map transformation exclusively through the parameter g in the Tukey transform. The inner part of the composite map, that is, the distribution function F , does not alter the marginal skewness of the driving process under the distortion used to produce the quantile process.

Corollary 3.3.1. *Assume the quantile process given by Eq. (3.3.2) has the reshaping transformation representation given in Proposition 3.3.1. The relative skewness between the marginal distribution of Z_t and that of Y_t , for each $t \in [t_0, \infty)$ is produced exclusively via the parameter g if, and only if, $F(t, y; \boldsymbol{\theta}) = -F(t, -y; \boldsymbol{\theta})$ for all $y \in \mathbb{R}$ and $t \in [t_0, \infty)$.*

Proof. Consider Eq. (3.3.2). We have $Z_t \stackrel{d}{=} r_g(\Phi^-(F(t, Y_t)))$ where $r_g(x) = xT_g(x)$ with $T_g(x)$ given by Eq. (2.1.8), and Φ is the CDF of the standard normal distribution. If $F = \Phi$ for all $t \in [t_0, \infty)$, then $Z_t \stackrel{d}{=} r_g(Y_t)$ and all symmetry is introduced to the distribution of Y_t via the parameter g , by definition of the Tukey- g distribution. If $F \neq \Phi$ for all $t \in [t_0, \infty)$, all symmetry is introduced in the Tukey- g transform part of the composite map if, and only if, the inner part of the composite map preserves symmetry, i.e., $\Phi^-(F(t, y)) = -\Phi^-(F(t, -y))$ for all $t \in [t_0, \infty)$ and $y \in \mathbb{R}$.

Note, that for any generic, symmetric random variable X with mean $\mu \in \mathbb{R}$, its quantile function satisfies the relation $Q_X(u) = 2\mu - Q_X(1 - u)$ for all $u \in [0, 1]$. For the standard normal distribution, $\mu = 0$ and so for all $u \in [0, 1]$, $\Phi^-(u) = -\Phi^-(1 - u)$. It follows that for all $t \in [t_0, \infty)$ and $y \in \mathbb{R}$, $\Phi^-(F(t, y)) = -\Phi^-(1 - F(t, y))$, and so $\Phi^-(F(t, y)) = -\Phi^-(F(t, -y))$ if, and only if, $F(t, y) = 1 - F(t, -y)$. By the definition and properties of a distribution function, $F(t, y) = 1 - F(t, -y)$ if, and only if, F is the CDF of a symmetric random variable for each $t \in [t_0, \infty)$, as required. \square

Remark 2. *One may still construct the class of Tukey- g quantile processes in the instance where the conditions given in Proposition 3.3.1 are not met, however the interpretability in regard to relative skewness between the driving process and the output quantile process, marginally, must be carefully considered. For consistent interpretability at each $t \in [t_0, \infty)$, it is advantageous to consider the skewness distortion relative to some fixed measure of centrality at all times, e.g., the mode as shown in Proposition 3.3.1. Else, the role of the composite map in introducing skewness to the marginal distributions of the driving process is not explicit, as intended with the Tukey- g distributional family, and one must check this case by case—see Proposition 3.1.1. Additionally, we note that one may also consider alternative measures of centrality to the mode, e.g., one could choose the composite map such that at each $t \in [t_0, \infty)$, the median of the quantile process is equal to the median of the driving process. Here, quantiles at levels $u > 0.5$ and $u < 0.5$ will be distorted through the composite map relative to the ‘anchor’ at $u = 0.5$ —see Example 3.1.1.*

In Proposition 3.3.1 and Corollary 3.3.1, we consider the one-parameter Tukey- g family, however more flexibility can be introduced by allowing the skew parameter to have a polynomial representation. Figure 3.4 shows the quantile function of the one-parameter Tukey- g distribution, for different values of the parameter g , relative to the quantile function of a standard normal random variable.

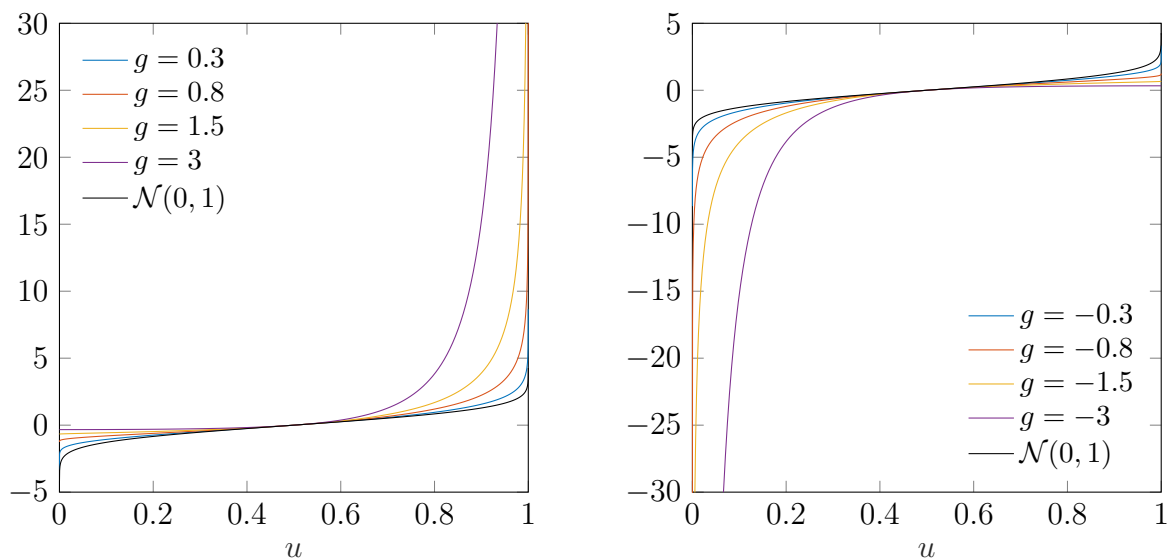


Figure 3.4: Quantile functions of the Tukey– g distribution for $g \in \{0.3, 0.8, 1.5, 3\}$ and $g \in \{-0.3, -0.8, -1.5, -3\}$, relative to the standard normal quantile function.

We now present the following examples of Tukey– g quantile processes.

Example 3.3.1. Consider a geometric Brownian motion (GBM) driving process, $Y_t = Y_0 \exp((\mu - 0.5\sigma^2)t + \sigma W_t)$, for all $t \in [0, \infty)$, with $Y_0 = y_0 \in \mathbb{R}^+$, $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$. Let $F = F_Y$ in Definition 3.1.1 be the lognormal law associated with this process, so that the quantile process construction is well-defined in regard to $\max_{t \in [t_0, \infty)} \text{ran}(Y_t) = \text{dom}(F)$. The skewed-GBM quantile process is then given by

$$\begin{aligned} Z_t &\stackrel{d}{=} \frac{1}{g} \left[\exp \left(\frac{g}{\sigma\sqrt{t}} \left(\log(Y_t) - \log(y_0) - \left(\mu - \frac{1}{2}\sigma^2 \right) t \right) \right) - 1 \right] \\ &\stackrel{d}{=} \frac{1}{g} \left[\exp \left(\frac{gW_t}{\sqrt{t}} \right) - 1 \right] \end{aligned} \quad (3.3.9)$$

and satisfies the SDE

$$dZ_t = \left(\frac{g}{2t} - \frac{\log(gZ_t + 1)}{2gt} \right) (gZ_t + 1) dt + \frac{(gZ_t + 1)}{\sqrt{t}} dW_t \quad (3.3.10)$$

for all $t \in [t_0, \infty)$ with $Z_{t_0} = z_{t_0} \in [-1/g, \infty)$ when $g > 0$ and $z_{t_0} \in (-\infty, -1/g]$ when $g < 0$. Here, we take $A = 0, B = 1$ in Eq. (3.3.1). Because the driving process is GBM, its mode at each $t \in [t_0, \infty)$ does not lie at the origin and so the result given in Proposition 3.3.1 is not applicable. Additionally, as $F = F_Y$ is the

lognormal law associated with the GBM driving process, used in the quantile process composite map, all symmetry is not introduced exclusively via the parameter g —see Corollary 3.3.1. Here, the role of F_Y in the inner part of the composite map is to ‘resymmetrise’ each marginal of the driving process, before the quantile function of the Tukey- g distribution, i.e., outer part of the composite map, is applied. By Proposition 3.1.1, the finite-dimensional distributions of (Z_t) are more positively skewed than those of (Y_t) at each $t \in [t_0, \infty)$ if, and only if, $g > \sigma\sqrt{t}$. This follows from the convexity of the composite map on $D_Y = \mathbb{R}^+$.

Figure 3.5 shows 30 sample paths of this quantile process for parameters $\mu = 0.1$, $\sigma = 0.05$ and $g \in \{0.6, 2, -0.8\}$. The corresponding sample paths of the GBM driving process and uniformly distributed process, $U_t \stackrel{d}{=} F_Y(t, Y_t)$, are also given, to visualise how the composite map distorts the paths of the process at each stage of transformation.

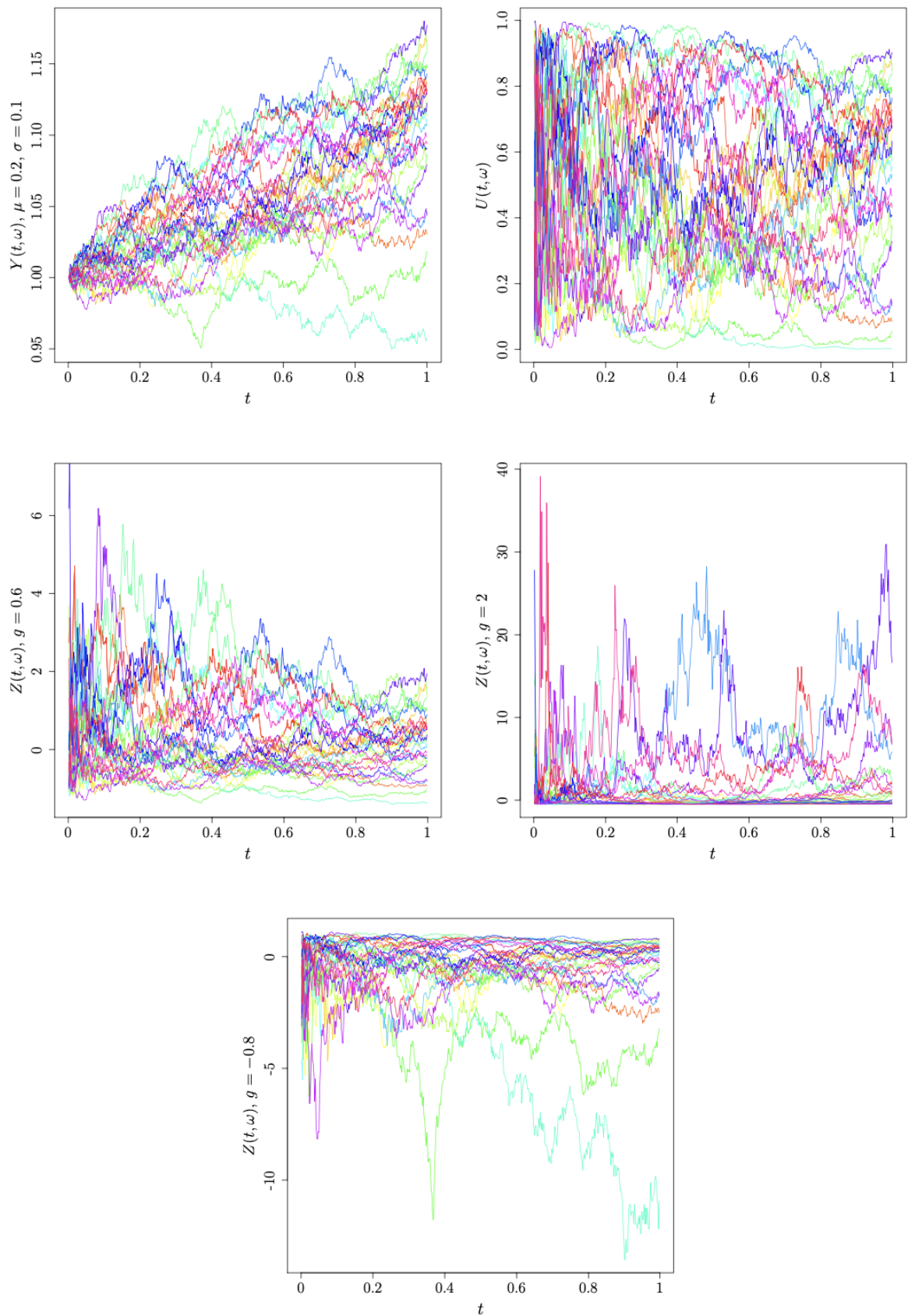


Figure 3.5: Sample paths of the GBM driving process with parameters $\mu = 0.1$, $\sigma = 0.05$, the uniformly distributed process (U_t), and the Tukey- g quantile process with parameter values $g = \{0.6, 2, -0.8\}$.

We remark that the SDE 3.1.31—where a Tukey– g quantile function, composed with the distribution function of a scaled Brownian motion, is applied to the scaled Brownian driving process—is indistinct to the SDE 3.3.10. The random–level quantile process construction is not unique and any two transformations may produce, in finite–dimensional law, the same quantile process. In the case where the driving process is a scaled Brownian motion, however, the composite map is interpretable with regard to the result presented in Proposition 3.3.1, when $\sigma \approx 1$, and Corollary 3.3.1 informs that all skewness in the quantile process, relative to the scaled Brownian driver, is introduced via the parameter g . We also note that, here, the composite map is convex on \mathbb{R} for each $t \in [t_0, \infty)$ and all $g > 0$ so, by Proposition 3.1.1, the finite–dimensional distributions of the quantile process are more positively skewed than those of the scaled Brownian driving process.

Example 3.3.2. Consider the driving process (Y_t) to be an Ornstein–Uhlenbeck (OU) process with time inhomogeneous parameters, satisfying the SDE

$$dY_t = \theta(t) (\mu(t) - Y_t) dt + \sigma(t) dW_t \quad (3.3.11)$$

for all $t \in [0, \infty)$ with $y_0, \mu(t) \in \mathbb{R}$, $\sigma(t) \in \mathbb{R}^+$, and the mean–reversion parameter $\theta(t) \in \mathbb{R}^+$ for all $t \in (0, \infty)$. The marginal law of the driving process at each time $t \in (0, \infty)$ is given by

$$F_Y(t, y; \boldsymbol{\vartheta}(t)) = \frac{1}{2} \left[1 + \operatorname{erf} \left(\frac{y - y_0 e^{-\int_0^t \theta(s) ds} - e^{-\int_0^t \theta(s) ds} \int_0^t e^{\int_0^s \theta(u) du} \theta(s) \mu(s) ds}{e^{-\int_0^t \theta(s) ds} \sqrt{2 \int_0^t \exp(2 \int_0^s \theta(u) du) \sigma^2(s) ds}} \right) \right], \quad (3.3.12)$$

where $\boldsymbol{\vartheta}(t) = (\theta(t), \mu(t), \sigma(t))$, and so the process $U_t := F_Y(t, Y_t; \boldsymbol{\vartheta}(t))$ is given by

$$U_t \stackrel{d}{=} \frac{1}{2} \left[1 + \operatorname{erf} \left(\frac{\int_0^t \exp(\int_0^s \theta(u) du) \sigma(s) dW_s}{\sqrt{2 \int_0^t \exp(2 \int_0^s \theta(u) du) \sigma^2(s) ds}} \right) \right], \quad (3.3.13)$$

for $t \in (0, \infty)$. It follows that the skewed–OU random–level quantile process is given by

$$Z_t \stackrel{d}{=} A + \frac{B}{g} \left[\exp \left(\frac{g \int_0^t \exp(\int_0^s \theta(u) du) \sigma(s) dW_s}{\sqrt{\int_0^t \exp(2 \int_0^s \theta(u) du) \sigma^2(s) ds}} \right) - 1 \right] \quad (3.3.14)$$

for $t \in [t_0, \infty)$ with $Z_{t_0} = z_{t_0} \in [A - B/g, \infty)$ when $g > 0$ and $z_{t_0} \in (-\infty, A - B/g]$ when $g < 0$. In the case where $\mu(t) = \mu \in \mathbb{R}$, $\sigma(t) = \sigma \in \mathbb{R}^+$, and $\theta(t) = \theta \in \mathbb{R}^+$ for all $t \in (0, \infty)$, the skewed-OU quantile process satisfies the SDE

$$dZ_t = \left(\frac{g\theta}{(1 - \exp(-2\theta t))} - \frac{\theta \log(gZ_t + 1)}{g(1 - \exp(-2\theta t))} \right) (gZ_t + 1) dt + \frac{\sqrt{2\theta}(gZ_t + 1)}{\sqrt{1 - \exp(-2\theta t)}} dW_t \quad (3.3.15)$$

for all $t \in [t_0, \infty)$, where we take $A = 0$, $B = 1$ without loss of generality, and with $Z_{t_0} = z_{t_0} \in [-1/g, \infty)$ when $g > 0$ and $z_{t_0} \in (-\infty, -1/g]$ when $g < 0$. We highlight that both SDEs, (3.3.10) and (3.3.15), can be written in the form

$$dZ_t = \frac{\sigma^2(gZ_t + 1)}{2\text{Var}(Y_t|Y_0 = y_0)} \left(g - \frac{\log(gZ_t + 1)}{g} \right) dt + \frac{\sigma}{\sqrt{\text{Var}(Y_t|Y_0 = y_0)}} (gZ_t + 1) dW_t \quad (3.3.16)$$

for all $t \in [t_0, \infty)$.

We now consider the Tukey- h random-level quantile process, where the parameter $h \in \mathbb{R}_0^+$ allows for flexible modelling of the heaviness of the tails of the distribution. The quantile function of a Tukey- h distributed random variable is given by

$$Q_\zeta(u; \boldsymbol{\xi}) = Q_{T_h}(u; A, B, h) = A + B\sqrt{2}\text{erf}^-(2u - 1)\exp\left(h(\text{erf}^-(2u - 1))^2\right) \quad (3.3.17)$$

for $u \in [0, 1]$, $A \in \mathbb{R}$, $B \in \mathbb{R}^+$ and $h \in \mathbb{R}_0^+$.

In the following proposition, we consider the Tukey- h quantile process, in the case where $F \neq F_Y$ in the composite map from which it is constructed. The assumption that $F \neq F_Y$ where $Y_t \sim F_Y(t, y)$ for all $t \in [t_0, \infty)$ is made for the same reasons discussed prior to Proposition 3.3.1.

Proposition 3.3.2. *Consider Definition 3.1.1 and let $Q_\zeta(u; \boldsymbol{\xi}) = Q_{T_h}(u; A, B, g)$ for $u \in [0, 1]$. Set $A = 0$, $B = 1$, and consider the Tukey- h random-level quantile process, given by*

$$Z_t \stackrel{d}{=} \sqrt{2}\text{erf}^-(2F(t, Y_t; \boldsymbol{\theta}) - 1)\exp\left(h(\text{erf}^-(2F(t, Y_t; \boldsymbol{\theta}) - 1))^2\right) \quad (3.3.18)$$

for each $t \in [t_0, \infty)$ and where the vector of parameters $\boldsymbol{\theta}$ does not include h . The quantile process has the representation $Z_t \stackrel{d}{=} T_E(Y_t)$ for all $h \in \mathbb{R}_0^+$ where T_E is an elongation map, given in Definition 2.1.6, if, and only if, for all $t \in [t_0, \infty)$, it holds

that: (1) the median of the random variable with distribution function $F(t, y; \boldsymbol{\theta})$ and the mode of Y_t are zero, and $\partial_y F(t, y; \boldsymbol{\theta}) =: f(t, y; \boldsymbol{\theta}) \approx 1/\sqrt{2\pi}$ for $y \approx 0$; (2) for all $y > 0$,

$$(i) \quad F(t, y; \boldsymbol{\theta}) = -F(t, -y; \boldsymbol{\theta}),$$

$$(ii) \quad \partial_y \{\sqrt{2} \operatorname{erf}^{-1}(2F(t, y) - 1)\} > 0,$$

$$(iii) \quad \partial_y^2 \{\sqrt{2} \operatorname{erf}^{-1}(2F(t, y) - 1)\} \geq 0.$$

Proof. The first part of the proof is analogous to that of Proposition 3.3.1, however where we now define the function

$$m(t, y) := \sqrt{2} \operatorname{erf}^{-1}(2F(t, y) - 1) \exp\left(h (\operatorname{erf}^{-1}(2F(t, y) - 1))^2\right) \quad (3.3.19)$$

and consider its Taylor series expansion around $y = 0$, given by

$$\begin{aligned} m(t, y) &\approx \sqrt{2} \operatorname{erf}^{-1}(2F(t, 0) - 1) \exp\left(h (\operatorname{erf}^{-1}(2F(t, 0) - 1))^2\right) \\ &\quad + \sqrt{2\pi} f(t, 0) \left[1 + 2h (\operatorname{erf}^{-1}(2F(t, 0) - 1))^2\right] \\ &\quad \times \exp\left((h + 1) (\operatorname{erf}^{-1}(2F(t, y) - 1))^2\right) y + \mathcal{O}(y^2) \end{aligned} \quad (3.3.20)$$

for each $t \in [t_0, \infty)$. From here we derive the requirements that the mode of Y_t lies at the origin, $F(t, 0.5) = 0$ so that the median of the random variable with distribution function $F(t, y)$ must equal zero, and that $f(t, 0) \approx 1/\sqrt{2\pi}$. To derive the remainder of the proof, we recall Definition 2.1.6 of an elongation map and it remains to show that for $y > 0$, (i)–(iii) hold, that is, $m(t, y)$ is a symmetric and convex function for each $t \in [t_0, \infty)$.

We have $m(t, y) = r_h(\Phi^-(F(t, y)))$ where $r_h(x) = xT_h(x)$ with $T_h(x)$ given by Eq. (2.1.5), and Φ is the CDF of the standard normal distribution. By the proof of Corollary 3.3.1, $r_h(\Phi^-(F(t, y))) = -r_h(\Phi^-(F(t, -y)))$ if, and only if, $F(t, y) = -F(t, -y)$ for each $t \in [t_0, \infty)$.

Since, by the definition of an elongation map, $r_h(x)$ is a convex, increasing function for $x > 0$. It follows that $r_h(\Phi^-(F(t, y)))$ is convex for $y > 0$ if, and only if, $\Phi^-(F(t, y))$ is convex or linear—see Section 3.2.4 by Boyd and Vandenberghe (2004). This is equivalent to the derivative conditions (ii)–(iii), which concludes the proof. \square

Considering Eq. (3.3.18), if F is the distribution function of the standard normal distribution at all $t \in [t_0, \infty)$, then we may write $Z_t \stackrel{d}{=} T_E(Y_t) = r_h(Y_t)$ when Y_t has mode equal to zero for all $t \in [t_0, \infty)$. This agrees with the result given in Proposition 3.3.2. The requirement that the composite map be convex for $y > 0$ is equivalent to the distribution function $F(t, y)$ being convex for $y > 0$, which is uncommon. Instead, one may wish to consider a (not necessarily standard) normal distribution function $F(t, y)$, to ensure that the map $\Phi^-(F(t, y))$ is linear, and thus the composite map used in the quantile process construction is convex for $y > 0$.

Figure 3.6 shows the quantile function of the Tukey- h distribution for varying values of the parameter h , with $A = 0, B = 1$, relative to the quantile function of a standard normal random variable. The plot on the right shows that for negative values of h beyond a certain threshold, the function $Q_{T_h}(u; h)$ is no longer monotonically increasing and hence we restrict to $h \in \mathbb{R}_0^+$, i.e., we introduce more kurtosis to the base random variable which, in the figure, and considered throughout this thesis, is a standard normal random variable.

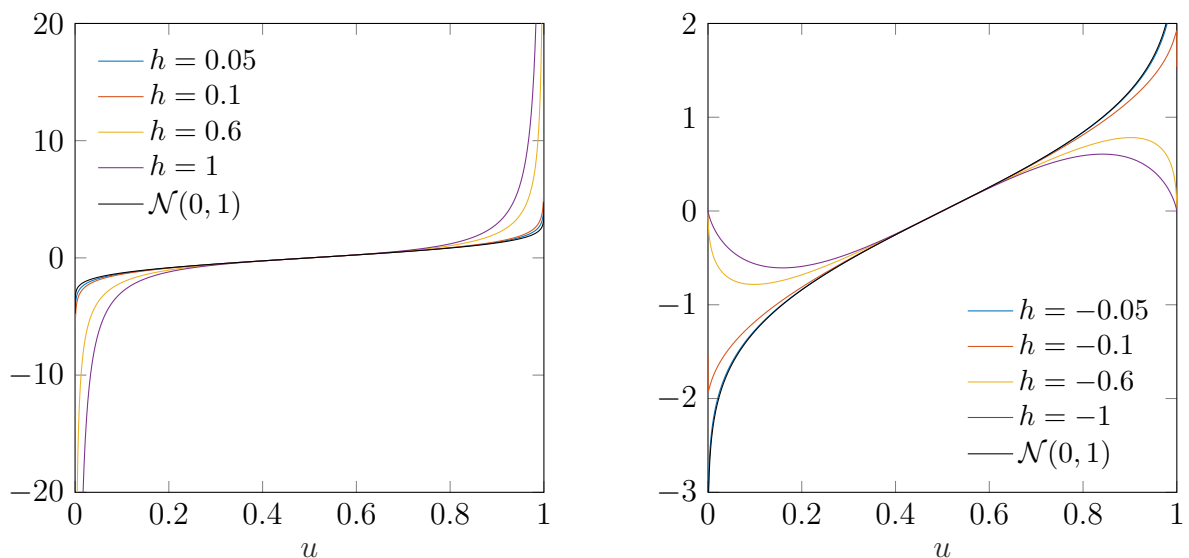


Figure 3.6: Quantile functions of the Tukey- h distribution, relative to the standard normal quantile function, for $h \in \{-0.05, -0.1, 0.05, 0.1, 0.6, 1\}$, and h -quantile transforms for $h \in \{-0.6, -1\}$.

We present the following example of a Tukey- h quantile process.

Example 3.3.3. Consider Example 3.3.1, where we have a GBM driving process and

let $F = F_Y$ in Definition 3.1.1. Now consider the quantile function of the Tukey– h distribution, given by Eq. (3.3.17), with $A = 0$, $B = 1$, so that the leptokurtic–GBM quantile process is given by

$$\begin{aligned} Z_t &\stackrel{d}{=} \left(\frac{1}{\sigma\sqrt{t}} (\log(Y_t) - \tilde{\mu}(t)) \right) \exp \left(\frac{h}{2} \left(\frac{1}{\sigma\sqrt{t}} (\log(Y_t) - \tilde{\mu}(t)) \right)^2 \right) \\ &\stackrel{d}{=} \frac{W_t}{\sqrt{t}} \exp \left(h \frac{W_t^2}{2t} \right) \end{aligned} \quad (3.3.21)$$

for all $t \in [t_0, \infty)$ and where $\tilde{\mu}(t) := \log(y_0) - (\mu - 0.5\sigma^2)t$. Because the driving process is GBM, its mode at each $t \in [t_0, \infty)$ does not lie at the origin, and its marginal distribution functions are not symmetric, and so the result given in Proposition 3.3.2 is not applicable. Much like Example 3.3.1, the role of F_Y in the inner part of the composite map is to ‘resymmetrise’ each marginal of the driving process, before the quantile function of the Tukey– h distribution is applied, producing a quantile process (Z_t) that is marginally distributed according to the Tukey– h distribution.

We remark that the quantile process in Eq. (3.3.21) is indistinct (in distribution) to that obtained by considering the canonical Tukey– h quantile process—see Definition 3.1.2. In this case, however, the composite map is symmetric and convex for $y > 0$, and so we can ensure, by Proposition 3.1.2, that the finite–dimensional distributions of the quantile process will have heavier tails than those of the driving Brownian motion, due to the parameter h .

Figure 3.7 shows 30 sample paths of the quantile process, for the same driving process parameters given in Example 3.3.1 and for $h \in \{0.02, 0.2\}$. Again, the corresponding sample paths of the driving GBM are also given, to visualise the effect of the quantile distortion on the paths.

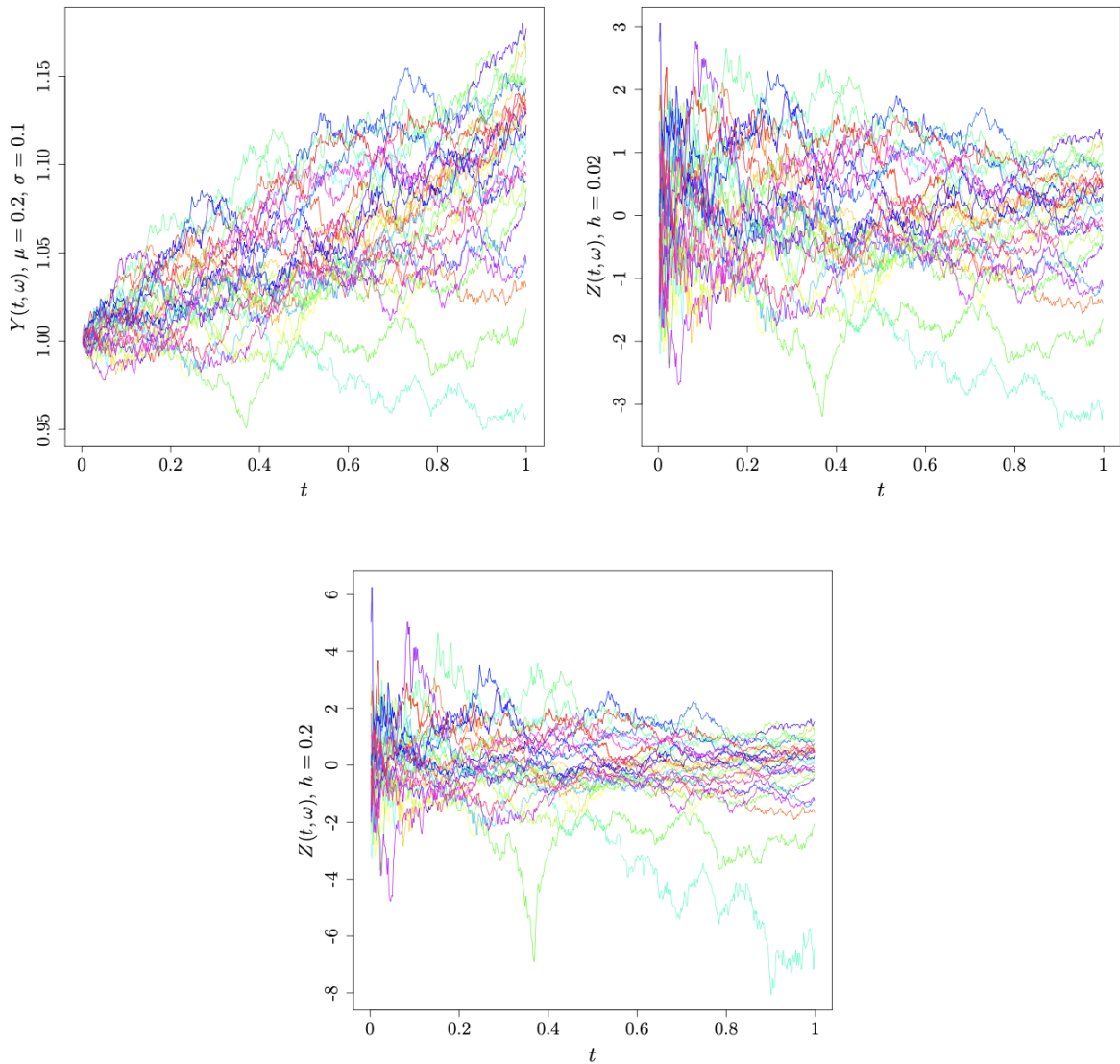


Figure 3.7: Sample paths of the GBM driving process with parameters $\mu = 0.1$, $\sigma = 0.05$, and the Tukey– h quantile process with parameter values $h = \{0.02, 0.2\}$.

We conclude this section by combining the above two Tukey families to consider the more general, two-parameter Tukey– gh distribution, with quantile function given by Eq. (2.1.11), that is,

$$Q_{T_{gh}}(u; A, B, g, h) = A + \frac{B}{g} \left[\exp \left(g\sqrt{2} \operatorname{erf}^{-1}(2u - 1) \right) - 1 \right] \exp \left(h \left(\operatorname{erf}^{-1}(2u - 1) \right)^2 \right) \quad (3.3.22)$$

for $u \in [0, 1]$. Here, the parameters $g \in \mathbb{R} \setminus 0$ and $h \in \mathbb{R}_0^+$ are responsible for controlling the skewness and kurtosis of the distribution, respectively, relative to the standard

normal distribution. It holds that $Q_{T_{gh}}(u; A, B, g, h) : [0, 1] \rightarrow \mathbb{R}$ and for $n \in \mathbb{N}$, the n^{th} moment of the distribution exists for $h < 1/n$ —see, e.g., Klein and Fischer (2002). Figure 2.2 shows the quantile function of the Tukey– gh distribution for varying values of the skewness and kurtosis parameters, relative to the quantile function of a standard normal random variable. We remark that this Tukey– gh class of quantile processes allows one to capture any skew–kurtosis range. Such features are often advantageous in loss modelling, e.g., in non–life insurance settings as discussed by Peters et al. (2016). Since the Tukey– gh transformation involves multiplying the Tukey– g and $-h$ transformations, it holds that the distribution will be asymmetric whenever $g \neq 0$. When $h = 0$, we recover the Tukey– g distribution in which tail elongation occurs only as a result of the introduced skewness, e.g., an elongated right (resp. left) tail when $g > 0$ (resp. $g < 0$). The functional form of the Tukey– gh transform allows one to first treat skewness and the tail elongation that comes with it (through the g –transform), and then account for excess tail–heaviness (through multiplying by the h –transform). In other words, the family provides a nice, practical way to treat both skewness and kurtosis in a distributional transformation, but allow for a disentanglement of the two. In regard to the quantile process construction, one may apply the results in Propositions 3.3.1 and 3.3.2, or Propositions 3.1.1 and 3.1.2 to each part of the transformation map to analyse the effect of the composite map on the marginals driving process, when producing marginals of the Tukey– gh quantile process.

As an example, we consider a univariate Gaussian process as the driving process, to produce a skewed–leptokurtic–Gaussian quantile process, or what is referred to, more simply, as a ‘Tukey process’ by Nagarajan et al. (2018). We first define a Gaussian process, as follows.

Definition 3.3.1. *The process $(Y_t)_{t \in [0, \infty)}$ is a Gaussian process if, and only if, for every finite set of indices $\{t_1, t_2, \dots, t_k\}$ for $1 \leq k < \infty$ and where $t_i \in [0, \infty)$ for all $i \in 1, \dots, k$, the random vector $\mathbf{Y}_{t_1, \dots, t_k} := (Y_{t_1}, Y_{t_2}, \dots, Y_{t_k})$ is a multivariate Gaussian (\mathcal{MVN}) random variable.*

In what follows, if $\mathbf{Y}_{t_1, \dots, t_k} \sim \mathcal{MVN}(\boldsymbol{\mu}_{1:k}, \boldsymbol{\Sigma}_{1:k})$, we write $Y_t \sim \mathcal{GP}(\boldsymbol{\mu}_{1:k}, \boldsymbol{\Sigma}_{1:k})$ where

$$\boldsymbol{\mu}_{1:k} := [\mathbb{E}[Y_{t_1}], \mathbb{E}[Y_{t_2}], \dots, \mathbb{E}[Y_{t_k}]]^\top = [\mu_{Y_{t_1}}, \mu_{Y_{t_2}}, \dots, \mu_{Y_{t_k}}]^\top, \quad (3.3.23)$$

$$\Sigma_{1:k} := \begin{bmatrix} \mathcal{C}(Y_{t_1}, Y_{t_1}) & \mathcal{C}(Y_{t_1}, Y_{t_2}) & \cdots & \mathcal{C}(Y_{t_1}, Y_{t_k}) \\ \mathcal{C}(Y_{t_2}, Y_{t_1}) & \mathcal{C}(Y_{t_2}, Y_{t_2}) & \cdots & \mathcal{C}(Y_{t_2}, Y_{t_k}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{C}(Y_{t_k}, Y_{t_1}) & \mathcal{C}(Y_{t_k}, Y_{t_2}) & \cdots & \mathcal{C}(Y_{t_k}, Y_{t_k}) \end{bmatrix} = \begin{bmatrix} \sigma_{Y_{t_1}}^2 & \sigma_{Y_{t_1}, Y_{t_2}} & \cdots & \sigma_{Y_{t_1}, Y_{t_k}} \\ \sigma_{Y_{t_2}, Y_{t_1}} & \sigma_{Y_{t_2}}^2 & \cdots & \sigma_{Y_{t_2}, Y_{t_k}} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{Y_{t_k}, Y_{t_1}} & \sigma_{Y_{t_k}, Y_{t_2}} & \cdots & \sigma_{Y_{t_k}}^2 \end{bmatrix}, \quad (3.3.24)$$

for $\mathcal{C}(\cdot, \cdot)$ the covariance function, are the mean and covariance matrices, respectively. In this example, we consider Definition 3.1.1 and take $F(t, y; \boldsymbol{\theta})$ to be the distribution function of the normal distribution with mean parameter $\gamma \in \mathbb{R}$ and standardised variance. We construct the *shifted* Tukey process as

$$Z_t \stackrel{d}{=} A + \frac{B}{g} (\exp(g(Y_t - \gamma)) - 1) \exp\left(\frac{h}{2}(Y_t - \gamma)^2\right) \quad (3.3.25)$$

for all $t \in [t_0, \infty)$. Each finite-dimensional distribution of the quantile process in Eq. (3.3.25) will belong to the Tukey- gh family. Similar processes, derived by applying a Tukey- gh transform to a Gaussian process are discussed by Nagarajan et al. (2018), Xu and Genton (2017) and Yan et al. (2020).

Lastly, we consider the class of function-valued quantile processes given in Section 3.2, more specifically Definition 3.2.1. Let $\zeta = T_{gh}$, and $(\boldsymbol{\xi}_t)_{t \in [0, \infty)}$ be the four-dimensional stochastic process with continuous-time marginals (A_t, B_t, g_t, h_t) where $A_t \in \mathbb{R}$, $B_t \in \mathbb{R}^+$, $g_t \in \mathbb{R} \setminus 0$ and $h_t \in \mathbb{R}_0^+$ for all $t \in [0, \infty)$. It follows that the Tukey- gh function-valued quantile process is given by

$$Z_t(u) = \begin{cases} A_t + B_t \frac{\exp(g_t \sqrt{2} \operatorname{erf}^{-1}(2u - 1)) - 1}{g_t} \exp\left(h_t (\operatorname{erf}^{-1}(2u - 1))^2\right), & g_t \neq 0 \\ A_t + B_t \sqrt{2} \operatorname{erf}^{-1}(2u - 1) \exp\left(h_t (\operatorname{erf}^{-1}(2u - 1))^2\right), & g_t = 0, \end{cases} \quad (3.3.26)$$

for all $u \in [0, 1]$ and $t \in [0, \infty)$. We note the relation between the quantile function-valued processes given in Eqs (2.2.7) and (3.3.26), where the difference is that the driving parameter processes are discrete- and continuous-time processes, respectively.

3.4 Multidimensional and multivariate extension

We conclude this chapter on the construction of quantile processes by introducing multidimensional and multivariate extensions of Definition 3.1.1. We distinguish between the notion of a *multidimensional* and a *multivariate* quantile process, where *multidimensional* means a univariate quantile process driven by a multivariate risk process, and *multivariate* means an n -dimensional process for $n \geq 2$. As such, the class of function-valued quantile processes given in Definition 3.2.1 produces a multidimensional class of (function-valued) quantile processes for each $u \in [0, 1]$ whenever the dimension of the driving parameter process is $d \geq 2$. The focus of this section will be on multidimensional random-level quantile processes, as these processes will be used in a data-driven example in Chapter 7, however we introduce multivariate random-level quantile processes as well. We note the distinction between the terminology *margin* and *marginal*: the margin of a n -dimensional multivariate process refers to one of the $i = 1, \dots, n$ univariate processes in each element of the vector process, and the marginal of a process refers to the random variable associated to the process at each point in time, $t \in (0, \infty)$. In what follows, we utilise a copula to capture the dependence structure between any univariate stochastic processes—see Nelsen (2007), where a d -dimensional copula $C : [0, 1]^d \rightarrow [0, 1]$, for $d \in \mathbb{N}$, is a multivariate CDF with standard uniform margins. We may now define the multidimensional random-level quantile process as follows.

Definition 3.4.1. *Let $Q_\zeta(u; \boldsymbol{\xi})$ and $F_j(t, y; \boldsymbol{\theta}_j)$ be continuous quantile and distribution functions, respectively, as per Definition 3.1.1, for $j = 1, \dots, m$ and $m > 1$. For all $t \in (0, \infty)$, let $C(u_1, \dots, u_m; t, \tilde{\boldsymbol{\theta}}) : \mathbb{R}^+ \times [0, 1]^m \rightarrow [0, 1]$ be an m -dimensional, time-inhomogeneous copula where $\tilde{\boldsymbol{\theta}} \in \mathbb{R}^{d''}$ is a d'' -dimensional vector of parameters, for $d'' \in \mathbb{N}$. Consider an \mathbb{R}^m -valued càdlàg process $(\mathbf{Y}_t)_{t \in [0, \infty)}$. At each time $t \in [t_0, \infty)$, for $t_0 > 0$, the random-level, multidimensional quantile process is defined by*

$$Z_t \stackrel{d}{=} Q_\zeta \left(C \left(F_1 \left(t, Y_t^{(1)}; \boldsymbol{\theta}_1 \right), \dots, F_m \left(t, Y_t^{(m)}; \boldsymbol{\theta}_m \right); t, \tilde{\boldsymbol{\theta}} \right); \boldsymbol{\xi} \right). \quad (3.4.1)$$

That is,

$Z(t, \omega) = Q_\zeta \left(C \left(F_1(t, Y^{(1)}(t, \omega); \boldsymbol{\theta}_1), \dots, F_m(t, Y^{(m)}(t, \omega); \boldsymbol{\theta}_m); t, \tilde{\boldsymbol{\theta}} \right); \boldsymbol{\xi} \right) : [t_0, \infty) \times \Omega \rightarrow [-\infty, \infty]$, and the map $t \mapsto Z(t, \omega)$ for each $\omega \in \Omega$ and $t \in [t_0, \infty)$ is \mathcal{F}_t -measurable.

The output quantile process in Eq. (3.4.1) is a univariate quantile process, similar to that in Definition 3.1.1, but driven by a multivariate base process. The dependence between each margin of the driving process is described by the choice of copula and, similarly to the univariate case, the choice of quantile function Q_ζ and distribution functions F_j for $j = 1, \dots, m$, are chosen to determine the statistical properties and behaviour of the quantile process, e.g., skewness and kurtosis. It is less clear in the multidimensional case than it is in the univariate case as to how the finite-dimensional distributions between each driving process; the choice of distribution function, copula and quantile function in the composite map; and the resulting finite-dimensional distributions of the quantile process relate to one another. To our knowledge, the literature on distributional distortions in the multivariate case is less dense than the univariate case, and there exists space in future work to explore the distortions of the above type in more detail. We remark, however that when $F_i = F_{Y^{(i)}}$ in Eq. (3.4.1), for $i = 1, \dots, m$, and the copula C is that implied by the joint distribution of the driving process marginals, then for $t \in [t_0, \infty)$, the random variable defined by

$$\tilde{U}_t := C_Y \left(F_{Y^{(1)}} \left(t, Y_t^{(2)}; \boldsymbol{\theta}_1 \right), \dots, F_{Y^{(m)}} \left(t, Y_t^{(m)}; \boldsymbol{\theta}_m \right); t, \tilde{\boldsymbol{\theta}} \right) \quad (3.4.2)$$

is analogous to Kendall's core, given in Definition 2 by Millosovich et al. (2021). Some properties of Kendall's core, related to the multivariate behaviour of the driving process (\mathbf{Y}_t) , are given therein.

As such, for the multidimensional quantile process construction to be more intuitive, we may consider the case in which the marginal distribution functions of each driving process are used in the construction so that the inputs to the m -dimensional copula are uniformly distributed, and a direct connection to Kendall's core may be made. We consider this case, in addition to considering the copula implied by the joint distribution of the driving process marginals, as follows.

Assume the joint law of the margins of the driving process, i.e., the processes $(Y_t^{(i)})$ for $i = 1, \dots, m$, is given by $F_{\mathbf{Y}}(t, y_1, \dots, y_m; \boldsymbol{\vartheta}) : \mathbb{R}^+ \times \mathbb{R}^m \rightarrow [0, 1]$ for all $t \in (0, \infty)$. By Sklar's theorem, there exists a copula $C_Y(u_1, \dots, u_m; t, \tilde{\boldsymbol{\theta}}) : \mathbb{R}^+ \times [0, 1]^m \rightarrow [0, 1]$ such that $F_{\mathbf{Y}}(t, y_1, \dots, y_m; \boldsymbol{\vartheta}) = C_Y(F_{Y^{(1)}}(t, y_1; \boldsymbol{\theta}_1), \dots, F_{Y^{(m)}}(t, y_m; \boldsymbol{\theta}_m); t, \tilde{\boldsymbol{\theta}})$ for all $t \in (0, \infty)$ and where $F_{Y^{(i)}}(t, y_i; \boldsymbol{\theta}_i)$ is the law associated to the i^{th} margin of (\mathbf{Y}_t) . When each $F_{Y^{(i)}}(t, y_i; \boldsymbol{\theta}_i)$ is continuous for $i = 1, \dots, m$, C_Y is unique. The copula

itself may be static, but we consider a time–inhomogeneous copula for generality. If $C(u_1, \dots, u_m; t, \tilde{\boldsymbol{\theta}}) = C_Y(u_1, \dots, u_m; t, \tilde{\boldsymbol{\theta}})$ and $F_j(t, y; \boldsymbol{\theta}_j) = F_{Y^{(j)}}(t, y; \boldsymbol{\theta}_j)$ for all $j = 1, \dots, m$, then $Z_t \stackrel{d}{=} Q_{\zeta}(F_{\mathbf{Y}}(t, Y_t^{(1)}, \dots, Y_t^{(m)}; \boldsymbol{\vartheta}); \boldsymbol{\xi})$. Here, we say that $C = C_Y$ is the *implicit* copula, determined by the joint distribution of the multivariate driving process. One may choose $C \neq C_Y$ to impose an alternative dependence structure between the margins of the multivariate driving process.

The multivariate quantile process is defined as follows, where we also employ a copula to describe the dependence structure between the quantile process margins.

Definition 3.4.2. *Let $Q_{\zeta_j}(u; \boldsymbol{\xi}_j)$ and $F_j(t, y; \boldsymbol{\theta}_j)$ be continuous quantile and distribution functions, respectively, as per Definition 3.1.1, for $j = 1, \dots, m$. Consider an \mathbb{R}^m –valued càdlàg process $(\mathbf{Y}_t)_{t \in [0, \infty)}$ where the joint law of the margins of the process is given by $F_{\mathbf{Y}}(t, y_1, \dots, y_m; \boldsymbol{\vartheta}) = C_Y(u_1, \dots, u_m; t, \tilde{\boldsymbol{\theta}})$ for $C_Y(u_1, \dots, u_m; t, \tilde{\boldsymbol{\theta}}) : \mathbb{R}^+ \times [0, 1]^m \rightarrow [0, 1]$ a copula and where $\tilde{\boldsymbol{\theta}} \in \mathbb{R}^{d''}$ is a d'' –dimensional vector of parameters. At each time $t \in [t_0, \infty)$, for $t_0 > 0$, the random–level, multivariate quantile process is given by the m –dimensional process $(\mathbf{Z}_t)_{t \in [t_0, \infty)}$ with each margin defined by*

$$Z_t^{(i)} \stackrel{d}{=} Q_{\zeta_i} \left(F_i \left(t, Y_t^{(i)}; \boldsymbol{\theta}_i \right); \boldsymbol{\xi}_i \right) \quad (3.4.3)$$

for $i = 1, \dots, m$. The joint law of the margins of the quantile process is given by $F_{\mathbf{Z}}(t, z_1, \dots, z_m) = C_Y(u_1, \dots, u_m; t, \tilde{\boldsymbol{\theta}})$ for any $t \in [t_0, \infty)$.

The fact that the joint law of the margins of the quantile process is described by the same copula that describes the joint law of the margins of the driving process follows from the fact that copulas are invariant under increasing transformations, see Nelsen (2007), and each margin transformation in Eq. (3.4.3) will be monotonically increasing.

In the subsequent chapters of this thesis, we focus largely on the analysis and application of univariate random–level quantile processes, with some attention placed on multidimensional random–level quantile processes, as given by Definition 3.4.1.

Chapter 4

Measure distortions induced by Quantile Processes

In this chapter, we define the distorted probability measures induced by random-level quantile processes. The construction of this class of quantile processes is motivated largely by the ability to flexibly build relative statistical properties into the process in a controlled manner, and so of particular interest are the measures (induced by these processes) that inherit such properties. For instance, if the composite map produces a heavy-tailed quantile process (i.e., with excess kurtosis over the driving process), then the distribution of some random variable or stochastic process under the induced measure will be more heavy-tailed than under the original measure. The concept of a probability distortion framework is well-explored in the literature, largely through the use of distortion operators, see e.g., Godin et al. (2012, 2019), Wang (1996, 2000, 2002) and Section 6.1, and predominantly in the context of economic decision making. The measure distortion framework we present in this chapter is distinct from that produced under distortion operators, despite the distorted probabilities arising from the application of nonlinear functions on the base measure in both cases. Additionally, we note that the framework presented herein extends the static case to produce dynamic distorted probability measures in continuous time, each characterised by the parametric form of the selected composite map and the underlying base process.

Consider Definition 3.1.1 for a random-level quantile process $(Z_t)_{t \in [t_0, \infty)}$ constructed from some D_Y -valued driving process $(Y_t)_{t \in [0, \infty)}$, where $D_Y \subseteq \mathbb{R}$ and where $F(t, y; \boldsymbol{\theta}) : [0, \infty) \times D_Y \rightarrow [0, 1]$ and $Q_\zeta(u; \boldsymbol{\xi}) : [0, 1] \rightarrow D_\zeta \subseteq D_Y$. It follows that (Z_t) is an (\mathcal{F}_t) -

adapted process on the measurable space $(D_\zeta, \mathcal{B}(D_\zeta))$ where $\mathcal{B}(D_\zeta) \subseteq \mathcal{F}$. We refer to Section 3.6 by Bogachev and Ruas (2007) to introduce the pushforward measure that defines the law of the quantile process, and subsequently its finite-dimensional distributions, as follows.

Definition 4.0.1. *Let \mathcal{D} denote the collection of all càdlàg functions from $[t_0, \infty)$ to D_ζ , that is, $t \mapsto Z_t(\omega)$ for all $t \in [t_0, \infty)$ and each $\omega \in \Omega$. The quantile process (Z_t) induces a measurable function $Z : \Omega \rightarrow \mathcal{D}$ where $(Z(\omega))(t) = Z_t(\omega)$. The law of the quantile process is defined to be the pushforward measure $\mathbb{P}^Z(A) := \mathbb{P}(Z^{-1}(A))$ for all $A \in \mathcal{B}(D_\zeta)$.*

Definition 4.0.2. *For $n \geq 1$, $t_0 \leq t_1 < \dots < t_n$ and all $A \in \mathcal{B}(D_\zeta^n)$, the finite-dimensional distributions of the quantile process (Z_t) are defined by the pushforward measures*

$$\mathbb{P}_{t_1, \dots, t_n}^Z(A) := \mathbb{P}\{(Z_{t_1}, \dots, Z_{t_n}) \in A\} = \int_{\{\omega \in \Omega : (Z_{t_1}(\omega), \dots, Z_{t_n}(\omega)) \in A\}} d\mathbb{P}(\omega). \quad (4.0.1)$$

Thus, for all $t \in [t_0, \infty)$, the ‘marginal distorted measure’ induced by the quantile process (Z_t) is given by Eq. (4.0.1) for $n = 1$, that is, $\mathbb{P}_t^Z(B)$ for all $B \in \mathcal{F}_t$. For $t_0 \leq s < t < \infty$, the ‘conditional distorted measure’ is defined as the restriction of \mathbb{P}_t^Z to the sub- σ -algebra \mathcal{F}_s . That is,

$$\mathbb{P}_{t|s}^Z(B) := \mathbb{P}_t^Z(B|\mathcal{F}_s) = \mathbb{P}\{Z_t \in B|\mathcal{F}_s\} = \int_{\{\omega \in \Omega : Z_t(\omega) \in B\}} d\mathbb{P}(\omega|\mathcal{F}_s) \quad (4.0.2)$$

for all $B \in \mathcal{F}_t$. It holds that $\mathbb{P}_t^Z = \mathbb{P}_{t|t_0}^Z$. By construction, if $B \in \mathcal{F}_t$ but $B \not\subseteq D_\zeta$, then $\mathbb{P}_{t|s}^Z(B) = 0$ for all $t_0 \leq s < t < \infty$.

The existence of \mathbb{P}^Z follows from the fact that each marginal distribution function induces a unique probability measure on the Borel σ -algebra $\mathcal{B}(D_\zeta)$. The mechanism that changes measure from \mathbb{P} to \mathbb{P}^Z is the quantile transformation (i.e., the composite map). In other words, for all $\omega \in \Omega$ and each $t \in [t_0, \infty)$, we view the probabilities associated to the distorted random variable $Z_t(\omega)$ under \mathbb{P} as those assigned to the random variable $Y_t(\omega)$ under \mathbb{P}^Z . The \mathbb{P}^Z -measure redistributes the probabilities in (Ω, \mathcal{F}) to account for properties, e.g., the skewness and kurtosis, we may factor in to the quantile transformation.

As an example, consider a driving process $Y_t = \mu t + W_t$ for $\mu \in \mathbb{R}$ and $Y_0 = y_0 \in \mathbb{R}$ and construct a Tukey– gh quantile process as per Definition 3.1.1 with $Q_\zeta = Q_{T_{gh}}$ given by Eq. (2.1.11) with $A = 0$, $B = 1$, and $F(t, y; \boldsymbol{\theta}) = F_Y(t, y; \mu) = \Phi((y - \mu t)/\sqrt{t})$ for all $t \in (0, \infty)$. Figures 4.1 and 4.2 illustrate how the finite–dimensional distribution (shown at time $t = 0.5$) of the driving process is distorted under the quantile transformation, to produce the finite–dimensional distribution of the quantile process, $F_Z(t, z; g, h, \mu)$ for all $t \in [t_0, \infty)$, or equivalently the finite–dimensional distribution of the driving process under the \mathbb{P}^Z –measure, for a range of g , h and μ parameters. Figure 4.3 shows the same result, considering the Tukey– gh quantile process construction again, with the same driving process, however now with $F(t, y; \lambda) = \Phi(x + \lambda)$ for Φ the CDF of the standard normal distribution and $\lambda \in \mathbb{R}$.

The emphasis we place here is that, by nature of the flexible quantile process construction, one has large levels of (directly parameterised) control over the properties of the measure induced by the quantile process.

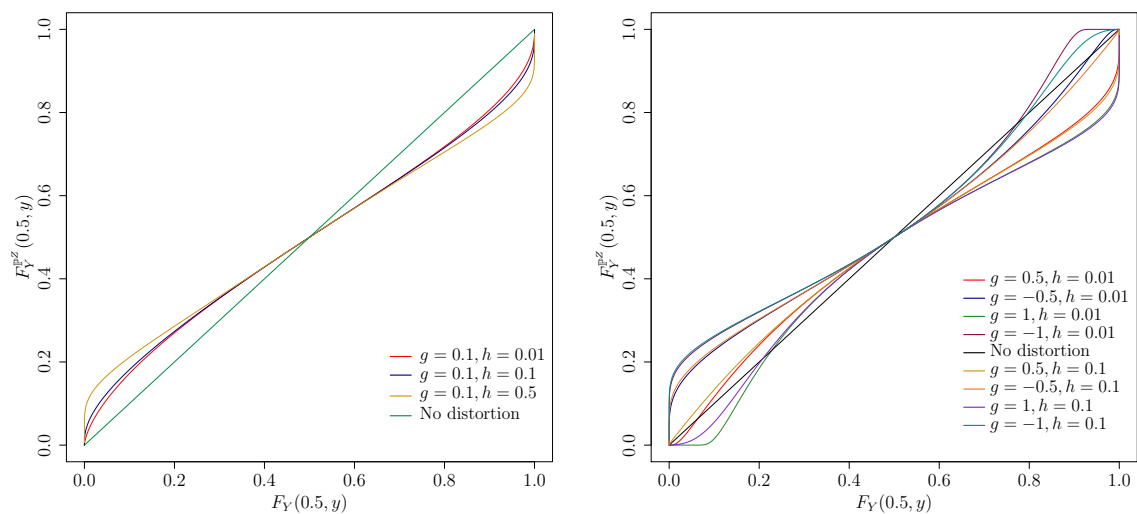


Figure 4.1: Relation between the marginal distribution functions of the driving process under the \mathbb{P} – and \mathbb{P}^Z –measures at $t = 0.5$ for $\mu = 0$ and a range of g and h parameters.

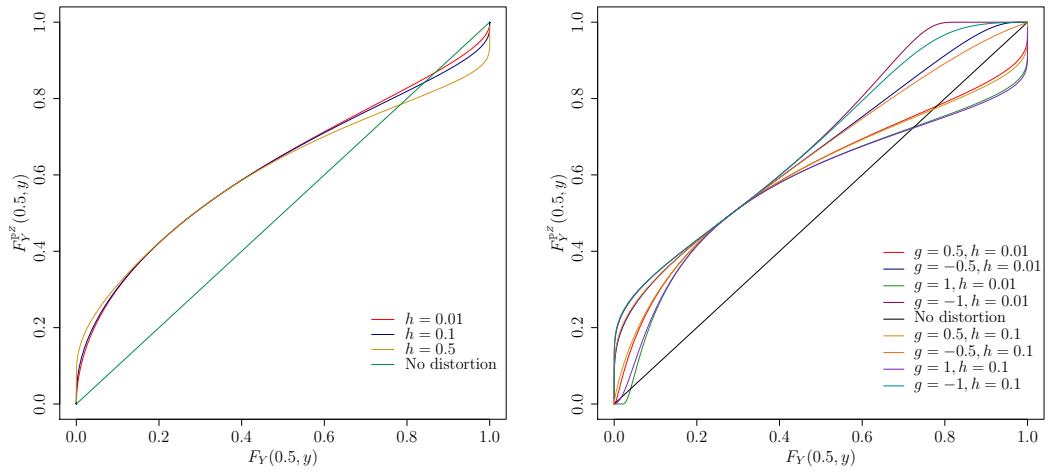


Figure 4.2: Relation between the marginal distribution functions of the driving process under the \mathbb{P} - and \mathbb{P}^Z -measures at $t = 0.5$ for $\mu = 0.8$ and a range of g and h parameters.

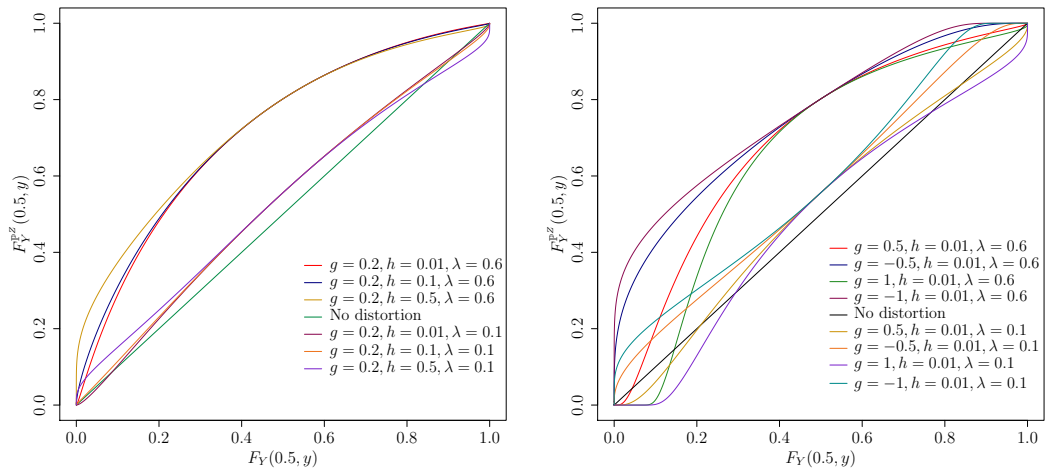


Figure 4.3: Relation between the marginal distribution functions of the driving process under the \mathbb{P} - and \mathbb{P}^Z -measures at $t = 0.5$ for $\mu = 0$ and a range of g , h and λ parameters.

4.1 Connection with the Radon–Nikodym derivative

Consider the distorted measure induced by the quantile process, given by Definition 4.0.2. The measure transform from \mathbb{P} to \mathbb{P}^Z implies a Radon–Nikodym derivative,

which follows from the assumption that $D_\zeta \subseteq D_Y$, and so \mathbb{P}^Z is absolutely continuous with respect to \mathbb{P} , i.e., $\mathbb{P}^Z \ll \mathbb{P}$. To consider the instance in which the measures \mathbb{P} and \mathbb{P}^Z are equivalent, that is, $\mathbb{P} \sim \mathbb{P}^Z$, we refer to e.g., Feller (2008), and define the sets

$$\begin{aligned}\mathcal{N}_{\mathbb{P}} &:= \{A \in \mathcal{F} \mid \mathbb{P}(A) = 0\}, \\ \mathcal{N}_{\mathbb{P}^Z} &:= \{A \in \mathcal{F} \mid \mathbb{P}^Z(A) = \mathbb{P}(Z^-(A)) = 0\}.\end{aligned}$$

Then, if $\mathcal{N}_{\mathbb{P}} = \mathcal{N}_{\mathbb{P}^Z}$, it holds that $\mathbb{P} \sim \mathbb{P}^Z$. For the application that follows in this thesis, we consider the less stringent case that $\mathbb{P}^Z \ll \mathbb{P}$, and the explication of the equivalence of the two probability measures is left for future work.

Now, let $\mathbb{P}|_{\mathcal{F}_t}$ denote the restriction of \mathbb{P} to the sub- σ -algebra \mathcal{F}_t , where $\mathbb{P}|_{\mathcal{F}_t}(A) = \mathbb{P}(A)$ for all $A \in \mathcal{F}_t$, and likewise $(\mathbb{P}|_{\mathcal{F}_t})|_{\mathcal{F}_s}$ the restriction of $\mathbb{P}|_{\mathcal{F}_t}$ to \mathcal{F}_s for all $t_0 \leq s < t < \infty$, where $(\mathbb{P}|_{\mathcal{F}_t})|_{\mathcal{F}_s}(A) = \mathbb{P}|_{\mathcal{F}_s}(A) = \mathbb{P}(A)$ for all $A \in \mathcal{F}_s$. By the same argument as above, for all $t \in [t_0, \infty)$, $\mathbb{P}_{t|s}^Z \ll \mathbb{P}|_{\mathcal{F}_t}$, and for all $t_0 \leq s < t < \infty$, $\mathbb{P}_{t|s}^Z \ll (\mathbb{P}|_{\mathcal{F}_t})|_{\mathcal{F}_s}$. As such, by the Radon–Nikodym theorem, there exists an \mathcal{F}_t -measurable function $\varrho_{t|s}(\omega) : \Omega \rightarrow \mathbb{R}_0^+$, such that

$$\mathbb{P}_{t|s}^Z(A) = \int_{\{\omega \in A\}} \varrho_{t|s}(\omega) d\mathbb{P}|_{\mathcal{F}_s}(\omega) \quad (4.1.1)$$

for all $A \in \mathcal{F}_t$.

By construction, it holds that $\mathbb{E}^{\mathbb{P}}[\varrho_{t|s}|\mathcal{F}_s] = 1$ for all $0 \leq s < t < \infty$, and for an \mathcal{F}_t -adapted random variable Y_t , we can write

$$\mathbb{E}^{\mathbb{P}^Z}[Y_t|\mathcal{F}_s] = \mathbb{E}^{\mathbb{P}_{t|s}^Z}[Y_t] = \mathbb{E}^{(\mathbb{P}|_{\mathcal{F}_t})|_{\mathcal{F}_s}}[\varrho_{t|s}Y_t] = \mathbb{E}^{\mathbb{P}}[\varrho_{t|s}Y_t|\mathcal{F}_s]. \quad (4.1.2)$$

We may now define the Radon–Nikodym derivative explicitly.

Definition 4.1.1. *Recall Definition 3.1.1 for a continuous quantile process $(Z_t)_{t \in [t_0, \infty)}$ with driving process $(Y_t)_{t \in [0, \infty)}$. Let $F_Y^{\mathbb{P}}$ denote the distribution associated to the driving process under the \mathbb{P} -measure. The Radon–Nikodym derivative in Eq. (4.1.1) is given by*

$$\varrho_{t|s}(\omega) = \frac{dF_Y^{\mathbb{P}}(t, Q(t, F_\zeta(Y_t(\omega)))|\mathcal{F}_s)}{dF_Y^{\mathbb{P}}(t, Y_t(\omega)|\mathcal{F}_s)} \quad (4.1.3)$$

for all $t_0 \leq s < t < \infty$ and all $\omega \in \Omega$, where the derivative is taken with respect to the

second argument, and so

$$\varrho_{t|s}(\omega) = \frac{f_Y^{\mathbb{P}}(t, Q(t, F_\zeta(Y_t(\omega))) | \mathcal{F}_s)}{f(t, Q(t, F_\zeta(Y_t(\omega))))} \frac{f_\zeta(Y_t(\omega))}{f_Y^{\mathbb{P}}(t, Y_t(\omega) | \mathcal{F}_s)}. \quad (4.1.4)$$

Assuming the existence of the derivatives $\partial_u Q_\zeta(u; \boldsymbol{\xi})$, $\partial_z F_\zeta(z; \boldsymbol{\xi}) = f_\zeta(z; \boldsymbol{\xi})$ and $\partial_y F(t, y; \boldsymbol{\theta}) = f(t, y; \boldsymbol{\theta})$, the (unconditional/ marginal) Radon–Nikodym derivative is given by

$$\varrho_t(\omega) = \frac{f_Y^{\mathbb{P}}(t, Q(t, F_\zeta(Y_t(\omega))))}{f(t, Q(t, F_\zeta(Y_t(\omega))))} \frac{f_\zeta(Y_t(\omega))}{f_Y^{\mathbb{P}}(t, Y_t(\omega))} \quad (4.1.5)$$

for $t_0 = s < t < \infty$ and for all $\omega \in \Omega$. The case where (Z_t) is a discrete quantile process is similar and one may consider the ratio of conditional probability mass functions of (Z_t) and (Y_t) under \mathbb{P} .

4.2 The multidimensional case

We conclude this chapter by considering the multidimensional random–level quantile process, given by Eq. (3.4.1). Whilst the driving process is multivariate, with dimension $m > 1$, the output quantile process is univariate by construction and so we may consider Definition 4.0.2 for the distorted measures induced by $(Z_t)_{t \in [t_0, \infty)}$. That is,

$$\mathbb{P}_t^Z(A) := \mathbb{P}\{Z_t \in A\} = \int_{\{\omega \in \Omega : Q_\zeta(C(t, F_1(t, Y_t^{(1)}(\omega)), \dots, F_m(t, Y_t^{(m)}(\omega)))) \in A\}} d\mathbb{P}(\omega) \quad (4.2.1)$$

for all $A \in \mathcal{F}_t$ and $t \in [t_0, \infty)$.

Consider the special case whereby $C(t, u_1, \dots, u_m; \tilde{\boldsymbol{\theta}}) = C_Y(t, u_1, \dots, u_m; \tilde{\boldsymbol{\theta}})$ and $F_j(t, y; \boldsymbol{\theta}_j) = F_{Y^{(j)}}(t, y; \boldsymbol{\theta}_j)$ for all $j = 1, \dots, m$, i.e., the implicit copula is used in the construction of the multidimensional quantile process—see the discussion following Definition 3.4.1. Here, we may simplify the expression for the probability measure induced by the quantile process, using the Kendall distribution function, defined as follows.

Definition 4.2.1. Consider Definition 3.4.1 where $Q_\zeta(u; \boldsymbol{\xi}) : [0, 1] \rightarrow D_\zeta \subseteq \mathbb{R}$ is a quantile function, and assume $F_j(t, y; \boldsymbol{\theta}_j) = F_{Y^{(j)}}(t, y; \boldsymbol{\theta}_j)$ for $j = 1, \dots, m$ is the distribution function associated to each margin of the multivariate driving process (\mathbf{Y}_t) so that $U_t^{(j)} \stackrel{d}{=} F_{Y^{(j)}}(t, Y_t^{(j)}; \boldsymbol{\theta}_j)$ is uniformly distributed on $[0, 1]$ for all $j = 1, \dots, m$.

For all $t \in [t_0, \infty)$, define the random variable

$$C_t \stackrel{d}{=} C_Y(t, F_{Y^{(1)}}(t, Y_t^{(1)}), \dots, F_{Y^{(m)}}(t, Y_t^{(m)})). \quad (4.2.2)$$

The distribution function of C_t for $t \in [t_0, \infty)$ is given by the Kendall distribution function $K_{C_Y}(t, v) : [t_0, \infty) \times [0, 1] \rightarrow [0, 1]$, where $K_{C_Y}(t, v) := \mathbb{P}(C_t \leq v)$ for all $v \in [0, 1]$.

The properties of the Kendall distribution function are discussed by Capéreaa et al. (1997), Genest and Rivest (1993, 2001) and Nelsen et al. (2009). It follows that for any $z \in D_\zeta$, the marginal distorted measure induced by the multidimensional quantile process is given by $\mathbb{P}_t^Z((\inf D_\zeta, z]) = K_{C_Y}(t, F_\zeta(z))$ for all $t \in [t_0, \infty)$. The distribution function $F_\zeta = Q_\zeta^-$ may be approximated if a closed-form or analytical expression for it does not exist. In this context, the Radon–Nikodym derivative in Eq. (4.1.1), where $\mathbb{P}_{t|s}^Z$ is the restriction of the measure \mathbb{P}^Z in Eq. (4.2.1) to the sub- σ -algebra \mathcal{F}_s , is given by

$$\varrho_{t|s}(\omega) = \frac{dK_C(t, Q_\zeta(Y_t(\omega)) | \mathcal{F}_s)}{dF_Y^{\mathbb{P}}(t, Y_t(\omega) | \mathcal{F}_s)} \quad (4.2.3)$$

for all $0 \leq s < t < \infty$ and all $\omega \in \Omega$.

Chapter 5

Stochastic ordering

In this chapter we consider random-level quantile processes, as given by Definition 3.1.1, and derive the conditions under which the composite map preserves stochastic ordering of the driving processes to produce a stochastic ordering of the output quantile processes. We consider first- and second-order stochastic dominance.

The stochastic ordering of risks plays an important role in decision making, risk measures and valuation frameworks. In decision theory—specifically, expected utility theory—the maximisation of von Neumann–Morgenstern, see Morgenstern and Von Neumann (1953), non-decreasing (resp. non-decreasing and concave) utility functions by rational market participants equates to the notion of first-order (resp. second-order) stochastic dominance. In the context of risk measures, if one risk is preferred to another under stochastic dominance (of some order), the risk measure for the preferred risk should be less than that of the other risk if the risk measure considered preserves stochastic ordering (of that order). The monotonicity of risk measures with respect to stochastic orderings has been considered, for example by Bäuerle and Müller (2006), De Giorgi (2005), de Vries et al. (2006), Goovaerts et al. (2004), Pflug (1999) and Wirch and Hardy (2001), and the references therein, largely in the characterisation of *consistent* risk measures. Drawing these together, valuation frameworks with preserved stochastic ordering (of some order) imply monotonic behaviour between the level of risk and the price associated to some contract dependent on the risk, e.g., an insurance premium. That is, assuming investors act rationally, it is desirable for valuation frameworks to preserve the stochastic dominance of risks. The results presented in this chapter thus lead to the establishment of desirable properties in the novel valuation

principle developed in the following chapter.

We begin with recalling the notion of stochastic ordering, as presented by Levy (1992), adapted to our context of quantile processes, given by Definition 3.1.1. We consider the compact supports $[z_i, \bar{z}_i] \subseteq \mathbb{R}$ and $[y_i, \bar{y}_i] \subseteq \mathbb{R}$ where $\underline{z}_i, \bar{z}_i \in \mathbb{R}$ with $\underline{z}_i < \bar{z}_i$ and $\underline{y}_i, \bar{y}_i \in \mathbb{R}$ with $\underline{y}_i < \bar{y}_i$, for $i = 1, 2$. We consider the quantile processes marginally on the timeline $[t_0, \infty)$, $t_0 > 0$, allowing us to apply static stochastic ordering results to the dynamic setting. That is, we wish to observe when families of quantile processes satisfy stochastic ordering results at all fixed times $t \in [t_0, \infty)$. In what follows, we drop the notational dependence of distribution and quantile functions on the vectors of parameters.

Definition 5.0.1. *Recall Definition 3.1.1, where $Q_{\zeta_i}(u) : [0, 1] \rightarrow [z_i, \bar{z}_i]$ are quantile functions and $F_i(t, y) : (0, \infty) \times [y_i, \bar{y}_i] \rightarrow [0, 1]$ are distribution functions for $i = 1, 2$. Consider the quantile processes $Z_t^{(i)} \stackrel{d}{=} Q_{\zeta_i}(F_i(t, Y_t^{(i)}))$ with marginal distributions $F_{Z^{(i)}}(t, z_i) = \mathbb{P}(Z_t^{(i)} \leq z_i)$, for $z_i \in [z_i, \bar{z}_i]$. We say that $(Z_t^{(1)})_{t \in [t_0, \infty)}$ dominates $(Z_t^{(2)})_{t \in [t_0, \infty)}$ by first-, or second-order stochastic dominance on*

$D_\zeta := [z_0(t), \max\{\bar{z}_1, \bar{z}_2\}]$, where $z_0(t) \in [\min\{z_1, z_2\}, \max\{\bar{z}_1, \bar{z}_2\})$ with $F_{Z^{(1)}}(t, z_0(t)) = F_{Z^{(2)}}(t, z_0(t))$, if, and only if, for all $t \in [t_0, \infty)$ the following hold, respectively:

$$\text{FOSD: } F_{Z^{(2)}}(t, z) - F_{Z^{(1)}}(t, z) \geq 0, \text{ for all } z \in D_\zeta,$$

$$\text{SOSD: } \int_{z_0(t)}^z [F_{Z^{(2)}}(t, x) - F_{Z^{(1)}}(t, x)] dx \geq 0, \text{ for all } z \in D_\zeta.$$

In either stochastic dominance criterion, strict inequality is required for at least one $z \in D_\zeta$ and all $t \in [t_0, \infty)$.

If $(Z_t^{(1)})$ dominates $(Z_t^{(2)})$ by first-order (resp. second-order) stochastic dominance on D_ζ , we write $Z_t^{(1)} \succ_{\text{FOSD}} Z_t^{(2)}$ (resp. $Z_t^{(1)} \succ_{\text{SOSD}} Z_t^{(2)}$) on D_ζ .

The relation between risk preferences via utility functions with the FOSD and SOSD criterion are given, with proofs, by Hadar and Russell (1969), Hanoch and Levy (1969) and Rothschild and Stiglitz (1970).

5.1 First-order stochastic dominance of univariate quantile processes

We begin by considering two distinct quantile processes, constructed by the transformation of two driving processes such that one driving process dominates the other in the first order. The conditions under which the pair of composite maps preserve the FOSD of the driving processes are given in the following Proposition, where the most general case is considered. Here, the quantile and distribution functions used in the two composite maps are distinct, and the distribution function is not that associated to the law of the driving process, i.e., $F_i \neq F_{Y^{(i)}}$. In the following two Corollaries, we consider the case where $F_i = F_{Y^{(i)}}$ for both composite maps, and then that where the two driving processes used to construct the two quantile processes are distributionally indistinct. This covers all possible ‘types’ of random-level quantile process constructions.

Proposition 5.1.1. *Consider $(Z_t^{(i)})_{t \in [t_0, \infty)}$, $i = 1, 2$, in Definition 5.0.1. Define*

$$y_0(t) := \{y_0 \in [\min\{\underline{y}_1, \underline{y}_2\}, \max\{\bar{y}_1, \bar{y}_2\}] : F_{Y^{(1)}}(t, y_0) = F_{Y^{(2)}}(t, y_0)\} \quad (5.1.1)$$

and assume $Y_t^{(1)} \succsim_{FOSD} Y_t^{(2)}$ on $D_Y := [y_0(t), \max\{\bar{y}_1, \bar{y}_2\}]$ for all $t \in [t_0, \infty)$. It holds that $Z_t^{(1)} \succsim_{FOSD} Z_t^{(2)}$ on $D_\zeta := [z_0(t), \max\{\bar{z}_1, \bar{z}_2\}]$, where

$$z_0(t) := \{\max z_0 \in \{[\underline{z}_1, \bar{z}_1] \cup [\underline{z}_2, \bar{z}_2]\} : z_0 \leq Q_{\zeta_2}(F_2(t, y_0(t))) \text{ \& } F_{Z^{(1)}}(t, z_0) = F_{Z^{(2)}}(t, z_0)\}, \quad (5.1.2)$$

if $Q_{\zeta_1}(F_1(t, y)) \geq Q_{\zeta_2}(F_2(t, y))$ for all $y \in D_Y$ and $t \in [t_0, \infty)$. That is, first-order stochastic dominance is preserved under the pair of composite maps at each time.

Proof. Since, by assumption, $Y_t^{(1)} \succsim_{FOSD} Y_t^{(2)}$ on D_Y , by the definition of FOSD it holds that $F_{Y^{(1)}}(t, y) \leq F_{Y^{(2)}}(t, y)$ for all $y \in D_Y$ and $t \in [t_0, \infty)$ with strict inequality for at least one $y \in D_Y$. The quantile processes are constructed as $Z_t^{(i)} \stackrel{d}{=} Q_{\zeta_i}(F_i(t, Y_t^{(i)}))$ for $i = 1, 2$, and so we may write $Y_t^{(i)} \stackrel{d}{=} Q_i(t, F_{\zeta_i}(Z_t^{(i)}))$. It follows that

$$\begin{aligned} F_{Y^{(i)}}(t, y) &= \mathbb{P}\left(Y_t^{(i)} \leq y\right) = \mathbb{P}\left(Q_i\left(t, F_{\zeta_i}\left(Z_t^{(i)}\right)\right) \leq y\right) \\ &= \mathbb{P}\left(Z_t^{(i)} \leq Q_{\zeta_i}\left(F_i(t, y)\right)\right) = F_{Z^{(i)}}\left(t, Q_{\zeta_i}\left(F_i(t, y)\right)\right) \end{aligned}$$

for all $y \in D_Y$, $t \in [t_0, \infty)$ and so, by assumption,

$$F_{Z^{(1)}}(t, Q_{\zeta_1}(F_1(t, y))) \leq F_{Z^{(2)}}(t, Q_{\zeta_2}(F_2(t, y))) \quad (5.1.3)$$

for all $y \in D_Y$, $t \in [t_0, \infty)$. Eq. (5.1.3) can be written as $F_{Z^{(1)}}(t, z_1(y)) \leq F_{Z^{(2)}}(t, z_2(y))$ where $z_i(y) = Q_{\zeta_i}(F_i(t, y))$, for $y \in D_Y$ and $i = 1, 2$. However, we need to show that $F_{Z^{(1)}}(t, z) \leq F_{Z^{(2)}}(t, z)$ for all $z \in D_\zeta$, for some $D_\zeta \subseteq [\min\{\underline{z}_1, \underline{z}_2\}, \max\{\bar{z}_1, \bar{z}_2\}]$, to be determined. We consider the following cases: First, we assume that $z_1(y) \geq z_2(y)$. Since $F_{Z^{(i)}}(t, z)$, $i = 1, 2$, are increasing functions in z for all $t \in [t_0, \infty)$, it holds that

$$F_{Z^{(1)}}(t, z_2(y)) \leq F_{Z^{(1)}}(t, z_1(y)) \leq F_{Z^{(2)}}(t, z_2(y)) \leq F_{Z^{(2)}}(t, z_1(y)), \quad (5.1.4)$$

so that $F_{Z^{(1)}}(t, z_2(y)) \leq F_{Z^{(2)}}(t, z_2(y))$ and $F_{Z^{(1)}}(t, z_1(y)) \leq F_{Z^{(2)}}(t, z_1(y))$ for all $y \in D_Y$ and $t \in [t_0, \infty)$. It then follows that $F_{Z^{(1)}}(t, z) \leq F_{Z^{(2)}}(t, z)$ for all

$$z \in [\min_i Q_{\zeta_i}(F_i(t, y_0(t))), \max\{\bar{z}_1, \bar{z}_2\}] = [Q_{\zeta_2}(F_2(t, y_0(t))), \max\{\bar{z}_1, \bar{z}_2\}], \quad (5.1.5)$$

with strict inequality for at least the values $z = z_i(y^*)$, $i = 1, 2$, where $y^* \in D_Y$ is any value such that $F_{Y^{(1)}}(t, y^*) < F_{Y^{(2)}}(t, y^*)$, and hence $Z_t^{(1)} \succ_{FOSD} Z_t^{(2)}$ on $D_\zeta \subseteq \mathbb{R}$ for $t \in [t_0, \infty)$. □

The situation considered next is that in which $F_i = F_{Y^{(i)}}$ is the distribution function applied in the composite map, where $F_{Y^{(i)}}$ is the marginal distribution of each driving process at $t \in [t_0, \infty)$ for $i = 1, 2$. Here, we construct the quantile processes

$$Z_t^{(i)} \stackrel{d}{=} Q_{\zeta_i}(F_{Y^{(i)}}(t, Y_t^{(i)})) \stackrel{d}{=} Q_{\zeta_i}(U_t^{(i)}) \quad (5.1.6)$$

where $U_t^{(i)}$ is uniformly distributed on $[0, 1]$ for each $t \in [t_0, \infty)$ and $i = 1, 2$.

Corollary 5.1.1. *Consider the case where $F_i(t, y_i) = F_{Y^{(i)}}(t, y_i)$ for $i = 1, 2$, and all $y_i \in [y_i, \bar{y}_i]$, $t \in (0, \infty)$. Assume $Y_t^{(1)} \succ_{FOSD} Y_t^{(2)}$ on D_Y where D_Y is defined as in Proposition 5.1.1. It holds that $Z_t^{(1)} \succ_{FOSD} Z_t^{(2)}$ on D_ζ , defined as in Proposition 5.1.1, for all $t \in [t_0, \infty)$, if $Q_{\zeta_1}(u) \geq Q_{\zeta_2}(u)$ for all $u \in [F_{Y^{(1)}}(t, y_0(t)), 1]$ and $t \in [t_0, \infty)$.*

Proof. Since, by assumption, $Y_t^{(1)} \succ_{FOSD} Y_t^{(2)}$ on D_Y for each $t \in (0, \infty)$, by the definition of FOSD it holds that $F_{Y^{(1)}}(t, y) \leq F_{Y^{(2)}}(t, y)$ for all $y \in D_Y$ and $t \in [t_0, \infty)$

with strict inequality for at least one $y \in D_Y$. The quantile processes are constructed as $Z_t^{(i)} \stackrel{d}{=} Q_{\zeta_i}(F_{Y^{(i)}}(t, Y_t^{(i)}))$ for $i = 1, 2$, and so we may write $Y_t^{(i)} \stackrel{d}{=} Q_{Y^{(i)}}(t, F_{\zeta_i}(Z_t^{(i)}))$. It follows that

$$\begin{aligned} F_{Y^{(i)}}(t, y) &= \mathbb{P}\left(Y_t^{(i)} \leq y\right) = \mathbb{P}\left(Q_{Y^{(i)}}\left(t, F_{\zeta_i}\left(Z_t^{(i)}\right)\right) \leq y\right) \\ &= \mathbb{P}\left(Z_t^{(i)} \leq Q_{\zeta_i}\left(F_{Y^{(i)}}(t, y)\right)\right) = F_{Z^{(i)}}\left(t, Q_{\zeta_i}\left(F_{Y^{(i)}}(t, y)\right)\right) \end{aligned}$$

for all $y \in D_Y$, $t \in [t_0, \infty)$ and so, by assumption,

$$F_{Z^{(1)}}\left(t, Q_{\zeta_1}\left(F_{Y^{(1)}}(t, y)\right)\right) \leq F_{Z^{(2)}}\left(t, Q_{\zeta_2}\left(F_{Y^{(2)}}(t, y)\right)\right) \quad (5.1.7)$$

for all $y \in D_Y$, $t \in [t_0, \infty)$. By the same argument as the proof to Proposition 5.1.1, under the assumption that the inequality 5.1.7 holds, $F_{Z^{(1)}}(t, z) \leq F_{Z^{(2)}}(t, z)$ for all $z \in D_\zeta$, with strict inequality for at least one $z \in D_\zeta$ whenever $Q_{\zeta_1}(F_{Y^{(1)}}(t, y)) \geq Q_{\zeta_2}(F_{Y^{(2)}}(t, y))$ for all $y \in D_Y$, $t \in [t_0, \infty)$.

Since $u_1(y) := F_{Y^{(1)}}(t, y) \leq F_{Y^{(2)}}(t, y) =: u_2(y)$ for all $y \in D_Y$, $t \in [t_0, \infty)$, by the increasing property of quantile functions, it holds that

$$Q_{\zeta_1}(u_2(y)) \geq Q_{\zeta_1}(u_1(y)) \geq Q_{\zeta_2}(u_2(y)) \quad (5.1.8)$$

so that $Q_{\zeta_1}(u_2(y)) \geq Q_{\zeta_2}(u_2(y))$, and that

$$Q_{\zeta_1}(u_1(y)) \geq Q_{\zeta_2}(u_2(y)) \geq Q_{\zeta_2}(u_1(y)) \quad (5.1.9)$$

so that $Q_{\zeta_1}(u_1(y)) \geq Q_{\zeta_2}(u_1(y))$ for all $y \in D_Y$ and $t \in [t_0, \infty)$. As such, the condition $Q_{\zeta_1}(F_{Y^{(1)}}(t, y)) \geq Q_{\zeta_2}(F_{Y^{(2)}}(t, y))$ reduces to $Q_{\zeta_1}(u) \geq Q_{\zeta_2}(u)$ for all $u \in [F_{Y^{(1)}}(t, y_0(t)), 1]$, as required. \square

Now we present the corollary for the case where the driving processes are equal in finite-dimensional distributions.

Corollary 5.1.2. *Assume $Y_t^{(1)} \stackrel{d}{=} Y_t^{(2)}$, i.e., $F_{Y^{(1)}}(t, y) = F_{Y^{(2)}}(t, y)$ for all $y \in [\min\{\underline{y}_1, \underline{y}_2\}, \max\{\bar{y}_1, \bar{y}_2\}]$ and $t \in (0, \infty)$. It holds that $Z_t^{(1)} \succ_{FOSD} Z_t^{(2)}$ on $\tilde{D}_Z := [\tilde{z}_0(t), \max\{\bar{z}_1, \bar{z}_2\}]$, if $Q_{\zeta_1}(F_1(t, y)) \geq Q_{\zeta_2}(F_2(t, y))$ for all $y \in \tilde{D}_Y := [\tilde{y}_0(t), \max\{\bar{y}_1, \bar{y}_2\}]$, for some $\tilde{y}_0(t) \in [\min\{\underline{y}_1, \underline{y}_2\}, \max\{\bar{y}_1, \bar{y}_2\})$, with strict inequality for at least one*

$y \in \tilde{D}_Y$, where

$$\begin{aligned} \tilde{z}_0(t) := \{ \max z_0 \in \{ [z_1, \bar{z}_1] \cup [z_2, \bar{z}_2] \} : z_0 \leq Q_{\zeta_2}(F_2(t, \tilde{y}_0(t))) \\ \& F_{Z^{(1)}}(t, z_0) = F_{Z^{(2)}}(t, z_0) \} \end{aligned} \quad (5.1.10)$$

for all $t \in [t_0, \infty)$. However, in the case that $F_i(t, y) = F_{Y^{(i)}}(t, y)$ for $i = 1, 2$, for all $y \in D_Y$, and $t \in (0, \infty)$, then $Z_t^{(1)} \succsim_{FOSD} Z_t^{(2)}$ on $\bar{D}_Z := [z_0, \max\{\bar{z}_1, \bar{z}_2\}]$ where

$$z_0 \in [\min\{z_1, z_2\}, \max\{\bar{z}_1, \bar{z}_2\}] \quad (5.1.11)$$

with $F_{\zeta_1}(z_0) = F_{\zeta_2}(z_0)$ for all $t \in [t_0, \infty)$ if $Q_{\zeta_1}(u) \geq Q_{\zeta_2}(u)$ for all $u \in [F_{\zeta_1}(z_0), 1]$, with strict inequality for at least one u .

Proof. The proof is similar to that of Proposition 5.1.1. However, to ensure

$$F_{Z^{(1)}}(t, z) < F_{Z^{(2)}}(t, z) \quad (5.1.12)$$

for at least one $z \in \tilde{D}_Z$ and all $t \in [t_0, \infty)$, see Definition 5.0.1 of FOSSD, $Q_{\zeta_1}(F_1(t, y)) > Q_{\zeta_2}(F_2(t, y))$ is required for at least one $y \in \tilde{D}_Y$. This follows from the fact that there does not exist some $y \in [\min\{y_1, y_2\}, \max\{\bar{y}_1, \bar{y}_2\}]$ such that $F_{Y^{(1)}}(t, y) < F_{Y^{(2)}}(t, y)$ to impose Inequality 5.1.12 whenever $Q_{\zeta_1}(F_1(t, y)) \geq Q_{\zeta_2}(F_2(t, y))$ without strict inequality for at least one $y \in \tilde{D}_Y$ holding almost surely.

In the case where $F_i(t, y) = F_{Y^{(i)}}(t, y)$, $i = 1, 2$, it holds that

$$F_{Z^{(i)}}(t, z) = \mathbb{P} \left(Z_t^{(i)} \leq z \right) = \mathbb{P} \left(Y_t^{(i)} \leq Q_{Y^{(i)}}(t, F_{\zeta_i}(z)) \right) = F_{\zeta_i}(z), \quad (5.1.13)$$

i.e., the quantile process is stationary. Thus, by the definition of FOSSD, $Z_t^{(1)} \succsim_{FOSSD} Z_t^{(2)}$ on $\bar{D}_Z := [z_0, \max\{\bar{z}_1, \bar{z}_2\}]$ when $F_{\zeta_1}(z) \leq F_{\zeta_2}(z)$ for all $z \in \bar{D}_Z$ with strict inequality for at least one z and where $F_{\zeta_1}(z_0) = F_{\zeta_2}(z_0)$. This is equivalent to $Q_{\zeta_1}(u) \geq Q_{\zeta_2}(u)$ for all $u \in [F_{\zeta_1}(z_0), 1]$ with strict inequality for at least one u . \square

This concludes the results for first-order stochastic dominance. The following example applies the result given in Corollary 5.1.2 for canonical Tukey- gh quantile processes, given by Definition 3.1.2 with quantile function $Q_\zeta = Q_{T_{gh}}$ given by Eq. (2.1.11).

Example 5.1.1. Let $(W_t^{(i)})_{t \in [0, \infty)}$ for $i = 1, 2$ be two independent Brownian motions and consider the canonical Tukey- gh quantile processes

$$Z_t^{(i)} \stackrel{d}{=} Q_{T_{gh,i}} \left(F_W \left(t, W_t^{(i)} \right); g_i, h_i \right) \stackrel{d}{=} \frac{1}{g_i} \left[\exp \left(\frac{g_i W_t^{(i)}}{\sqrt{t}} \right) - 1 \right] \exp \left(\frac{h}{2t} \left(W_t^{(i)} \right)^2 \right) \quad (5.1.14)$$

for $i = 1, 2$, $g_i \in \mathbb{R} \setminus 0$, $h_i \in \mathbb{R}_0^+$ and all $t \in [t_0, \infty)$. The location and scale parameters A_i and B_i , respectively, are standardised. As the (indistinct) distribution functions of the driving processes (Brownian motions) are used in the composite maps producing such quantile processes, we consider Corollary 5.1.2 with $F_i(t, y) = F_{W^{(i)}}(t, y)$ for all $y \in \mathbb{R}$. We consider the following cases:

- (i) $g_1 > g_2$, $h_1 = h_2$. For fixed $h \in \mathbb{R}_0^+$, there is monotonicity with respect to the projection $g \mapsto Q_{T_{gh}}(u; g, h)$ since $\partial Q_{T_{gh}}(u; g, h) / \partial g \geq 0$ for all $g \in \mathbb{R} \setminus 0$. It follows that $Z_t^{(1)} \succsim_{FOSD} Z_t^{(2)}$ on $D_{T_{gh}} = \mathbb{R}$ for $t \in [t_0, \infty)$.
- (ii) $g_1 = g_2$, $h_1 > h_2$. Here, $Q_{T_{gh,1}}(u; g_1, h_1) \geq Q_{T_{gh,2}}(u; g_2, h_2)$ for all $u \in [0.5, 1]$ with equality at $u = 0.5$ and so $Z_t^{(1)} \succsim_{FOSD} Z_t^{(2)}$ on $[Q_{T_{gh,1}}(0.5), 1]$.
- (iii) $g_1 > g_2$, $h_1 > h_2$. Then there exists some $0 \leq u^* < 0.5$ such that $Z_t^{(1)} \succsim_{FOSD} Z_t^{(2)}$ on $[Q_{T_{gh,1}}(u^*), 1]$. Here, $Q_{T_{gh,1}}(u^*) = Q_{T_{gh,2}}(u^*)$ and $Q_{T_{gh,1}}(u) > Q_{T_{gh,2}}(u)$ for all $u \in (u^*, 1]$. The larger $g_1 - g_2$, or the smaller $h_1 - h_2$, the closer u^* is to 0. When $u^* = 0$, $Z_t^{(1)} \succsim_{FOSD} Z_t^{(2)}$ on \mathbb{R} . The rate of change in u^* as $g_1 - g_2$ increases for fixed values of $h_1 - h_2$ is illustrated in Figure 5.1.
- (iv) $g_1 > g_2$, $h_1 < h_2$. Then there exists some $0.5 < u^* \leq 1$ such that $Z_t^{(2)} \succsim_{FOSD} Z_t^{(1)}$ on $[Q_{T_{gh,1}}(u^*), 1]$. Here, $Q_{T_{gh,1}}(u^*) = Q_{T_{gh,2}}(u^*)$ and $Q_{T_{gh,2}}(u) > Q_{T_{gh,1}}(u)$ for all $u \in (u^*, 1]$. The larger $g_1 - g_2$, or the smaller $h_2 - h_1$, the closer u^* is to 1. When $u^* = 1$, $Z_t^{(1)} \succsim_{FOSD} Z_t^{(2)}$ on \mathbb{R} .

As an example, cases (i)–(iv) are illustrated for values of the g and h parameters in Table 5.1.

g_1	g_2	h_1	h_2	u^*	Outcome
2	0.8	0.4	0.05	0.0218	$Z_t^{(1)} \succsim_{FOSD} Z_t^{(2)}$ on $[-1.109, \infty)$
3	0.5	0.2	0.05	0	$Z_t^{(1)} \succsim_{FOSD} Z_t^{(2)}$ on \mathbb{R}
2	0.8	0.05	0.4	1	$Z_t^{(1)} \succsim_{FOSD} Z_t^{(2)}$ on \mathbb{R}
2	1.5	0.05	0.2	0.985	$Z_t^{(2)} \succsim_{FOSD} Z_t^{(1)}$ on $[42.36, \infty)$

Table 5.1: FOSD results for Tukey- gh quantile processes for different values of the skewness and kurtosis parameters.

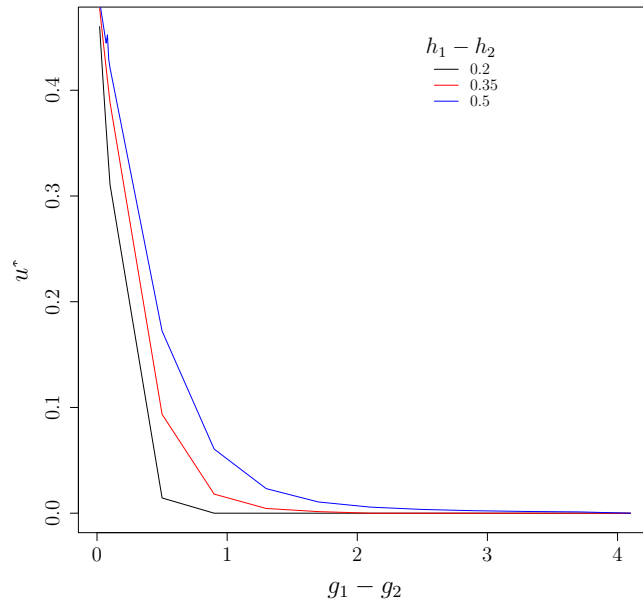


Figure 5.1: Values of u^* as $g_1 - g_2$ is increased for $g_2 = 0.1$ fixed, and $h_1 - h_2 \in \{0.2, 0.35, 0.5\}$ with $h_2 = 0.05$ fixed.

5.2 Second-order stochastic dominance of univariate quantile processes

We now proceed to show under what conditions the composite maps used in the construction of two quantile processes preserve the SOSD of the driving processes. The same cases considered in Section 5.1 are considered in this section. We use the shorthand notation $\partial_z \equiv \partial/\partial z$.

Proposition 5.2.1. Consider $(Z_t^{(i)})_{t \in [t_0, \infty)}$, $i = 1, 2$, in Definition 5.0.1. Define

$$y_0(t) := \left\{ y_0 \in [\min\{\underline{y}_1, \underline{y}_2\}, \max\{\bar{y}_1, \bar{y}_2\}] : F_{Y^{(1)}}(t, y_0) = F_{Y^{(2)}}(t, y_0) \right\} \quad (5.2.1)$$

and assume $Y_t^{(1)} \succ_{SOSD} Y_t^{(2)}$ on $D_Y := [y_0(t), \max\{\bar{y}_1, \bar{y}_2\}]$ for all $t \in [t_0, \infty)$. It holds that $Z_t^{(1)} \succ_{SOSD} Z_t^{(2)}$ on $D_\zeta := [z_0(t), \max\{\bar{z}_1, \bar{z}_2\}]$ where $z_0(t) := \min_i Q_{\zeta_i}(F_i(t, y_0(t)))$, for $i = 1, 2$, and for all $t \in [t_0, \infty)$, if any of the following conditions are satisfied for all $z \in D_\zeta$ and $t \in [t_0, \infty)$:

$$(i) \quad \partial_z Q_2(t, F_{\zeta_2}(z)) \leq 1 \leq \partial_z Q_1(t, F_{\zeta_1}(z)).$$

$$(ii) \quad \partial_z Q_2(t, F_{\zeta_2}(z)) \geq 1, \quad \partial_z Q_1(t, F_{\zeta_1}(z)) \geq 1, \quad \text{and} \quad \frac{F_{Z^{(2)}}(t, z)}{F_{Z^{(1)}}(t, z)} \leq \frac{\partial_z Q_1(t, F_{\zeta_1}(z)) - 1}{\partial_z Q_2(t, F_{\zeta_2}(z)) - 1}.$$

$$(iii) \quad \partial_z Q_2(t, F_{\zeta_2}(z)) \leq 1, \quad \partial_z Q_1(t, F_{\zeta_1}(z)) \leq 1, \quad \text{and} \quad \frac{F_{Z^{(2)}}(t, z)}{F_{Z^{(1)}}(t, z)} \geq \frac{1 - \partial_z Q_1(t, F_{\zeta_1}(z))}{1 - \partial_z Q_2(t, F_{\zeta_2}(z))}.$$

Proof. Since, by assumption, $Y_t^{(1)} \succ_{SOSD} Y_t^{(2)}$ for all $t \in (0, \infty)$, by the definition of SOSD it holds that $\int_{y_0(t)}^y [F_{Y^{(2)}}(t, x) - F_{Y^{(1)}}(t, x)] dx \geq 0$ for all $y \in D_Y$ and $t \in [t_0, \infty)$, with strict inequality for at least one $y \in D_Y$. The quantile processes are constructed as $Z_t^{(i)} \stackrel{d}{=} Q_{\zeta_i}(F_i(t, Y_t^{(i)}))$ for $i = 1, 2$, and so we may write $Y_t^{(i)} \stackrel{d}{=} Q_i(t, F_{\zeta_i}(Z_t^{(i)}))$. It follows that

$$\begin{aligned} F_{Y^{(i)}}(t, y) &= \mathbb{P}\left(Y_t^{(i)} \leq y\right) = \mathbb{P}\left(Q_i\left(t, F_{\zeta_i}\left(Z_t^{(i)}\right)\right) \leq y\right) \\ &= \mathbb{P}\left(Z_t^{(i)} \leq Q_{\zeta_i}\left(F_i(t, y)\right)\right) = F_{Z^{(i)}}\left(t, Q_{\zeta_i}\left(F_i(t, y)\right)\right) \end{aligned} \quad (5.2.2)$$

for all $y \in D_Y$, $t \in [t_0, \infty)$ and so, by assumption,

$$\int_{y_0(t)}^y [F_{Z^{(2)}}(t, Q_{\zeta_2}(F_2(t, x))) - F_{Z^{(1)}}(t, Q_{\zeta_1}(F_1(t, x)))] dx \geq 0 \quad (5.2.3)$$

for all $y \in D_Y$ and $t \in [t_0, \infty)$, with strict inequality for at least one $y \in D_Y$. By making the changes of variables $v_i := Q_{\zeta_i}(F_i(t, x))$ for $i = 1, 2$ so that $dv_i/dx = f_i(t, x)/f_{\zeta_i}(Q_{\zeta_i}(F_i(t, x)))$, Eq. (5.2.3) can be rewritten as

$$\begin{aligned} &\int_{Q_{\zeta_2}(F_2(t, y_0(t)))}^{Q_{\zeta_2}(F_2(t, y))} F_{Z^{(2)}}(t, v_2) \frac{f_{\zeta_2}(v_2)}{f_2(t, Q_2(t, F_{\zeta_2}(v_2)))} dv_2 \\ &\quad - \int_{Q_{\zeta_1}(F_1(t, y_0(t)))}^{Q_{\zeta_1}(F_1(t, y))} F_{Z^{(1)}}(t, v_1) \frac{f_{\zeta_1}(v_1)}{f_1(t, Q_1(t, F_{\zeta_1}(v_1)))} dv_1 \geq 0. \end{aligned} \quad (5.2.4)$$

If $Q_{\zeta_1}(F_1(t, y)) \geq Q_{\zeta_2}(F_2(t, y))$ for all $y \in D_Y$ and $t \in [t_0, \infty)$, then by Proposition 5.1.1, $Z_t^{(1)} \succ_{FOSD} Z_t^{(2)}$ which implies $Z_t^{(1)} \succ_{SOSD} Z_t^{(2)}$. We consider the case where $Q_{\zeta_1}(F_1(t, y)) \leq Q_{\zeta_2}(F_2(t, y))$ for at least one $y \in D_Y$. It holds that

$$\begin{aligned} 0 \leq & \int_{Q_{\zeta_2}(F_2(t, y_0(t)))}^{Q_{\zeta_2}(F_2(t, y))} F_{Z^{(2)}}(t, v_2) \frac{f_{\zeta_2}(v_2)}{f_2(t, Q_2(t, F_{\zeta_2}(v_2)))} dv_2 \\ & - \int_{Q_{\zeta_1}(F_1(t, y_0(t)))}^{Q_{\zeta_1}(F_1(t, y))} F_{Z^{(1)}}(t, v_1) \frac{f_{\zeta_1}(v_1)}{f_1(t, Q_1(t, F_{\zeta_1}(v_1)))} dv_1 \end{aligned} \quad (5.2.5)$$

which, since it holds that $Q_{\zeta_2}(F_2(t, y_0(t))) \geq \min_i Q_{\zeta_i}(F_i(t, y_0(t)))$ and $Q_{\zeta_2}(F_2(t, y)) \geq \min_i Q_{\zeta_i}(F_i(t, y))$, we have

$$\begin{aligned} \leq & \int_{\min_i Q_{\zeta_i}(F_i(t, y_0(t)))}^{\max_i Q_{\zeta_i}(F_i(t, y))} F_{Z^{(2)}}(t, v) \frac{f_{\zeta_2}(v)}{f_2(t, Q_2(t, F_{\zeta_2}(v)))} dv \\ & - \left\{ \int_{\min_i Q_{\zeta_i}(F_i(t, y_0(t)))}^{\max_i Q_{\zeta_i}(F_i(t, y))} F_{Z^{(1)}}(t, v) \frac{f_{\zeta_1}(v)}{f_1(t, Q_1(t, F_{\zeta_1}(v)))} dv \right. \\ & - \int_{\min_i Q_{\zeta_i}(F_i(t, y_0(t)))}^{Q_{\zeta_1}(F_1(t, y_0(t)))} F_{Z^{(1)}}(t, v) \frac{f_{\zeta_1}(v)}{f_1(t, Q_1(t, F_{\zeta_1}(v)))} dv \\ & \left. - \int_{Q_{\zeta_1}(F_1(t, y))}^{\max_i Q_{\zeta_i}(F_i(t, y))} F_{Z^{(1)}}(t, v) \frac{f_{\zeta_1}(v)}{f_1(t, Q_1(t, F_{\zeta_1}(v)))} dv \right\}. \end{aligned} \quad (5.2.6)$$

Rearranging terms, Eq. (5.2.6) is equal to

$$\begin{aligned} & \int_{\min_i Q_{\zeta_i}(F_i(t, y_0(t)))}^{\max_i Q_{\zeta_i}(F_i(t, y))} \left[F_{Z^{(2)}}(t, v) \frac{f_{\zeta_2}(v)}{f_2(t, Q_2(t, F_{\zeta_2}(v)))} - F_{Z^{(1)}}(t, v) \frac{f_{\zeta_1}(v)}{f_1(t, Q_1(t, F_{\zeta_1}(v)))} \right] dv \\ & + \int_{\min_i Q_{\zeta_i}(F_i(t, y_0(t)))}^{Q_{\zeta_1}(F_1(t, y_0(t)))} F_{Z^{(1)}}(t, v) \frac{f_{\zeta_1}(v)}{f_1(t, Q_1(t, F_{\zeta_1}(v)))} dv \\ & + \int_{Q_{\zeta_1}(F_1(t, y))}^{\max_i Q_{\zeta_i}(F_i(t, y))} F_{Z^{(1)}}(t, v) \frac{f_{\zeta_1}(v)}{f_1(t, Q_1(t, F_{\zeta_1}(v)))} dv \\ \leq & \int_{\min_i Q_{\zeta_i}(F_i(t, y_0(t)))}^{\max_i Q_{\zeta_i}(F_i(t, y))} \left[F_{Z^{(2)}}(t, v) \frac{f_{\zeta_2}(v)}{f_2(t, Q_2(t, F_{\zeta_2}(v)))} - F_{Z^{(1)}}(t, v) \frac{f_{\zeta_1}(v)}{f_1(t, Q_1(t, F_{\zeta_1}(v)))} \right] dv, \end{aligned} \quad (5.2.7)$$

where the last inequality follows from $F_{Z^{(i)}}(t, z), f_{\zeta_i}(z), f_i(t, Q_i(F_{\zeta_i}(z))) \geq 0$ for all

$z \in D_\zeta$, $t \in [t_0, \infty)$ and so

$$\int_{\min_i Q_{\zeta_i}(F_i(t, y_0(t)))}^{Q_{\zeta_1}(F_1(t, y_0(t)))} F_{Z^{(1)}}(t, v) \frac{f_{\zeta_1}(v)}{f_1(t, Q_1(t, F_{\zeta_1}(v)))} dv \geq 0,$$

$$\int_{Q_{\zeta_1}(F_1(t, y))}^{\max_i Q_{\zeta_i}(F_i(t, y))} F_{Z^{(1)}}(t, v) \frac{f_{\zeta_1}(v)}{f_1(t, Q_1(t, F_{\zeta_1}(v)))} dv \geq 0$$

for all $y \in D_Y$, $t \in [t_0, \infty)$, with strict inequality for at least one $y \in D_Y$.

For all $z \in D_\zeta := [z_0(t), \max\{\bar{z}_1, \bar{z}_2\}]$, define

$$A(z) := \frac{f_{\zeta_2}(z)}{f_2(t, Q_2(t, F_{\zeta_2}(z)))}, \quad (5.2.8)$$

$$B(z) := \frac{f_{\zeta_1}(z)}{f_1(t, Q_1(t, F_{\zeta_1}(z)))}. \quad (5.2.9)$$

It holds that

$$A(z)F_{Z^{(2)}}(t, z) - B(z)F_{Z^{(1)}}(t, z) \leq F_{Z^{(2)}}(t, z) - F_{Z^{(1)}}(t, z)$$

whenever either

- (i) $A(z) \leq 1$ and $B(z) \geq 1$,
- (ii) $A(z), B(z) \geq 1$ and $A(z)F_{Z^{(2)}}(t, z) - F_{Z^{(2)}}(t, z) \leq B(z)F_{Z^{(1)}}(t, z) - F_{Z^{(1)}}(t, z)$, i.e., $F_{Z^{(2)}}(t, z)/F_{Z^{(1)}}(t, z) \leq (B(z) - 1)/(A(z) - 1)$,
- (iii) $A(z) \geq 1$, $B(z) \leq 1$ and $F_{Z^{(2)}}(t, z) - A(z)F_{Z^{(2)}}(t, z) \geq F_{Z^{(1)}}(t, z) - B(z)F_{Z^{(1)}}(t, z)$, i.e., $F_{Z^{(2)}}(t, z)/F_{Z^{(1)}}(t, z) \geq (1 - B(z))/(1 - A(z))$

for all $z \in D_\zeta$, $t \in [t_0, \infty)$. As such, whenever either condition (i)–(iii) hold with $A(z), B(z)$ given by Eq. (5.2.8) and (5.2.9), respectively, it follows that

$$0 \leq \int_{z_0(t)}^z \left[F_{Z^{(2)}}(t, v) \frac{f_{\zeta_2}(v)}{f_2(t, Q_2(t, F_{\zeta_2}(v)))} - F_{Z^{(1)}}(t, v) \frac{f_{\zeta_1}(v)}{f_1(t, Q_1(t, F_{\zeta_1}(v)))} \right] dv$$

$$\leq \int_{z_0(t)}^z [F_{Z^{(2)}}(t, v) - F_{Z^{(1)}}(t, v)] dv \quad (5.2.10)$$

where $z_0(t) := \min_i Q_{\zeta_i}(t, y_0(t))$ for all $t \in [t_0, \infty)$ and $z := \max_i Q_{\zeta_i}(F_i(t, y))$ for all

$y \in D_Y$ and $t \in [t_0, \infty)$. If $\bar{y} := \max y \in D_Y$, then $\max_i Q_{\zeta_i}(F_i(t, \bar{y})) = \max\{\bar{z}_1, \bar{z}_2\} := \max z \in D_\zeta$. Since the inequality (5.2.3) is strict for at least one $y \in D_Y$, that in Eq. (5.2.10) will be strict for at least one $z \in D_\zeta$, corresponding to the composite map transformation of such y value. Here, if the above conditions are satisfied, $Z_t^{(1)} \succ_{SOSD} Z_t^{(2)}$ on $D_\zeta := [z_0(t), \max\{\bar{z}_1, \bar{z}_2\}]$ by the definition of SOSD, as required. \square

In the next corollary, the driving processes are distinct and the distribution functions $F = F_{Y^{(i)}}$, corresponding to the marginals of the driving processes at $t \in (0, \infty)$, are used in each composite map.

Corollary 5.2.1. *Consider the case where $F_i(t, y_i) = F_{Y^{(i)}}(t, y_i)$ for $i = 1, 2$, and all $y_i \in [y_i, \bar{y}_i]$, $t \in (0, \infty)$. Assume $Y_t^{(1)} \succ_{SOSD} Y_t^{(2)}$ on D_Y , where D_Y is defined in Proposition 5.2.1. It holds that $Z_t^{(1)} \succ_{SOSD} Z_t^{(2)}$ on D_ζ for all $t \in [t_0, \infty)$, if any of the following conditions is satisfied for all $z \in D_\zeta$, defined in Proposition 5.2.1, and $t \in [t_0, \infty)$:*

$$(i) \quad \partial_z Q_{Y^{(2)}}(t, F_{\zeta_2}(z)) \leq 1 \leq \partial_z Q_{Y^{(1)}}(t, F_{\zeta_1}(z)).$$

$$(ii) \quad \partial_z Q_{Y^{(2)}}(t, F_{\zeta_2}(z)) \geq 1, \partial_z Q_{Y^{(1)}}(t, F_{\zeta_1}(z)) \geq 1, \text{ and } \frac{F_{\zeta_2}(z)}{F_{\zeta_1}(z)} \leq \frac{\partial_z Q_{Y^{(1)}}(t, F_{\zeta_1}(z)) - 1}{\partial_z Q_{Y^{(2)}}(t, F_{\zeta_2}(z)) - 1}.$$

$$(iii) \quad \partial_z Q_{Y^{(2)}}(t, F_{\zeta_2}(z)) \leq 1, \partial_z Q_{Y^{(1)}}(t, F_{\zeta_1}(z)) \leq 1, \text{ and } \frac{F_{\zeta_2}(z)}{F_{\zeta_1}(z)} \geq \frac{1 - \partial_z Q_{Y^{(1)}}(t, F_{\zeta_1}(z))}{1 - \partial_z Q_{Y^{(2)}}(t, F_{\zeta_2}(z))}.$$

Proof. The proof is analogous to that of Proposition 5.2.1, however it now holds that

$$\begin{aligned} F_{Z^{(i)}}(t, z) &= \mathbb{P}\left(Z_t^{(i)} \leq z\right) = \mathbb{P}\left(Q_{\zeta_i}\left(F_{Y^{(i)}}(t, Y_t^{(i)})\right) \leq z\right) \\ &= \mathbb{P}\left(Y_t^{(i)} \leq Q_{Y^{(i)}}(t, F_{\zeta_i}(z))\right) = F_{Y^{(i)}}(t, Q_{Y^{(i)}}(t, F_{\zeta_i}(z))) = F_{\zeta_i}(z) \end{aligned} \quad (5.2.11)$$

for $i = 1, 2$ and all $z \in [\min\{z_1, z_2\}, \max\{\bar{z}_1, \bar{z}_2\}]$ and $t \in [t_0, \infty)$, and so we replace $F_{Z^{(i)}}(t, z)$ and $Q_i(t, u)$ with $F_{\zeta_i}(z)$ and $Q_{Y^{(i)}}(t, u)$, respectively, in conditions (i)–(iii) in the proposition. \square

Analogous to the FOSD analysis in Section 5.1, we next give conditions for SOSD in the case where the two quantile processes are generated by driving process that are equal in distribution, however under different composite maps.

Corollary 5.2.2. *Assume $Y_t^{(1)} \stackrel{d}{=} Y_t^{(2)}$, that is $F_{Y^{(1)}}(t, y) = F_{Y^{(2)}}(t, y)$ for all $y \in [\min\{\underline{y}_1, \underline{y}_2\}, \max\{\bar{y}_1, \bar{y}_2\}]$ and $t \in (0, \infty)$. Define $D_\zeta := [z_0(t), \max\{\bar{z}_1, \bar{z}_2\}]$ where*

$$z_0(t) := \{z_0 \in [\min\{\underline{z}_1, \underline{z}_2\}, \max\{\bar{z}_1, \bar{z}_2\}) : F_{Z^{(1)}}(t, z_0) = F_{Z^{(2)}}(t, z_0)\} \quad (5.2.12)$$

for all $t \in [t_0, \infty)$. It holds that $Z_t^{(1)} \succ_{SOSD} Z_t^{(2)}$ on D_ζ for all $t \in [t_0, \infty)$ if either of the conditions (i)–(iii) in Proposition 5.2.1 hold for all $z \in D_\zeta$ and $t \in [t_0, \infty)$, with strict inequality for at least one $z \in D_\zeta$. If, however, $F_i(t, y) = F_{Y^{(i)}}(t, y)$ for $i = 1, 2$ and for all $y \in D_Y$ and $t \in (0, \infty)$, then $Z_t^{(1)} \succ_{SOSD} Z_t^{(2)}$ on D_ζ for all $t \in [t_0, \infty)$ if either of the conditions (i)–(iii) in Corollary 5.2.1 hold for all $z \in D_\zeta$, with strict inequality for at least one z , and all $t \in [t_0, \infty)$.

Proof. The proof is similar to that of Proposition 5.2.1, however it holds that $F_{Y^{(1)}}(t, y) = F_{Y^{(2)}}(t, y)$ for all $y \in [\min\{\underline{y}_1, \underline{y}_2\}, \max\{\bar{y}_1, \bar{y}_2\}]$ and $t \in (0, \infty)$, and so

$$\int_{\min\{\underline{y}_1, \underline{y}_2\}}^y [F_{Y^{(2)}}(t, x) - F_{Y^{(1)}}(t, x)] dx = 0 \quad (5.2.13)$$

for all $y \in [\min\{\underline{y}_1, \underline{y}_2\}, \max\{\bar{y}_1, \bar{y}_2\}]$ and $t \in [t_0, \infty)$. By Eq. (5.2.2), this can equivalently be written as

$$\int_{y_0(t)}^y [F_{Z^{(2)}}(t, Q_{\zeta_2}(F_2(t, x))) - F_{Z^{(1)}}(t, Q_{\zeta_1}(F_1(t, x)))] dx = 0 \quad (5.2.14)$$

for all $y \in [\min\{\underline{y}_1, \underline{y}_2\}, \max\{\bar{y}_1, \bar{y}_2\}]$, where $y_0(t) \in [\min\{\underline{y}_1, \underline{y}_2\}, \max\{\bar{y}_1, \bar{y}_2\})$. Consider the case where $Q_{\zeta_1}(F_1(t, y)) \leq Q_{\zeta_2}(F_2(t, y))$ for all $t \in [t_0, \infty)$ and at least one $y \in [\min\{\underline{y}_1, \underline{y}_2\}, \max\{\bar{y}_1, \bar{y}_2\}]$ so that, by Proposition 5.1.1, $Z_t^{(1)} \not\prec_{FOSD} Z_t^{(2)}$. Making the changes of variables $v_i := Q_{\zeta_i}(F_i(t, x))$ for $i = 1, 2$, Eq. (5.2.14) can be rewritten as

$$\begin{aligned} 0 = & \int_{Q_{\zeta_2}(F_2(t, y_0(t)))}^{Q_{\zeta_2}(F_2(t, y))} \frac{f_{\zeta_2}(v_2)}{f_2(t, Q_2(t, F_{\zeta_2}(v_2)))} F_{Z^{(2)}}(t, v_2) dv_2 \\ & - \int_{Q_{\zeta_1}(F_1(t, y_0(t)))}^{Q_{\zeta_1}(F_1(t, y))} \frac{f_{\zeta_1}(v_1)}{f_1(t, Q_1(t, F_{\zeta_1}(v_1)))} F_{Z^{(1)}}(t, v_1) dv_1 \end{aligned} \quad (5.2.15)$$

which, since it holds that $Q_{\zeta_2}(F_2(t, y_0(t))) \geq \min_i Q_{\zeta_i}(F_i(t, y_0(t)))$ and $Q_{\zeta_2}(F_2(t, y)) \geq$

$\min_i Q_{\zeta_i}(F_i(t, y))$, we have

$$\begin{aligned}
0 \leq & \int_{\min_i Q_{\zeta_i}(F_i(t, y_0(t)))}^{\max_i Q_{\zeta_i}(F_i(t, y))} \frac{f_{\zeta_2}(v_2)}{f_2(t, Q_2(t, F_{\zeta_2}(v_2)))} F_{Z^{(2)}}(t, v_2) dv_2 \\
& - \left\{ \int_{\min_i Q_{\zeta_i}(F_i(t, y_0(t)))}^{\max_i Q_{\zeta_i}(F_i(t, y))} \frac{f_{\zeta_1}(v_1)}{f_1(t, Q_1(t, F_{\zeta_1}(v_1)))} F_{Z^{(1)}}(t, v_1) dv_1 \right. \\
& - \int_{Q_{\zeta_1}(F_1(t, y))}^{\max_i Q_{\zeta_i}(F_i(t, y))} \frac{f_{\zeta_1}(v_1)}{f_1(t, Q_1(t, F_{\zeta_1}(v_1)))} F_{Z^{(1)}}(t, v_1) dv_1 \\
& \left. - \int_{\min_i Q_{\zeta_i}(F_i(t, y_0(t)))}^{Q_{\zeta_1}(F_1(t, y_0(t)))} \frac{f_{\zeta_1}(v_1)}{f_1(t, Q_1(t, F_{\zeta_1}(v_1)))} F_{Z^{(1)}}(t, v_1) dv_1 \right\}. \tag{5.2.16}
\end{aligned}$$

Rearranging terms, Eq. (5.2.16) is equal to

$$\begin{aligned}
& \int_{\min_i Q_{\zeta_i}(F_i(t, y_0(t)))}^{\max_i Q_{\zeta_i}(F_i(t, y))} \left[\frac{f_{\zeta_2}(v)}{f_2(t, Q_2(t, F_{\zeta_2}(v)))} F_{Z^{(2)}}(t, v) - \frac{f_{\zeta_1}(v)}{f_1(t, Q_1(t, F_{\zeta_1}(v)))} F_{Z^{(1)}}(t, v) \right] dv \\
& + \int_{Q_{\zeta_1}(F_1(t, y))}^{\max_i Q_{\zeta_i}(F_i(t, y))} \frac{f_{\zeta_1}(v)}{f_1(t, Q_1(t, F_{\zeta_1}(v)))} F_{Z^{(1)}}(t, v) dv \\
& + \int_{\min_i Q_{\zeta_i}(F_i(t, y_0(t)))}^{Q_{\zeta_1}(F_1(t, y_0(t)))} \frac{f_{\zeta_1}(v)}{f_1(t, Q_1(t, F_{\zeta_1}(v)))} F_{Z^{(1)}}(t, v) dv \\
\leq & \int_{\min_i Q_{\zeta_i}(F_i(t, y_0(t)))}^{\max_i Q_{\zeta_i}(F_i(t, y))} \left[\frac{f_{\zeta_2}(v)}{f_2(t, Q_2(t, F_{\zeta_2}(v)))} F_{Z^{(2)}}(t, v) - \frac{f_{\zeta_1}(v)}{f_1(t, Q_1(t, F_{\zeta_1}(v)))} F_{Z^{(1)}}(t, v) \right] dv \tag{5.2.17}
\end{aligned}$$

for all $y \in [y_0(t), \max\{\bar{y}_1, \bar{y}_2\}]$ and $t \in [t_0, \infty)$. By the same argument given in the proof of Proposition 5.2.1,

$$\frac{f_{\zeta_2}(z)}{f_2(t, Q_2(t, F_{\zeta_2}(z)))} F_{Z^{(2)}}(t, z) - \frac{f_{\zeta_1}(z)}{f_1(t, Q_1(t, F_{\zeta_1}(z)))} F_{Z^{(1)}}(t, z) \leq F_{Z^{(2)}}(t, z) - F_{Z^{(1)}}(t, z) \tag{5.2.18}$$

for all $z \in D_{\zeta}$, with strict inequality for at least one $z \in D_{\zeta}$, and all $t \in [t_0, \infty)$ if the inequalities (i)–(iii) hold for all $z \in D_{\zeta}$, with strict inequality for at least one $z \in D_{\zeta}$,

and all $t \in [t_0, \infty)$. Here,

$$\begin{aligned} 0 &\leq \int_{z_0(t)}^z \left[\frac{f_{\zeta_2}(v)}{f_2(t, Q_2(t, F_{\zeta_2}(v)))} F_{Z^{(2)}}(t, v) - \frac{f_{\zeta_1}(v)}{f_1(t, Q_1(t, F_{\zeta_1}(v)))} F_{Z^{(1)}}(t, v) \right] dv \\ &\leq \int_{z_0(t)}^z [F_{Z^{(2)}}(t, v) - F_{Z^{(1)}}(t, v)] dv \end{aligned} \quad (5.2.19)$$

where $z_0(t) := \min_i Q_{\zeta_i}(F_i(t, y_0(t)))$ for all $t \in [t_0, \infty)$ and $z := \max_i Q_{\zeta_i}(F_i(t, y))$ for all $y \in D_Y$, and by the definition of SOSD, $Z_t^{(1)} \succ_{SOSD} Z_t^{(2)}$ on $D_\zeta := [z_0(t), \max\{\bar{z}_1, \bar{z}_2\}]$ as required. In the case where $F_1(t, y) = F_{Y^{(1)}}(t, y) = F_{Y^{(2)}}(t, y) = F_2(t, y)$ for all $y \in [\min\{\underline{y}_1, \underline{y}_2\}, \max\{\bar{y}_1, \bar{y}_2\}]$ and $t \in (0, \infty)$, we replace $F_{Z^{(i)}}(t, z)$ and $Q_i(t, u)$ with $F_{\zeta_i}(z)$ and $Q_{Y^{(i)}}(t, u)$ in the inequalities. \square

The following example considers the case where SOSD is satisfied, but FOSD does not hold, for canonical Tukey- g quantile processes, given by Definition 3.1.2 with quantile function $Q_\zeta = Q_{T_g}$ given by Eq. (2.1.11) with $h = 0$.

Example 5.2.1. Consider two canonical Tukey- g quantile processes,

$$Z_t^{(i)} \stackrel{d}{=} Q_{T_{g_i}}(F_W(t, W_t^{(i)}); g_i) \stackrel{d}{=} \frac{1}{g} \left[\exp\left(\frac{gW_t^{(i)}}{\sqrt{t}}\right) - 1 \right] \quad (5.2.20)$$

for $i = 1, 2$, $g_i \in \mathbb{R}^+$ and all $t \in [t_0, \infty)$. By the same argument given in Example 5.1.1, it holds that $Z_t^{(1)} \succ_{FOSD} Z_t^{(2)}$ if, and only if, $g_1 > g_2$. As such, $Z_t^{(1)} \succ_{SOSD} Z_t^{(2)}$ if, and only if, $g_1 > g_2$. If $g_1 < g_2$, then by the same argument, that is $Z_t^{(2)} \succ_{FOSD} Z_t^{(1)} \implies Z_t^{(2)} \succ_{SOSD} Z_t^{(1)}$, and so $Z_t^{(1)} \not\succeq_{SOSD} Z_t^{(2)}$. As such, when considering the induced measure of a canonical Tukey- g quantile process, we have a risk ordering in the skewness parameter. To treat the case where SOSD is satisfied but FOSD does not hold, we consider the canonical Tukey- g case, see Definition 3.1.2, as follows.

Assume now that the skewness parameter is state-dependent. Let $g_2 \in \mathbb{R}^+ \setminus 0$ and

$$g_1 = g_1(z) = \begin{cases} g_1^a > g_2, & z \leq 0 \\ g_1^b < g_2, & z > 0, \end{cases} \quad (5.2.21)$$

for all $t \in [t_0, \infty)$ and where $g_1^a, g_1^b \in \mathbb{R}^+ \setminus 0$. It holds that $Z_t^{(1)} \not\succeq_{FOSD} Z_t^{(2)}$ on $D_{T_g} := [-1/g_2, \infty)$. We have $F_{Z^{(i)}}(t, z) = 0.5[1 + \text{erf}(\log(g_i z + 1)/(g_i \sqrt{2}))]$. It follows

that if g_1^a, g_1^b, g_2 are such that

$$\begin{aligned} & \int_{-1/g_2}^0 \left[\operatorname{erf} \left(\frac{\log(g_2 x + 1)}{g_2 \sqrt{2}} \right) - \operatorname{erf} \left(\frac{\log(g_1^a x + 1)}{g_1^a \sqrt{2}} \right) \right] dx \\ & \geq \int_0^\infty \left[\operatorname{erf} \left(\frac{\log(g_1^b x + 1)}{g_1^b \sqrt{2}} \right) - \operatorname{erf} \left(\frac{\log(g_2 x + 1)}{g_2 \sqrt{2}} \right) \right] dx, \end{aligned} \quad (5.2.22)$$

then, by Definition 5.0.1, $Z_t^{(1)} \succ_{\text{SOSD}} Z_t^{(2)}$ on D_{T_g} . For example, take $g_1^a = 0.8$, $g_1^b = 0.2$ and $g_2 = 0.3$. The left-hand integral in Eq. (5.2.22) is equal to 0.1341347, and the right-hand is equal to 0.0660684 and the inequality is satisfied so that, here, $Z_t^{(1)} \succ_{\text{SOSD}} Z_t^{(2)}$ on D_{T_g} .

In the above propositions and corollaries, we derive the conditions under which, in various constructed cases, we have FOSD and SOSD for the marginals of quantile processes, that is at each $t \in [t_0, \infty)$. As such, the results utilise the marginal (finite-dimensional) distributions of the driving and quantile processes. If one wishes to consider *conditional stochastic dominance*, that is conditional on the information contained in the filtration up to some time $t_0 \leq s < t < \infty$, the conditional distribution functions are considered, e.g., $F_Z(t, z | \mathcal{F}_s)$ for the quantile process. Here, we may say $Z_t^{(1)} \succ Z_t^{(2)}$ conditional on the sub- σ -algebra \mathcal{F}_s , where $t_0 \leq s < t < \infty$, for all $t \in [t_0, \infty)$, if the necessary conditions are satisfied for the statement to hold true.

5.3 First-order stochastic dominance of multidimensional quantile processes

We conclude this section by providing a result on the stochastic ordering of multidimensional random-level quantile processes, given by Definition 3.4.1. Recall, we have a multivariate, \mathbb{R}^m -valued driving process $(\mathbf{Y}_t)_{t \in [0, \infty)}$ that produces a univariate quantile process $(Z_t)_{t \in [t_0, \infty)}$, $t_0 > 0$. We consider the case in which the implicit copula is used in the quantile process construction. Additionally, we refer to Capérea et al. (1997) and Nelsen et al. (2009) for the definition of Kendall stochastic ordering.

Definition 5.3.1. Let $(\mathbf{Y}_t)_{t \in [0, \infty)}$ and $(\mathbf{X}_t)_{t \in [0, \infty)}$ be m -dimensional stochastic processes for $m > 1$, with the joint distribution function of the margins of each process given by $F_{\mathbf{Y}}(t, y_1, \dots, y_m; \boldsymbol{\vartheta}_{\mathbf{Y}}) : \mathbb{R}^+ \times \mathbb{R}^m \rightarrow [0, 1]$ and $F_{\mathbf{X}}(t, x_1, \dots, x_m; \boldsymbol{\vartheta}_{\mathbf{X}}) : \mathbb{R}^+ \times \mathbb{R} \rightarrow [0, 1]$,

respectively. By Sklar's theorem, see Nelsen (2007), there exists copulas C_Y and C_X such that $F_{\mathbf{Y}}(t, y_1, \dots, y_m; \boldsymbol{\vartheta}_Y) = C_Y(t, u_1, \dots, u_m; \tilde{\boldsymbol{\theta}}_Y)$ and $F_{\mathbf{X}}(t, x_1, \dots, x_m; \boldsymbol{\vartheta}_X) = C_X(t, u_1, \dots, u_m; \tilde{\boldsymbol{\theta}}_X)$ for all $t \in (0, \infty)$ and where $\boldsymbol{\vartheta}_Y, \boldsymbol{\vartheta}_X, \tilde{\boldsymbol{\theta}}_Y, \tilde{\boldsymbol{\theta}}_X$ are vectors of parameters. At each $t \in (0, \infty)$, the Kendall distribution functions K_{C_Y} and K_{C_X} of (\mathbf{Y}_t) and (\mathbf{X}_t) , respectively are given by Definition 4.2.1. We say (\mathbf{Y}_t) dominates (\mathbf{X}_t) by Kendall stochastic order on \mathbb{R}^m if, and only if, $K_{C_Y}(t, v) \leq K_{C_X}(t, v)$ for all $v \in [0, 1]$ and $t \in (0, \infty)$, with strict inequality for at least one v . Here, we write $\mathbf{Y}_t \succ_K \mathbf{X}_t$ for all $t \in (0, \infty)$.

Equivalently, consider the definition of FOSD, given by Definition 5.0.1; it holds that $\mathbf{Y}_t \succ_K \mathbf{X}_t$ if, and only if, $F_{\mathbf{Y}}(t, Y_t^{(1)}, \dots, Y_t^{(m)}; \boldsymbol{\vartheta}_Y) \succeq_{FOSD} F_{\mathbf{X}}(t, X_t^{(1)}, \dots, X_t^{(m)}; \boldsymbol{\vartheta}_X)$ for all $t \in (0, \infty)$ —see Nelsen et al. (2009). Using the notion of Kendall stochastic ordering, we present the following result on the stochastic ordering of multidimensional quantile process, where we omit any notional dependence on the vectors of parameters.

Proposition 5.3.1. *Let (\mathbf{Y}_t) and (\mathbf{X}_t) be m -dimensional D^m -valued, $D \subseteq \mathbb{R}$, stochastic processes with marginal laws $F_{Y^{(i)}}(t, y)$ and $F_{X^{(i)}}(t, x)$ for $i = 1, \dots, m$ and $t \in (0, \infty)$, respectively, and joint distributions $F_{\mathbf{Y}}(t, y_1, \dots, y_m)$ and $F_{\mathbf{X}}(t, x_1, \dots, x_m)$, respectively. Let $Q_{\zeta_1}(u) : [0, 1] \rightarrow D_\zeta \subseteq \mathbb{R}$ and $Q_{\zeta_2}(u) : [0, 1] \rightarrow D_\zeta$ be quantile functions so that*

$$Z_t^{(1)} \stackrel{d}{=} Q_{\zeta_1} \left(C_Y \left(t, F_{Y^{(1)}} \left(t, Y_t^{(1)} \right), \dots, F_{Y^{(m)}} \left(t, Y_t^{(m)} \right) \right) \right), \quad (5.3.1)$$

$$Z_t^{(2)} \stackrel{d}{=} Q_{\zeta_2} \left(C_X \left(t, F_{X^{(1)}} \left(t, X_t^{(1)} \right), \dots, F_{X^{(m)}} \left(t, X_t^{(m)} \right) \right) \right) \quad (5.3.2)$$

are multidimensional quantile processes, given by Definition 3.4.1, for $t \in [t_0, \infty)$, and where C_Y and C_X are the implicit copulas determined by $F_{\mathbf{Y}}$ and $F_{\mathbf{X}}$, respectively. Assume $\mathbf{Y}_t \succ_K \mathbf{X}_t$ on D^m for all $t \in (0, \infty)$, then $Z_t^{(1)} \succeq_{FOSD} Z_t^{(2)}$ on D_ζ for all $t \in [t_0, \infty)$ if, and only if, $Q_{\zeta_1}(u) \geq Q_{\zeta_2}(u)$ for all $u \in [0, 1]$.

Proof. The quantile processes are constructed as $Z_t^{(1)} \stackrel{d}{=} Q_{\zeta_1}(C_t^{(1)})$ and $Z_t^{(2)} \stackrel{d}{=} Q_{\zeta_2}(C_t^{(2)})$, where

$$C_t^{(1)} \stackrel{d}{=} C_Y \left(t, F_{Y^{(1)}} \left(t, Y_t^{(1)} \right), \dots, F_{Y^{(m)}} \left(t, Y_t^{(m)} \right) \right), \quad (5.3.3)$$

$$C_t^{(2)} \stackrel{d}{=} C_X \left(t, F_{X^{(1)}} \left(t, X_t^{(1)} \right), \dots, F_{X^{(m)}} \left(t, X_t^{(m)} \right) \right) \quad (5.3.4)$$

for all $t \in (0, \infty)$. It follows that

$$\begin{aligned} K_{C_Y}(t, v) &= \mathbb{P}\left(C_t^{(1)} \leq v\right) = \mathbb{P}\left(F_{\zeta_1}\left(Z_t^{(1)}\right) \leq v\right) \\ &= \mathbb{P}\left(Z_t^{(1)} \leq Q_{\zeta_1}(v)\right) = F_{Z^{(1)}}(t, Q_{\zeta_1}(v)) \end{aligned} \tag{5.3.5}$$

and, similarly, $K_{C_X}(t, v) = F_{Z^{(2)}}(t, Q_{\zeta_2}(v))$ for all $v \in [0, 1]$ and $t \in (0, \infty)$ with strict inequality for at least one $v \in [0, 1]$. As such, by assumption, $F_{Z^{(1)}}(t, Q_{\zeta_1}(v)) \leq F_{Z^{(2)}}(t, Q_{\zeta_2}(v))$ for all $v \in [0, 1]$, $t \in [t_0, \infty)$ and the rest of the proof follows by the same argument as in the proof of Proposition 5.1.1, following Eq. (5.1.3). \square

Chapter 6

Distortion–based pricing and valuation

In this chapter, we present a novel stochastic valuation principle, based on random–level quantile processes, given by Definition 3.1.1. We utilise the distorted measures induced by such processes, discussed in Chapter 4, and the stochastic ordering results discussed in Chapter 5 to present some useful and, in many financial and insurance settings, desirable properties of the valuation principle. Considering quantile processes in such a setting is one of the many ways to interpret their potential in applications, and utilise the properties that the theoretical construction given in Section 2.2 presents. Additionally, the valuation framework provides a means to add interpretation to each constructive component of random–level quantile processes, as discussed in this chapter. We begin by presenting some background material on distortion–based pricing methods, so to draw connections between such work and the new ideas presented in this thesis. For the majority of the following section, the static case is considered, in line with the literature.

6.1 Existing frameworks for decision–based valuation of risks

Existing methods for actuarial pricing of insurance risks are largely reliant on theories of decision under uncertainty, whereby many well–established techniques exist to

capture and quantify such uncertainty. As insurance markets are often incomplete, contingent claims may carry risk that cannot be hedged (as in complete financial market models), and so (unique) prices cannot be obtained by arbitrage arguments. In this case, investor preferences are explicitly characterised in order to determine prices and strategies in light of such risks. In expected utility (EU) theory, see Morgenstern and Von Neumann (1953), investor preferences (e.g., risk aversion) are modelled by a nonlinear utility function that is applied to the random cash flow of the risky financial or insurance contract. Expectations of the transformed cash flow are taken under some objective, or ‘real-world’ probability measure and so, by Jensen’s inequality, concavity of the utility function is a sufficient condition for investor risk aversion. Extending EU theory, Savage (1954) described and axiomatised investor behaviour and preferences through subjective expected utility theory. Here, nonlinear utility functions to describe preferences are combined with subjective probability measures that reflect each agent’s beliefs on the possible outcomes of each considered decision situation. Similarly, the dual theory of choice under risk was introduced by Yaari (1987) and considers a nonlinear, increasing (and, for risk averse agents, convex) function applied to the probability distribution of the cash flow, where the choice of function characterises risk attitudes of market participants. The premium proposed is then expressed as the Choquet integral, see Choquet (1954), with respect to the distorted probability distribution. A review on further alternative theories to EU theory, e.g., rank-dependent expected utility, see Quiggin (1982, 1993) and Schmeidler (1989), and Choquet expected utility theory (CEU), see Choquet (1954), is given by Sugden (1997). The approach discussed by Schmeidler (1989) and Quiggin (1993) combines both EU theory and the dual theory of Yaari (1987), characterising preferences through both a utility function and a distortion on the probability distribution of the underlying risk—such an approach is discussed by Tsanakas and Desli (2003), and employed in the proposal of a new class of risk measures, or premium principle. A general framework for distributional transforms, and the conditions under which they are probability distortions, is given by Liu et al. (2021), with a connection to convex and coherent risk measures also discussed.

Following the approach of Yaari (1987), distortion premium principles (in the context of insurance markets) were derived by Denneberg (1990) and Wang (2000) via an indifference argument, i.e., the premium should be set so to offset potential losses from the contract. A set of axioms for such premiums is proposed, under which they

can be represented as the Choquet integral with respect to the distorted probability distribution, see Eq. (6.1.1). The prices are shown to be directly related to coherent risk measures, in the sense of Artzner et al. (1999), leading to the notion of *distortion risk measures*. More generally, risk measures, under the likeness of premium principles, and their axiomatic properties have been studied by Bühlmann (1970) and Goovaerts et al. (1984), and in more recent work, the reader may refer to Laeven et al. (2008) and the references therein for a detailed survey on premium calculation principles (PCPs). We define PCPs and risk measures as follows, referring to Laeven et al. (2008) and Föllmer and Schied (2004), respectively.

Definition 6.1.1. *Let \mathcal{X} be a set of random variables on the measurable space (Ω, \mathcal{F}) with each $X \in \mathcal{X}$ representing a risk. A premium calculation principle (or pricing principle) is defined as the functional $\pi : \mathcal{X} \rightarrow \mathbb{R}$, where $\pi(X)$ represents the premium charged by an insurer to insure the risk X . If $X > 0$, it is considered a loss, and $\pi(X) \in \mathbb{R}^+$.*

The premium calculation principle gives the minimum amount that the insurer must charge for it to be beneficial to sell the contract. As such, there is a clear correspondence between the interpretation of a premium principle and a risk measure, which represents the amount of additional capital that, in an insurance context, the insurer must hold to make his aggregate position acceptable to a regulator. Whilst in the insurance literature, a loss is most commonly denoted by a positive random variable, in the general risk management literature, a loss is denoted by a negative random variable. We adhere to these conventions, considering $\pi(X) = \rho(-X)$ for any $X \in \mathcal{X}$ and where ρ is a risk measure. Mathematically, a risk measure is defined as follows.

Definition 6.1.2. *Let $\rho : \mathcal{X} \rightarrow \mathbb{R}$ be a mapping such that for all $X, Y \in \mathcal{X}$ and all $\omega \in \Omega$, $X(\omega) \leq Y(\omega)$ implies that $\rho(X) \geq \rho(Y)$, and for any $m \in \mathbb{R}$, $\rho(Y + m) = \rho(Y) - m$. Then ρ is called a risk measure.*

Further axiomatic properties may be imposed on the risk measure, depending on the required interpretation in terms of factors such as the aggregation of risky positions, portfolio diversification and scaling of portfolios sizes, among others, and how capital requirements should respond. The widely accepted axiomatic characterisation

of coherent risk measures was proposed by Artzner et al. (1999), and characterises risk measures as follows. For all $X, Y \in \mathcal{X}$, the following properties are satisfied:

- (i) *Monotonicity*: If $X \leq Y$ almost surely, then $\rho(X) \geq \rho(Y)$.
- (ii) *Translation invariance*: For $m \in \mathbb{R}$, $\rho(X + m) = \rho(X) - m$.
- (iii) *Subadditivity*: $\rho(X + Y) \leq \rho(X) + \rho(Y)$.
- (iv) *Positive homogeneity*: For $m \in \mathbb{R}^+$, $\rho(mX) = m\rho(X)$.

Convex risk measures were proposed by Föllmer and Schied (2002a,b) whereby the requirement of positive homogeneity and subadditivity were replaced with a convexity condition, that is for all $X, Y \in \mathcal{X}$ the following property is satisfied:

Convexity: For $\lambda \in [0, 1]$, $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$.

The class of distortion risk measures, which we focus on next, is a subclass of coherent risk measures, also satisfying the following property:

Additivity for comonotonic risks: If $X, Y \in \mathcal{X}$ are comonotonic, then $\rho(X + Y) = \rho(X) + \rho(Y)$,

see Denneberg (1990), Wang (1996) and Wang et al. (1997). Here, these risk measures are defined in the context of insurance pricing, and termed *distortion principles*. It is shown that any premium principle satisfying monotonicity, translation invariance, positive homogeneity, subadditivity, comonotonic additivity and law invariance has the representation

$$\pi_\nu(Y) = \int_{-\infty}^0 [\nu(1 - F_Y(y)) - 1] dy + \int_0^{\infty} \nu(1 - F_Y(y)) dy, \quad (6.1.1)$$

for $Y \in \mathcal{X}$ a risk with probability distribution function $F_Y(y) = \mathbb{P}(Y \leq y)$, and where $\nu : [0, 1] \rightarrow [0, 1]$ is an increasing, concave function such that $\nu(0) = 0$ and $\nu(1) = 1$. The function ν is called a distortion operator, acting on the probability distribution of the underlying risky random variable Y . The requirement that the distortion operator be a concave function is to ensure the resulting risk measure satisfies the axioms of coherence, as shown by Denneberg (2013). Since insurance risks (liabilities) are often

denoted by positive random variables, i.e., considered to be losses as per Definition 6.1.1, here, Eq. (6.1.1) becomes

$$\pi_\nu(Y) = \int_0^\infty \nu(1 - F_Y(y)) dy. \quad (6.1.2)$$

The corresponding (coherent) distortion risk measure is given by $\rho_\nu(Y) = \pi_\nu(-Y)$. As an example, value at risk (VaR) at quantile level $\alpha \in (0, 1)$ can be represented as a distortion risk measure with distortion function

$$\nu_\alpha(u) := \begin{cases} 0, & 0 \leq u \leq 1 - \alpha \\ 1, & 1 - \alpha < u \leq 1. \end{cases} \quad (6.1.3)$$

We remark that the distortion operator given by Eq. (6.1.3) is not concave, and so VaR is not a coherent risk measure.

In what follows, we focus largely on the characterisation of preferences, and how this feeds into the valuation of risk and risk measures, through probability distortions. Bühlmann's economic principle, see Bühlmann (1980, 1984), however, states that premiums must also reflect conditions of the market where risks are traded, and the interactions of market participants. One may also refer to Aase (1993), Borch (1962) and the references cited therein for discussions on market equilibrium pricing models involving risk measures. For those obtained under the distortion principle, see Tsanakas and Christofides (2006) and the references therein. The following subsection presents distortion operators in further detail.

6.1.1 Distortion operators

We now discuss some well-known distortion operators that have been developed in the literature. Motivated in part by the argument by Venter (1991) that no-arbitrage implications of insurance pricing (by layer) implies a distributional transformation, a distortion operator termed the Proportional Hazards (PH) transform was introduced by Wang (1995, 1996), and given by $\nu_\gamma(u) := u^{1/\gamma}$ for $\gamma > 1$. Subsequently, the well-known one-factor Wang transform was introduced, see Wang (2000), given by

$$\nu_{\lambda,1}(u) := \Phi[\Phi^-(u) + \lambda], \quad (6.1.4)$$

where $\Phi(x)$ is the CDF of the standard normal distribution, and $\lambda \in \mathbb{R}$ is a parameter of risk aversion, or market price of risk. It is shown by Wang (2003) that, under certain conditions on the aggregate environment, the one-factor Wang transform can be derived from Bühlmann's economic model. However, whilst the one-factor Wang transform is motivated from the perspective of its ability to recover CAPM and the Black-Scholes model, see Wang (2002), it has received criticism for its inability to produce a distorted measure that accounts for higher-order moments, e.g., heavy-tailed features or skewness, that are often observed in financial returns data. The two-factor Wang transform, see Wang (2002), given by

$$\nu_{\lambda,2}(u) := T_k(\Phi^-(u) + \lambda), \quad (6.1.5)$$

where $T_k(x)$ is the CDF of the Student-t distribution with location parameter $\mu = 0$ and $k \in \mathbb{N}$ degrees of freedom, overcomes such limitations, as well as accounting for parameter uncertainty that may arise when pricing under such a model. However, this model is no longer consistent with the risk-neutral CAPM model or Bühlmann's pricing principle. Figure 6.1 illustrates the effect of the one- and two-factor Wang transforms on a generic input distribution $F(x)$ for a range of $\lambda \in \mathbb{R}$ and $k \in \mathbb{N}$ parameter values.

A further extension of the Wang transform (the generalised Wang transform) proposed by Kijima and Muromachi (2008), is consistent with Bühlmann's principle and also provides more flexibility in incorporating higher-order moments. A multivariate extension is given by Kijima (2006). Additionally, Hamada and Sherris (2003) and Pelsser (2008), among others, consider the Wang transform in the context of the pricing of contingent claims.

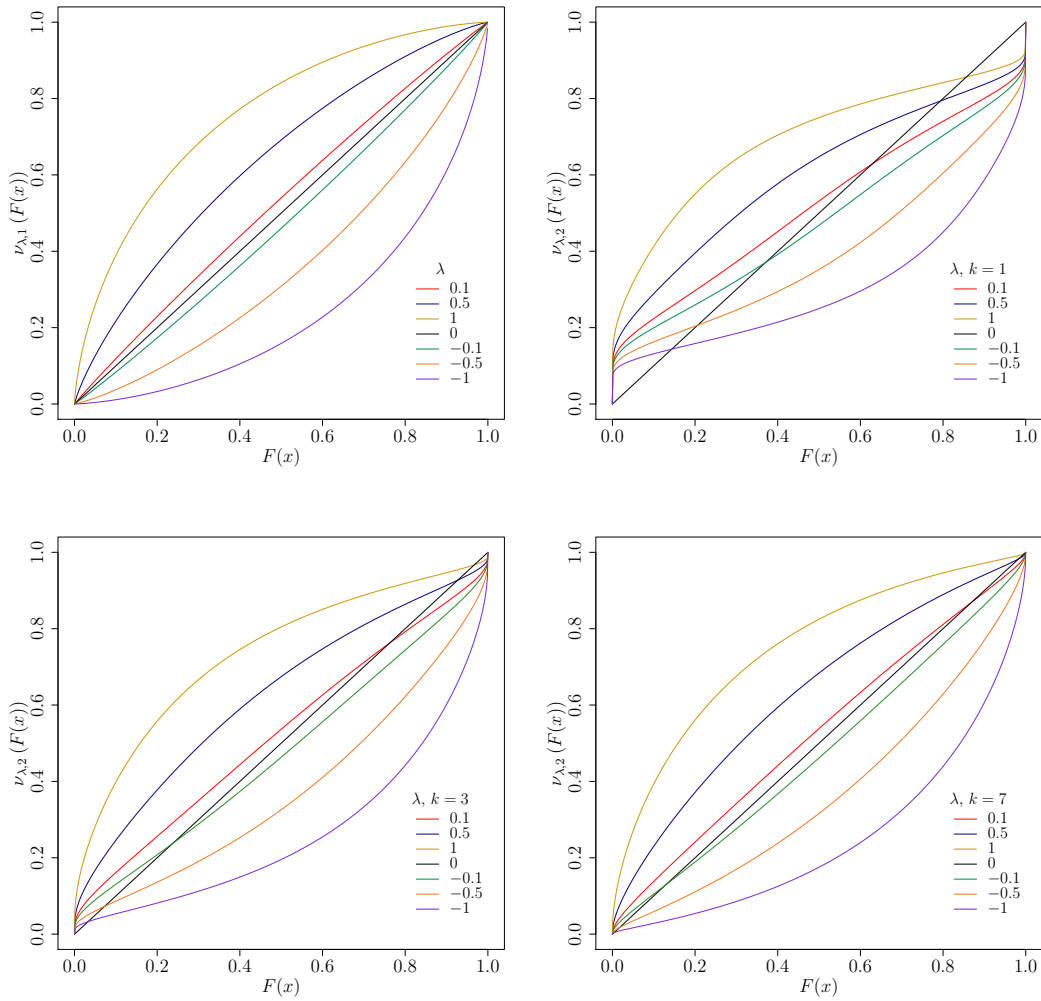


Figure 6.1: The one- and two-factor Wang transforms, $\nu_{\lambda,1}(u) = \Phi(\Phi^{-}(u) + \lambda)$ and $\nu_{\lambda,2}(u) = T_k(\Phi^{-}(u) + \lambda)$, respectively, applied to some base CDF $F(x)$ for risk aversion parameter values $\lambda \in \{0.1, 0.5, 1, 0, -0.1, -0.5, -1\}$ and degrees of freedom parameter (in the two-factor transform) values $k = \{1, 3, 7\}$.

We now consider the generalised class of distortion operators, introduced by Godin et al. (2019). Here, the existence of a probability measure \mathbb{Q} on (Ω, \mathcal{F}) , that is equivalent to \mathbb{P} , is assumed, such that one may derive the distribution of some random variable X under both probability measures, that is $F_X^{\mathbb{P}}$ and $F_X^{\mathbb{Q}}$. Then, under the conditions on X given in the paper, the general distortion operator implied by the

random variable is defined by

$$\nu_X^{\mathbb{Q}\mathbb{P}}(u) := 1 - F_X^{\mathbb{Q}}(Q_X^{\mathbb{P}}(1 - u)), \quad (6.1.6)$$

for $u \in [0, 1]$. This class of distortion operators is flexible in its ability to incorporate higher-order distribution features such as skewness and kurtosis, and allows one to produce prices that are consistent with no-arbitrage models, equilibrium models and actuarial pricing principles. In the case that Eq. (6.1.6) is used to price financial derivative contracts (with continuous and increasing payoff functions) written on the random variable X , the distorted \mathbb{P} -distribution of the contract coincides with its \mathbb{Q} -distribution, where \mathbb{Q} is a given pricing measure such as a risk-neutral measure. The distortion operator used is that implied by the random variable modelling financial risk that underlies the derivative contract, i.e., Eq. (6.1.6) when the value of the asset underlying the financial derivative is modelled by the random variable X . The connection with the Radon-Nikodym derivative is also given, and the distortion operator is employed in the pricing of CAT bonds.

A further discussion on the properties of distortion risk measures and premium principles is given by Wang et al. (1997) and Wirch and Hardy (2001). We conclude this section by making a remark on the preservation of stochastic ordering of risks under distortion operators—recall Definition 5.0.1 of first- and second-order stochastic dominance. It is shown in these papers that the FOSD of any two risks is preserved under distortion operators, as they are increasing functions, and SOSD is preserved if the distortion operator is concave. That is, for any $X, Y \in \mathcal{X}$ and $\nu : [0, 1] \rightarrow [0, 1]$ a distortion operator,

$$X \succsim_{FOSD} Y \iff \pi_\nu(X) \geq \pi_\nu(Y) \ \& \ \rho_\nu(X) \leq \rho_\nu(Y), \quad (6.1.7)$$

and if ν is concave,

$$X \succsim_{SOSD} Y \iff \pi_\nu(X) \geq \pi_\nu(Y) \ \& \ \rho_\nu(X) \leq \rho_\nu(Y). \quad (6.1.8)$$

There is a direct connection between SOSD and risk aversion, in that when $X \succsim_{SOSD} Y$ a risk averse investor will prefer X to Y and so the requirement that $\pi_\nu(X) \geq \pi_\nu(Y)$ indicates that this investors preferences may be reflected through the choice of a concave

distortion operator. Referring to Figure 6.1, we see that the Wang transform produces a concave distortion operator for $\lambda > 0$, i.e., λ is considered to be a parameter of risk aversion.

6.2 A general stochastic valuation principle

The purpose of this section is now to characterise a general, stochastic risk valuation principle, motivated by the literature discussed in the preceding section. We consider a finite timeline $t \in [0, T]$, where $0 < T < \infty$, and recall the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \infty)}, \mathbb{P})$ whereby $(Y_t)_{t \in [0, T]}$ is an (\mathcal{F}_t) -adapted càdlàg process with law F_Y . Henceforth, this process will represent the time dynamic of a risk, including non-tradable risks. We emphasise that we consider the case in which losses are assumed to be positive, however risk measures are calculated under the usual convention in risk management literature that losses are negative, i.e., in the calculation of risk measures, the negative of any loss process is considered. We assume the filtration contains all information about the development of the risk process and the market. The principle and its properties are characterised by a collection of \mathcal{F}_t -measurable stochastic mappings, or operators, to which a direct connection is made to the quantile process-induced distorted measures, given in Section 6.3. The valuation principle is also shown to be directly related to time-consistent, dynamic risk measures.

Definition 6.2.1. *Let $\Pi_t : \mathbb{R} \rightarrow \mathbb{R}$ denote a collection of (\mathcal{F}_t) -measurable, continuous and monotonically increasing mappings, or operators, satisfying the properties of positive homogeneity and translational invariance, and where $\Pi_t(0) = 0$, for all $t \in [0, T]$. The stochastic valuation principle for the risk (Y_t) is defined as the (\mathcal{F}_t) -adapted process $(\Pi_{t,u})_{0 \leq t \leq u \leq T}$ where $\Pi_{t,u} := \Pi_t(Y_u)$ for all $0 \leq t \leq u \leq T$ and such that $\Pi_{s,u} = \Pi_s(\Pi_{t,u})$ for all $0 \leq s \leq t \leq u \leq T$. In the context of a financial or insurance contract, we refer to $(\Pi_{t,u})$ as the stochastic pricing principle for the contract written on the risk (Y_t) .*

Consider Definition 6.1.2. The definition of a risk measure may also be extended to that of a conditional risk measure to capture riskiness at some future point in time. It follows that a sequence of conditional risk measures defines a dynamic risk measure. One may refer to, e.g., Definition 1 by Acciaio and Penner (2011) or Bion-Nadal (2008), and Definition 2.3 by Detlefsen and Scandolo (2005), for the following definition of a convex, conditional risk measure.

Definition 6.2.2. Consider the filtered probability space and let $L_t^\infty := L_t^\infty(\Omega, \mathcal{F}_t, \mathbb{P})$ denote the space of all essentially bounded, (\mathcal{F}_t) -measurable random variables, for all $t \in [0, \infty)$. A map $\rho_{t,u}(X) := L_u^\infty \rightarrow L_t^\infty$ is called a convex, conditional risk measure if it satisfies the following properties for all $0 \leq t < u < \infty$ and $X_u, Y_u \in L_u^\infty$:

- (i) *Conditional translation invariance:* For all $Y_t \in L_t^\infty$, $\rho_{t,u}(Y_u + Y_t) = \rho_{t,u}(Y_u) - Y_t$;
- (ii) *Monotonicity:* $X_u \leq Y_u \Rightarrow \rho_{t,u}(X_u) \geq \rho_{t,u}(Y_u)$;
- (iii) *Conditional convexity:* For all $\lambda \in L_t^\infty$ where $0 \leq \lambda \leq 1$, $\rho_{t,u}(\lambda Y_u + (1 - \lambda)X_u) \leq \lambda \rho_{t,u}(Y_u) + (1 - \lambda)\rho_{t,u}(X_u)$;
- (iv) *Normalisation:* $\rho_{t,u}(0) = 0$.

One may now refer to, e.g., Definitions 5 and 6 by Bion-Nadal (2006), Definitions 1 and 2 by Bion-Nadal (2009), and Definition 5.1, as well as Remark 5.2, by Detlefsen and Scandolo (2005), for the definition of a time-consistent, dynamic (in continuous-time), convex risk measure, given as follows.

Definition 6.2.3. A dynamic, convex risk measure on the filtered probability space is a family $(\rho_{t,u})_{t \in [0, \infty)}$ of conditional convex risk measures for all $0 \leq t \leq u < \infty$. A dynamic risk measure is time-consistent if $\rho_{s,u} = \rho_{s,t}(-\rho_{t,u})$, for all $0 \leq s < t < u < \infty$.

We may now present the following proposition, that characterises the setting in which the stochastic valuation principle in Definition 6.2.1 relates to a time-consistent dynamic risk measure.

Proposition 6.2.1. Consider Definition 6.2.1 and an (\mathcal{F}_t) -adapted risk process (Y_t) . If, and only if, $\Pi_t(\cdot)$ is a concave mapping for all $t \in [0, T]$, then $-\Pi_t(Y_u) : L_u^\infty \rightarrow L_t^\infty$ is a convex, conditional risk measure. The family of mappings $(-\Pi_{t,u})_{0 \leq t \leq u \leq T}$ is a time-consistent dynamic risk measure.

Proof. If $\Pi_t(\cdot)$ is a concave mapping for all $t \in [0, T]$, then $-\Pi_t(\cdot)$ will be a convex mapping. By Definition 6.2.1, $-\Pi_t(\cdot)$ is translation invariant and for any real-valued $x \leq y$, $-\Pi_t(y) \leq -\Pi_t(x)$ for all $t \in [0, T]$ as $\Pi_t(\cdot)$ is monotonically increasing. Thus, under the assumption that $\Pi_t(\cdot)$ is concave, for some (\mathcal{F}_t) -adapted risk (Y_t) , $-\Pi_t(Y_u)$

adheres to Definition 6.2.2 of a convex, conditional risk measure. It follows, by Definition 6.2.3, that the family of maps $(-\Pi_{t,u})$ is a dynamic risk measure. The dynamic risk measure $(-\Pi_{t,u})$ is time-consistent, by Definition 6.2.3, if for all $0 \leq s \leq t \leq u \leq T$, $-\Pi_{s,u} = -\Pi_{s,t}(\Pi_{t,u})$. For all $0 \leq s \leq t \leq u \leq T$ it holds that $\Pi_{s,t} = \Pi_{s,t}(Y_t) = \Pi_s(Y_t)$ and so, since $\Pi_{s,u} = \Pi_s(\Pi_{t,u})$, it follows that $\Pi_{s,u} = \Pi_{s,t}(\Pi_{t,u})$ and $-\Pi_{s,u} = -\Pi_{s,t}(\Pi_{t,u})$, as required. \square

For all $0 \leq t \leq u \leq T$, the map $-\Pi_t(\cdot)$ assigns to each random variable Y_u an \mathcal{F}_t -measurable random variable $-\Pi_t(Y_u)$ that quantifies the risk given the information at time t . For each $t \in [0, T]$, if the map $\Pi_t(\cdot)$ is linear, it fulfills Assumption 2.3 by Wüthrich and Merz (2013) and thus is consistent with the definition of a valuation functional given therein and by Bühlmann (1980) and Wüthrich et al. (2010). If $-\Pi_0(\cdot)$ is a sub-additive map, then for each $t \in (0, T]$, $-\Pi_{0,t}$ is a coherent risk measure in the sense of the axiomatisation by Artzner et al. (1999), and thus incorporates static pricing frameworks built by concave distortion operators—see, for example, Godin et al. (2012, 2019), Kijima (2006), Kijima and Muromachi (2008), Wang (1996, 2000, 2002), Wang et al. (1997). For $t = 0, u = T$, Definition 6.2.1 covers a large number of premium calculation principles (PCPs) as given by Definition 6.1.1—also, see Laeven et al. (2008) and the references therein.

In what follows, we may assume the existence of a money-market, or ‘risk-free’, asset with price process $(B_t)_{t \in [0, \infty)}$ that offers a positive rate of return. We now give the definition of a dynamically consistent risk-loading.

Definition 6.2.4. *Consider Definition 6.2.1. A stochastic pricing principle $(\Pi_{t,u})$ for a risk (Y_t) produces a dynamically consistent risk-loading if $\Pi_{t,u} \geq \mathbb{E}[Y_u | \mathcal{F}_t]$ for all $0 \leq t \leq u \leq T$. If the pricing principle accounts for discounting using the money-market process (B_t) , then we require $\Pi_{t,u} \geq B_t \mathbb{E}[Y_u / B_u | \mathcal{F}_t]$ for all $0 \leq t \leq u \leq T$.*

The choice of mappings $\Pi_t(\cdot)$ in the valuation principle given in Definition 6.2.1 should ensure that time-consistent prices, or premiums, in line with the treatment of different risks in the literature, and the observed behaviour in financial or insurance markets, are produced. For example, if the framework is applied to the valuation of insurance products, prices ought to be at least higher than the expected value of the loss process to ensure a risk-loading. In the context of a traded financial asset in complete markets, the valuation framework should be consistent with the principle of

no–arbitrage. Hereafter, we consider an insurance context whereby markets are not assumed to be complete. We remark, however, that in general, insurance liabilities may be a combination of both traded and non–traded risks, and thus neither entirely replicable or non–replicable. Whilst we remain in the setting of totally incomplete markets for the purpose of presenting the quantile process–based valuation principle, consider the paper by Barigou et al. (2022), and the references given therein, for literature on pricing insurance liabilities in the aforementioned context. Here, insurance liabilities are decomposed into “hedgeable” and “residual” parts, for which hedging arguments and risk measures are applied, respectively, to obtain insurance premiums. We leave the application of the valuation principle presented in this thesis, in the context of such a decomposition of the considered liabilities, for future work.

In the following proposition, we present a special case of the stochastic valuation principle thus introduced, that is characterised by a probability measure. The superscript of the valuation process denotes the measure that it is characterised by.

Proposition 6.2.2. *Consider a σ –finite probability measure on the measurable space (Ω, \mathcal{F}) , denoted $\tilde{\mathbb{P}}$, and define the stochastic operator $\Pi_t^{\tilde{\mathbb{P}}}(\cdot) := \mathbb{E}^{\tilde{\mathbb{P}}}[\cdot | \mathcal{F}_t]$ for all $t \in [0, T]$ where $0 < T < \infty$. Then for a risk process (Y_t) , and by Definition 6.2.1, $\Pi_{t,u}^{\tilde{\mathbb{P}}} := \Pi_t^{\tilde{\mathbb{P}}}(Y_u) = \mathbb{E}^{\tilde{\mathbb{P}}}[Y_u | \mathcal{F}_t]$ is a time–consistent stochastic valuation principle for all $0 \leq t \leq u \leq T$.*

Proof. By the properties of the conditional expectation, $\Pi_t^{\tilde{\mathbb{P}}}(\cdot) := \mathbb{E}^{\tilde{\mathbb{P}}}[\cdot | \mathcal{F}_t]$ is a collection of (\mathcal{F}_t) –measurable, continuous and monotonically increasing maps, satisfying the properties in Definition 6.2.1 for all $t \in [0, T]$. For all $0 \leq s \leq t \leq u \leq T$, we have $\Pi_{s,u}^{\tilde{\mathbb{P}}} = \mathbb{E}^{\tilde{\mathbb{P}}}[Y_u | \mathcal{F}_s]$ and, by the tower property of conditional expectations,

$$\Pi_s^{\tilde{\mathbb{P}}}(\Pi_{t,u}^{\tilde{\mathbb{P}}}) = \mathbb{E}^{\tilde{\mathbb{P}}}[\mathbb{E}^{\tilde{\mathbb{P}}}[Y_u | \mathcal{F}_t] | \mathcal{F}_s] = \mathbb{E}^{\tilde{\mathbb{P}}}[Y_u | \mathcal{F}_s] = \Pi_{s,u}^{\tilde{\mathbb{P}}}. \quad (6.2.1)$$

Therefore, $\Pi_{s,u}^{\tilde{\mathbb{P}}} = \Pi_s^{\tilde{\mathbb{P}}}(\Pi_{t,u}^{\tilde{\mathbb{P}}})$, and so the stochastic valuation principle in Proposition 6.2.2 is time–consistent in the sense of Definition 6.2.1. \square

Corollary 6.2.1. *Consider an (\mathcal{F}_t) –adapted risk (Y_t) and the pricing principle given in Proposition 6.2.2, that is, $\Pi_{t,u}^{\tilde{\mathbb{P}}} := \mathbb{E}^{\tilde{\mathbb{P}}}[Y_u | \mathcal{F}_t]$. If $\tilde{\mathbb{P}}$ is such that $\tilde{\mathbb{P}}(Y_u > y | \mathcal{F}_t) \geq \mathbb{P}(Y_u > y | \mathcal{F}_t)$ for $y \in D_Y$ and all $0 \leq t \leq u \leq T$, then the pricing principle produces a dynamically consistent risk–loading, as per Definition 6.2.4.*

Proof. For all $0 \leq u \leq t \leq T$, $\Pi_{t,u}^{\tilde{\mathbb{P}}} = \mathbb{E}^{\tilde{\mathbb{P}}}[Y_u | \mathcal{F}_t]$, so under the assumption that $\tilde{\mathbb{P}}(Y_u > y | \mathcal{F}_t) \geq \mathbb{P}(Y_u > y | \mathcal{F}_t)$ for all $y \in D_Y$, $0 \leq t \leq u \leq T$, it follows that $\mathbb{E}^{\tilde{\mathbb{P}}}[Y_u | \mathcal{F}_t] \geq \mathbb{E}^{\mathbb{P}}[Y_u | \mathcal{F}_t]$. As such, $\Pi_{t,u}^{\tilde{\mathbb{P}}} \geq \mathbb{E}^{\mathbb{P}}[Y_u | \mathcal{F}_t]$, as required. \square

The stochastic valuation principle given in Proposition 6.2.2 is a general valuation, or pricing, principle that depends not only on the underlying risk (Y_t), but also on the measure $\tilde{\mathbb{P}}$ which may, for instance, be chosen to incorporate investor preferences or systemic risk factors. If $(\Pi_{t,u}^{\tilde{\mathbb{P}}})$ in Proposition 6.2.2 produces a risk-loading, (Y_t) becomes, for example, more heavy tailed or skewed, or both, under $\tilde{\mathbb{P}}$; that is, the riskiness of (Y_t) is amplified under $\tilde{\mathbb{P}}$ relative to \mathbb{P} . We emphasise that the risk-loading in Corollary 6.2.1 is attributed to the choice of distorted measure $\tilde{\mathbb{P}}$. It is common in many valuation settings to define prices with respect to a pricing kernel (financial mathematics), state price deflator (actuarial mathematics) or state price density and stochastic discount factor (economic theory)—see e.g., Bühlmann (1980), Wüthrich and Merz (2013) and Wüthrich et al. (2010). Here, for a given deflator, or pricing kernel, $(\varphi_t)_{t \in [0, \infty)}$ such that $\varphi_t > 0$ almost surely for all $t \geq 0$ with $\varphi_0 = 1$, the valuation process is given by

$$\Pi_{t,u} = \frac{1}{\varphi_t} \mathbb{E}^{\mathbb{P}}[\varphi_u Y_u | \mathcal{F}_t]. \quad (6.2.2)$$

In this thesis, we do not require that prices should *necessarily* satisfy no-arbitrage unless markets are assumed complete (which in general they are not, in an insurance setting) However, this is not saying that trading in such markets *must* lead to arbitrage, i.e., if all market participants do agree on a specific pricing kernel/equivalent martingale measure then we have no-arbitrage prices.

The valuation processes $(\Pi_{t,u})_{0 \leq t \leq u \leq T}$, given by Eq. (6.2.2), defines a pricing system for a given state price deflator (φ_t). In general, there are infinitely many deflators which map from the space of (insurable or financial) risks to corresponding prices. Such deflators may be determined by factors such as market risk aversion, individual risk preferences, market completeness or incompleteness, legal constraints or tail behaviour of underlying risks, among others. It is, in general, not assumed that (φ_t) be independent of (Y_t).

We note that there is an explicit connection between the pricing kernel and the Radon–Nikodym derivative process that induces the measure change from \mathbb{P} to $\tilde{\mathbb{P}}$ in

Proposition 6.2.2. Thus, the task pertains to modelling (φ_t) by the Radon–Nikodym derivative process to capture both external risk and risk preferences in the desired way. We present this in the following section, in the context of quantile process–induced distorted measures, given by Definition 4.0.2 and discussed in Chapter 4.

6.3 A valuation principle based on measure distortions induced by quantile processes

We now present the quantile process–based stochastic valuation principle. Consider a finite time horizon $[0, T]$, for $0 < T < \infty$, and Definition 3.1.1 for a random–level quantile process $(Z_t)_{t \in [t_0, T]}$ constructed from some D_Y –valued driving process $(Y_t)_{t \in [0, T]}$ for $D_Y \subseteq \mathbb{R}$. As per Chapter 4, we consider $Q_\zeta(u; \boldsymbol{\xi}) : [0, 1] \rightarrow D_\zeta \subseteq D_Y$ and so the law of the quantile process is given by Definition 4.0.1 and the marginal and conditional distorted measures induced by the quantile process by Definition 4.0.2. We recall, for the purpose of this section, that we define the conditional distorted measure by

$$\mathbb{P}_{t|s}^Z(B) := \int_{\{\omega \in \Omega : Z_t(\omega) \in B\}} d\mathbb{P}(\omega | \mathcal{F}_s) \quad (6.3.1)$$

for all $0 \leq s < t \leq T$ and $B \in \mathcal{F}_t$, and that the marginal distorted measure is given by $\mathbb{P}_t^Z = \mathbb{P}_{t|0}^Z$.

Definition 6.3.1. Recall Proposition 6.2.2 and let $\tilde{\mathbb{P}} = \mathbb{P}^Z$ be the law of the quantile process (Z_t) , given by Definition 4.0.1. The process $(\Pi_{t,u}^{\mathbb{P}^Z, \zeta})_{0 \leq t \leq u \leq T}$, defined by

$$\Pi_{t,u}^{\mathbb{P}^Z, \zeta} := B_t \mathbb{E}^{\mathbb{P}^Z} \left[\frac{1}{B_u} Y_u | \mathcal{F}_t \right], \quad (6.3.2)$$

is the (discounted) quantile process–based stochastic valuation, or pricing, principle (QPVP or QPPP) for the risk (Y_t) and money–market process (B_t) , corresponding to the constructive choice of (Z_t) . The second argument in the superscript of the valuation principle process denotes the random variable ζ that characterises (Z_t) —see Definition 3.1.1.

Since (B_t) is the money–market process, Eq. (6.3.2) is akin to the quantile process–induced measure \mathbb{P}^Z being a risk–neutral measure. We note that as markets are not

assumed complete, there may exist infinitely many risk-neutral measures, i.e., there is a risk-neutral measure corresponding to each construction of the quantile process (Z_t) that induces the measure \mathbb{P}^Z . If all market participants do agree on a specific risk-neutral (martingale) measure, however, this equates to all of their preferences corresponding to the same composite map that produces the quantile process from any given driving risk process. In such a case, the measure \mathbb{P}^Z is unique and so by the Fundamental Theorem of Asset Pricing, the market is complete.

We also highlight that the transformation from \mathbb{P} to each \mathbb{P}^Z involves more than just a drift transformation; the distorted measure is constructed so to account for risk associated to higher-order moments (e.g., skewness, kurtosis), and as such the Girsanov theorem (where the risk adjustment is captured by a first-order/ drift correction) does not apply. Additionally, we do not, in general, assume equivalence of the probability measures \mathbb{P} and \mathbb{P}^Z . If one were to apply the QPVP in the context of arbitrage-free asset pricing (i.e., under the assumption of complete markets), however, the quantile process that induces the distorted measure must be constructed so to ensure that $\mathbb{P} \sim \mathbb{P}^Z$ —see Section 4.1.

Now consider the Radon–Nikodym derivative between the conditional distorted measure and the \mathbb{P} -measure, given in Eq. (4.1.3). There is a natural connection between the pricing kernel representation in Eq. (6.2.2) and the QPVP given in Definition 6.3.1 when the pricing kernel is defined by the relation $\varphi_u/\varphi_t := \varrho_{u|t}B_t/B_u$ for all $0 \leq t < u < \infty$, and where

$$\varrho_{t|s}(\omega) = \frac{dF_Y^{\mathbb{P}}(t, Q(t, F_\zeta(Y_t(\omega))) | \mathcal{F}_s)}{dF_Y^{\mathbb{P}}(t, Y_t(\omega) | \mathcal{F}_s)} \quad (6.3.3)$$

is given in Definition 4.1.1.

Proposition 6.3.1. *Let $(B_t)_{t \in [0, \infty)}$ be the money–market process and consider the pricing kernel associated with the \mathbb{P} -measure, defined by*

$$\frac{\varphi_u}{\varphi_t} := \frac{\varrho_{u|t}B_t}{B_u} \quad (6.3.4)$$

for all $0 \leq t < u < \infty$ with $\varphi_0 = 1$. The process $(\varphi_t B_t)_{t \in [0, \infty)}$ is a \mathbb{P} -martingale.

Proof. By construction, $(\varphi_t B_t)_{t \in [0, \infty)}$ is (\mathcal{F}_t) -adapted for all $t \in (0, \infty)$. Recall that

$\mathbb{E}^{\mathbb{P}}[\varrho_{t|s}|\mathcal{F}_s] = 1$ for all $0 \leq s < t < \infty$ and so

$$\mathbb{E}^{\mathbb{P}}[\varphi_u B_u|\mathcal{F}_t] = \mathbb{E}^{\mathbb{P}}[\varrho_{u|t}\varphi_t B_t|\mathcal{F}_t] = \varphi_t B_t \mathbb{E}^{\mathbb{P}}[\varrho_{u|t}|\mathcal{F}_t] = \varphi_t B_t. \quad (6.3.5)$$

For $0 \leq s < t < u < \infty$,

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}[\varphi_u B_u|\mathcal{F}_s] &= \mathbb{E}^{\mathbb{P}}[\varrho_{u|t}\varphi_t B_t|\mathcal{F}_s] = \mathbb{E}^{\mathbb{P}}[\varrho_{u|t}\varrho_{t|s}\varphi_s B_s|\mathcal{F}_s] \\ &= \varphi_s B_s \mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}[\varrho_{u|t}\varrho_{t|s}|\mathcal{F}_t]|\mathcal{F}_s] = \varphi_s B_s \mathbb{E}^{\mathbb{P}}[\varrho_{t|s}\mathbb{E}^{\mathbb{P}}[\varrho_{u|t}|\mathcal{F}_t]|\mathcal{F}_s] \\ &= \varphi_s B_s \mathbb{E}^{\mathbb{P}}[\varrho_{t|s}|\mathcal{F}_s] = \varphi_s B_s, \end{aligned} \quad (6.3.6)$$

where the third and fourth equalities follow from the tower property and since $\varrho_{t|s}$ is \mathcal{F}_t -adapted, respectively. \square

Given that (φ_t) is the pricing kernel associated with the \mathbb{P} -measure, we may write the QPVP as

$$\Pi_{t,u}^{\mathbb{P}^Z,\zeta} = B_t \mathbb{E}^{\mathbb{P}^Z} \left[\frac{1}{B_u} Y_u | \mathcal{F}_t \right] = B_t \mathbb{E}^{\mathbb{P}} \left[\frac{\varrho_{u|t}}{B_u} Y_u | \mathcal{F}_t \right] = \frac{1}{\varphi_t} \mathbb{E}^{\mathbb{P}} [\varphi_u Y_u | \mathcal{F}_t] \quad (6.3.7)$$

for all $0 \leq t \leq u \leq T$.

6.3.1 Properties of the quantile process–based valuation principle

We now discuss properties of the valuation principle supported by quantile processes in the context of varying risk profiles and preferences. The quantile process–induced measure allows one to incorporate and parameterise higher–order risk behaviour, investor risk preferences and auxiliary factors (e.g., systemic risk, economic and market conditions) to the valuation principle in Definition 6.3.1. Such factors may be determined, for instance, by how investors react to other risky contracts or market risks. We view the distorted measure as subjective; in an insurance setting, given some risk attitude, the valuation principle captures the price an investor would be willing to pay for a contract written on the risk process (Y_t) . The challenge thus lies in selecting a suitable composite map in Definition 3.1.1 to construct the quantile process that induces the distorted measure. The map must be chosen so that the valuation principle in Definition 6.3.1 is appropriate given the situation or market under consideration.

We emphasise that each investor's preferences correspond to a different, but not necessarily unique, composite map used in the construction of a quantile process; the valuation process will capture the preferences of the investor through the induced measure; and the valuation process may be considered relative to those under no distortion. We consider the \mathbb{P} measure to be some objective baseline to which the distorted, 'subjective' risk-neutral measure \mathbb{P}^Z can be compared. The subjectivity of \mathbb{P}^Z means each market participant determines their own risk-neutral measure used for their valuation of financial, insurance, or other risks. The notion of a subjective probability measure was considered and axiomatised by Savage (1954). First, we present the following result regarding the risk-loading produced by the valuation principle.

Proposition 6.3.2. *Recall Definition 6.3.1 and let the càdlàg risk process (Y_t) be the stochastic driver used to construct a quantile process (Z_t) . If $Z_u \succsim_{FOSD} Y_u$, conditional on the sub- σ -algebra \mathcal{F}_t for all $0 \leq t < u \leq T$, then the valuation principle based on a quantile process produces a dynamically consistent risk-loading, i.e., $\Pi_{t,u}^{\mathbb{P}^Z, \zeta} \geq B_t \mathbb{E}^{\mathbb{P}}[Y_u/B_u | \mathcal{F}_t]$ for all $0 \leq t \leq u \leq T$.*

Proof. By construction, $\mathbb{P}\{Z_t \in B\} = \mathbb{P}\{Y_t \in Q(t, F_\zeta(B))\} = \mathbb{P}\{Y_t \in Z^-(B)\} = \mathbb{P}^Z\{Y_t \in B\}$ and $\mathbb{P}\{Z_t \in B | \mathcal{F}_s\} = \mathbb{P}\{Y_t \in Z^-(B) | \mathcal{F}_s\} = \mathbb{P}^Z\{Y_t \in B | \mathcal{F}_s\}$ for all $B \in \mathcal{F}_t$, i.e., the marginal and conditional distributions of the driving process under the pushforward measure \mathbb{P}^Z coincide with the distributions of the quantile process under \mathbb{P} . As such, it follows that

$$\Pi_{t,u}^{\mathbb{P}^Z, \zeta} = \mathbb{E}^{\mathbb{P}^Z} \left[\frac{B_t}{B_u} Y_u | \mathcal{F}_t \right] = \mathbb{E}^{\mathbb{P}} \left[\frac{B_t}{B_u} Z_u | \mathcal{F}_t \right] \geq \mathbb{E}^{\mathbb{P}} \left[\frac{B_t}{B_u} Y_u | \mathcal{F}_t \right] \quad (6.3.8)$$

where the last inequality follows from the definition of FOSSD. \square

In the context of insurance pricing, it is desirable that the valuation principle produces a risk-loading, i.e., so premiums are mean-value exceeding or satisfy the 'no rip-off' condition. As such, here, one may consider the following result.

Proposition 6.3.3. *Consider Proposition 6.3.2, such that the quantile process is defined by $Z_t \stackrel{d}{=} Q_\zeta(F(t, Y_t))$ for all $t \in [t_0, \infty)$. It holds that $Z_u \succsim_{FOSD} Y_u$, conditional on the sub- σ -algebra \mathcal{F}_t for all $0 \leq t < u \leq T$ if, and only if, $Q_\zeta(F(u, y)) > y$ for all $u \in (t, \infty)$ and $y \in D_Y \subseteq \mathbb{R}$.*

Proof. Recall Definition 5.0.1 of FOSD. Consider the conditional distributions of the driving and quantile processes, given by $F_Z(u, z|\mathcal{F}_t)$ and $F_Y(u, y|\mathcal{F}_t)$, respectively, for $0 \leq t < u \leq T$, so that $Z_u \succsim_{FOSD} Y_u$, conditional on \mathcal{F}_t when

$$F_Z(u, y|\mathcal{F}_t) \leq F_Y(u, y|\mathcal{F}_t) \quad (6.3.9)$$

for all $0 \leq t < u \leq T$ and $y \in D_Y$, with strict inequality for at least one y . We have

$$\begin{aligned} F_Z(u, y|\mathcal{F}_t) &= \mathbb{P}(Z_u \leq y|\mathcal{F}_t) = \mathbb{P}(Q_\zeta(F(u, Y_u)) \leq y|\mathcal{F}_t) \\ &= \mathbb{P}(Y_u \leq Q(u, F_\zeta(y))|\mathcal{F}_t) = F_Y(u, Q(u, F_\zeta(y))|\mathcal{F}_t). \end{aligned} \quad (6.3.10)$$

Since distribution functions are increasing, it holds that $F_Y(u, x|\mathcal{F}_t) \leq F_Y(u, y|\mathcal{F}_t)$ if, and only if, $x < y$ and for all $0 \leq t < u \leq T$. As such, $F_Y(u, Q(u, F_\zeta(y))|\mathcal{F}_t) \leq F_Y(u, y|\mathcal{F}_t)$ if, and only if, $Q(u, F_\zeta(y)) < y$ or, equivalently, $Q_\zeta(F(u, y)) > y$ for all $u \in (t, \infty)$ and $y \in D_Y$, as required. \square

We remark on a similar result, given by Wirch and Hardy (2001), in the context of distortion operators: A distortion risk measure is bounded from below by the mean loss, that is, $\rho_\nu(Y) \geq \mathbb{E}[Y]$ for some loss random variable Y and distortion operator $\nu(u)$ if, and only if, $\nu(u) \geq u$ for all $u \in [0, 1]$.

As the operator characterising the QPVP in Definition 6.3.1 is the conditional expectation, the principle satisfies the property of law invariance, as follows. For any risks $(Y_t^{(1)})$ and $(Y_t^{(2)})$, and quantile processes

$$Z_t^{(1)} \stackrel{d}{=} Q_{\zeta_1} \left(F_1 \left(t, Y_t^{(1)} \right) \right), \quad (6.3.11)$$

$$Z_t^{(2)} \stackrel{d}{=} Q_{\zeta_2} \left(F_2 \left(t, Y_t^{(2)} \right) \right), \quad (6.3.12)$$

where the pushforward measures are \mathbb{P}^{Z_1} and \mathbb{P}^{Z_2} , respectively, if the \mathbb{P}^{Z_1} distribution of $Y_t^{(1)}$ is equal to the \mathbb{P}^{Z_2} distribution of $Y_t^{(2)}$ for all $t \in (0, \infty)$, then it holds that

$$\Pi_{t,u}^{\mathbb{P}^{Z_1}, \zeta_1} = B_t \mathbb{E}^{\mathbb{P}^{Z_1}} \left[\frac{1}{B_u} Y_u^{(1)} | \mathcal{F}_t \right] = B_t \mathbb{E}^{\mathbb{P}^{Z_2}} \left[\frac{1}{B_u} Y_u^{(2)} | \mathcal{F}_t \right] = \Pi_{t,u}^{\mathbb{P}^{Z_2}, \zeta_2}. \quad (6.3.13)$$

We may refer to this as *distorted law invariance*. We note that if $Y_t^{(1)} \stackrel{d}{=} Y_t^{(2)}$ under the \mathbb{P} -measure and the finite-dimensional distribution is denoted F_Y , then if $\zeta_1 = \zeta_2 =: \zeta$

and $F_1 = F_2 =: F_Y$, it will hold that the \mathbb{P}^{Z_1} distribution of $Y_t^{(1)}$ will be equal to the \mathbb{P}^{Z_2} distribution of $Y_t^{(2)}$, i.e., the distorted law invariance property holds. This follows from the fact that for each $t \in [t_0, \infty)$ the random variables defined by $U_t^{(i)} = F_Y(t, Y_t^{(i)})$, for $i = 1, 2$, are uniformly distributed on $[0, 1]$ and so, marginally,

$$Z_t^{(1)} \stackrel{d}{=} Q_\zeta(U_t^{(1)}) \stackrel{d}{=} Q_\zeta(U_t^{(2)}) \stackrel{d}{=} Z_t^{(2)}. \quad (6.3.14)$$

As such, the finite-dimensional distribution of the risk process (Y_t) is assumed to fully characterise (i.e., contain all information required to measure) its riskiness. We now present the following result on the ranking, or ordering, of valuation principles under different distorted measures.

Proposition 6.3.4. *Consider Definition 5.0.1, where $Z_t^{(i)} = Q_{\zeta_i}(F_i(t, Y_t^{(i)}))$ are quantile processes for $i = 1, 2$. Let \mathbb{P}^{Z_i} be the pushforward measure induced by each quantile process, as given in Definition 4.0.1. Define*

$$\Pi_{t,u}^{\mathbb{P}^{Z_i}, \zeta} = B_t \mathbb{E}^{\mathbb{P}^{Z_i}} \left[\frac{1}{B_u} Y_u^{(i)} \middle| \mathcal{F}_t \right], \quad (6.3.15)$$

as per Definition 6.3.1, where the risk process $(Y_t^{(i)})$ is taken to be the quantile process driver for $i = 1, 2$. Then, $\Pi_{t,u}^{\mathbb{P}^{Z_1}, \zeta_1} \geq \Pi_{t,u}^{\mathbb{P}^{Z_2}, \zeta_2}$ for all $0 \leq t < u < \infty$ if either $Z_u^{(1)} \succsim_{FOSD} Z_u^{(2)}$, or $Z_u^{(1)} \succsim_{SOSD} Z_u^{(2)}$, conditional on \mathcal{F}_t .

Proof. Since the risk process underlying each valuation principle is given by the driver of each quantile process, it holds that

$$\Pi_{t,u}^{\mathbb{P}^{Z_i}, \zeta} = B_t \mathbb{E}^{\mathbb{P}^{Z_i}} \left[\frac{1}{B_u} Y_u^{(i)} \middle| \mathcal{F}_t \right] = B_t \mathbb{E}^{\mathbb{P}} \left[\frac{1}{B_u} Z_u^{(i)} \middle| \mathcal{F}_t \right], \quad (6.3.16)$$

for $i = 1, 2$ —see the proof of Proposition 6.3.2. It is known, see e.g., Hanoch and Levy (1969) and Levy (1992), that a necessary condition for both FOSD and SOSD, as per Definition 5.0.1, is that $\mathbb{E}^{\mathbb{P}}[Z_t^{(1)}] \geq \mathbb{E}^{\mathbb{P}}[Z_t^{(2)}]$ for all $t \in [t_0, \infty)$, $t_0 > 0$. We consider the conditional distribution functions of the quantile processes: for all $0 \leq t < u < \infty$, $Z_u^{(1)} \succsim_{FOSD} Z_u^{(2)}$ conditional on \mathcal{F}_t when $F_{Z^{(1)}}^{\mathbb{P}}(u, z | \mathcal{F}_t) - F_{Z^{(2)}}^{\mathbb{P}}(u, z | \mathcal{F}_t) \geq 0$, and $Z_u^{(1)} \succsim_{SOSD} Z_u^{(2)}$ conditional on \mathcal{F}_t when $\int_{z_0(t)}^z F_{Z^{(1)}}^{\mathbb{P}}(u, x | \mathcal{F}_t) - F_{Z^{(2)}}^{\mathbb{P}}(u, x | \mathcal{F}_t) dx \geq 0$, for all $z \in D_\zeta$. As such, $\mathbb{E}^{\mathbb{P}}[Z_u^{(1)} | \mathcal{F}_t] \geq \mathbb{E}^{\mathbb{P}}[Z_u^{(2)} | \mathcal{F}_t]$ if either $Z_u^{(1)} \succsim_{FOSD} Z_u^{(2)}$ or

$Z_u^{(1)} \succ_{SOSD} Z_u^{(2)}$, conditional on \mathcal{F}_t . It follows that

$$\Pi_{t,u}^{\mathbb{P}^{Z_1,\zeta}} = B_t \mathbb{E}^{\mathbb{P}} \left[\frac{1}{B_u} Z_u^{(1)} | \mathcal{F}_t \right] \geq B_t \mathbb{E}^{\mathbb{P}} \left[\frac{1}{B_u} Z_u^{(2)} | \mathcal{F}_t \right] = \Pi_{t,u}^{\mathbb{P}^{Z_2,\zeta}} \quad (6.3.17)$$

if either $Z_u^{(1)} \succ_{FOSD} Z_u^{(2)}$ or $Z_u^{(1)} \succ_{SOSD} Z_u^{(2)}$, conditional on \mathcal{F}_t , as required. \square

We note that FOSD implies SOSD, and so one may first check for FOSD between the two considered quantile processes when considering when we observe ordered valuation principles under the corresponding distorted probability measures. If FOSD is not observed, one may check for SOSD between the quantile processes. We also remark that, although we do not consider it in this thesis, third-order stochastic dominance (TOSD) between the considered quantile processes produces the necessary, and sufficient, equivalent to Proposition 6.3.4.

Stochastic dominance results for quantile processes are given in Chapter 5. Such results can be applied in the context of Proposition 6.3.4 to produce stochastically ordered valuation principles, where the order is characterised by the quantile process composite map, and thus risk preferences or profiles that are embedded into the map. We emphasise that stochastic ordering in any of the quantile process parameters implies such parameters capture levels of risk-aversion—the more risk-averse a market participant, the higher they value some given risk. We draw attention to the pivot quantile process given in Definition 3.1.4. When considering valuation principles for different underlying risk factors from the same family, this construction may be advantageous: One may map from each risk factor to a common pivot process, which will be used as the quantile process driver, with distribution under the ‘reference measure’ to which all distorted measures may be compared. The pivot process may be constructed in such a way to account for all baseline risk factors that are common to all market participants and independent of their preferences. This setting allows for a ‘relativised’ system of valuation principle processes across market participants.

The flexibility and generality of the quantile process approach to constructing a valuation principle allows the framework to be applicable to a large number of risk-based situations, not restricted to the pricing of financial or insurance contracts. One may consider non-monetary risks, for example those in the context of disaster modelling or climate and environmental science—we mention risks in these contexts as extreme events may often be related to skewness and heavy-tailedness, and one has direct con-

trol over such factors in the quantile process-induced measure. The composite map also need not necessarily capture purely risk preferences, e.g., risk aversion, and can be used, for instance, to reflect the variability of risks or risk profiles involving external market factors (e.g., economical development, government policy, geopolitical factors, technological developments etc.), or historical behaviours (e.g., levels of historical emissions of an economy or business), among other things. We illustrate this in the following qualitative, toy example.

Example 6.3.1. *We consider the QPVP in the context of carbon tariffs, where the goal is to cut global emissions that are not controlled under domestic emissions trading schemes (ETS), see, e.g., The World Bank (2021). The aim of the cost-adjustment (tariff) on carbon-intensive imports is to prevent ‘carbon leakage’, that is domestic firms taking production to countries with looser environmental standards, and to ensure a level-playing field for foreign and domestic production. Exporters are also incentivised by such schemes to switch to greener production methods. It is important, however, for the fairness of trade that the tariff is not used as an instrument that unfairly hits imports from a country reliant on exporting such goods. Tariffs correspond to the price that would have been paid had the good been produced under domestic carbon pricing rules, however given that not all countries have access to the same levels of ‘green production’ methods, an indiscriminate carbon tariff scheme could lead to regional inequality and negatively impact export-led development of nations. Further details on the impact carbon tariffs may have on vulnerable nations under the scheme recently proposed in the EU for implementation in 2026 are given by Durant et al. (2021). We consider an illustrative framework for the determination of tariffs that accounts for the emissions of imported goods through the use of the QPVP. The idea is to relativise carbon tariffs by adjusting the monitored emissions levels that determine the tariff, in line with a number of production-based and economic factors. Monitoring absolute levels of emissions involved in the production of exported goods is difficult for reasons including, but not limited to, (greener) technological access and advancements, GDP, inflation, geopolitical factors, historical emissions, and whether exporters have paid a domestic carbon price. The goal is for the mechanism to prevent tariffs drastically impacting less developed or technologically advanced exporters that are disadvantaged in the area of greener production.*

Consider a setting with one importer, $n \in \mathbb{N}$ exporters of goods produced within a

given industry sector, and $d \in \mathbb{N}$ factors that must be considered in the relativisation of tariffs charged to each exporter. For $i = 1, \dots, n$, the càdlàg process $(Y_t^{(i)})$, equipped with \mathbb{P} -law $F_{Y^{(i)}}$, corresponds to the absolute level of CO2 emissions (in tonnes) of the production sector for each exporter through time. Consider Definition 3.1.1: We aim to produce a set of ordered maps $Q_{\zeta_i}(F_i(t, y; \boldsymbol{\theta}_i); \boldsymbol{\xi}_i)$ such that if each exporter were to produce the same levels of CO2 emissions, the tariffs paid would be ranked fairly. The distributional families and the choice of parameters will be determined by the d ‘relativising’ factors that impact the level of sophistication in emissions management of each exporter. One may start, for instance, by considering $F_i(t, y; \boldsymbol{\theta}) = F_{Y^{(i)}}(t, y - \gamma_i; \boldsymbol{\theta})$ where $\gamma_i \in [0, 1]$ quantifies the domestic carbon cost already paid, if any. The exporter with the highest average amount of carbon price paid per tonne of CO2 has parameter $\gamma_i = 1$, and the lowest has $\gamma_i = 0$. The remaining ‘relativising’ factors, e.g., those discussed above, would then determine the parameters of the distributional family characterised by ζ . If $\zeta_i = T_g$, $\boldsymbol{\xi}_i = g_i$ so that

$$Z_t^{(i)} \stackrel{d}{=} \frac{1}{g_i} \left[\exp \left(g_i \sqrt{2} \operatorname{erf}^{-1} \left(2F_{Y^{(i)}} \left(t, Y_t^{(i)} - \gamma_i; \boldsymbol{\theta}_i \right) - 1 \right) \right) - 1 \right] \quad (6.3.18)$$

for all $i \in 1, \dots, n$, the larger the skewness parameter g_i , the larger the tariff that will be paid by the i^{th} exporter. One could, for example, construct a weighted index of all relativising factors that determines the skewness parameter of the composite map corresponding to each exporter. Assume GBP $C \in \mathbb{R}^+$ is the domestic price per tonne of CO2 emissions. It follows that the stochastic, relative cost of exporting for the i^{th} exporter, with emission levels modelled by the process $(Y_t^{(i)})$, is given by $\Pi_{t,u}^{\mathbb{P}^{Z_i}, \zeta} = B_t \mathbb{E}^{\mathbb{P}^{Z_i}} [CY_u^{(i)} / B_u | \mathcal{F}_t]$ for each $i \in 1, \dots, n$ and all $0 \leq t < u < \infty$. One may refer to Proposition 6.3.4 for the conditions under which the prices will be ordered.

6.4 Connection to insurance premium calculation principles

The purpose of this section is to draw comparison between the quantile process-based valuation principle, in the context of insurance premiums, and existing premium calculation principles. We consider a risk process $(Y_t)_{t \in [0, \infty)}$ with marginal distribution function $F_Y(t, y)$ at all $t \in (0, \infty)$, omitting the vector of parameters. First, we present

a general setup in which the QPVP recovers premiums derived under the Wang and Proportional Hazards (PH) transforms, and then we consider an explicit Tukey- g QPVP to compare premiums in terms of the parameters involved.

First, recall the one-factor Wang transform, given by Eq. (6.1.4) such that for any distribution function $F(t, y)$, the distorted distribution function is given by

$$F^*(t, y) := \nu_{\lambda,1}(F(t, y)) = \Phi [\Phi^- (F(t, y)) + \lambda]. \quad (6.4.1)$$

Now, consider Definition 3.1.1 for some quantile function Q_ζ , stationary distribution function $F(t, y; \lambda) = \Phi(y + \lambda)$ for $\lambda \in \mathbb{R}$, and driving process (Y_t) that is marginally standard normally distributed, that is $F_Y(t, y) = \Phi(y)$, for all $t \in (0, \infty)$. Such a (Y_t) may be obtained from the pivotal quantile process construction in Definition 3.1.4 for instance. Then, for each $t \in [t_0, \infty)$, we have $Z_t \stackrel{d}{=} Q_\zeta(\Phi(Y_t + \lambda); \boldsymbol{\xi})$ and so

$$\begin{aligned} F_Z^{\mathbb{P}}(t, z; \lambda, \boldsymbol{\xi}) &= \mathbb{P}(Z_t \leq z) = \mathbb{P}(Q_\zeta(\Phi(Y_t + \lambda); \boldsymbol{\xi}) \leq z) \\ &= \mathbb{P}(Y_t \leq \Phi^-(F_\zeta(z; \boldsymbol{\xi})) + \lambda) = \Phi(\Phi^-(F_\zeta(z; \boldsymbol{\xi})) + \lambda). \end{aligned} \quad (6.4.2)$$

Therefore, the distorted distribution, i.e., that induced by the quantile process, is equivalent to the distortion induced by a one-factor Wang transform acting on the base distribution function $F_\zeta = Q_\zeta^-$. The two factor Wang transform is replicated similarly if we consider a driving process with $F_Y(t, y; \boldsymbol{\theta}) = T_k(y)$ for all $t \in (0, \infty)$. Similarly, consider the PH transform, characterised by the distortion operator $\nu_\gamma(u) = u^{1/\gamma}$ for $\gamma > 1$. Let the distortion operator act on the decumulative distribution function, so that the distorted distribution is given by $F^*(t, y; \gamma) = 1 - (1 - F(t, y))^{1/\gamma}$ for each $t \in (0, \infty)$. Now, consider Definition 3.1.1 for some quantile function Q_ζ , stationary distribution function $F(t, y) = 1 - 1/y$ for $y > 1$ (i.e., that of the inverse standard uniform distribution) and driving process (Y_t) that is marginally distributed according to the Pareto distribution with scale parameter 1 and shape parameter $1/\gamma$, that is $F_Y(t, y; 1, \gamma) = 1 - (1/y)^{1/\gamma}$ and $Y_t > 1$ for all $t \in (0, \infty)$. It follows that $Z_t \stackrel{d}{=} Q_\zeta(1 - 1/Y_t; \boldsymbol{\xi})$ and so for each $t \in [t_0, \infty)$,

$$\begin{aligned} F_Z^{\mathbb{P}}(t, z; \gamma, \boldsymbol{\xi}) &= \mathbb{P}(Z_t \leq z) = \mathbb{P}(Q_\zeta(1 - 1/Y_t; \boldsymbol{\xi}) \leq z) \\ &= \mathbb{P}(Y_t \leq 1/(1 - F_\zeta(z; \boldsymbol{\xi}))) = 1 - (1 - F_\zeta(z; \boldsymbol{\xi}))^{1/\gamma}. \end{aligned} \quad (6.4.3)$$

Therefore, the distorted distribution induced by the quantile process is equivalent to the distortion induced by the PH transform acting on the distribution function $F_\zeta := Q_\zeta^-$. The above two results confirm that the distortion produced by the composite map in the random-level quantile process construction does not imply a distortion operator, as the output (distorted) distributions cannot be obtained by defining a function acting on the base (driving process) distribution. Instead, we recover the Wang and PH transforms by specifying the different components of the composite map construction.

For the following numerical example, consider Example 1 by Wang (2000). Let a positive-valued risk process $(Y_t)_{t \in [0, T]}$ be such that at $T = 1$ the random variable Y_T has finite-dimensional distribution given by the Pareto distribution function

$$F_Y(T, y) = F_Y(y) = 1 - \left(\frac{2000}{2000 + y} \right)^{1.2}, \quad (6.4.4)$$

for $y > 0$. We note that whilst, most generally in the literature (and in this example), the Wang transform is considered at some fixed time $t = T$, the distortions we present lead naturally to a dynamic structure. We note that whilst we do not consider the following example dynamically, such a structure will be meaningful in the context of applications that wish to consider price or premium processes that describe, for instance, how premiums evolve over time, or when one may wish to compute forward premiums. Additionally, such a setting leads naturally to the consideration of dynamic (distortion) risk measures.

Now, assume for the risk (Y_t) , the maximum insured loss is given by some $\bar{Y} \in \mathbb{R}^+$, so that the insured risk is limited and refers to the interval $(0, \bar{Y}]$. Often, the insured risk is divided into layers $(a_i, a_i + h_i]$ for $1 \leq i \leq n$, that is $\cup_{i=1}^n (a_i, a_i + h_i] = (0, \bar{Y}]$, each of which characterises some range in $(0, \bar{Y}]$ to which an insurance product refers. It follows that a layer at $(a_i, a_i + h_i]$, $1 \leq i \leq n$, of the risk (Y_t) for all $t \in (0, \infty)$ is defined as the loss from an excess-of-loss cover, given by the random variable

$$Y_{t, (a_i, a_i + h_i]} = \begin{cases} 0, & 0 \leq Y_t \leq a_i \\ (Y_t - a_i) & a_i < Y_t \leq a_i + h_i \\ h_i & Y_t > a_i + h_i \end{cases} \quad (6.4.5)$$

with marginal distribution function

$$F_{Y_{t,(a_i, a_i+h_i]}}(t, y) = \begin{cases} F_Y(t, y + a_i) & y < h_i \\ 1 & y \geq h_i, \end{cases} \quad (6.4.6)$$

where a_i is the attachment point and $a_i + h_i$ is the exhaustion point of the i^{th} layer.

Now consider a Tukey- g quantile process, given by Definition 3.1.1 with $F(t, y) = \Phi(y + \kappa t)$ for $\kappa \in \mathbb{R}$, i.e.,

$$Z_t \stackrel{d}{=} \frac{B}{g} \exp(gY_t + g\kappa t) \quad (6.4.7)$$

for all $t \in [t_0, T]$ and $B, g > 0, \kappa \in \mathbb{R}$. By Proposition 3.1.1, the finite-dimensional distributions of the quantile process are more positively skewed than those of the driving process, and by Corollary 3.3.1, all skewness is controlled via the parameter g . Let \mathbb{P}^Z be the measure induced by the distribution of the random variable Z_T , that is,

$$\mathbb{P}^Z(A) := \int_{\{\omega \in \Omega: Z_T(\omega) \in A\}} d\mathbb{P}(\omega) \quad (6.4.8)$$

for all $A \in \mathcal{F}$. We compare the Tukey- g risk-adjusted premium by layer to that produced under the Wang transform with parameter $\lambda = 0.1$ and PH transform with index $\gamma = 0.9245$. Premiums are obtained using the QPVP, given by Definition 6.3.1 with $t = 0, u = T$, and where we consider $B_t = 1$ for all $t \in [0, \infty)$, for simplicity. As such,

$$\Pi_{0,T}^{\mathbb{P}^Z, T_g} = \mathbb{E}^{\mathbb{P}^Z} [Y_{T,(a_i, a_i+h_i]}] \quad (6.4.9)$$

for the layer $(a_i, a_i + h_i]$.

We allow $g > 0$ to introduce positive skewness relative to the risk distribution to capture investor risk-aversion levels, to account for skewness-related systemic risk and to place probabilistic emphasis on the occurrence of higher losses under the transformation. The results are presented in Table 6.1, where in the third column, the parameter κ is selected to match the premium with the Wang and PH premiums for the basic limit layer ($\$0, \50000], as shown in bold, illustrating how the quantile transform influences the premiums for subsequent layers relative to this figure. The same is done in the fourth column, however κ is now chosen so to match the premiums for the layer ($\$200, \300], shown in bold. We observe that prior to this layer, the Tukey- g premium

is lower than the Wang and PH premiums, and higher in subsequent layers with significantly larger layer-by-layer increases towards the higher layers resulting from the introduction of relative skewness under the quantile transform. Overall, we observe that the PH transform premium increases faster than the Wang transform, and the Tukey- g measure distortion produces a premium that increases much faster than the PH transform with the rate of increase determined by the magnitude of g . Figure 6.2 illustrates further how the rate of increase of premiums at higher layers is determined by the magnitude of the skewness parameter g , as expected.

Layer in 000's	PH premium $\gamma = 0.9245$	Wang premium $\alpha = 0.1$	$\Pi_{0,T}^{\mathbb{P}^Z, T_g}, B = 0.01$	
			$g = 0.8, \kappa = 6.96$	$g = 0.08, \kappa = -10.25$
(0,50]	5,487.0	5,487.0	5,487.2	261.9
(50,100]	910	845.0	4,632.6	228.5
(100,200]	857	769.9	8,642.6	431.7
(200,300]	475	414.2	8,219.7	414.2
(300,400]	325	278.4	7,965.5	403.5
(400,500]	246	207.3	7,785.9	395.9
(500,1000]	728	598.0	37,287.4	1,909.3
(1000,2000]	675	533.2	70,349.5	3,635.2
(2000,5000]	819	616.6	197,478.0	10,306.4
(5000,10000]	567	405.7	310,290.9	16,331.72

Table 6.1: Risk-adjusted premiums by layer under the Wang transform and Tukey- g distorted measure. The figures in bold correspond to the premiums matched across the different distortions, through the choice of κ .

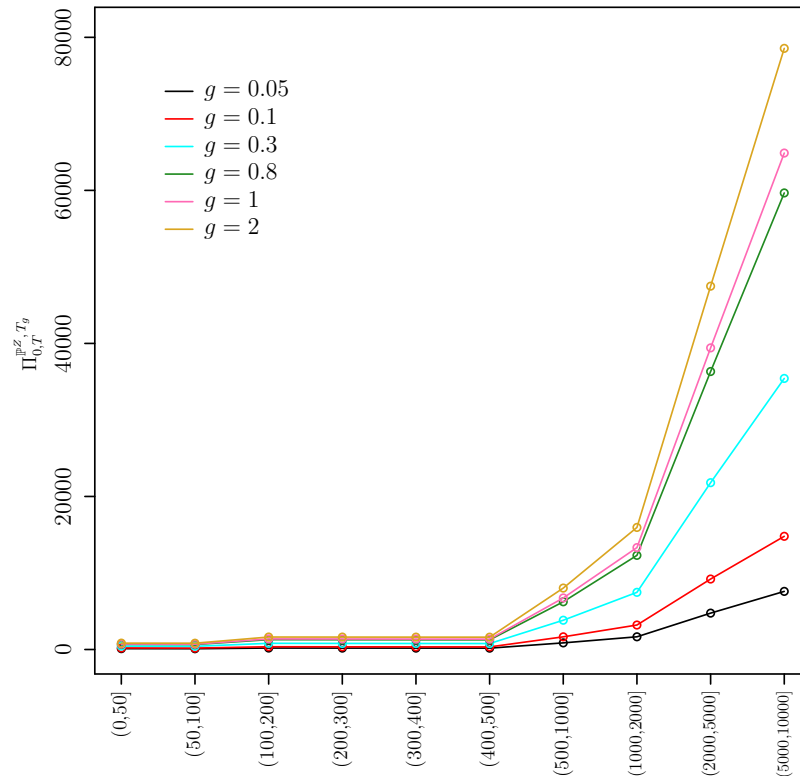


Figure 6.2: Premiums by layer under the Tukey- g QPVP, with $B = 0.01$, $\kappa = -100$ and $g \in \{0.05, 0.1, 0.3, 0.8, 1, 2\}$

We now present an outline of the multidimensional QPVP in the context of determining the correct premium to charge for property and casualty (P&C) insurance. The reader may refer to Young (2006) for a discussion on the properties, and examples, of premium principles considered in the literature. It is important that the premium cover the expected loss of the risk, as well as some risk-loading to account for uncertainty and involved costs. The risk-loading induced by the QPVP can be derived from Corollary 3.1.1. The multidimensional premium QPVP for an excess of loss insurance policy (or layer contract) is defined as follows, where we may consider the risk process for which we wish to determine a premium for to be given by $Y_t = Y_t^{(1)}$ where $\mathbf{Y}_t = (Y_t^{(1)}, \dots, Y_t^{(m)})$. Intuitively, the multidimensional QPVP is analogous to that given by Definition 6.3.1 in the univariate case, however one has a constructive, and flexible, mechanism for the incorporation of auxiliary risk factors, that is $(Y_t^{(2)}, \dots, Y_t^{(m)})$.

Definition 6.4.1. Consider the multidimensional quantile process given by Definition

3.4.1 where (\mathbf{Y}_t) is an m -dimensional, positive-valued risk process. Let \mathbb{P}^Z be the distorted measure induced by (Z_t) , given by Equation 4.2.1. Recall Proposition 6.2.2 and let $\tilde{\mathbb{P}} = \mathbb{P}^Z$. Consider a function $V(y)$ that defines the payoff of a layer contract covering losses between the predefined attachment point a and exhaustion point b , with maximum payout $b - a$, i.e., $V(y) = (y - a)\mathbb{1}_{\{a \leq y < b\}} + (b - a)\mathbb{1}_{\{y \geq b\}}$ for $0 < a < b < \infty$. Then, for a univariate risk process (Y_t) and money-market process (B_t) , the premium QPVP is given by $\Pi_{t,u}^{\mathbb{P}^Z} := B_t \mathbb{E}^{\mathbb{P}^Z} [V(Y_u) / B_u | \mathcal{F}_t]$ for $0 \leq t < u \leq T$.

The multidimensional premium QPVP for a limited stop-loss contract can be computed similarly, using Definition 6.4.1 with $V(y) = \min\{(y - a)^+, b\}$ for $a, b > 0$.

Chapter 7

Application

The purpose of this chapter is to exemplify the novel stochastic valuation principle based on quantile processes, introduced in Section 6.3, in the context of pricing a stop-loss insurance contract linked to greenhouse gas emissions. We consider the univariate quantile process, given in Section 3.1, and the induced probability measure, given in Chapter 4. The example presented is empirical, apart from the choice of parametric quantile function in the composite map, and serves the purpose of illustrating the properties of the QPVP in regard to the inputs to its construction (e.g., choices of parameters).

We consider an application in the context of insuring the regulatory risk associated with excess greenhouse gas emissions under an emissions trading scheme (ETS). We employ the univariate quantile process, given in Section 3.1, for the purpose of pricing excess greenhouse gas (GHG) emissions in geographical regions that implement an ETS, see, e.g., The World Bank (2021). We consider insurance contracts from the perspective of the insurer, where the contract issued insures the regulatory risk (in the forms of fines, or ‘civil penalties’) faced by market participants (‘installations’), e.g., those in the power sector, manufacturing industry or airlines, exceeding the level of emissions for which they’ve purchased allowances for. We highlight the following to provide context:

- (i) The EU ETS works on the ‘cap and trade’ principle whereby a cap (that reduces over time to contribute to meeting legally binding carbon reduction commitments, e.g., the EU’s 2030 Climate Target plan to reduce emissions to below 55% of 1990 levels by 2030) is set on the total amount of certain GHGs that can be emitted

by the installations covered by the system. Details of the EU ETS is given in EC Europa (2015).

- (ii) The primary means of introducing allowances into the market is auctioning, with participants also able to trade allowances on a secondary market. The auction calendar confirms the volume of annual allowances available in each auction and is published in the second half of the year prior. Free allowances are also allocated to operators of eligible installations to protect international competitiveness of domestic producers and reduce the risk of carbon leakage—see Example 6.3.1.
- (iii) Following each year, an installation must submit an emissions report that must be verified by an accredited verifier by 31 March of the following year. Once verified, installations must cover fully their reportable emissions by surrendering the equivalent number of allowances by 30 April, or face heavy fines (‘civil penalties’).
- (iv) Installations can trade in EU ETS futures markets, where the futures contract is ‘a deliverable contract where each Clearing Member with a position open at cessation of trading for a contract month is obliged to make or take delivery of EUAs to or from a Trading Account within the EUA Delivery Period and in accordance with the Rules’, see ICE (2021). That is, the delivery of one EUA entitles the installation to emit one tonne of carbon dioxide equivalent gas; the minimum trading size is one lot, equivalent to 1000 EUAs.

Consider the timeline $[0, T]$. For the purpose of this example we assume that at auction date, the installations purchase the required amount of allowances to cover their expected emissions at the reporting date ($t = T$), i.e., market participants act rationally to avoid civil penalties. Additionally, we assume that EU ETS futures prices reflect expectations of future GHG emissions in the price formulation. This assumption lies on the basis of installations trading in the futures market when realised production levels appear to be driving up emissions and so EUA futures are purchased to prevent civil penalties. Associated with exceeding the emissions levels for which an installation can cover with allowances is a regulatory risk (in the form of civil penalties). Such a risk may be insured by purchasing a stop-loss contract issued by an insurance firm, that pays out in the instance of such an event occurring at the reporting date. The purpose of this example is to value the premium associated to this stop-loss insurance

contract. Whilst the contract considered is of a simple nature, we see it as most fitting for the purpose of this illustrative example.

As an alternative to purchasing a stop-loss contract, installations can consider covering the cost of emitting over their allowances by trading on the open market, namely buying EUA futures with expiry on or before the date at which allowances must be surrendered following the reporting date. However since the market is highly illiquid (due to caps on allowances), in the case of unexpected market shocks driving increases in demand of goods and services provided by the installations, the price impact of buying allowances in the open market (particularly in the case of significantly large producers) may be so substantial that it's no longer equitable to cover excess emissions in such a way. As such, the installation may purchase the insurance contract where the insurer is pricing the excess risk associated with potential price impacts associated to the occurrence of such an event.

In this context, we motivate the use of the QPVP for pricing the insurance contract as follows. The distortion map allows the insurer to account for the excess risk that may be associated with an individual installation's behaviour, i.e., the price impact they would have if covering excess emissions by buying allowances on the open market. Historical emissions prices (and thus the real-world or market probability measures) don't reflect such potential price impacts, and so the insurer may adjust premiums accordingly by pricing in skew or leptokurtosis that may materialise in the event of this scenario (through the quantile process-induced distorted measure). If the scenario plays out at time $t = T$, the excess skew or kurtosis ensures the insurer is solvent when the insurance contract pays out.

Consider the case where the insurer is pricing the premium for an EU-based installation. We consider the case where the driving process (Y_t) represents the EUA futures with a December 2022 expiry date. We have stop-loss payoff function $V(y) = (y - a)^+$ where $a > 0$ is the threshold beyond which the installation may wish to insure against the price going beyond. We consider the threshold to be directly related to civil penalties that may be incurred in the case that realised emissions exceed the levels for which the installation has purchased allowances for. As such, this may also be partially determined by the level of emissions allowances held by the institution. The composite map accounts for the price impact of the individual installation, with the potential to build in other factors, e.g., historical behaviour of exceeding emissions allowances, realised

efforts to make production ‘greener’, weather indexes, macroeconomic factors such as GDP or inflation, government policies such as imposed national lockdowns. We omit the consideration of such factors for the purpose of this somewhat *simple*, illustrative example, but the case study may be further developed under their consideration.

We choose the empirical distribution function of the driving process, and the Tukey–*gh* quantile function. The installation can enter into the insurance contract at any time prior to the reporting date, and after the auction date (so that futures on the allowances begin trading on the market) i.e., any $t \in (0, T)$. The expiry date of the contract is assumed to be the reporting date for the calendar year, 31 December. If the contract pays out, the installation is equipped with financial means to cover the associated cost of higher than expected realised emissions. These costs may materialise in the form of civil penalties, or purchasing allowances on the secondary market (which are now inflated as a result of rising emissions and thus demand for the futures contracts) which may be surrendered, in line with their level of reported emissions, by 30 April the following year.

We consider the Tukey–*gh* random–level quantile process,

$$Z_t \stackrel{d}{=} Q_{T_{gh}}(F_Y(t, Y_t); A, B, g, h), \quad (7.0.1)$$

given by Definition 3.1.1, for all $t \in (0, \infty)$. Here, $Q_{T_{gh}}$ is given by Eq. (2.1.11) with $A \in \mathbb{R}$, $B \in \mathbb{R}^+$, $g \in \mathbb{R} \neq 0$, $h \in \mathbb{R}_0^+$, and F_Y is the marginal distribution of the driving process. It follows that the price of the insurance contract (premium), obtained under the QPVP, is given by

$$\Pi_{t,T}^{\mathbb{P}^Z} = \mathbb{E}^{\mathbb{P}^Z} \left[\frac{B_t}{B_T} (Y_T - a)^+ | \mathcal{F}_t \right] = \int_{\{\omega \in \Omega : Y_T(\omega) \in [a, \infty)\}} (Y_T(\omega) - a) d\mathbb{P}_{T|t}^Z(\omega), \quad (7.0.2)$$

where $(B_t)_{t \in (0, \infty)}$ is the money–market process with which we discount. Here,

$$\mathbb{P}_{T|t}^Z(A) = \int_{\omega \in \Omega : Z_t(\omega) \in A} d\mathbb{P}(\omega | \mathcal{F}_t) \quad (7.0.3)$$

is the probability measure induced by the quantile process in Eq. (7.0.1). The premium accounts for the effect of the distortion map through the choice of the A, B, g and h parameters in the quantile function $Q_{T_{gh}}$, as well as F_Y . We refer to Chapter 5, in

particular, Example 5.1.1, in which stochastic ordering results of quantile processes are derived, and Proposition 6.3.4 where such results are applied in the context of the QPVP. There will be a first-order stochastic ordering in the skewness parameter g for all $h \in \mathbb{R}_0^+$, and so the insurer may, for example, adjust (increase) g to guard against excess payments due to shocks. The extent to which the parameter is increased will differ between insurers and their risk appetites. The location and scale parameters, A and B , respectively, may be set by the insurer based on whether they wish to preserve some measure of centrality of the driving risk process under the distortion map—we discuss this further in Section 7.4. We remark that an exploration into the calibration of these flexible classes of quantile process-based premium principles is left for future work. We discuss the data, next.

7.1 Description of the data

We consider the daily closing price of the ICE EUA December 2022 (with symbol CKZ22), over the period ranging 27 February 2020 to 25 April 2022, in Euros and Euro cents per metric tonne. The period is set based on the data available. We consider 25 April 2022 to be the time $t \in (0, T)$, so that the filtration $(\mathcal{F}_t)_{t \in (0, T)}$ is assumed to contain all information about such prices over the period for which the data is obtained. The closing value from Friday is carried over to Saturday and Sunday to account for missing daily values on weekends; otherwise, no daily values are missing from the data. The time series of the data sets are plotted in Figure 7.1. We observe an upward linear trend, where the steepness of the trend increases from 2021 onwards, most likely as a result of the easing of Covid-19 restrictions that were put in place in 2020. Shortly after January 2020, a price drop of approximately 40% occurred, the cause of which may be linked to the Russian invasion of Ukraine.

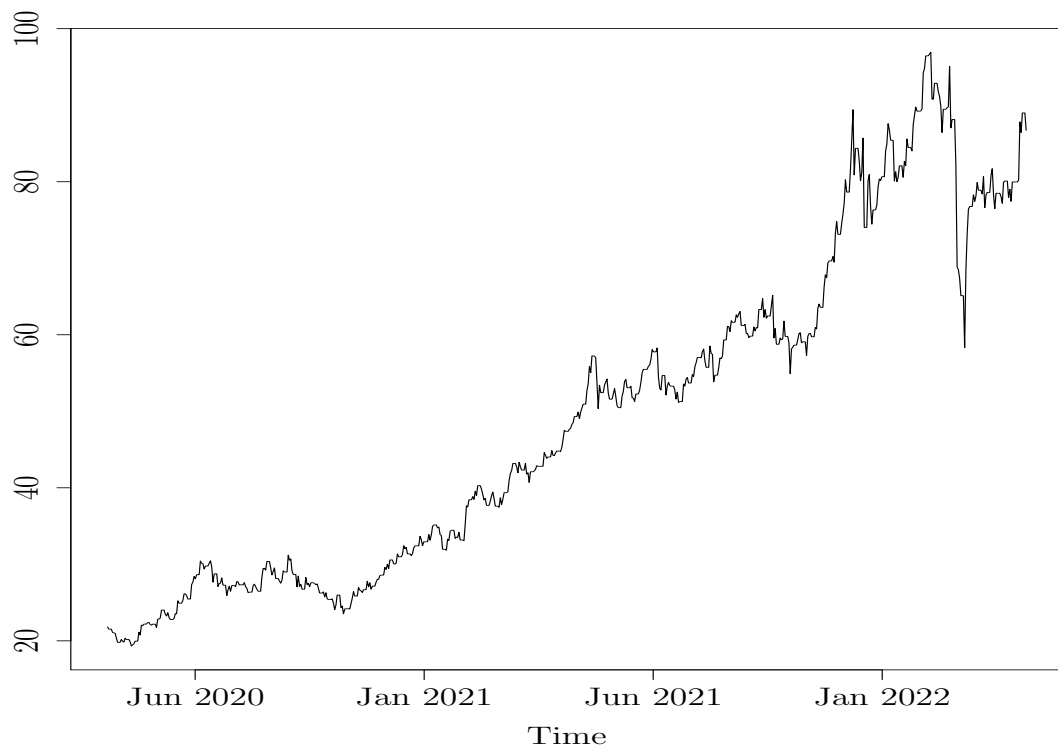


Figure 7.1: Time series plot of the EUA December 2022 futures daily closing price.

We use the R function `ndiffs()` to estimate the number of differences required to make the time series stationary, and perform an Augmented Dickey–Fuller (unit–root) test using the function `adf.test()`, with the null hypothesis being that the differenced time series is non–stationary. The output of the `ndiffs()` function is one, and we reject the null hypothesis (with a p –value of less than 0.01) and conclude that the once–differenced time series is stationary. Figure 7.2 shows the time series after one difference has been taken, and Figure 7.3 shows the autocorrelation functions at various lags of this de–trended time series.

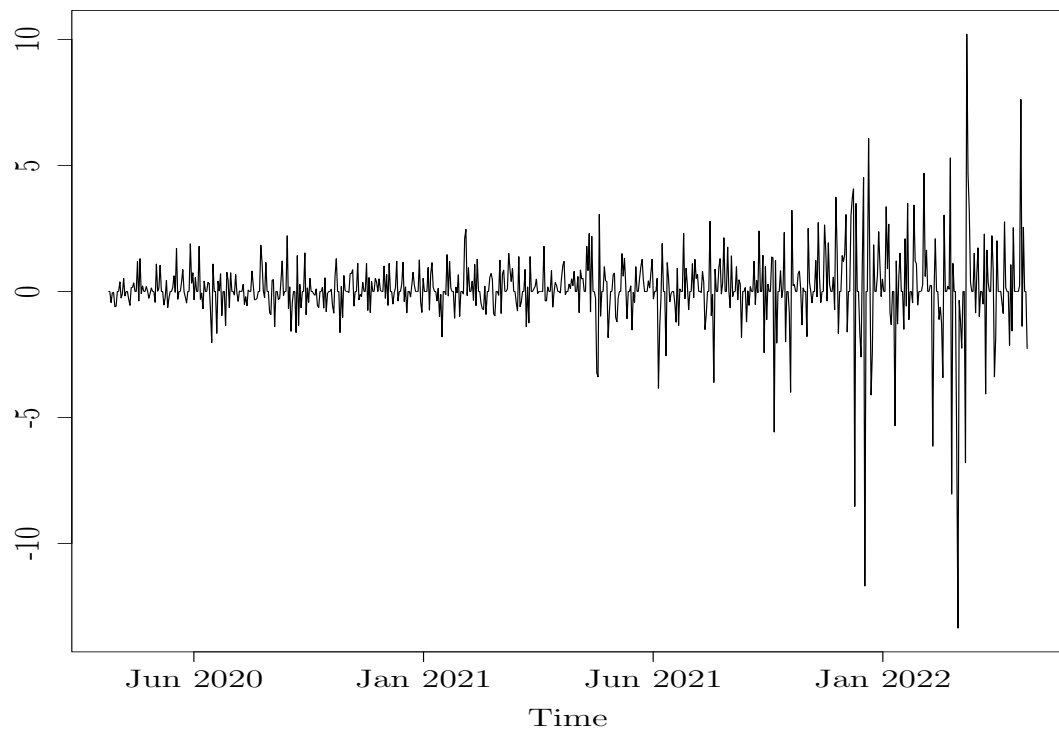


Figure 7.2: Time series plot of the de-trended (once differenced) EUA December 2022 futures daily closing price.

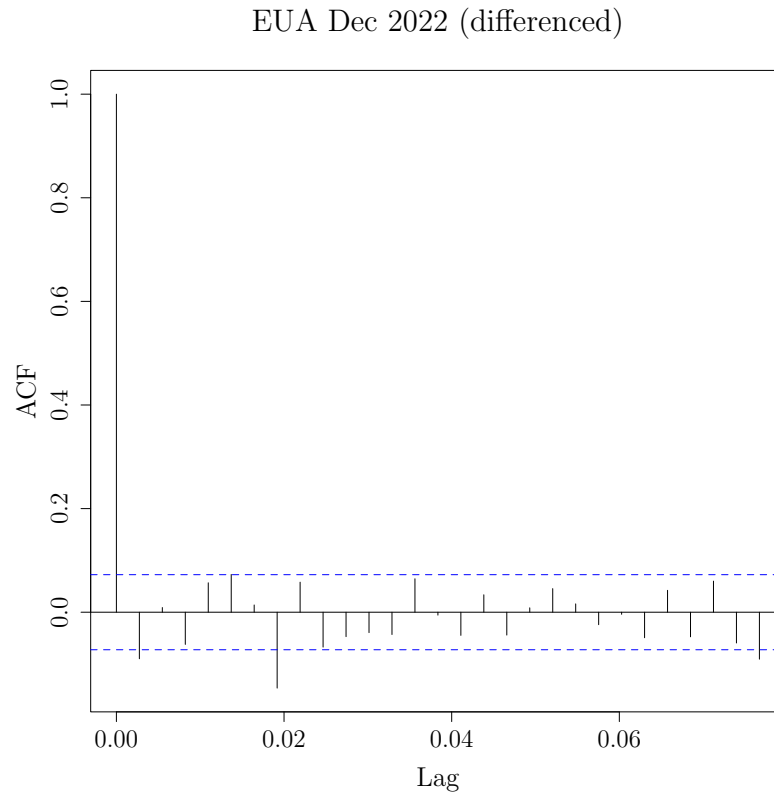


Figure 7.3: Plots of the ACF of the once differenced EUA December 2022 futures daily close time series.

7.2 Model calibration

A non-seasonal autoregressive integrated moving average (ARIMA) model is fitted to the data for the purpose of forecasting through the use of the `auto.arima()` function in R, see Hyndman and Khandakar (2008). A non-seasonal ARIMA(p,d,q) model, for $p, d, q \geq 0$, is given by

$$\left(1 - \sum_{i=1}^p \phi_i B^i\right) (1 - B)^d Y_t = c + \left(1 + \sum_{i=1}^q \theta_i B^i\right) \epsilon_t, \quad (7.2.1)$$

where B is the backshift (or lag) operator, that is $B^i Y_t = Y_{t-i}$ for $t > 1, i \geq 0$, ϕ_i are the autoregressive parameters, θ_i are the moving average parameters, $c \in \mathbb{R}$ is a drift, and ϵ_t is a white noise sequence with mean 0 and variance σ^2 , $\sigma \in \mathbb{R}^+$. The modelling task pertains to selecting the most appropriate values of p, q, d , where the criterion for ‘most

appropriate' will be some chosen forecast accuracy measure such as mean square error (MSE), mean absolute percentage error (MAPE), Akaike's Information Criteria (AIC), among others. The `auto.arima()` function, which belongs to the 'forecast' package, see Hyndman et al. (2022), selects the best models according to the AIC. Since it is not feasible to fit every possible model and observe which has the lowest AIC, the function fits an $\text{ARIMA}(p,d,q)$ model by using the following step-wise algorithm:

1. Choose d based on successive KPSS unit-root tests, see Kwiatkowski et al. (1992). It is assumed $0 \leq d \leq 2$ so that the approach does not lead to over-differencing (which can lead to inaccurate forecasts and wider prediction intervals).
2. Four initial models are fitted:
 - (i) $\text{ARIMA}(2,d,2)$,
 - (ii) $\text{ARIMA}(0,d,0)$,
 - (iii) $\text{ARIMA}(1,d,0)$,
 - (iv) $\text{ARIMA}(0,d,1)$,

where $c \in \mathbb{R}$, a drift constant, is included unless $d = 2$. If $0 \leq d \leq 1$, an $\text{ARIMA}(0,d,0)$ model without a constant is also fitted. Of the considered models, that with the lowest AIC is set as the 'current model'.

3. Variations on the current model are considered by:
 - (i) Varying p and/ or q from the current model by ± 1 ,
 - (ii) Including or excluding c , the drift constant, if $c = 0$ or $c \neq 0$, respectively,and if a model with a lower AIC is found it becomes the new 'current' model, for which the above procedure is repeated (until no model with a lower AIC can be found).

Maximum bounds may be set on either of the parameters, with default values of 5 on p and q . We keep these default upper bounds on p and q , and use the `auto.arima()` function to find the best fitting model on the data sets discussed in Section 7.1. The output of the function applied to the EUA December 2022 data is given in Table 7.1, and the in-sample fitted model is shown against the data in Figure 7.4.

Series: Dec22ts.EU			ARIMA(2,1,3) with drift				
Coefficients	AR1	AR2	MA1	MA2	MA3	Drift	
	-0.2489	-0.8724	0.1656	0.8751	-0.1502	0.0889	
s.e.	0.0520	0.0425	0.0606	0.0378	0.0386	0.0489	
σ^2 estimated as 2.224			log likelihood = -1328.47				
AIC = 2670.93		AICc = 2671.09		BIC = 2703.1			
Training set error measures:							
	ME	RMSE	MAE	MPE	MAPE	MASE	ACF1
Training set	-1.4953e-06	1.4843	0.8082	-0.0710	1.6065	0.0220	-0.00175

Table 7.1: Output of the `auto.arima()`, function applied to the EUA December 2022 futures time series data

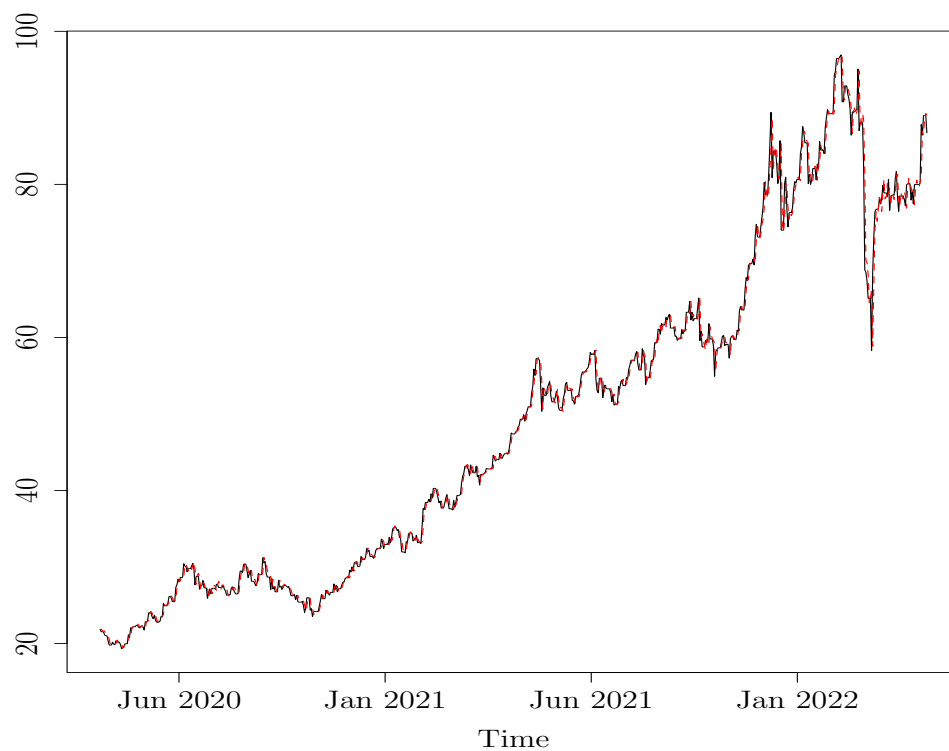


Figure 7.4: Time series for the EUA December 2022 (black line) against the fitted ARIMA(2,1,3) with drift $c = 0.0889$ model (red dashed line).

We compute the Ljung–Box test statistic on the residuals of the fitted model; the p -values are shown for lags 1–5 and 10 in Table 7.2. The purpose of this test is to determine whether any of the autocorrelations of the residuals at given lag are different from zero; the null hypothesis is that the data are independently distributed, and any observed correlations result from randomness. In all cases, we do not reject the null hypothesis at a 95% significance level (as $p > 0.05$). In agreement, Figure 7.5 shows the ACF of the residuals.

p -value (Ljung–Box statistic)	Lag					
	1	2	3	4	5	10
EUA Dec.22, ARIMA(2,1,3) w.drift	0.9621	0.9846	0.9797	0.7788	0.8326	0.0553

Table 7.2: p -values for the Ljung–Box test statistic for the residuals of the fitted model at lags 1–5 and 10.

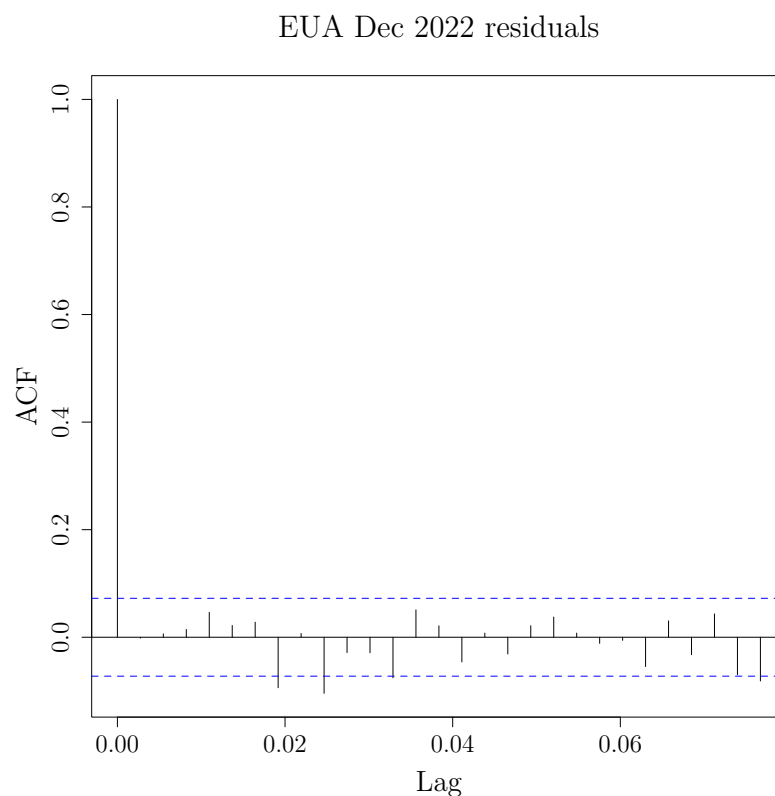


Figure 7.5: Plots of the ACF of the residuals of the fitted ARIMA(2,1,3) with drift $c = 0.0889$ model for the EUA December 2022 futures daily close time series.

7.3 Forecasting

We use the fitted ARIMA model given in Table 7.1 to obtain a forecast for each time series up to time $T = 31$ December 2022, i.e., we choose a forecast length of 249 days. The forecasted time series, with the 80% and 95% prediction intervals, is shown in Figure 7.6, and the forecast accuracy measures are shown in Table 7.3. The measures given in the tables are defined and discussed by Hyndman and Koehler (2006).

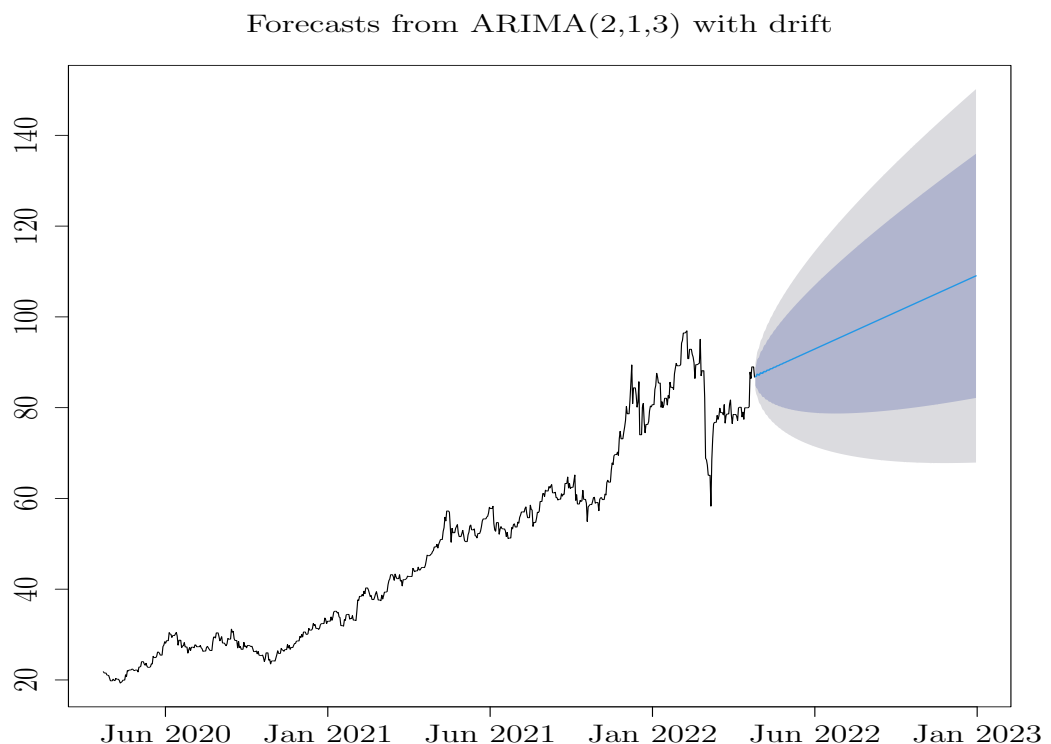


Figure 7.6: Forecasts from the fitted ARIMA(2,1,3), with drift $c = 0.0889$, model for a forecast horizon $h = 249$ days.

	ME	RMSE	MAE	MPE	MAPE	MASE	ACF1
Training set	-1.4953e-06	1.4843	0.8082	-0.0710	1.6065	0.0220	-0.0018

Table 7.3: Forecast accuracy measures of the ARIMA(2,1,3) with drift $c = 0.0889$ model fitted to the EUA December 2022 data.

Based on the data and forecasts, we construct the empirical Tukey- gh quantile

process, given in Eq. (7.0.1), as follows.

7.4 Construction of the empirical quantile process

Consider the quantile process in Eq. (7.0.1) where (Y_t) is a process modelling the EUA December 2022 daily futures prices. We want to construct the process (Z_t) from the data and the forecasts obtained from the fitted models, to obtain the quantile process–induced distorted measure, given in Eq. (7.0.3), so that we can price the insurance contract at time $T = 31$ December 2022, using Eq. (7.0.2). We consider a half–yearly, overlapping sliding window (that is, of length 182 days) that shifts by one day to produce the next window, and use the `runner()` function in R to compute the pseudo–observations, that is the process defined by $U_t \stackrel{d}{=} F_Y(t, Y_t)$, on each window. It follows that the empirical quantile process at time t corresponding to each sliding window is given by

$$Z_t \stackrel{d}{=} Q_{T_{gh}}(U_t; A, B, g, h) \quad (7.4.1)$$

for $A \in \mathbb{R}$, $B \in \mathbb{R}^+$, $g \in \mathbb{R} \setminus 0$ and $h \in \mathbb{R}_0^+$. We show the empirical distribution of the quantile process at $T = 31$ December 2022 (which corresponds to the last sliding window) for $A = 92.9430$, $B \in \{10, 30\}$ and a range of g and h parameters in Figure 7.7. The grey, vertical dashed lines lie at the values of the thresholds of the stop–loss contract that we consider, next.

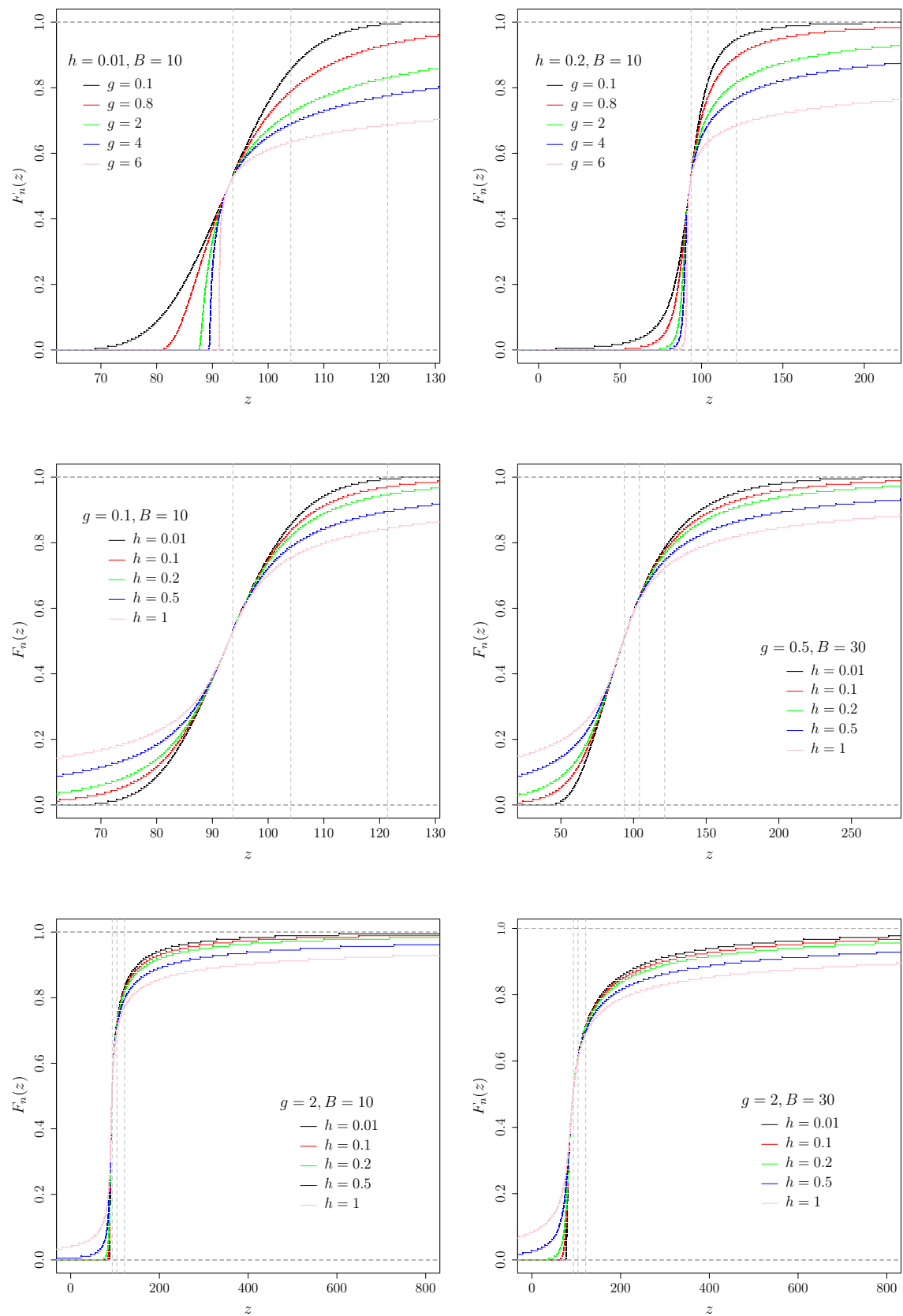


Figure 7.7: Empirical CDFs of the forecasted quantile process at time $T = 31$ December 2022 for a range of parameters in the Tukey- gh quantile function. The grey dashed lines lie at the considered threshold values $a \in \{93.6476, 104.064, 121.408\}$.

It is imperative in the quantile process construction to choose the parameters intuitively (with regard to the application at hand), particularly in the context of the discussion in Section 3.1 following Proposition 3.1.2, where one may wish to preserve some measure of centrality under the distortion induced by the composite map. We employ such an argument to set the parameter A . We consider the instance that $h = 0$ and $g \rightarrow 0$ in Eq. (7.4.1), where it would hold that $Z_t \approx A$ at any $t \in (0, \infty)$. If we consider such a case to represent ‘no distortion’, we may set A equal to the mode of the forecasted samples of Y_T , that is, $Z_T = \text{mode}(Y_T)$ where $T = 31$ December 2022 corresponds to the last sliding window, for which we have 182 samples of the forecasted driving processes. We vary the remaining parameters in the pricing case study, below.

7.5 Pricing results and sensitivities

We now employ the results given in the preceding subsections for the purpose of pricing the stop-loss contract. We recall that the premium is given by

$$\begin{aligned} \Pi_{t,T}^{\mathbb{P}^Z} &= e^{-r(T-t)} \mathbb{E}^{\mathbb{P}^Z} [(Y_T - a)^+ | \mathcal{F}_t] \\ &= e^{-r(T-t)} \int_{\{\omega \in \Omega : Y_T(\omega) \in [a, \infty)\}} (Y_T(\omega) - a) d\mathbb{P}_{T|t}^Z(\omega) \end{aligned} \quad (7.5.1)$$

where we have chosen the discount factor $B_t = e^{-rt}$ for $r \geq 0$ the interest rate, and

$$\mathbb{P}_{T|t}^Z(A) = \int_{\omega \in \Omega : Z_t(\omega) \in A} d\mathbb{P}(\omega | \mathcal{F}_t) \quad (7.5.2)$$

is the probability measure induced by the quantile process in Eq. 7.4.1 at time $T = 31$ December 2022, conditional on the information (in the form of the daily futures price data) available at time $t \in (0, T)$. The purpose of this empirical example is largely to illustrate the sensitivity of the insurance premium to the parameters in the quantile distortion, that is B , g and h , as well as the choice of copula. We set $A = \text{mode}(Y_T) = 92.9430$, as discussed in the previous section. The parameter $B \in \mathbb{R}^+$ ‘spreads’ out the empirical distribution of the quantile process; we consider two values of B for comparison of its impact on the output premiums. We also consider various thresholds $a > 0$, which the installation wishes to insure against the futures price going beyond. Such a threshold is likely to differ between market participants (e.g.,

installations and/ or the insurers), with those that are more risk averse setting a lower threshold, and vice versa. Whilst the threshold will likely be related to the quantity of allowances held by each installation and the potential civil penalties they may face if their reported emissions exceed such allowances, we consider, for the purpose of this case study, that $a > 0$ is some proportion of the price of the EUA December 2022 futures price at $t = 25$ April 2022. In other words, the insurance contract covers the possibility of the futures price rising, if $a > 1$, (or decreasing, if $a < 1$) by some percentage. The EUA December 2022 futures price at $t = 25$ April 2022 is 86.72. The pricing results for a range of distortion parameters, scale values $B \in \{10, 30\}$, and thresholds $a = k \times 87.72$ for $k \in \{1.08, 1.2, 1.4\}$, are given in Tables 7.4 and 7.5. We set $r = 0$ in line with the Euro Area interest rate as of 25 April 2022.

We consider the pricing results in regards to the plots of the empirical CDFs of the quantile process, for a range of parameter values, given in Figure 7.7, where the grey dashed lines lie at the considered threshold values $a \in \{93.6576, 104.064, 121.408\}$. In line with the stochastic ordering results given in Chapter 5 and Proposition 6.3.4, which relates such results to the QPVP, we observe ordered prices in both the g and h parameters. As such, the characterisation of the preferences of more risk averse agents will correspond to higher values of these skewness and kurtosis parameters. We also observe that, as expected, prices are higher for a lower threshold value a , as again, more risk averse agents will prefer insuring against a less extreme price rise, as well as the probability of the contract paying out with a lower threshold being more likely. As the threshold increases, i.e., to $a = 121.408$, the installation is insuring against more extreme events which are less probabilistically likely, and so we observe lower prices of the insurance contract. Additionally, referring to Figure 7.7, the closer the threshold a is to the median of the empirical distribution of Z_T , the choice of distortion parameters have less effect on the marginal distribution of the quantile process. The theoretical median of the quantile process is $A = 92.9430$, which is equal to the mode of the empirical distribution of the forecasted driving process at $T = 31$ December 2022. We construct the quantile process in such a way that the distortion has less effect around the mode of the driving process, A , since as the threshold a converges to A , the payoff of the insurance contract converges to 0.

B= 10		g						
a= 93.6576		0.1	0.5	0.8	1	2	4	6
h	0.01	1.8570	2.5007	3.2226	3.8794	12.3277	382.06	28476.16
	0.1	2.2630	3.1074	4.0776	4.9769	17.1972	615.65	48727.70
	0.2	2.9032	4.0844	5.48001	6.8000	25.8210	1065.92	89182.13
	0.5	7.6856	11.7318	16.8868	22.0272	107.54	6047.95	566704.52
a= 104.064		0.1	0.5	0.8	1	2	5	6
h	0.01	0.13088	0.3807	0.73720	1.07176	6.2655	269.64	21928.35
	0.1	0.2754	0.6477	1.1459	1.6656	9.1490	435.32	37524.36
	0.2	0.5368	1.1060	1.8799	2.6297	14.5382	754.70	68678.92
	0.5	2.7723	5.0498	7.9612	11.1110	66.3937	4358.76	436426.05
a= 121.408		0.1	0.5	0.8	1	2	4	6
h	0.01	7.4535e-05	0.01513	0.08964	0.1976	3.1693	210.82	18653.63
	0.1	0.0089	0.0768	0.2419	0.4496	5.0618	341.19	31921.87
	0.2	0.06633	0.2456	0.5671	0.9572	8.7647	604.83	59451.52
	0.5	1.0822	2.3251	4.1337	6.0892	45.8372	3513.67	377799.76

Table 7.4: Premiums obtained under the QPVP for $A = 92.9430$, $B = 10$, and range of distortion parameters $g \in \{0.1, 0.5, 0.8, 1, 2, 4, 6\}$ and $h \in \{0.01, 0.1, 0.2, 0.5\}$. The thresholds of the stop-loss contract that are considered are $a \in \{93.6576, 104, 064, 121.408\}$.

B= 30		g						
a= 93.6576		0.1	0.5	0.8	1	2	4	6
h	0.01	6.1034	8.1021	10.3433	12.3827	38.6117	1186.50	87392.69
	0.1	7.3639	9.9853	12.9978	15.7899	53.7299	1911.73	149543.96
	0.2	9.3514	13.0188	17.3517	21.4500	80.5039	3309.6575	273697.22
	0.5	24.1992	36.7612	52.7657	68.7252	334.1997	18777.12	1739196.96
a= 104.064		0.1	0.5	0.8	1	2	5	6
h	0.01	2.8778	4.4578	6.2278	7.7804	29.0234	997.94	75607.41
	0.1	3.7967	5.8742	8.2548	10.5352	41.0832	1609.09	129378.73
	0.2	5.3272	8.1563	11.5801	14.9223	62.4413	2787.12	236792.23
	0.5	16.3330	26.4142	39.2024	51.5704	268.56	15821.50	1504696.50
a= 121.408		0.1	0.5	0.8	1	2	4	6
h	0.01	0.6264	1.5631	2.7639	3.9048	20.3220	836.01	66767.60
	0.1	1.1674	2.4499	4.1521	5.7739	29.4855	1349.37	114253.96
	0.2	2.0552	4.0146	6.4773	8.9337	46.5678	2376.04	209112.63
	0.5	9.4324	16.8713	26.3506	35.8227	207.0160	13498.69	1348361.76

Table 7.5: Premiums obtained under the QPVP for $A = 92.9430$, $B = 30$, and range of distortion parameters $g \in \{0.1, 0.5, 0.8, 1, 2, 4, 6\}$ and $h \in \{0.01, 0.1, 0.2, 0.5\}$. The thresholds of the stop-loss contract that are considered are $a \in \{93.6576, 104, 064, 121.408\}$.

7.6 Multidimensional extension

We now consider the multidimensional quantile process, given in Section 3.4, and its induced probability measure, for the purpose of incorporating multiple risk drivers in the valuation principle. This extension of the above example serves the purpose of illustrating the comparison between the univariate and multidimensional QPVP. We consider the driving processes $(Y_t^{(1)})$ and $(Y_t^{(2)})$, where $(Y_t^{(1)})$ represents the EUA futures with a December 2022 expiry date (i.e., the risk driver considered in the univariate case), and $(Y_t^{(2)})$ the EUA futures price process with a different expiry date, that is March 2023 (symbol CKH2023). This allows us to observe how premiums are impacted by including

of more ‘information’ regarding the EUA futures market, and the effect of considering different maturities. We choose the empirical distribution functions of each marginal driving process and, the empirical copula (the nonparametric Bernstein copula, see, e.g., Definition 1 by Sancetta and Satchell (2004)), and the Tukey– gh quantile function. It follows that the Tukey– gh multidimensional random–level quantile process, given by Definition 3.4.1, is defined by

$$\begin{aligned} Z_t &\stackrel{d}{=} Q_{T_{gh}} \left(C_Y \left(t, F_Y^{(1)} \left(t, Y_t^{(1)} \right), F_Y^{(2)} \left(t, Y_t^{(2)} \right) \right); A, B, g, h \right) \\ &\stackrel{d}{=} Q_{T_{gh}} \left(F_Y \left(t, Y_t^{(1)}, Y_t^{(2)} \right); A, B, g, h \right), \end{aligned} \quad (7.6.1)$$

for all $t \in (0, \infty)$. Here, $F_Y^{(i)}$ is the marginal distribution of the i^{th} driving process, and F_Y is the marginal joint distribution function of the two driving processes. The premium obtained under the QPVP is again given by Eq. (7.0.2) where, now, \mathbb{P}^Z is the probability measure induced by the quantile process in Eq. (7.6.1). The premium accounts for the dependence between the underlying risk and auxiliary risk factors through the copula C , and the effect of the distortion map through the choice of the A, B, g and h parameters in the quantile function $Q_{T_{gh}}$, as well as the choice F_Y . The data is treated analogously to the EUA December 2022 data in Sections 7.1 and 7.2, and the `auto.arima()` function is applied to fit an ARIMA(2,1,3) model, as shown in Table 7.6.

Series: Mar23ts.EU		Model: ARIMA(2,1,3)					
Coefficients		AR1	AR2	MA1	MA2	MA3	
		-0.2636	-0.8715	0.1846	0.8806	-0.1449	
s.e.		0.0792	0.0559	0.0899	0.0552	0.0540	
σ^2 estimated as 3.757		log likelihood = -823.78					
AIC = 1659.56		AICc = 1659.78			BIC = 1683.47		
Training set error measures:							
	ME	RMSE	MAE	MPE	MAPE	MASE	ACF1
Training set	0.1325	1.9237	1.0984	0.1708	1.6152	0.0297	-0.0055

Table 7.6: Output of the `auto.arima()`, function applied to the EUA March 2023 futures time series data

Using the fitted model, we obtain forecasts with a forecast length of $h = 249$ days. The forecasted time series, with the 80% and 95% prediction intervals, is shown in Figure 7.8, and the forecast accuracy measures are shown in Table 7.7.

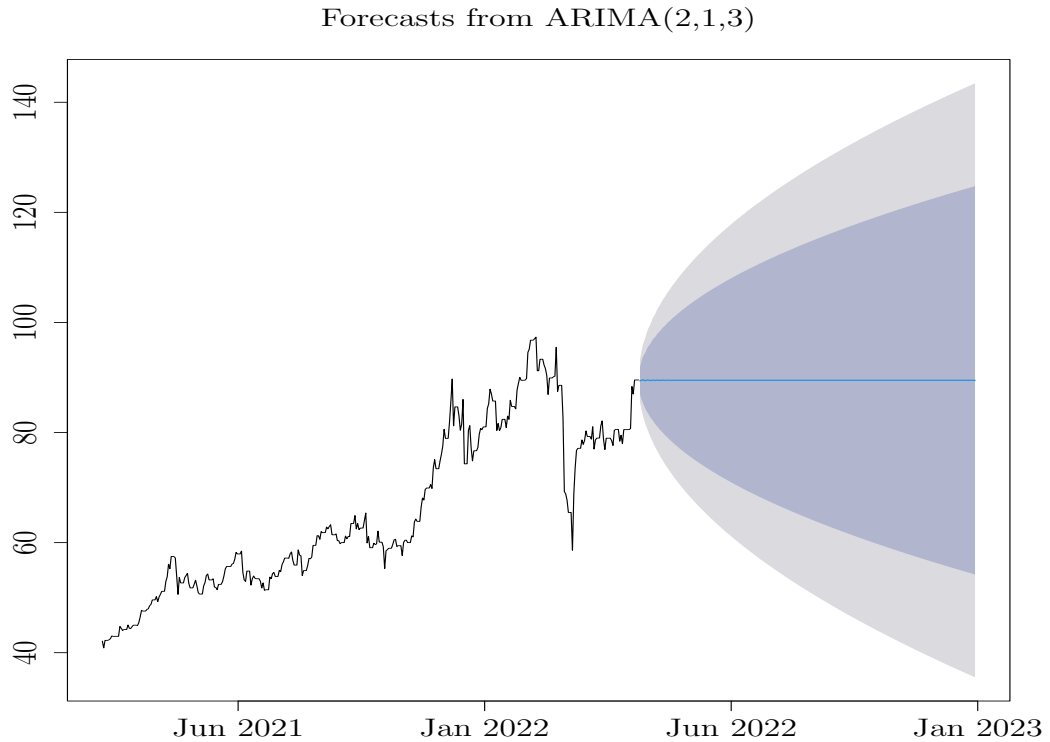


Figure 7.8: Forecasts from the fitted ARIMA(2,1,3) model for a forecast horizon $h = 249$ days.

	ME	RMSE	MAE	MPE	MAPE	MASE	ACF1
Training set	0.1325	1.9237	1.0984	0.1708	1.6153	0.0297	-0.0055

Table 7.7: Forecast accuracy measures of the ARIMA(2,1,3) model fitted to the EUA March 2023 data.

Since the data is bivariate, we use the function `Bcopula()` from the package ‘subcopem2D’, see Erdely (2019), to produce the Bernstein copula approximation from the empirical subcopula of the data, which we denote $C_B(t, \cdot)$. We do this on each window, as we do not necessarily assume that the joint dependence structure between the data

sets (modelled by the two driving processes) is stationary. It follows that the empirical quantile process at time t corresponding to each sliding window is given by

$$\begin{aligned} Z_t &\stackrel{d}{=} Q_{T_{gh}} \left(C_B \left(t, U_t^{(1)}, U_t^{(2)} \right); A, B, g, h \right) \\ &\stackrel{d}{=} A + \frac{B}{g} \left[\exp \left(g\sqrt{2} \operatorname{erf}^{-1} \left(2C_B \left(t, U_t^{(1)}, U_t^{(2)} \right) - 1 \right) \right) - 1 \right] \\ &\quad \times \exp \left(h \left(\operatorname{erf}^{-1} \left(2C_B \left(t, U_t^{(1)}, U_t^{(2)} \right) - 1 \right) \right)^2 \right). \end{aligned} \quad (7.6.2)$$

Figure 7.9 shows the contour plots of the empirical copula between the EUA December 2022 and EUA March 2023 data sets at windows $i \in \{182, 239, 296, 353, 400, 466\}$, where $i = 182$ corresponds to the first full length (half-yearly) window, and $i = 466$ corresponds to the last window, i.e., that at time T . In Figure 7.10, we show the empirical distribution of the multidimensional quantile process at $T = 31$ December 2022 (which corresponds to the last sliding window) for $A \in \{92.9430, 106.8844\}$, $B = 10$ and a range of g and h parameters. The grey, vertical dashed lines lie at the value of the thresholds of the stop-loss contract that we consider in the univariate QPVP example above. The effect of distributional distortions in the multivariate setting is less transparent than the univariate setting. For example, the output process Z_T is no longer distributed according to the Tukey- gh distribution, however ‘inherits’ properties (e.g., increasing heavy-tailedness in the h parameter, as shown in Figure 7.10) from the distribution, when the Tukey- gh quantile function is used in the composite map. We also see from the empirical CDF plots that considering the same parameter $A = 92.9430$ (which is equal to the empirical mode of $Y_T^{(1)}$), for the given thresholds of the insurance contract, may not lead to ordered prices, as desired. The thresholds, for many of the considered combinations of parameters, sit in the area where the empirical CDF is equal to one. This is less so the case in the second two plots in Figure 7.10, where we consider $A = 106.8844$, that is 15% higher than the previously considered value, as the increased value of A shifts the location of the distribution to the right. Intuitively, it makes sense that we need not necessarily consider $A = \operatorname{mode}(Y_T^{(1)})$ in the multidimensional quantile process construction, as the composite map is not defined in such a way that we distort around this measure of centrality, as in the univariate case. The lack of transparency of the impact of multivariate distortions, relative to the univariate case, gives rise to interesting future work in better understanding the impact of distortion maps and

copulas when applied to multivariate driving processes. We further discuss the choice of parameters of the multidimensional quantile process construction in regard to the pricing results given in Table 7.8.

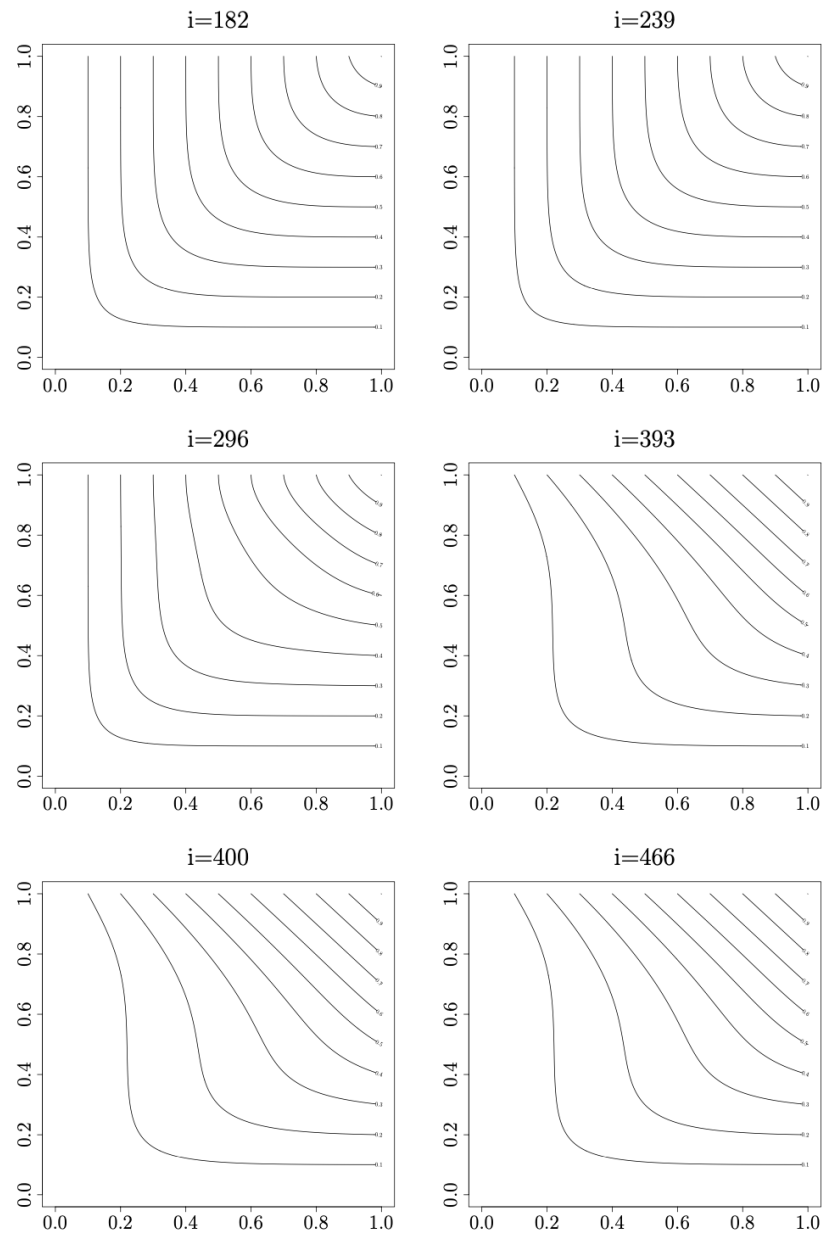


Figure 7.9: Contour plots of the Bernstein copula based on the empirical EUA December 2022 and EUA March 2023 futures data at time windows $i \in \{1, 94, 188, 282, 376, 466\}$, where $i = 466$ corresponds to the last window, i.e., that at time T .

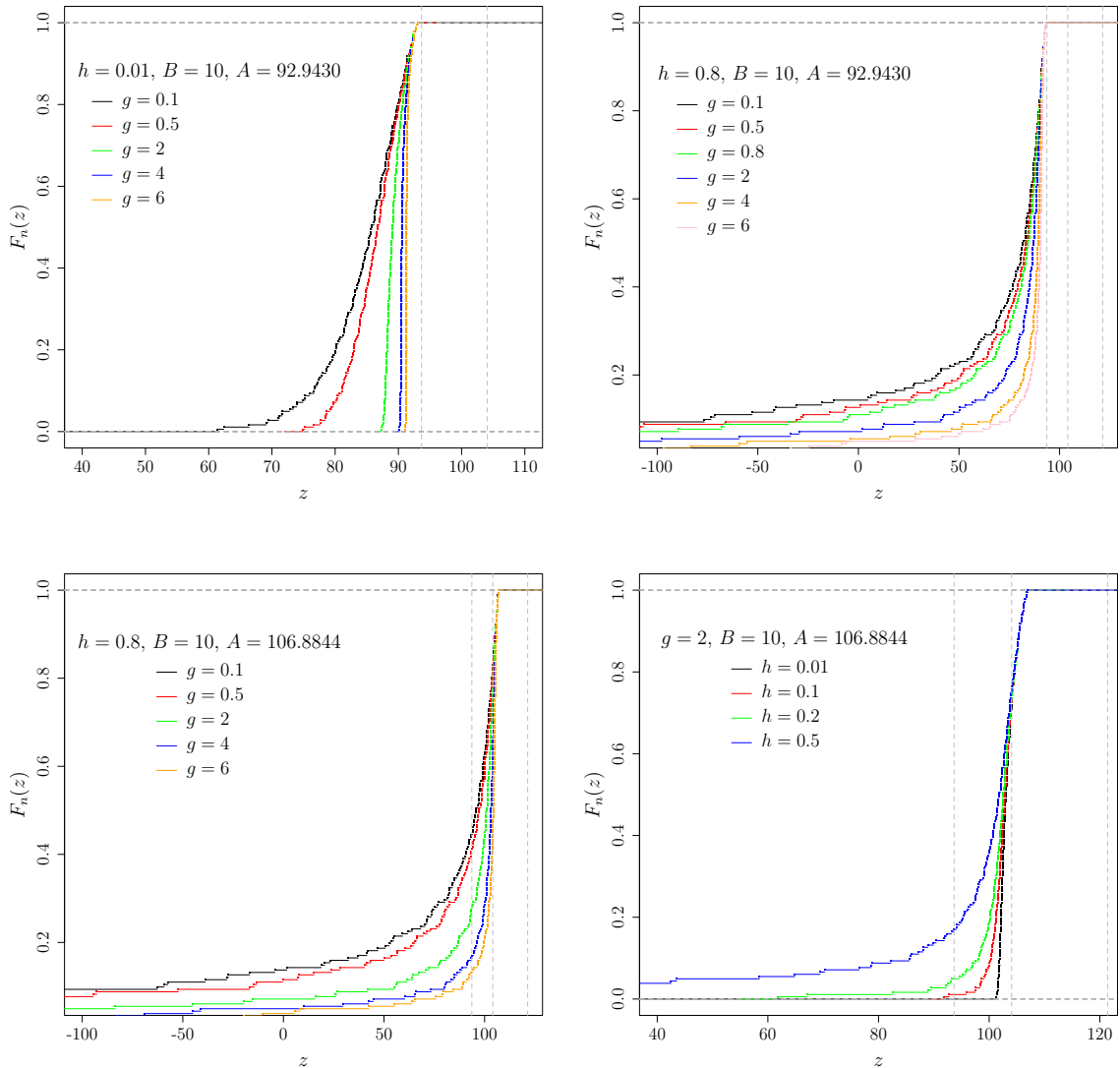


Figure 7.10: Empirical CDFs of the forecasted quantile process at time $T = 31$ December 2022, for a range of parameters in the Tukey– gh quantile function. The grey dashed lines lie at the considered threshold values $a \in \{93.6476, 104, 064, 121, 408\}$.

We present some pricing results obtained under the multidimensional QPVP in Table 7.8. We consider the same values of the g and h parameters as those considered in the univariate case, and we increase the location parameter by 15% to $A = 106.8844$. If $A = 92.9430$, we have $\mathbb{P}^Z(Y_T^{(1)} > a) = 0$ for most cases of the considered distortion parameters, as shown in the distribution plots in Figure 7.10, and so $\Pi_{t,T}^{\mathbb{P}^Z, \zeta} = 0$ and the insurer does not sell the contract. Considering the results in Table 7.8, we again

observe an ordering in the g parameter, however this is no longer the case with regard to the h parameter. In Table 7.9, we set $A = 126.9485$ to match the premium given in Table 7.5 for $B = 30$, $g = 0.1$, $h = 0.01$. This provides a direct comparison as to how the prices increase with regard to increasing g and h from this reference point. In this example, the increase in information (in the form of price data from the EUA March 2023 futures contract) reduces the sensitivity of premiums to changes in the parameters.

A=106.8844		g						
a= 93.6576		0.1	0.5	0.8	1	2	4	6
h	0.01	2.8288	3.2741	3.6618	3.8993	4.6703	5.3403	5.6352
	0.2	2.5580	2.8509	3.0962	3.2662	4.0531	4.9474	5.3480
	0.5	2.3010	2.5047	2.6732	2.7874	3.3817	4.2738	4.7623
a= 104.064		0.1	0.5	0.8	1	2	4	6
h	0.01	0.0328	0.0332	0.03352	0.03371	0.0348	0.0376	0.0411
	0.2	0.0328	0.0332	0.0335	0.0337	0.0347	0.0375	0.0410
	0.5	0.0328	0.0332	0.0335	0.0336	0.0347	0.0374	0.0409

Table 7.8: Premiums obtained under the QPVP for $A = 106.8844$, $B = 10$, and range of distortion parameters $g \in \{0.1, 0.5, 0.8, 1, 2, 4, 6\}$ and $h \in \{0.01, 0.2, 0.5\}$. The thresholds of the stop-loss contract that are considered are $a \in \{93.6576, 104, 064\}$.

A=126.9485		g						
a= 93.6576		0.1	0.5	0.8	1	2	4	6
h	0.01	6.1034	7.0506	7.9580	8.6268	10.9397	12.9497	13.8344
	0.1	5.8352	6.5996	7.2817	7.7772	10.1028	12.4794	13.5182
	0.2	5.6041	6.2451	6.8033	7.2034	9.2535	11.8136	13.0066
	0.5	5.1062	5.5590	5.9373	6.2021	7.6497	10.0325	11.4248

Table 7.9: Premiums obtained under the QPVP for $A = 126.9485$, $B = 30$, and range of distortion parameters $g \in \{0.1, 0.5, 0.8, 1, 2, 4, 6\}$ and $h \in \{0.01, 0.1, 0.2, 0.5\}$. The thresholds of the stop-loss contract that are considered are $a \in \{93.6576, 104, 064\}$.

Chapter 8

Conclusions

In this thesis, two new classes of stochastic processes are developed, that model the continuous-time evolution of quantiles and quantile functions. The first of these constructions is predominantly focused on. Here, the statistical properties of the finite-dimensional distributions of the process are characterised by a composite map for which direct interpretation is provided, and a focus is placed on such properties *relative* to those of the base driving process from which it is constructed. This allows one to take a stochastic process (we focus on those with càdlàg paths), e.g., a process that may already be calibrated to some data set, and transform it in such a way that all introduced statistical properties (e.g., skewness and kurtosis) are controlled via the parametric distortion. We consider the Tukey family of transformations in the quantile process construction, and motivate such a choice by the above reason, being that skewness and kurtosis are directly parameterised in the transformation.

Following the quantile process construction, we focus on the probability measures induced by the processes, and become intrigued by the properties that such measures may bring to dynamic risk analysis and a pricing application. It is not novel to study distorted measures in the context of risk valuation, however we consider the constructive approach of the quantile process-induced measures in this thesis, and the properties they exhibit, as an innovative way to extend this area of literature. With such a construction, one may develop a probability measure that produces a wide range of statistical properties—covering an extensive skew-kurtosis range—in the evaluation of distributions under the measure. Necessary and sufficient conditions are derived under which the quantile processes satisfy first- and second-order stochastic dominance, thus

producing ordered parametric families of measure distortions. We consider the conditional expectation under the distorted measure as a member of the time-consistent class of dynamic valuation principles, and extend it to the setting where the driving risk process is multivariate. This requires the introduction of a copula function in the composite map for the construction of quantile processes, which presents another new element in the risk quantification and modelling framework based on probability measure distortions induced by quantile processes.

The second quantile process construction, which models the dynamic evolution of the entire quantile function through time, is presented as a continuous-time extension to the dynamic quantile function (DQM) models presented by Chen et al. (2022). Here, a quantile function is considered (more complex quantile functions may be obtained by applying the discussed quantile-preserving maps to simpler quantile functions), and its parameters are modelled by continuous-time càdlàg processes. There exists much room for exploration of this class of models and what they can achieve, in further work.

In summary, we consider the work presented in this thesis as the start of a framework that has vast potential in various applications, as well as to further examine theoretically. It opens up many interesting questions in connection with additional families of such models that may be developed and the breadth of statistical behaviours they may inherit, as well as a deeper exploration into their dynamic properties. We believe, as a next step, it would be of particular interest to examine serial dependence structures, and the parametric control one may have over them, in the quantile process constructions. Additionally, there is much room for development of the multidimensional random-level quantile processes in application, as well as in regard to their stochastic orderings and the distorted measures they induce. This thesis is of a theoretical nature, and so going forward, one may also wish to take a more statistical approach in the estimation and calibration of the models, and their descriptive capabilities of large data sets.

Bibliography

- K. K. Aase. Equilibrium in a reinsurance syndicate; existence, uniqueness and characterization. *ASTIN Bulletin: The Journal of the IAA*, 23(2):185–211, 1993.
- M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55. United States Department of Commerce, National Bureau of Standards, Washington, DC, 1964.
- B. Acciaio and I. Penner. Dynamic risk measures. In *Advanced mathematical methods for finance*, pages 1–34. Springer, Berlin, Heidelberg, 2011.
- J. Akahori. Some formulae for a new type of path-dependent option. *The Annals of Applied Probability*, 5(2):383–388, 1995.
- R. B. Arellano-Valle, M. D. Branco, and M. G. Genton. A unified view on skewed distributions arising from selections. *Canadian Journal of Statistics*, 34(4):581–601, 2006.
- P. Artzner, F. Delbaen, J.-M. Eber, and D. Heath. Coherent measures of risk. *Mathematical Finance*, 9(3):203–228, 1999.
- A. Azzalini. The skew-normal distribution and related multivariate families. *Scandinavian Journal of Statistics*, 32(2):159–188, 2005.
- A. Azzalini. References on the skew-normal distribution and related ones, 2008. URL <http://azzalini.stat.unipd.it/SN/list-publ.pdf>.
- A. Azzalini and A. Capitanio. Distributions generated by perturbation of symmetry with emphasis on a multivariate skew-t distribution. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 65(2):367–389, 2003.

- R. R. Bahadur. A note on quantiles in large samples. *The Annals of Mathematical Statistics*, 37(3):577–580, 1966.
- K. P. Balanda and H. L. MacGillivray. Kurtosis and spread. *Canadian Journal of Statistics*, 18(1):17–30, 1990.
- K. Barigou, V. Bignozzi, and A. Tsanakas. Insurance valuation: A two-step generalised regression approach. *ASTIN Bulletin: The Journal of the IAA*, 52(1):211–245, 2022.
- O. Barndorff-Nielsen and D. R. Cox. Edgeworth and saddle-point approximations with statistical applications. *Journal of the Royal Statistical Society: Series B (Methodological)*, 41(3):279–299, 1979.
- N. Bäuerle and A. Müller. Stochastic orders and risk measures: consistency and bounds. *Insurance: Mathematics and Economics*, 38(1):132–148, 2006.
- B. Beranger, H. Lin, and S. A. Sisson. New models for symbolic data analysis. *Preprint arXiv:1809.03659*, 2018.
- E. Bibbona, L. Sacerdote, and E. Torre. A copula-based method to build diffusion models with prescribed marginal and serial dependence. *Methodology and Computing in Applied Probability*, 18(3):765–783, 2016.
- P. J. Bickel. Edgeworth expansions in nonparametric statistics. *The Annals of Statistics*, pages 1–20, 1974.
- L. Billard. Brief overview of symbolic data and analytic issues. *Statistical Analysis and Data Mining: The ASA Data Science Journal*, 4(2):149–156, 2011.
- L. Billard and E. Diday. From the statistics of data to the statistics of knowledge: symbolic data analysis. *Journal of the American Statistical Association*, 98(462):470–487, 2003.
- J. Bion-Nadal. Dynamic risk measuring: discrete time in a context of uncertainty, and continuous time on a probability space. *École polytechnique*, 2006.
- J. Bion-Nadal. Dynamic risk measures: time consistency and risk measures from BMO martingales. *Finance and Stochastics*, 12(2):219–244, 2008.

- J. Bion-Nadal. Time consistent dynamic risk processes. *Stochastic Processes and Their Applications*, 119(2):633–654, 2009.
- V. I. Bogachev and M. A. S. Ruas. *Measure theory*, volume 1. Springer, Berlin, Heidelberg, 2007.
- K. Borch. Equilibrium in a reinsurance market. *Econometrica*, 30(3):424–444, 1962.
- S. P. Boyd and L. Vandenberghe. *Convex optimization*. Cambridge University Press, Cambridge, England, 2004.
- H. Brannelly, A. Macrina, and G. Peters. Stochastic measure distortions induced by quantile processes for risk quantification and valuation. *Available at SSRN 3982716*, 2021a.
- H. Brannelly, A. Macrina, and G. W. Peters. Quantile diffusions for risk analysis. *Available at SSRN 3508702*, 2021b.
- M. Buchinsky. Recent advances in quantile regression models: a practical guideline for empirical research. *Journal of Human Resources*, 33(1):88–126, 1998.
- H. Bühlmann. *Mathematical methods in risk theory*. Springer, Berlin, Heidelberg, 1970.
- H. Bühlmann. An economic premium principle. *ASTIN Bulletin: The Journal of the IAA*, 11(1):52–60, 1980.
- H. Bühlmann. The general economic premium principle. *ASTIN Bulletin: The Journal of the IAA*, 14(1):13–21, 1984.
- P. Capérea, A.-L. Fougères, and C. Genest. A stochastic ordering based on a decomposition of Kendall’s tau. In *Distributions with given marginals and moment problems*, pages 81–86. Springer, Berlin, Heidelberg, 1997.
- C. V. L. Charlier. *Über das Fehlergesetz*. Arkiv för matematik, astronomi och fysik. Almqvist & Wiksells Boktryckery-A.-B., Stockholm, Sweden, 1905.
- W. Y. Chen, G. W. Peters, R. H. Gerlach, and S. A. Sisson. Dynamic quantile function models. *Quantitative Finance*, pages 1–27, 2022.
- G. Choquet. Theory of capacities. *Annales de l’institut Fourier*, 5:131–295, 1954.

- E. A. Cornish and R. A. Fisher. Moments and cumulants in the specification of distributions. *Revue de l'Institut international de Statistique*, 5(4):307–320, 1938.
- M. G. Cruz, G. W. Peters, and P. V. Shevchenko. *Fundamental aspects of operational risk and insurance analytics: A handbook of operational risk*. John Wiley & Sons, Hoboken, NJ, 2015.
- M. Csörgő. *Quantile processes with statistical applications*. SIAM, Philadelphia, Pennsylvania, 1983.
- M. Csörgő and P. Révész. Strong approximations of the quantile process. *The Annals of Statistics*, 6(4):882–894, 1978.
- M. Csörgő, S. Csörgő, L. Horváth, and D. M. Mason. Weighted empirical and quantile processes. *The Annals of Probability*, 14(1):31–85, 1986.
- H. E. Daniels. Saddlepoint approximations in statistics. *The Annals of Mathematical Statistics*, pages 631–650, 1954.
- A. Dassios. The distribution of the quantile of a Brownian motion with drift and the pricing of related path-dependent options. *The Annals of Applied Probability*, 5(2):389–398, 1995.
- A. Dassios. Sample quantiles of stochastic processes with stationary and independent increments. *The Annals of Applied Probability*, 6(3):1041–1043, 1996.
- E. G. De Giorgi. Reward-risk portfolio selection and stochastic dominance. *Journal of Banking & Finance*, 29(4):895–926, 2005.
- C. G. de Vries, M. Sarma, B. N. Jorgensen, J.-P. Zigrand, J. Danielsson, et al. Consistent measures of risk. Technical report, Financial Markets Group, 2006.
- M. Degen, P. Embrechts, and D. D. Lambrigger. The quantitative modeling of operational risk: between g-and-h and EVT. *ASTIN Bulletin: The Journal of the IAA*, 37(2):265–291, 2007.
- D. Denneberg. Distorted probabilities and insurance premiums. *Methods of Operations Research*, 63(3):3–5, 1990.

- D. Denneberg. *Non-additive measure and integral*, volume 27. Springer Science & Business Media, Berlin, Heidelberg, 2013.
- K. Detlefsen and G. Scandolo. Conditional and dynamic convex risk measures. *Finance and Stochastics*, 9(4):539–561, 2005.
- I. Durant, C. Contreras, R. Hamwey, G. Mott, A. Nicita, R. Peters, C. Razo, D. Vivas, M. Chepeliev, and E. Corong. A European Union carbon border adjustment mechanism: Implications for developing countries, 2021.
- EC Europa. EU Emissions Trading System (EU ETS), 2015. URL https://ec.europa.eu/clima/eu-action/eu-emissions-trading-system-eu-ets_en.
- F. Edgeworth. On the representation of statistical frequency by a series. *Journal of the Royal Statistical Society*, 70(1):102–106, 1907.
- P. Embrechts and M. Hofert. A note on generalized inverses. *Mathematical Methods of Operations Research*, 77(3):423–432, 2013.
- P. Embrechts, L. C. G. Rogers, and M. Yor. A proof of Dassios’ representation of the α -quantile of Brownian motion with drift. *The Annals of Applied Probability*, 5(3):757–767, 1995.
- A. Erdely. *subcopem2D: Bivariate Empirical Subcopula*, 2019. URL <https://CRAN.R-project.org/package=subcopem2D>. R package version 1.3.
- F. Esscher. On the probability function in the collective theory of risk. *Skand. Aktuarie Tidsskr.*, 15(3):175–195, 1932.
- W. Feller. *An introduction to probability theory and its applications, vol 2*. John Wiley & Sons, Hoboken, NJ, 2008.
- C. Field and M. G. Genton. The multivariate g-and-h distribution. *Technometrics*, 48(1):104–111, 2006.
- M. Fischer and I. Klein. Kurtosis modelling by means of the j-transformation. *Allgemeines Statistisches Archiv*, 88(1):35–50, 2004.

- H. Föllmer and A. Schied. Robust preferences and convex risk measures. *Advances in Finance and Stochastics*, pages 39–56, 2002a.
- H. Föllmer and A. Schied. Convex measures of risk and trading constraints. *Finance and Stochastics*, 6(4):429–447, 2002b.
- H. Föllmer and A. Schied. *Stochastic finance: An introduction in discrete time*. de Gruyter Studies in Mathematics, Berlin, Germany, 2nd edition, 2004.
- E. Furman and R. Zitikis. Weighted premium calculation principles. *Insurance: Mathematics and Economics*, 42(1):459–465, 2008.
- F. Galton. Statistics by intercomparison, with remarks on the law of frequency of error. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 49(322):33–46, 1875.
- F. Galton. Report of the anthropometric committee. In *Report of British Association for the Advancement of Science*, volume 51, pages 225–72, 1882.
- F. Galton. *Inquiries into human faculty and its development*. Macmillan, London, England, 1883.
- F. Galton. The most suitable proportion between the value of first and second prizes. *Biometrika*, 1(4):385–399, 1902.
- C. Genest and L.-P. Rivest. Statistical inference procedures for bivariate Archimedean copulas. *Journal of the American Statistical Association*, 88(423):1034–1043, 1993.
- C. Genest and L.-P. Rivest. On the multivariate probability integral transformation. *Statistics & Probability Letters*, 53(4):391–399, 2001.
- M. G. Genton. Discussion of “the skew-normal”. *Scandinavian Journal of Statistics*, 32(2):189–198, 2005.
- W. Gilchrist. *Statistical modelling with quantile functions*. CRC Press, Boca Raton, Florida, 2000.
- F. Godin, S. Mayoral, and M. Morales. Contingent claim pricing using a normal inverse Gaussian probability distortion operator. *Journal of Risk and Insurance*, 79(3):841–866, 2012.

- F. Godin, V. S. Lai, and D.-A. Trottier. A general class of distortion operators for pricing contingent claims with applications to CAT bonds. *Scandinavian Actuarial Journal*, 2019(7):558–584, 2019.
- M. Goovaerts, F. E. De Vylder, and J. Haezendonck. *Insurance premiums*. North-Holland, Amsterdam, NL, 1984.
- M. J. Goovaerts, R. Kaas, J. Dhaene, and Q. Tang. Some new classes of consistent risk measures. *Insurance: Mathematics and Economics*, 34(3):505–516, 2004.
- J. Hadar and W. R. Russell. Rules for ordering uncertain prospects. *The American Economic Review*, 59(1):25–34, 1969.
- M. Hamada and M. Sherris. Contingent claim pricing using probability distortion operators: methods from insurance risk pricing and their relationship to financial theory. *Applied Mathematical Finance*, 10(1):19–47, 2003.
- G. Hanoch and H. Levy. The efficiency analysis of choices involving risk. *The Review of Economic Studies*, 36(3):335–346, 1969.
- M. A. Haynes, H. MacGillivray, and K. Mengersen. Robustness of ranking and selection rules using generalised g-and-k distributions. *Journal of Statistical Planning and Inference*, 65(1):45–66, 1997.
- D. C. Hoaglin. Summarizing shape numerically: The g-and-h distributions. In *Exploring data tables, trends, and shapes*, pages 461–513. John Wiley & Sons, Hoboken, NJ, 1985.
- J. R. Hosking. L-moments: Analysis and estimation of distributions using linear combinations of order statistics. *Journal of the Royal Statistical Society: Series B (Methodological)*, 52(1):105–124, 1990.
- M. A. Hossain and S. S. Hossain. Numerical maximum likelihood estimation for the g-and-k distribution using ranked set sample. *Journal of Statistics*, 16(1):45–58, 2009.
- R. Hyndman, G. Athanasopoulos, C. Bergmeir, G. Caceres, L. Chhay, M. O’Hara-Wild, F. Petropoulos, S. Razbash, E. Wang, and F. Yasmeeen. *forecast: Forecasting*

- functions for time series and linear models*, 2022. URL <https://pkg.robjhyndman.com/forecast/>. R package version 8.16.
- R. J. Hyndman and Y. Khandakar. Automatic time series forecasting: the forecast package for R. *Journal of Statistical Software*, 27(1):1–22, 2008.
- R. J. Hyndman and A. B. Koehler. Another look at measures of forecast accuracy. *International Journal of Forecasting*, 22(4):679–688, 2006.
- ICE. EUA futures, 2021. URL <https://www.theice.com/products/197/EUA-Futures>.
- N. Ikeda and S. Watanabe. *Stochastic differential equations and diffusion processes*. Elsevier, Amsterdam, NL, 2014.
- J. A. Jiménez and V. Arunachalam. Using Tukey’s g and h family of distributions to calculate value-at-risk and conditional value-at-risk. *The Journal of Risk*, 13(4):95–116, 2011.
- J. A. Jiménez and V. Arunachalam. A mixture of generalized Tukey’s g distributions. *Journal of Probability and Statistics*, 2016(4):1–7, 2016.
- J. A. Jiménez, V. Arunachalam, and G. Serna. A generalization of Tukey’s g – h family of distributions. *Journal of Statistical Theory and Applications*, 14(1):28–44, 2015.
- M. Jorge and I. Boris. Some properties of the Tukey g and h family of distributions. *Communications in Statistics–Theory and Methods*, 13(3):353–369, 1984.
- I. Karatzas and S. Shreve. *Brownian motion and stochastic calculus*, volume 113. Springer Science & Business Media, Berlin, Heidelberg, 2012.
- M. Kendall. Note on the distribution of quantiles for large samples. *Supplement to the Journal of the Royal Statistical Society*, 7(1):83–85, 1940.
- M. Kijima. A multivariate extension of equilibrium pricing transforms: The multivariate Esscher and Wang transforms for pricing financial and insurance risks. *ASTIN Bulletin: The Journal of the IAA*, 36(1):269–283, 2006.

- M. Kijima and Y. Muromachi. An extension of the Wang transform derived from Bühlmann's economic premium principle for insurance risk. *Insurance: Mathematics and Economics*, 42(3):887–896, 2008.
- J. Kingman. *Poisson Processes*. Oxford University Press, Oxford, England, 1993.
- I. Klein and M. Fischer. gh -Transformation of symmetrical distributions. In *Contributions to Modern Econometrics*, pages 119–134. Springer, Boston, MA, 2002.
- S. A. Klugman, H. H. Panjer, and G. E. Willmot. *Loss models: from data to decisions*, volume 715. John Wiley & Sons, Hoboken, NJ, 2012.
- R. Koenker. Quantile regression for longitudinal data. *Journal of Multivariate Analysis*, 91(1):74–89, 2004.
- R. Koenker. Quantile regression: 40 years on. *Annual Review of Economics*, 9:155–176, 2017.
- R. Koenker and G. Bassett Jr. Regression quantiles. *Econometrica: Journal of the Econometric Society*, 4(1):33–50, 1978.
- R. Koenker and K. F. Hallock. Quantile regression. *Journal of Economic Perspectives*, 15(4):143–156, 2001.
- R. Koenker and Z. Xiao. Quantile autoregression. *Journal of the American Statistical Association*, 101(475):980–990, 2006.
- D. Kwiatkowski, P. C. Phillips, P. Schmidt, and Y. Shin. Testing the null hypothesis of stationarity against the alternative of a unit root. *Journal of Econometrics*, 54: 159–178, 1992.
- R. J. Laeven, M. J. Goovaerts, et al. Premium calculation and insurance pricing. In *Encyclopedia of quantitative risk analysis and assessment*, pages 1302–1314. John Wiley & Sons, Hoboken, NJ, 2008.
- A. M. Legendre. Recherches sur l'attraction des sphéroïdes homogènes. *Mémoires de Mathématiques et de Physique, Présentés à l'Académie Royale des Sciences, par Divers Savans, et lus dans ses Assemblées*, 10:411–435, 1785.

- H. Levy. Stochastic dominance and expected utility: Survey and analysis. *Management Science*, 38(4):555–593, 1992.
- G. Li, Y. Li, and C.-L. Tsai. Quantile correlations and quantile autoregressive modelling. *Journal of the American Statistical Association*, 110(509):246–261, 2015.
- P. Liu, A. Schied, and R. Wang. Distributional transforms, probability distortions, and their applications. *Mathematics of Operations Research*, 46(4):1490–1512, 2021.
- H. MacGillivray. The mean, median, mode inequality and skewness for a class of densities. *Australian Journal of Statistics*, 23(2):247–250, 1981.
- H. MacGillivray. Skewness and asymmetry: measures and orderings. *The Annals of Statistics*, pages 994–1011, 1986.
- H. MacGillivray. Shape properties of the g -and- h and Johnson families. *Communications in Statistics—Theory and Methods*, 21(5):1233–1250, 1992.
- H. MacGillivray and K. Balanda. The relationships between skewness and kurtosis. *Australian Journal of Statistics*, 30(3):319–337, 1988.
- P. Millosovich, A. Tsanakas, and R. Wang. A theory of multivariate stress testing. Available at SSRN 3966204, 2021.
- R. Miura. A note on look-back options based on order statistics. *Hitotsubashi Journal of Commerce and Management*, 27(1):15–28, 1992.
- O. Morgenstern and J. Von Neumann. *Theory of games and economic behavior*. Princeton University Press, Princeton, NJ, 1953.
- S. G. Nagarajan, G. Peters, and I. Nevat. Spatial field reconstruction of non-Gaussian random fields: The Tukey g -and- h random process. Available at SSRN 3159687, 2018.
- R. B. Nelsen. *An introduction to copulas*. Springer Science & Business Media, Berlin, Heidelberg, 2007.
- R. B. Nelsen, J. J. Quesada-Molina, J. A. Rodriguez-Lallena, and M. Ubeda-Flores. Kendall distribution functions and associative copulas. *Fuzzy Sets and Systems*, 160(1):52–57, 2009.

- H. Oja. On location, scale, skewness and kurtosis of univariate distributions. *Scandinavian Journal of Statistics*, 8(3):154–168, 1981.
- B. Øksendal. *Stochastic differential equations: an introduction with applications*. Springer Science & Business Media, Berlin, Heidelberg, 2013.
- E. Parzen. Density quantile estimation approach to statistical data modelling. In *Smoothing Techniques for Curve Estimation*, pages 155–180. Springer, 1979a.
- E. Parzen. Nonparametric statistical data modeling. *Journal of the American Statistical Association*, 74(365):105–121, 1979b.
- E. Parzen. Change P–P plot and continuous sample quantile function. *Optimization*, 22(12):3287–3304, 1993.
- E. Parzen. Quantile probability and statistical data modeling. *Statistical Science*, 19(4):652–662, 2004.
- A. Pelsser. On the applicability of the Wang transform for pricing financial risks. *ASTIN Bulletin: The Journal of the IAA*, 38(1):171–181, 2008.
- G. W. Peters. General quantile time series regressions for applications in population demographics. *Risks*, 6(3):97, 2018.
- G. W. Peters and S. Sisson. Bayesian inference, Monte Carlo sampling and operational risk. *Journal of Operational Risk*, 1(3):27–50, 2006.
- G. W. Peters, W. Y. Chen, and R. H. Gerlach. Estimating quantile families of loss distributions for non-life insurance modelling via L-moments. *Risks*, 4(2):14, 2016.
- G. C. Pflug. How to measure risk? In *Modelling and decisions in economics*, pages 39–59. Springer-Verlag, Berlin, Heidelberg, 1999.
- J. Quiggin. A theory of anticipated utility. *Journal of Economic Behavior & Organization*, 3(4):323–343, 1982.
- J. Quiggin. *Generalized expected utility theory: The rank-dependent model*. Kluwer Academic Publishers, Boston, MA, 1993.

- G. D. Rayner and H. L. MacGillivray. Numerical maximum likelihood estimation for the g -and- k and generalized g -and- h distributions. *Statistics and Computing*, 12(1):57–75, 2002.
- M. Rothschild and J. E. Stiglitz. Increasing risk: I. A definition. *Journal of Economic Theory*, 2(3):225–243, 1970.
- L. Rüschemdorf and V. de Valk. On regression representations of stochastic processes. *Stochastic Processes and their Applications*, 46(2):183–198, 1993.
- A. Sancetta and S. Satchell. The bernstein copula and its applications to modeling and approximations of multivariate distributions. *Econometric Theory*, 20(3):535–562, 2004.
- L. J. Savage. *The foundations of statistics*. John Wiley & Sons, Hoboken, NJ, 1954.
- D. Schmeidler. Subjective probability and expected utility without additivity. *Econometrica: Journal of the Econometric Society*, 57(3):571–587, 1989.
- J. Shao. *Mathematical statistics: exercises and solutions*. Springer Science & Business Media, Berlin, Heidelberg, 2006.
- W. T. Shaw and I. R. Buckley. Distributional alchemy: a new approach to skew-normal, kurtotic-normal & related distributions, 2007. URL <https://pdfs.semanticscholar.org/1ac0/acbf34aea7aaa550f2c320e6c1cb5ed24afe.pdf>.
- W. T. Shaw and I. R. Buckley. The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map. *Preprint arXiv:0901.0434*, 2009.
- W. T. Shaw, T. Luu, and N. Brickman. Quantile mechanics II: changes of variables in Monte Carlo methods and GPU-optimized normal quantiles. *Available at SSRN 1969331*, 2011.
- A. Sklar. Fonctions de répartition à dimensions et leurs marges. *Publications de l'Institut Statistique de l'Université de Paris*, 8:229–231, 1959.
- G. Steinbrecher and W. T. Shaw. Quantile mechanics. *European Journal of Applied Mathematics*, 19(2):87–112, 2008.

- R. Sugden. Alternatives to expected utility theory: Foundations and concepts. In *Handbook of Expected Utility Theory*. Kluwer, Boston, MA, 1997.
- The World Bank. *State and trends of carbon pricing 2021*. World Bank, Washington, DC, 2021. doi: 10.1596/978-1-4648-1728-1.
- A. Tsanakas and N. Christofides. Risk exchange with distorted probabilities. *ASTIN Bulletin: The Journal of the IAA*, 36(1):219–243, 2006.
- A. Tsanakas and E. Desli. Risk measures and theories of choice. *British Actuarial Journal*, 9(4):959–991, 2003.
- J. W. Tukey. *Exploratory data analysis*, volume 2. Addison–Wesley, Reading, MA, 1977.
- W. R. van Zwet. *Convex transformations of random variables*. Mathematisch Centrum, Amsterdam, NL, 1964.
- G. G. Venter. Premium calculation implications of reinsurance without arbitrage. *ASTIN Bulletin: The Journal of the IAA*, 21(2):223–230, 1991.
- D. Vicari and S. Kotz. Survey of developments in the theory of continuous skewed distributions. *Metron*, 63(2):225–261, 2005.
- D. L. Wallace. Asymptotic approximations to distributions. *The Annals of Mathematical Statistics*, 29(3):635–654, 1958.
- M. P. Wand, J. S. Marron, and D. Ruppert. Transformations in density estimation. *Journal of the American Statistical Association*, 86(414):343–353, 1991.
- S. Wang. Insurance pricing and increased limits ratemaking by proportional hazards transforms. *Insurance: Mathematics and Economics*, 17(1):43–54, 1995.
- S. Wang. Premium calculation by transforming the layer premium density. *ASTIN Bulletin: The Journal of the IAA*, 26(1):71–92, 1996.
- S. S. Wang. A class of distortion operators for pricing financial and insurance risks. *Journal of Risk and Insurance*, 67(1):15–36, 2000.

- S. S. Wang. A universal framework for pricing financial and insurance risks. *ASTIN Bulletin: The Journal of the IAA*, 32(2):213–234, 2002.
- S. S. Wang. Equilibrium pricing transforms: new results using Bühlmann’s 1980 economic model. *ASTIN Bulletin: The Journal of the IAA*, 33(1):57–73, 2003.
- S. S. Wang, V. R. Young, and H. H. Panjer. Axiomatic characterization of insurance prices. *Insurance: Mathematics and Economics*, 21(2):173–183, 1997.
- J. L. Wirch and M. R. Hardy. Distortion risk measures: Coherence and stochastic dominance. In *International congress on insurance: Mathematics and economics*, pages 15–17, 2001.
- M. V. Wüthrich and M. Merz. *Financial modeling, actuarial valuation and solvency in insurance*. Springer, Berlin, Heidelberg, 2013.
- M. V. Wüthrich, H. Bühlmann, and H. Furrer. *Market-consistent actuarial valuation*, volume 2. Springer, Berlin, Heidelberg, 2010.
- G. Xu and M. G. Genton. Tukey g-and-h random fields. *Journal of the American Statistical Association*, 112(519):1236–1249, 2017.
- M. E. Yaari. The dual theory of choice under risk. *Econometrica: Journal of the Econometric Society*, 55:95–115, 1987.
- Y. Yan and M. G. Genton. The Tukey g-and-h distribution. *Significance*, 16(3):12–12, 2019.
- Y. Yan, J. Jeong, and M. G. Genton. Multivariate transformed Gaussian processes. *Japanese Journal of Statistics and Data Science*, 3(1):129–152, 2020.
- M. Yor. The distribution of Brownian quantiles. *Journal of Applied Probability*, 32(2):405–416, 1995.
- V. R. Young. Premium principles. In *Encyclopedia of Actuarial Science*, volume 3. John Wiley & Sons, Hoboken, NJ, 2006.
- W. Zhu, K. S. Tan, and L. Porth. Agricultural insurance ratemaking: Development of a new premium principle. *North American Actuarial Journal*, 23(4):512–534, 2019.