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On the geometry of the orthogonal momentum amplituhedron

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Abstract: In this paper we focus on the orthogonal momentum amplituhedron $\mathcal{O}_k$, a recently introduced positive geometry that encodes the tree-level scattering amplitudes in ABJM theory. We generate the full boundary stratification of $\mathcal{O}_k$ for various k and conjecture that its boundaries can be labelled by so-called orthogonal Grassmannian forests (OG forests). We determine the generating function for enumerating these forests according to their dimension and show that the Euler characteristic of the poset of these forests equals one. This provides a strong indication that the orthogonal momentum amplituhedron is homeomorphic to a ball. This paper is supplemented with the Mathematica package `orthitroids` which contains useful functions for exploring the structure of the positive orthogonal Grassmannian and the orthogonal momentum amplituhedron.

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Contents

1 Introduction

In recent years we have seen tremendous interest in positive geometries [1] that encode observables in quantum field theories, and in particular the ones that can be employed to study scattering amplitudes. In four dimensions, a particularly fruitful theory where positive geometries can be defined is planar $\mathcal{N} = 4$ super Yang-Mills (SYM). Scattering amplitudes in planar $\mathcal{N} = 4$ SYM were shown to be connected to the positive Grassmannian [2]. This lead to the discovery the amplituhedron $\mathcal{A}_{n,k}$ [3] and the momentum amplituhedron $\mathcal{M}_{n,k}$ [4]. Both are conjecturally positive geometries defined as the image of the positive Grassmannian through a linear map, and they encode the tree-level scattering amplitudes of $\mathcal{N} = 4$ SYM. Many of the remarkable properties of $\mathcal{N} = 4$ SYM have an analogue in three-dimensional $\mathcal{N} = 6$ Chern-Simons matter theory, also known as Aharony-Bergman-Jafferis-Maldacena (ABJM) theory [5], whose scattering amplitudes are closely related to the positive orthogonal Grassmannian [6–9]. ABJM theory has proven to be a fruitful theory to study, as it is holographically dual to M-theory on $AdS_4 \times S^7$, and it has been used as a toy model for problems in condensed matter physics [5]. Recently, the analogy between $\mathcal{N} = 4$ SYM and ABJM theory has been extended by the introduction of a positive geometry [10, 11], which is similarly defined as the image of the positive orthogonal Grassmannian through a linear map and describes scattering amplitudes in supersymmetry reduced ABJM theory. This geometry is called orthogonal momentum amplituhedron and it is denoted as $\mathcal{O}_k$ in this paper. It is $(2k-3)$-dimensional and can be thought of as a deformation of the Arkani-Hamed-Bai-He-Yan (ABHY) associahedron A_{2k-3} [12] where faces corresponding to even-particle planar Mandelstam variables are squashed to lower dimensional boundaries.

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There is also a loop-level connection between $\mathcal{N}=4$ SYM and ABJM theory; the all-loop integrand for the ABJM four-point amplitude can be obtained from the all-loop amplituhedron for $\mathcal{N}=4$ SYM by projecting to three-dimensional kinematics [13]. However, there is no known analogue of the loop momentum amplituhedron [14] for ABJM theory.

In this paper we study properties of $\mathcal{O}_k$, and in particular conjecture a complete classification of its boundaries. To this end we use the algorithm developed in [15] where it was successfully applied to conjecture all boundaries of the amplituhedron $\mathcal{A}^{(2)}_{n,k}$, and which has been subsequently used to conjecture all boundaries of the momentum amplituhedron $\mathcal{M}_{n,k}$ in [16]. Applying this algorithm to the orthogonal momentum amplituhedron $\mathcal{O}_k$, we observe that all boundaries can be labelled by a particular class of graphs, which we call orthogonal Grassmannian forests, that correspond to all possible factorizations of ABJM amplitudes. This observation is analogous to the one that has been made for $\mathcal{N}=4$ SYM in [16], where the boundaries of the momentum amplihedron $\mathcal{M}_{n,k}$ can be labelled using Grassmannian forests, see [17]. Both form a subset of the Grassmannian graphs introduced in [18]. In this paper we summarise our explorations of the boundaries for $\mathcal{O}_k$ for $k \leq 7$, and provide a conjecture on the boundary stratification for all k. In particular, using the methods developed in [17], we derive a generating function for orthogonal Grassmannian forests. This conjecturally enumerates the boundaries of $\mathcal{O}_k$ of a given dimension and allows us to argue that the Euler characteristic for the orthogonal momentum amplituhedron equals one. This story parallels the one developed for the momentum amplituhedron.

Moreover, [11] provided strong evidence that both the interior of the ABHY associahedron A_{2k-3} and the interior of the orthogonal momentum amplituhedron $\mathcal{O}_k$ are diffeomorphic to the positive part of the moduli space of n points on the Riemann sphere and therefore are diffeomorphic to each other. This is however not true for their closures. Nevertheless, in this paper we show that there is a simple diagrammatic way to understand how the boundaries of the associahedron naturally reduce to the boundaries of the orthogonal momentum amplituhedron.

This paper is organised as follows. In section 2 we recall the definition of the positive orthogonal Grassmannian and collect its basic properties. In section 3 we define the orthogonal momentum amplituhedron following [10, 11] and find its boundary stratification. Then, in section 4 we discuss the generating function for the number of boundaries of $\mathcal{O}_k$. Section 5 provides details on the diagrammatic map from boundaries of the associahedron A_{2k-3} to those of the orthogonal momentum amplituhedron $\mathcal{O}_k$. We provide details on the Mathematica package `orthitroids` in section A.

2 Positive orthogonal Grassmannian

Let us start by recalling a few basic facts about Grassmannian spaces and their positive parts. The *real Grassmannian* $G(k,n)$ is the space of k-dimensional linear subspaces of $\mathbb{R}^n$. Elements of $G(k,n)$ can be represented by $k \times n$ matrices of maximal rank, where matrices related by $\mathrm{GL}(k)$ transformations are identified. The *positive Grassmannian* $G_+(k,n)$, extensively studied by Postnikov in [19], is the subset of elements of $G(k,n)$ represented by matrices whose ordered maximal minors are all non-negative. The boundary stratification of the positive Grassmannian $G_+(k,n)$ is well known [19] and its boundary strata are called *positroid cells*.[1] Positroid cells of $G_+(k,n)$ are in a bijection with various combinatorial objects, including decorated permutations of type (k,n), i.e. permutations on $[n] = \{1,\ldots,n\}$ with k anti-exceedances where fixed points are coloured either black or white, equivalence classes of reduced plabic (*pla*nar *bi*coloured) graphs, and, more generally, refinement-equivalence classes of reduced Grassmannian graphs [18]. Since we will be interested in a special subset of these Grassmannian graphs, we review their definition below.

Grassmanian graphs were formally introduced by Postnikov in [18], but had already appeared implicitly in [2] as generalizations of plabic graphs. The following definitions relating to Grassmannian graphs are taken from section 4 of [18]. A *Grassmannian graph* is a finite planar graph Γ with vertices $\mathcal{V}(\Gamma)$ and edges $\mathcal{E}(\Gamma)$, embedded in a disk. It has n boundary vertices $\mathcal{V}_{\text{ext}}(\Gamma) = \{b_1,\ldots,b_n\}$, each of degree one, on the boundary of the disk which are labelled counterclockwise. All *internal vertices* $\mathcal{V}_{\text{int}} = \mathcal{V}(\Gamma) \setminus \mathcal{V}_{\text{ext}}(\Gamma)$ must be connected to the boundary of the disk via a path in Γ and none of them can have degree two. We denote by $\mathcal{E}_{\text{int}}(\Gamma)$ the set of *internal edges* in Γ, i.e. those which are not adjacent to boundary vertices, and by Γ_{int} the *internal subgraph* of Γ. Each $v \in \mathcal{V}_{\text{int}}(\Gamma)$ is assigned a nonnegative integral *helicity* $h(v)$ where $\min\{1, \deg(v)-1\} \le h(v) \le \max\{1, \deg(v)-1\}$. We refer to v as coloured *white* if $h(v) = 1$, coloured *black* if $h(v) = \deg(v)-1$, and *generic* otherwise. The *helicity* of Γ is the number $h(\Gamma)$ given by

$$h(\Gamma) \coloneqq \sum_{v\in\mathcal{V}_{\text{int}}(\Gamma)} \left(h(v) - \frac{\deg(v)}{2}\right) + \frac{n}{2}\,. \tag{2.1}$$

Such a Grassmannian graph is said to be of *type* (k,n) where n is the number of boundary vertices and $k = h(\Gamma)$ is the helicity of Γ.

An acyclic Grassmannian graph is called a *Grassmannian forest* while a connected acyclic Grassmannian graph is called a *Grassmannian tree* [17]. Given a Grassmanian forest F, we denote the set of Grassmannian trees in F by $\text{Trees}(F)$.

To each Grassmannian graph Γ, one can assign a collection of one-way strands. A *one-way strand* α in Γ is a directed walk along edges of Γ that either starts and ends at some boundary vertices, or is a closed walk in Γ_{int}, satisfying the following rules-of-the-road. For each internal vertex $v \in \mathcal{V}_{\text{int}}(\Gamma)$ with adjacent edges labelled clockwise as $a_1,\ldots,a_d$ (where $d = \deg(v)$), if α enters v through the edge a_i, it leaves v through the edge a_j where $j = i + h(v) \pmod d$.

A Grassmannian graph Γ is said to be *reduced* if its one-way strands satisfy the following conditions:

1. There are no strands forming closed loops in Γ_{int}.
2. All strands are simple curves without self-intersections.
3. Any pair of strands $\alpha \neq \beta$ cannot have a *bad double crossing*, i.e. a pair of vertices $u \neq v$ such that both α and β pass from u to v.

A Grassmannian forest is always reduced since it contains no internal cycles [17].

[1] The boundary stratification of $G_+(k,n)$ can be generated and studied using the Mathematica package `positroids` [20].

Given a reduced Grassmannian graph Γ of type (k, n), one can define the *decorated strand permutation* σ_Γ which is a permutation $\sigma_\Gamma : [n] \to [n]$ with fixed points coloured either black or white such that:

1. $\sigma_\Gamma(i) = j$ if the one-way strand that starts at the boundary vertex b_i ends at the boundary vertex b_j.

2. For a boundary leaf v connected to b_i, the decorated permutation σ_Γ has the fixed point $\sigma_\Gamma(i) = i$ coloured according to $h(v) \in \{0, 1\}$, i.e. white if $h(v) = 1$ and black if $h(v) = \deg(v) - 1 = 0$.

A *complete* reduced Grassmannian graph Γ of type (k, n) with $1 \leq k \leq n-1$, is a reduced Grassmannian graph whose decorated strand permutation is given by $\sigma_\Gamma(i) = i+k \pmod n$.

Given two Grassmannian graphs Γ and Γ', we say that Γ *refines* Γ' (and that Γ' coarsens Γ) if Γ can be obtained from Γ' by a sequence of the following operation: replace an internal vertex of degree d and helicity h by a complete reduced Grassmannian graph of type (h, d). The *refinement order* on Grassmannian graphs is the partial order $\preceq_{\text{ref}}$ where $\Gamma \preceq_{\text{ref}} \Gamma'$ if Γ refines Γ'. We say that Γ and Γ' are *refinement-equivalent* if they are in the same connected component of the refinement order $\preceq_{\text{ref}}$, i.e. they can be obtained from each other by a sequence of refinements and coarsenings. Moreover, Γ and Γ' are refinement-equivalent if and only if their decorated strand permutations coincide: $\sigma_\Gamma = \sigma_{\Gamma'}$.

In this paper we will focus on the *positive orthogonal Grassmannian* $OG_+(k)$ introduced in [6, 7]. This is defined as a $\frac{1}{2}k(k-1)$-dimensional slice of the positive Grassmannian $G_+(k, 2k)$ satisfying the following orthogonality conditions:

$$OG_+(k) = \left\{ C \in G_+(k, 2k) : C \cdot \eta \cdot C^T = 0 \right\}, \tag{2.2}$$

where $\eta \coloneqq \text{diag}(-, +, -, \dots, +)$. The boundary stratification of the positive orthogonal Grassmannian $OG_+(k)$ was conjectured in [8] in terms of cells which we call *orthitroid cells*, and the Eulerian poset structure of $OG_+(k)$ was studied in [7]. Orthitroid cells of $OG_+(k)$ are in a bijection with various combinatorial objects, including permutations on $[2k]$ given as the product of k disjoint two-cycles, a special class of marked Young diagrams called *folded OG tableaux* [8], and refinement-equivalence classes of reduced orthogonal Grassmannian graphs. A permutation on $[2k]$ given as the product of k disjoint two-cycles naturally defines a *matching* on $2k$ points and thus can be represented by (usually many) *strand diagrams* or *medial graphs* [21]. A strand diagram is called *lensless* if any two strands intersect at most once. We will restrict our attention to lensless strand diagrams whose strands are straight lines as depicted in figure 1 (left) and which are generated by the function `oposPermToStrand` included in our MATHEMATICA package, see section A. For a precise definition of folded OG tableaux, we refer the reader to section 4.1 of [8]. The folded OG tableau of an orthitroid cell can be generated using the function `oposPermToYoungReducedNice`; see figure 1 (right) for an example. However, the combinatorial objects which we are most interested in are the orthogonal Grassmannian graphs.

We define an *orthogonal Grassmannian graph* or *OG graph* of *type* $\underline{k}$ to be a Grassmannian graph Γ of type $(k, 2k)$ for which every $v \in \mathcal{V}_{\text{int}}(\Gamma)$ has $\deg(v) = 2h(v)$ and $h(v) > 1$,

Figure 1. A lensless strand diagram (left) and folded OG tableau (right) for the permutation (1 3)(2 6)(4 8)(5 7).

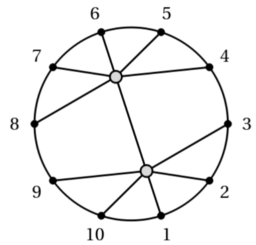

Figure 2. Example of an OG forest of type $\underline{5}$ with strand permutation (1 6)(2 9)(3 10)(4 7)(5 8).

i.e. every internal vertex has an even degree larger than two and its helicity is fixed to be precisely half of its degree. This requirement is clearly compatible with (2.1) since it implies $h(\Gamma) = \frac{2k}{2} = k$. All properties of Grassmannian graphs automatically descend to properties of OG graphs. An *orthogonal Grassmannian forest* or *OG forest* is an OG graph which is also a Grassmannian forest (i.e. acyclic) and an *orthogonal Grassmannian tree* or *OG tree* is an OG graph which is also a Grassmannian tree (i.e. connected and acyclic). Given a reduced OG graph Γ of type $\underline{k}$, the associated strand permutation $\sigma_\Gamma : [2k] \to [2k]$ is always a product of k disjoint two-cycles and hence carries no decoration. All OG forests are automatically reduced; this follows from the same property for Grassmannian forests. In addition, each OG forest is always unique with respect to the refinement order $\preceq_{\text{ref}}$ and hence uniquely defines a strand permutation. The OG forest associated with some strand permutation can be generated using the function `omomPermToForest`. Figure 2 shows an example of an OG forest with its strand permutation.

As studied in [8], orthitroid cells of $OG_+(k)$ are in bijection with permutations σ on $[2k]$ given as the product of k disjoint two-cycles, and therefore with refinement-equivalence classes of reduced OG graphs Γ of type $\underline{k}$. Consequently, these combinatorial objects provide unambiguous labels for orthitroid cells in $OG_+(k)$. We denote the orthitroid cell labelled by σ (resp., Γ) as S_σ (resp., S_Γ). The *top cell* (i.e. top-dimensional orthitroid cell) of $OG_+(k)$ is labelled by the permutation $(1, k+1)(2, k+2)\dots(k, 2k)$.

It is conjectured that each element in a orthitroid cell of $OG_+(k)$ can be represented by a matrix which is orthogonal, real and non-negative (for non-negative parameters) using the *BCFW construction* presented in [8]. This algorithm, which is described below, is implemented in the function `oposPermToMat`.

Let S_σ be some d-dimensional orthitroid cell in $OG_+(k)$ labelled by the permutation σ. Then there exists a k-element subset $I \coloneqq \{i_1, \dots, i_k\}$ of $[2k]$ with $i_1 < \cdots < i_k$ and a

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permutation $\tilde{\sigma} \in S_k$ such that

$$\sigma = (i_1, j_{\tilde{\sigma}(1)})(i_2, j_{\tilde{\sigma}(2)}) \cdots (i_k, j_{\tilde{\sigma}(k)}), \tag{2.3}$$

where $J \coloneqq \{j_1, \ldots, j_k\} = [2k] \setminus I$ with $j_1 < \cdots < j_k$ and $i_\ell < j_{\tilde{\sigma}(\ell)}$ for each $\ell \in [k]$. The labels in I (resp., J) are called the *pivots* (resp., the *sinks*) of σ where $i_{\tilde{\sigma}^{-1}(\ell)}$ is the pivot corresponding to the sink j_ℓ and $j_{\tilde{\sigma}(\ell)}$ is the sink corresponding to the pivot i_ℓ for each $\ell \in [k]$.

There is a unique zero-dimensional orthitroid cell in the boundary stratification of S_σ labelled by a permutation σ_0 which has the same pivots (and hence sinks) as σ. The permutation σ_0 is called the *zero-permutation* corresponding to σ and can be found using the function `oposZeroPerm`. One can define a matrix $C(\sigma_0)$ which parametrizes S_{σ_0} according to the prescription for *bottom cells* given in [8]: define $C(\sigma_0)$ to be a sparse $k \times 2k$ matrix whose only non-zero entries are $C_{\ell,i_\ell}(\sigma_0) = 1$ and $C(\sigma_0)_{\ell,j_{\tilde{\sigma}_0(\ell)}} = (-1)^{(j_{\tilde{\sigma}_0(\ell)} - i_\ell - 1)/2}$ for pivots i_ℓ and sinks $j_{\tilde{\sigma}_0(\ell)}$ of σ_0. By construction, $C(\sigma_0)$ is orthogonal, real and non-negative.

Suppose $d > 0$. To construct the matrix $C(\sigma)$, we first find a sequence of transpositions $\tau_1, \ldots, \tau_d$ on sinks such that

$$\sigma = \tau_d \circ \tau_{d-1} \circ \cdots \circ \tau_1 \circ \sigma_0 . \tag{2.4}$$

These transpositions are found using the following recursive algorithm. Suppose that we already know $\pi = \tau_{m-1} \circ \cdots \circ \tau_1 \circ \sigma_0$ for some $m \in [d]$ and that we want to determine τ_m. Then there exists a permutation $\tilde{\pi} \in S_k$ such that

$$\pi = (i_1, j_{\tilde{\pi}(1)})(i_2, j_{\tilde{\pi}(2)}) \cdots (i_k, j_{\tilde{\pi}(k)}), \tag{2.5}$$

where $i_\ell < j_{\tilde{\pi}(\ell)}$ for each $\ell \in [k]$. Define f, α, β via

$$f = \min \{\ell \in [k] : \tilde{\sigma}^{-1}(\ell) \neq \tilde{\pi}^{-1}(\ell)\} = \tilde{\sigma}(\alpha), \tag{2.6}$$

$$\beta = \min \{\ell \in [k] : \alpha < \ell \text{ and } f \leq \tilde{\pi}(\ell) < \tilde{\pi}(\alpha)\} . \tag{2.7}$$

Then $\tau_m = (j_{\tilde{\pi}(\beta)}, j_{\tilde{\pi}(\alpha)})$. An example of this algorithm is given in figure 3. The interpretation of the numbers f, α, β are as follows. Firstly, f (which stands for "floor") labels the smallest sink j_f for which $(i_{\tilde{\pi}^{-1}(f)}, j_f)$ *is not* one of the disjoint two-cycles in σ. Secondly, α labels the pivot i_α for which (i_α, j_f) *is* one of the disjoint two-cycles in σ; it's corresponding sink $j_{\tilde{\pi}(\alpha)}$ in π needs to be swapped out to eventually become j_f. Clearly f and α are well-defined for $\sigma \neq \pi$. Finally, β labels the pivot i_β whose corresponding sink $j_{\tilde{\pi}(\beta)}$ in π needs to be swapped with $j_{\tilde{\pi}(\alpha)}$. We have explicitly checked that

$$j_{\tilde{\pi}(\alpha)} - j_{\tilde{\pi}(\beta)} - 1 \equiv 0 \pmod 2, \tag{2.8}$$

for all orthitroid cells in $OG_+(k)$ for $k \leq 5$, and we conjecture that this is true for all k.

Having determined the transpositions in (2.4), we then construct a *BCFW rotation matrix* $R(\tau_m) \in \mathrm{SO}(2k, 2k)$ for each transposition τ_m according to the prescription given in [8]. Here

$$\mathrm{SO}(2k, 2k) \coloneqq \left\{ R \in \mathbb{R}^{2k \times 2k} : R \cdot \eta \cdot R^T = \eta \right\} . \tag{2.9}$$

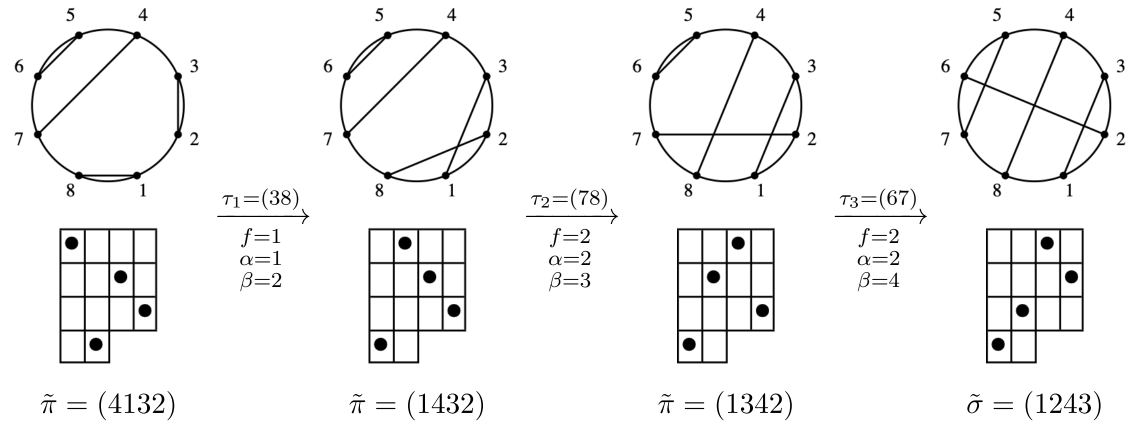


Figure 3. BCFW construction for writing $\sigma = (13)(26)(48)(57)$ as $\sigma = \tau_3 \circ \tau_2 \circ \tau_1 \circ \sigma_0$ where $\sigma_0 = (18)(23)(47)(56)$.

For each transposition $\tau_m = (j_{\tilde{\pi}(\beta)}, j_{\tilde{\pi}(\alpha)})$ define $R(\tau_m)$ to be a sparse matrix which has the following non-trivial 2×2 submatrix

$$R_{j_{\tilde{\pi}(\alpha)} j_{\tilde{\pi}(\alpha)}}(\tau_m) = R_{j_{\tilde{\pi}(\beta)} j_{\tilde{\pi}(\beta)}}(\tau_m) = \cosh\theta_m \,, \tag{2.10}$$

$$R_{j_{\tilde{\pi}(\beta)} j_{\tilde{\pi}(\alpha)}}(\tau_m) = R_{j_{\tilde{\pi}(\alpha)} j_{\tilde{\pi}(\beta)}}(\tau_m) = (-1)^{(j_{\tilde{\pi}(\alpha)} - j_{\tilde{\pi}(\beta)} - 1)/2} \sinh\theta_m \,, \tag{2.11}$$

together with ones on the remaining diagonal entries and zeros everywhere else. The reality of $R(\tau)$ is guaranteed by (2.8). Moreover, $R(\tau_m)$ is orthogonal by construction. Finally, the matrix $C(\sigma)$ which parametrises S_σ is constructed as

$$C(\sigma) = C(\sigma_0) R(\tau_1) R(\tau_2) \cdots R(\tau_d) \,. \tag{2.12}$$

It is automatically real and orthogonal. This algorithm for $C(\sigma)$ is implemented in the function `oposPermToMat`. We have explicitly checked that the matrices $C(\sigma)$ are non-negative for non-negative θ_i for all orthitroid cells S_σ in $OG_+(k)$ for $k \leq 5$, and we conjecture that this is true for all k.

3 Orthogonal momentum amplituhedron and its boundary stratification

The orthogonal momentum amplituhedron $\mathcal{O}_k$ was recently introduced by the authors of [10] and [11][2] independently. The canonical form of $\mathcal{O}_k$ conjecturally encodes tree-level scattering amplitudes of ABJM theory with reduced supersymmetry in a similar way to how the canonical form of the momentum amplituhedron encodes tree-level scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory [4]. The orthogonal momentum amplituhedron $\mathcal{O}_k$ is defined as the image of the map

$$\begin{aligned} \widetilde{\Phi}_\Lambda \colon OG_+(k) &\to G(k, k+2) \\ c_{\alpha i} &\mapsto Y^A_\alpha := c_{\alpha i}\Lambda^A_i \,, \end{aligned} \tag{3.1}$$

[2] What we refer to as the 'orthogonal momentum amplituhedron' was named the 'ABJM momentum amplituhedron' in [11], and they denoted it by $\mathcal{M}^{3d}(k, 2k)$.

where $\alpha = 1, \ldots, k$, $i = 1, \ldots, 2k$ and $A = 1, \ldots, k+2$. Moreover, Λ is a positive $(2k) \times (k+2)$ matrix whose entries are constrained to lie on the moment curve: $\Lambda_i^A = x_i^{A-1}$, for generic $x_1 < x_2 < \cdots < x_{2k}$. Importantly, it was shown in [10, 11] that the image of the map $\tilde{\Phi}_\Lambda$ is not full-dimensional, and instead it lives inside a codimension-three subspace defined by the momentum conservation constraint

$$\sum_{i=1}^{2k} (-1)^i (Y^\perp \Lambda^T)_i^\alpha (Y^\perp \Lambda^T)_i^\beta = 0\,. \tag{3.2}$$

The orthogonal momentum amplituhedron $\mathcal{O}_k$ is therefore $(2k-3)$-dimensional. It was also conjectured that the combinatorics of $\mathcal{O}_k$ are independent of the particular choice of positive matrix Λ, as long as it lives on the moment curve. Consequently, we will omit explicit references to Λ in what follows.

In this paper we conjecture a combinatorial description for the boundary stratification of the orthogonal momentum amplituhedron $\mathcal{O}_k$. We begin by generating the boundary stratification for small values of k using the algorithm introduced in [15] which was successfully applied to the momentum amplituhedron in [22]. For the purposes of explaining the algorithm, we introduce the following notation.

Given an orthitroid cell $S_\sigma \in OG_+(k)$, we denotes its dimension by $\dim(S_\sigma)$ and this can be calculated using `oposDimension`. Let ∂S_σ denote the boundaries of S_σ, i.e. the set of all orthitroid cells $S_{\sigma'} \in OG_+(k)$ with $S_{\sigma'} \neq S_\sigma$ which are contained in the closure of S_σ, and let $\partial^{-1} S_\sigma$ denote the inverse boundaries of S_σ, i.e. the set of all orthitroid cells which have S_σ as a boundary. We denote the image of S_σ through the map $\widetilde{\Phi}$ as $\widetilde{\Phi}^\circ_\sigma = \widetilde{\Phi}(S_\sigma)$, referred to as a *stratum* of $\mathcal{O}_k$, and we denote its closure by $\widetilde{\Phi}_\sigma$. The dimension of $\widetilde{\Phi}^\circ_\sigma$ is denoted by $\dim(\widetilde{\Phi}^\circ_\sigma)$ and can be calculated using `omomDimension`. Lastly, the statum associated with the top-cell of $OG_+(k)$ is denoted by $\mathcal{O}_k^\circ$ and its dimension is $\dim(\mathcal{O}_k^\circ) = \dim(\mathcal{O}_k) = 2k-3$.

Following the definition for boundaries of the momentum amplituhedron given in [17], we define boundaries of the orthogonal momentum amplituhedron as follows: given an orthitroid cell S_σ of $OG_+(k)$, we say that $\widetilde{\Phi}^\circ_\sigma$ is a *boundary stratum* or *boundary* of $\mathcal{O}_k$ if $\widetilde{\Phi}^\circ_\sigma \cap \mathcal{O}_k^\circ = \emptyset$ and for every orthitroid cell $S_{\sigma'} \neq S_\sigma$ whose closure contains S_σ, $\dim(\widetilde{\Phi}^\circ_{\sigma'}) > \dim(\widetilde{\Phi}^\circ_\sigma)$. This definition succinctly summarises the algorithm presented in [15] which we review below for completeness.

The algorithm starts with the known codimension-one boundaries of $\mathcal{O}_k$. Following [10], we define the planar Mandeslstam variables

$$S_{i,i+1,\ldots,i+p} \coloneqq \sum_{i \leq j_i < j_2 \leq i+p} (-1)^{j_1+j_2+1} \langle Y j_1 j_2 \rangle^2, \tag{3.3}$$

where $\langle Yij \rangle \coloneqq \epsilon_{l_1,l_2,\ldots,l_k,l_{k+1},l_{k+2}} Y_1^{l_1} Y_2^{l_2} \cdots Y_k^{l_k} \Lambda_i^{l_{k+1}} \Lambda_j^{l_{k+2}}$. It has been conjectured that the planar Mandelstam variables are positive for all $Y \in \mathcal{O}_k$. Moreover, it is easy to check that the codimension-one boundaries of $\mathcal{O}_k$ correspond to planar Mandelstam invariants involving an odd-number of particles, corresponding to factorization poles of the scattering amplitude. On the other hand, even-particle Mandelstam variables correspond to factorization into two odd-particle scattering amplitudes. It is known that ABJM theory only

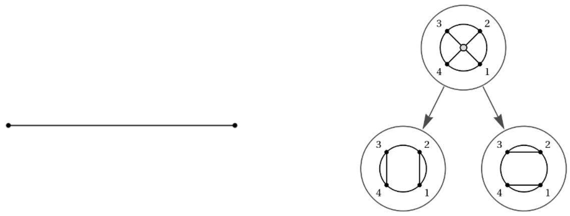


Figure 4. The orthogonal momentum amplituhedron $\mathcal{O}_2$ and its boundary poset.

has even-particle scattering amplitudes, and these limits can thus not be codimension-one boundaries and must correspond to higher codimension boundaries.

Suppose that we have determined all boundaries of $\mathcal{O}_k$ of dimension larger than d where $0 < d < 2k-3$. Then we can determine all boundaries of $\mathcal{O}_k$ of dimension d as follows. Firstly, we consider all d-dimensional strata $\widetilde{\Phi}^\circ_\sigma$ of $\mathcal{O}_k$ for which all inverse boundaries $S_{\sigma'} \in \partial^{-1} S_\sigma$ of S_σ correspond to higher-dimensional strata of $\mathcal{O}_k$, i.e. $\dim(\widetilde{\Phi}^\circ_{\sigma'}) > d$. Secondly, we further restrict to strata $\widetilde{\Phi}^\circ_\sigma$ where there is more than one $(d{+}1)$-dimensional boundary $\widetilde{\Phi}^\circ_{\sigma'}$ of $\mathcal{O}_k$ for which S_σ is contained in the closure of $S_{\sigma'}$. The latter restriction excludes strata which correspond to "spurious boundaries" in a tiling/-triangulation of some higher-dimensional strata, including those which lie inside $\mathcal{O}^\circ_k$. In this way we are able to recursively generate the boundary stratification of $\mathcal{O}_k$ as implemented in `omomStratificationAll[k]`. Moreover, we are able to identify the boundary stratification of $\mathcal{O}_k$ as an induced subposet of the orthitroid stratification of $OG_+(k)$.

It was observed in [22], and later clarified in [17], that the boundaries of the momentum amplituhedron $\mathcal{M}_{n,k}$ are in bijection with (contracted) Grassmannian forests of type (k,n). We conjecture that an analogous statement is true for the orthogonal momentum amplituhedron: boundaries of the orthogonal momentum amplituhedron $\mathcal{O}_k$ are in bijection with OG forests of type $\underline{k}$. More precisely, $\widetilde{\Phi}^\circ_\Gamma$ is a boundary of $\mathcal{O}_k$ if and only if Γ is an OG forest of type $\underline{k}$. We have explicitly verified this characterization for the boundaries of $\mathcal{O}_k$ for $k \leq 7$. Starting from the simplest example, one finds that the orthogonal momentum amplituhedron $\mathcal{O}_2$ is a segment, and its boundaries are labelled as in figure 4. A less trivial example is the three-dimensional $\mathcal{O}_3$ (which is equal to $OG_+(3)$); its shape and boundary poset are depicted in figure 5. A glimpse into the structure of the $k=4$ orthogonal momentum amplituhedron is given in table 1, where we have listed all (up to cyclic relabelling) OG forests which appear in the boundary stratification of $\mathcal{O}_4$. The complete boundary poset for $\mathcal{O}_k$ for $k = 4,5,6,7$ can be generated using the `orthitroids` package, see section A.

Before concluding this section, we provide an alternative way of calculating the dimension of boundaries of $\mathcal{O}_k$, i.e. the dimension of orthogonal momentum amplituhedron strata labelled by OG forests. Given an OG forest Γ of type $\underline{k}$, we define $\dim_{\mathcal{O}}(\Gamma)$ by

$$\dim_{\mathcal{O}}(\Gamma) = \sum_{T \in \mathrm{Trees}(\Gamma)} \dim_{\mathcal{O}}(T)\,, \tag{3.4}$$

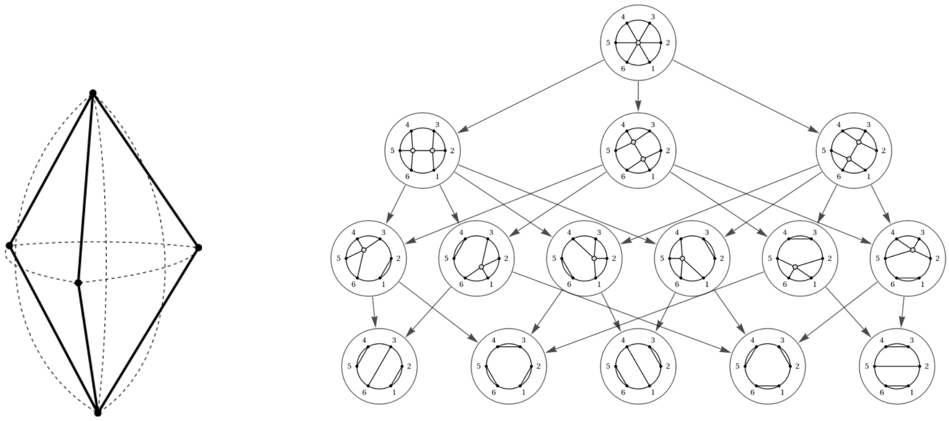


Figure 5. The orthogonal momentum amplituhedron $\mathcal{O}_3$ and its boundary poset.

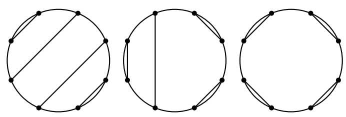
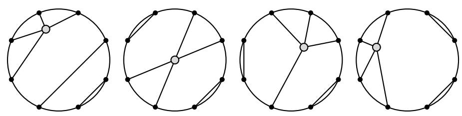
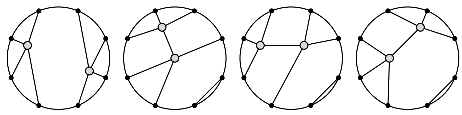
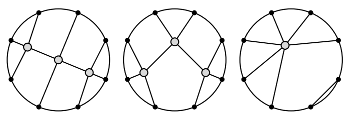
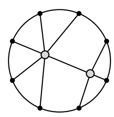
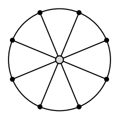
$\dim_{\mathcal{O}}$	$\mathcal{O}_4$	number of boundaries
0		4+8+2=14
1		8+4+8+8=28
2		4+8+8+8=28
3		4+8+8=20
4		8
5		1

Table 1. All types of OG forests that appear in the boundary stratification of $\mathcal{O}_4$. We find the Euler characteristic of $\mathcal{O}_4$: $\chi = 14 - 28 + 28 - 20 + 8 - 1 = 1$.

where for each $T \in \text{Trees}(\Gamma)$

$$\dim_{\mathcal{O}}(T) = \begin{cases} 0\,, & \text{if } |\mathcal{V}_{\text{ext}}(T)| = 0\,, \\ \sum_{v \in \mathcal{V}_{\text{int}}(T)} (\deg(v) - 3)\,, & \text{if } |\mathcal{V}_{\text{ext}}(T)| > 0\,. \end{cases} \tag{3.5}$$

Here $\deg(v)$ denotes the degree of the internal vertex v. We have checked that for every OG forest Γ in $\mathcal{O}_k$ for $k \leq 7$ we have $\dim(\widetilde{\Phi}^{\circ}_{\Gamma}) = \dim_{\mathcal{O}}(\Gamma)$, and we believe the equality holds true for all OG forests.

In [17], an analogous formula for the momentum amplituhedron dimension $\dim_{\mathcal{M}}(\Gamma)$ of a Grassmannian forest Γ was given. Since all OG forests are Grassmannian forests, it is natural to ask if there is any relationship between the two dimension formulae. Given an OG forest Γ, its momentum amplituhedron dimension is given by [17]

$$\dim_{\mathcal{M}}(\Gamma) = \sum_{T\in \mathrm{Trees}(\Gamma)} \dim_{\mathcal{M}}(T)\,, \tag{3.6}$$

where

$$\dim_{\mathcal{M}}(T) = \begin{cases} 1\,, & \text{if } |\mathcal{V}_{\mathrm{ext}}(T)| = 0\,, \\ 1 + \sum_{v\in\mathcal{V}_{\mathrm{int}}(T)}(2\deg(v) - 5)\,, & \text{if } |\mathcal{V}_{\mathrm{ext}}(T)| > 0\,. \end{cases} \tag{3.7}$$

By comparing these formulae with those of (3.4) and (3.5), we find that

$$\dim_{\mathcal{M}}(\Gamma) = 2\dim_{\mathcal{O}}(\Gamma) + |\mathrm{Trees}(\Gamma)| + |\mathcal{V}_{\mathrm{int}}(\Gamma)|\,. \tag{3.8}$$

Finally, we conjecture that the orthogonal momentum amplituhedron $\mathcal{O}_k$ is a CW complex with CW decomposition given by $\mathcal{O}_k = \sqcup_{\Gamma\in\mathcal{F}_k}\Phi^{\circ}_{\Gamma}$ where $\mathcal{F}_k$ denote the set of OG forests of type $\underline{k}$.

4 Generating function and Euler characteristic

In the previous section we observed that the boundaries of the orthogonal momentum amplituhedron appear to be labelled by OG forests. Assuming the validity of this conjecture, one can find the f-vector and the Euler characteristic χ of $\mathcal{O}_k$ for all k by enumerating all OG forests. We will enumerate all OG trees and forests according to their type and orthogonal momentum amplituhedron dimension following the recipe presented in [17] where (contracted) Grassmannian trees and forests were enumerated according to their type and momentum amplituhedron dimension.

To enumerate all OG trees, one uses the *series-reduced planar tree analogue of the Exponential Formula* presented in [17]: if a class of combinatorial objects is built by choosing a series-reduced planar tree and putting a structure on each internal vertex independently (depending only on its degree and encoded by some function f), then the generating function for this class can be obtained from the generating function F for the function f. Therefore, in order to enumerate OG trees, we need to determine the functions f and F. In this case, f is determined simply by considering the contribution made by each internal vertex to the orthogonal momentum amplituhedron dimension of an OG tree. From (3.5), it is clear that each internal vertex v of an OG tree T contributes $\deg(v) - 3$ to $\dim_{\mathcal{O}}(T)$. Therefore, we define the statistic $f : 2\mathbb{Z}_{\geq 2} \to \mathbb{Q}(q)$ which takes the degree $d \in 2\mathbb{Z}_{\geq 2}$ of an internal vertex and maps it to $f(d) = q^{d-3}$, where the exponent of q is precisely the above-mentioned dimension contribution. Thereafter, the generating function for enumerating the dimension contributions from an internal vertex of any degree is given by

$$F(x,q) := \sum_{d\in 2\mathbb{Z}_{\geq 2}} f(d)x^d = q^{-3}\sum_{k=2}^{\infty}(xq)^{2k} = \frac{x^4 q}{1-(xq)^2}\,, \tag{4.1}$$

i.e. the coefficient of x^d in $F(x,q)$, denoted by $[x^d]F(x)$, is $f(d)=q^{d-3}$. Then the number of OG trees of type $\underline{k}$ and with orthogonal momentum amplituhedron dimension r is given by $[x^{2k}q^r]\mathcal{G}^{(\mathcal{O})}_{\text{tree}}(x,q)$ where

$$\mathcal{G}^{(\mathcal{O})}_{\text{tree}}(x,q) := x\left(x-\frac{1}{x}F(x,q)\right)^{\langle -1\rangle}_x\,, \tag{4.2}$$

and $(\ldots)^{\langle -1\rangle}_x$ denotes the compositional inverse of $(\ldots)$ with respect to the variable x. One can explicitly compute the compositional inverse in (4.2) using the well-known *Lagrange inversion formula*[3] to find

$$\mathcal{G}^{(\mathcal{O})}_{\text{tree}}(x,q) = x^2\left(1+\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{k}\binom{k}{\ell}\binom{2k+\ell}{2k+1}x^{2k}q^{2k-\ell}\right). \tag{4.3}$$

Since an OG forest can be thought of as a non-crossing partition with an OG tree assigned to each block, one can compute the generating function $\mathcal{G}^{(\mathcal{O})}_{\text{forest}}(x,q)$ which enumerates all OG forests according to their type and orthogonal momentum amplituhedron dimension in terms of $\mathcal{G}^{(\mathcal{O})}_{\text{tree}}$ using *Speicher's analogue of the Exponential Formula for non-crossing partitions* [23]. In particular,

$$\mathcal{G}^{(\mathcal{O})}_{\text{forest}}(x,q) := \frac{1}{x}\left(\frac{x}{1+\mathcal{G}^{(\mathcal{O})}_{\text{tree}}(x,q)}\right)^{\langle -1\rangle}_x\,, \tag{4.4}$$

or

$$[x^n]\mathcal{G}^{(\mathcal{O})}_{\text{forest}}(x,q) := \frac{1}{n+1}[x^n]\left(1+\mathcal{G}^{(\mathcal{O})}_{\text{tree}}(x,q)\right)^{n+1}. \tag{4.5}$$

Using this formula, one can find the f-vector, namely the vector whose i-th component counts the number of boundaries of codimension $i-1$, for the orthogonal momentum amplituhedron $\mathcal{O}_k$. The results for the first few values of k are displayed in table 2.

One can also easily compute the Euler characteristic of $\mathcal{O}_k$ using this formula. Recall that for a CW complex, its *Euler characteristic* is defined by the alternating sum $\chi = n_0 - n_1 + n_2 - n_3 + \ldots$ where n_r denotes the number of boundaries with dimension r. Consequently, the Euler characteristic for $\mathcal{O}_k$ can be computed as $[x^{2k}]\mathcal{G}^{(\mathcal{O})}_{\text{forest}}(x,-1)$. To this end, first we use (4.3) to evaluate $\mathcal{G}^{(\mathcal{O})}_{\text{tree}}(x,q)$ at $q=-1$:

$$\mathcal{G}^{(\mathcal{O})}_{\text{tree}}(x,-1) = x\left(\frac{x}{1-x^2}\right)^{\langle -1\rangle}_x = -\frac{1}{2}\left(1+\sqrt{1+4x^2}\right). \tag{4.6}$$

[3] Series reversion can be performed in Mathematica using the built-in function `InverseSeries`.

k	f-vector	χ
2	$(1,2)$	1
3	$(1,3,6,5)$	1
4	$(1,8,20,28,28,14)$	1
5	$(1,15,65,145,195,180,120,42)$	1
6	$(1,24,168,562,1131,1518,1430,990,495,132)$	1
7	$(1,35,364,1764,5019,9436,12558,12285,9009,5005,2002,429)$	1

Table 2. The f-vector and Euler characteristic χ of $\mathcal{O}_k$ for $k \leq 7$.

Then

$$\frac{x}{1+\mathcal{G}^{(\mathcal{O})}_{\text{tree}}(x,-1)} = \frac{2x}{1-\sqrt{4x^2+1}} = -\frac{1}{2x}\left(1+\sqrt{1+4x^2}\right) = \left(\frac{x}{1-x^2}\right)^{\langle -1\rangle}_x , \tag{4.7}$$

and using (4.4) we obtain

$$\mathcal{G}^{(\mathcal{O})}_{\text{forest}}(x,-1) = \frac{1}{x}\left(\frac{x}{1+\mathcal{G}^{(\mathcal{O})}_{\text{tree}}(x,q)}\right)^{\langle -1\rangle}_x = \frac{1}{x}\frac{x}{1-x^2} = \frac{1}{1-x^2} = \sum_{k=0}^{\infty} x^{2k} . \tag{4.8}$$

Consequently, the Euler characteristic of $\mathcal{O}_k$ is $[x^{2k}]\mathcal{G}^{(\mathcal{O})}_{\text{forest}}(x,-1) = 1$.

5 Diagrammatic map

In this section we highlight an interesting relation between the associahedron A_{2k-3} and the orthogonal momentum amplituhedron $\mathcal{O}_k$. First notice that both geometries are $(2k-3)$-dimensional. Moreover, it was proposed in [11] that one can map the positive moduli space $\mathcal{M}^+_{0,2k} := \{\sigma_1 < \sigma_2 < \cdots < \sigma_{2k}\}/\mathrm{SL}(2,\mathbb{R})$ directly to $\mathcal{O}_k$ using the 'twistor-string map'

$$\begin{aligned} \Phi_\Lambda \colon \mathcal{M}^+_{0,2k} &\to G(k,k+2) \\ \sigma &\mapsto Y^A_\alpha = C_{\alpha i}(\sigma)\Lambda^A_i . \end{aligned} \tag{5.1}$$

Here $\sigma = \{\sigma_1,\sigma_2,\cdots,\sigma_{2k}\} \in \mathcal{M}^+_{0,2k}$ is intermediately mapped to an element $C_{\alpha i}(\sigma)$ of $OG_+(k)$ through the Veronese map

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \sigma_1 & \sigma_2 & \cdots & \sigma_{2k} \end{pmatrix} \mapsto C_{\alpha i}(\sigma) = \begin{pmatrix} t_1 & t_2 & \cdots & t_{2k-1} & 1 \\ t_1\sigma_1 & t_2\sigma_2 & \cdots & t_{2k-1}\sigma_{2k-1} & \sigma_{2k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t_1\sigma_1^{k-1} & t_2\sigma_2^{k-1} & \cdots & t_{2k-1}\sigma_{2k-1}^{k-1} & \sigma_{2k}^{k-1} \end{pmatrix} , \tag{5.2}$$

with $t_i^2 = (-1)^i \prod_{j\neq 2k}(\sigma_{2k}-\sigma_j)\prod_{j\neq i}(\sigma_i-\sigma_j)^{-1}$. The map in (5.1) is conjectured to provide a diffeomorphism between $\mathcal{M}^+_{0,2k}$ and the interior of $\mathcal{O}_k$. On the other hand, there is a partial compactification $\mathcal{M}'_{0,2k}(\mathbb{R})$ of $\mathcal{M}^+_{0,2k}$, called the *worldsheet associahedron*, whose boundary stratification is combinatorially equivalent to that of an associahedron [24]. It is further conjectured that $\mathcal{M}'_{0,2k}(\mathbb{R})$ is diffeomorphic to the ABHY associahedron A_{2k-3}

through the scattering equations [12]. Since the boundary structure of the orthogonal momentum amplituhedron differs from that of an associahedron, it implies that the map in (5.1) does not extend to a diffeomorphism between $\mathcal{M}'_{0,2k}(\mathbb{R})$ and the closure of $\mathcal{O}_k$. It is well-known that the boundaries of the ABHY associahedron can be labelled by planar tree Feynman diagrams by which we mean series-reduced planar trees (i.e. planar trees with no internal vertices of degree two). In this section, we propose a diagrammatic, partial-order-preserving map from the boundaries of A_{2k-3} to the boundaries of $\mathcal{O}_k$ and conjecture that (5.1) maps boundaries in a manner compatible with our diagrammatic map.

We start by reviewing the boundary structure of the associahedron. Given a planar tree Feynman diagram whose boundary leaves are labelled $1, 2, \ldots, n$ counterclockwise, each internal edge divides the leaves into two disjoint cyclic sets A and B, $A \sqcup B = [n]$. We label this edge by the planar Mandelstam variables $s_A = s_B$,[4] each planar tree Feynman diagram is then uniquely determined by the set of all planar Mandelstam variables corresponding to its internal edges. The boundary structure of A_n can be described using the following covering relation: given two planar tree Feynman diagrams F_1 and F_2, we say that F_2 is covered by (i.e. is a codimension-1 boundary of) F_1 if F_2 can be obtained from F_1 by taking some vertex $v \in \mathcal{V}_{\text{int}}(F_1)$ of degree $\deg(v) > 3$, and dissolving it into a vertex of degree $\deg(v) - p + 1$ connected by an internal edge to a vertex of degree $p + 1$, for $p = 2, 3, \ldots, \deg(v) - 2$. We denote this covering relation $F_2 \prec_A F_1$. This extends transitively to a partial order $\preceq_A$.

Next, to make a connection to the orthogonal momentum amplituhedron, we define the following covering relation on OG forests. Given two OG forests Γ_1 and Γ_2, we write that $\Gamma_2 \prec_{\mathcal{O}} \Gamma_1$, if Γ_2 can be obtained from Γ_1 by 'dissolving' one of the internal vertices $v \in \mathcal{V}_{\text{int}}(\Gamma_1)$. If $\deg(v) > 4$, then 'dissolving' means to replace v by a vertex of degree $\deg(v) - p + 1$ connected by an internal edge to a vertex of degree $p + 1$, where the allowed values for p are $3, 5, \ldots, \deg(v) - 3$. If $\deg(v) = 4$, 'dissolving' means to replace v by two non-intersecting lines. From the definition of the orthogonal momentum amplituhedron dimension (3.4), we see that $\dim_{\mathcal{O}}(\Gamma_2) = \dim_{\mathcal{O}}(\Gamma_1) - 1$. These covering relations are depicted in figure 6. These covering relations extend transitively to a partial order $\preceq_{\mathcal{O}}$ on OG forests. We conjecture that the set of all OG forests of type $\underline{k}$ equipped with the partial order $\preceq_{\mathcal{O}}$ gives the boundary poset of $\mathcal{O}_k$.

We now propose the following map from planar tree Feynman diagrams to OG forests: given a planar tree Feynman diagram F, we define $OG(F)$ as the graph obtained by applying the following three operations in order: remove all internal edges which correspond to even-particle planar Mandelstam invariants, replace all vertices of degree 2 with a single edge, and remove all subgraphs that are not connected to a boundary vertex through any path. It is not difficult to convince oneself that the resulting diagram is an OG forest, once we assign each vertex v a helicity $h(v) = \deg(v)/2$. In particular, every vertex in $OG(F)$ is of even degree. This can be seen as follows: consider some internal vertex v of F with incident edges labelled by the Mandelstam variables $s_{A_1}, \ldots, s_{A_p}$. Since there are an

[4] They can equivalently be represented by chords of an n-gon, the chord connecting vertices i and j corresponding to $s_{i,i+1,\ldots,j-1}$.

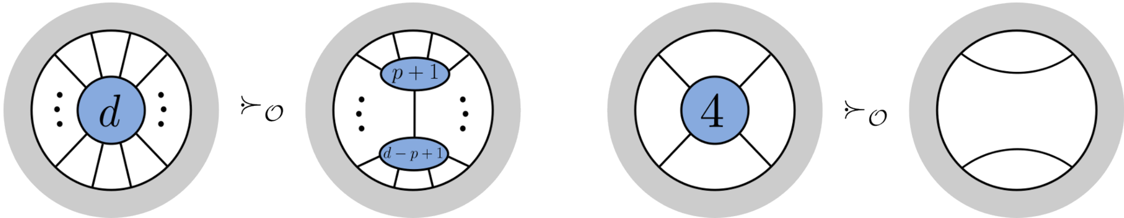


Figure 6. A graphical illustration of the covering relations $\prec_{\mathcal{O}}$ in the case when $\deg(v) > 4$ (left), and $\deg(v) = 4$ (right). The grey boundary indicates that these are part of some larger graph, and the numbers inside the vertices indicate their degree.

even number of boundary leaves, $|A_1| + \ldots + |A_p|$ is even, which means that the number of odd $|A_i|$'s needs to be even. After removing the edges corresponding to even-particle Mandelstam variables in $OG(F)$, the vertex will thus be of even degree. Furthermore, every OG forest can be obtained from at least one planar tree Feynman diagram as follows. Starting from an OG forest, add a vertex inside each of the faces bounding disconnected components. Add an edge from this vertex to an internal edge (by adding a new vertex along this edge) of each adjacent disconnected component, taking care to ensure that the resulting graph is still planar. Finally, replace all degree two vertices by a single edge. The resulting graph is a planar tree Feynman diagram on $2k$-leaves. Since all planar tree Feynman diagrams on $2k$ leaves appear in the boundary stratification of A_{2k-3}, and given our knowledge of the boundary stratification of $\mathcal{O}_k$ discussed in section 3, it is clear that the map defined above is surjective from boundary elements of A_{2k-3} to boundary elements of $\mathcal{O}_k$.

From the definitions given above, it follows that this map preserves partial order. That is, for any two planar tree Feynman diagrams F_1 and F_2 such that $F_1 \succeq_A F_2$, we have that $OG(F_1) \succeq_{\mathcal{O}} OG(F_2)$. This can be proved as follows. Take a planar tree Feynman diagram F, and consider any of its codimension-1 boundaries $\tilde{F} \prec_A F$. By showing that $OG(F) \succeq_{\mathcal{O}} OG(\tilde{F})$, the above statement follows from transitivity. To show that $OG(F) \succeq_{\mathcal{O}} OG(\tilde{F})$, we note that $\tilde{F}$ can be obtained from F by dissolving some internal vertex $v \in \mathcal{V}_{\text{int}}(F)$ of degree $d = \deg(v) > 3$ into a vertex v_1 of degree $d - p + 1$ connected through an internal edge e to a vertex v_2 of degree $p+1$, for some $p \geq 2$. We now map both F and $\tilde{F}$ to the space of OG forests through the map OG. It is sufficient to restrict our focus to the subgraphs containing vertices v, v_1, v_2 and their incident edges, since the rest of the graphs F and $\tilde{F}$ are identical. Applying OG to F, the vertex v transforms into a vertex v' of degree $d' \leq d$ where all the incident edges corresponding to even-particle Mandelstams have been removed. Applying OG to $\tilde{F}$ we find two scenarios. If the edge e corresponds to an odd-particle Mandelstam variable, then $OG(\tilde{F})$ is equivalent to $OG(F)$ with the vertex v' replaced by a vertex of degree $d' - p' + 1$ connected through an internal edge to a vertex of degree $p' + 1$ for some $0 \leq p' \leq p$. Alternatively, if the edge e corresponds to an even-particle Mandelstam variable, then $OG(\tilde{F})$ is equivalent to $OG(F)$ with the vertex v' replaced by two unconnected vertices of degree $d' - p'$ and p'. In the first case, it is clear from the definition of $\prec_{\mathcal{O}}$ that $OG(F) \succeq_{\mathcal{O}} OG(\tilde{F})$. In the second case, we can establish

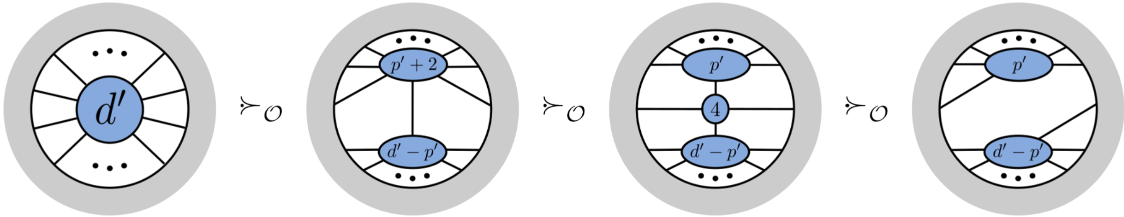


Figure 7. A sequence of covering relations which establishes that $OG(F) \succeq_{\mathcal{O}} OG(\tilde{F})$.

that $OG(F) \succeq_{\mathcal{O}} OG(\tilde{F})$ through a sequence of covering relations which is schematically illustrated in figure 7. In the case where e corresponds to an even-particle Mandelstam variable, there are a few edge cases that need extra consideration. The case where $d' = 4$ implies that $p' = d' - p' = 2$, and $OG(F) \succeq_{\mathcal{O}} OG(\tilde{F})$ follows from the definition of $\prec_{\mathcal{O}}$. In the case where $p' = 2$ and $d' - p' \geq 4$, we follow the same sequence of covering relations as in figure 7, except we skip the second diagram and replace the vertices p' with a single line. By symmetry, this of course also works for $d' - p' = 2$, $p' \geq 4$. Together with the statement for general $p', d' - p' \geq 4$, we conclude that $OG(F) \succeq_{\mathcal{O}} OG(\tilde{F})$ holds for general $F \succeq_A \tilde{F}$.

Starting from the full boundary poset of A_{2k-3} and applying the map OG to each of its elements, the above arguments show that the resulting poset contains all OG forests of type $\underline{k}$, and the inherited poset structure is included in $\preceq_{\mathcal{O}}$. To argue that the resulting poset is exactly the conjectured boundary poset of $\mathcal{O}_k$, we would further have to show that if $OG(F_1) \succeq_{\mathcal{O}} OG(F_2)$, then $F_1 \succeq_A F_2$. We do not have a general proof of this statement, but we have checked that it is true for all $k \leq 5$.

From these considerations, there is strong evidence that the map OG maps the boundary poset of A_{2k-3} to the boundary poset of $\mathcal{O}_k$. The simplest example is the case when $k = 2$, where both the associahedron A_1 and the orthogonal momentum amplituhedron $\mathcal{O}_2$ are segments. The first non-trivial example is the one for $k = 3$ where the f-vector for the associahedron A_3 given by $(1, 9, 21, 14)$ reduces to the f-vector of $\mathcal{O}_3$ given by $(1, 3, 6, 5)$. This is depicted in figure 8 at the level of geometry. To clarify this statement, consider the Hasse diagram of the boundary poset of A_{2k-3} and apply the map OG to each of the nodes. Next, identify all equivalent nodes (labelled by the same OG forest). By "identify" we mean that we connect all equivalent nodes by arrows, we take the transitive closure, and then remove all but one of these equivalent nodes including all arrows attached to it before transitively reducing the resulting graph. The resulting Hasse diagram is the Hasse diagram of the boundary poset of $\mathcal{O}_k$. This is illustrated for the case $k = 3$ in figure 9.

The reduction discussed above also allows us to explain some of the statements about physical boundaries of $\mathcal{O}_k$ made in [10, 11]. In particular, let us take a closer look at the codimension-one boundaries of the associahedron. Recall that the interior of the associahedron A_{2k-3} can be represented by a Feynman diagram with a single degree $2k$ vertex. Going to one of its codimension-one boundaries corresponds to dissolving the degree $2k$ vertex into two vertices of degree $(2k - p + 1)$ and $(p + 1)$, with $2 \leq p \leq 2k - 2$, connected via a single edge. For odd (resp., even) p, the internal edge corresponds to an odd-particle

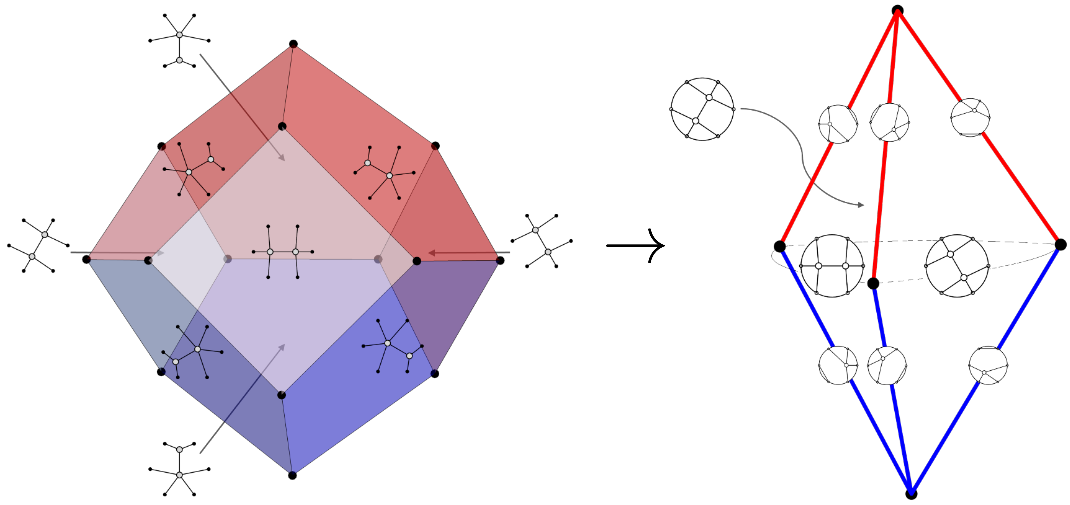


Figure 8. Reduction of the three-dimensional associahedron to the orthogonal momentum amplituhedron $\mathcal{O}_3$.

(resp., even-particle) Mandelstam variable. Via the map defined above, Feynman diagrams with a single edge corresponding to an odd-particle Mandelstam variable will remain unchanged, and we thus interpret the Feynman diagram as an OG tree without changing anything. From (3.5), the resulting OG tree will have an orthogonal momentum amplituhedron dimension of $(2k-p+1-3)+(p+1-3) = 2k-4$, and therefore labels a codimension-one boundary of $\mathcal{O}_k$. On the other hand, Feynman diagrams with a single edge corresponding to an even-particle Mandelstam variable will be mapped to OG forests consisting of two disconnected OG trees, each consisting of a single vertex of degree $2k-p$ and degree p. From (3.4) we find that these OG trees have orthogonal momentum amplituhedron dimension $(2k-p-3)+(p-3) = 2k-6$, with an exception for the cases where $p = 2$ or $p = 2k-2$ when the dimension of the OG forest is $2k-5$. These results agree with the discussions about the physical boundaries of $\mathcal{O}_k$ found in [10, 11].

6 Conclusions and outlook

In this paper, we investigated the poset of boundaries of the orthogonal momentum amplituhedron $\mathcal{O}_k$, a recently proposed positive geometry [10, 11] whose canonical form encodes the tree-level scattering amplitude for $2k$ particles in 3-dimensional ABJM theory with reduced supersymmetry. In section 3 we generated the entire boundary poset of $\mathcal{O}_k$ for $k \leq 7$ using the algorithm developed in [15]. In every example, we observed that the boundaries of $\mathcal{O}_k$ are enumerated by a special class of type $(k, 2k)$ Grassmannian forests which in this paper we called orthogonal Grassmannian forests or OG forests of type $\underline{k}$. This parallels the story for the momentum amplituhedron $\mathcal{M}_{n,k}$, whose boundaries are conjecturally enumerated by refinement-equivalence classes of Grassmanian forests of type (k, n) [17]. Using the generating function techniques described in [17], in section 4 we derived the generating

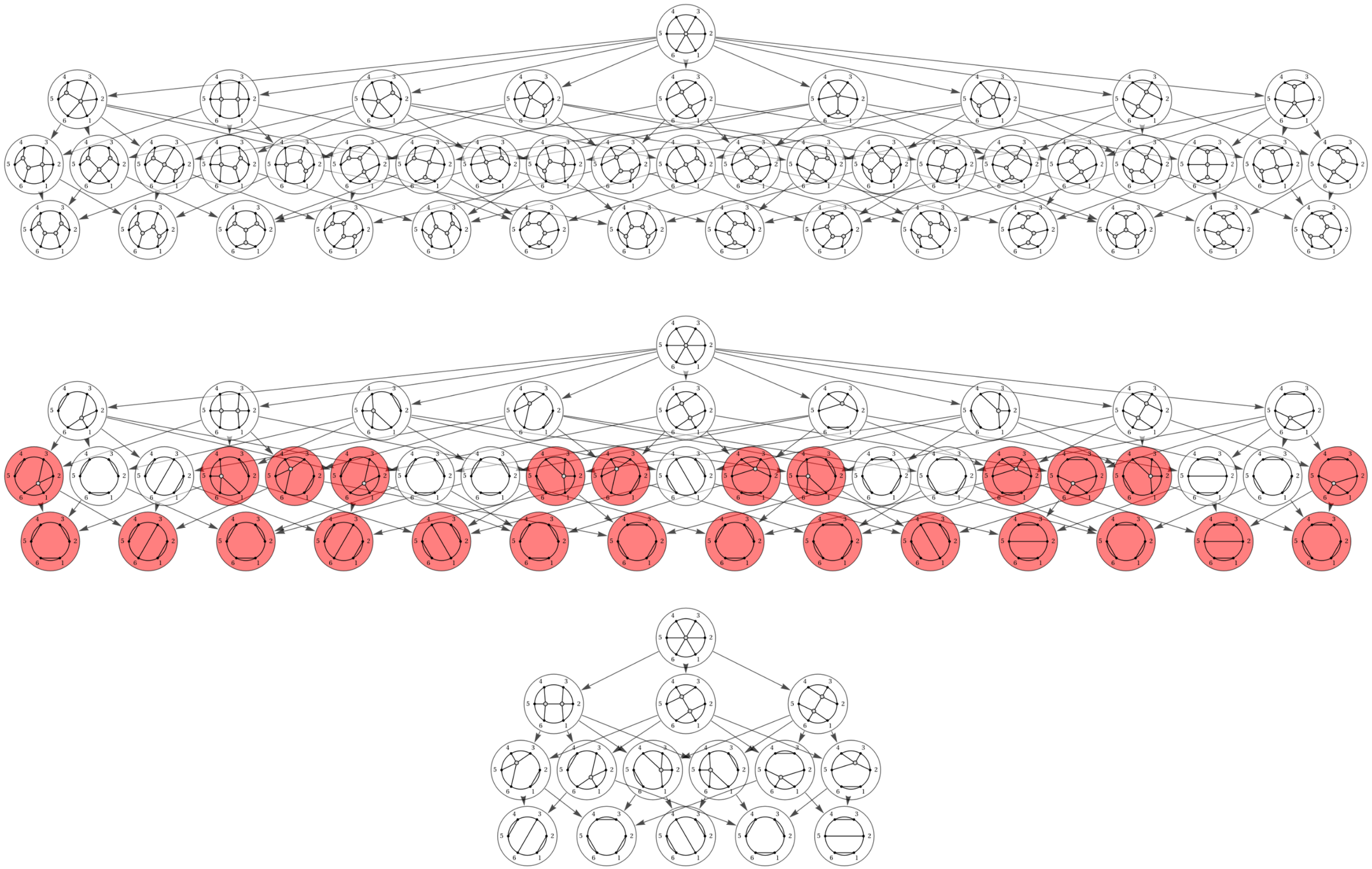


Figure 9. The top and middle Hasse diagrams depict the poset of boundaries of A_3 with nodes labelled by planar trees on 6 leaves (top) and OG forests of type 3 (middle). In the middle Hasse diagram, if a node has a parent labelled by the same OG forest then we colour it red. The bottom Hasse diagram is obtained from the middle one by identifying nodes labelled by the same OG forest.

function $\mathcal{G}^{(\mathcal{O})}_{\text{forest}}(x,q)$ given by (4.4) for enumerating OG forests according to their type and orthogonal momentum amplituhedron dimension. Moreover, we used this generating function to prove that the poset of OG forests of a given type always has Euler characteristic one. This result hints that the orthogonal momentum amplituhedron is homeomorphic to a ball.

In addition to these findings, in section 5 we defined a diagrammatic map from $2k$-particle planar tree Feynman diagrams (i.e. series-reduced planar trees on $2k$ leaves) to OG forests of type $\underline{k}$. The former label boundaries of the ABHY associahedron, a positive geometry which captures the complete tree-level S-matrix of bi-adjoint ϕ^3 theory [12]. We argued that our diagrammatic map is partial-order preserving. We further conjectured that the twistor-string map defined in [11] maps the boundaries of the worldsheet associahedron $\mathcal{M}'_{0,2k}(\mathbb{R})$ to the boundaries of the orthogonal momentum amplituhedron $\mathcal{O}_k$ in a manner compatible with our diagrammatic map.

This paper opens up multiple directions for future research. On the mathematics front, it should be possible to prove the conjectures made in this paper: that the boundaries of $\mathcal{O}_k$ are enumerated by OG forests of type $\underline{k}$ and that the twistor-string map introduced in [11] maps boundaries of $\mathcal{M}'_{0,2k}(\mathbb{R})$ to those of $\mathcal{O}_k$ in a manner compatible with the diagrammatic map which we introduced in section 5. Moreover, it would be interesting to explore whether the boundary poset of the orthogonal momentum amplituhedron is thin and shellable. Proving these conditions would not only imply that the poset is Eulerian, but that it is the face poset of some regular CW complex homeomorphic to a sphere [25].

From the point of view of physics, OG forests appear to correctly capture the underlying singularity structure of the tree-level S-matrix of ABJM theory with reduced supersymmetry as demonstrated by the covering relations given in figure 6. Indeed, the existence of these combinatorial labels for the boundaries of the orthogonal momentum amplituhedron strongly supports the claim that the orthogonal momentum amplituhedron is the correct positive geometry for the theory at hand. This suggests that combinatorics might be a good starting point for defining new, physically relevant positive geometries (or possible generalizations thereof): i.e. given a proper combinatorial description for the singularities of some S-matrix supplemented with some covering relations, is it possible to define a positive geometry in the relevant kinematic space whose boundary stratification realizes this combinatorial description? In fact, combinatorics is also the starting point for the definition of *so-called (dual) surfacehedra* [26, 27].

Furthermore, a projection of the loop amplituhedron was recently proposed to describe loop integrands in ABJM theory at four points [13]. This is similar to tree level for SUSY reduced ABJM, where the orthogonal momentum amplituhedron can be seen as a projection of the momentum amplituhedron. The momentum amplituhedron has recently been extended to the "loop momentum amplituhedron" [14], which conjecturally describes loop integrands of planar $\mathcal{N}=4$ SYM. It is therefore natural to look for a positive geometry which describes loop integrands in SUSY reduced ABJM as a projection of the loop momentum amplituhedron. Presumably, the boundaries of the loop momentum amplituhedron, as well as the conjectural "orthogonal loop momentum amplituhedron", can be described by combinatorial objects which capture the correct singularity structure. The definition of a suitable extension of $\mathcal{O}_k$ to loop integrands and the combinatorial description of its boundaries are exciting prospect which we leave for future research.

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Lastly, given the existence of positive geometries for supersymmetric theories in 3 and 4 dimensions, namely the orthogonal momentum amplituhedron and the momentum amplituhedron, respectively, it is natural to consider whether there exists a 6-dimensional theory whose tree-level S-matrix can be encoded as a positive geometry. It is known that the tree-level S-matrix for some 6-dimensional theories can be expressed as an integral over the symplectic Grassmannian [28–30]. In this case, can its singularities be captured by a symplectic analogue of Grassmannian forests, and can this be used to define a "symplectic momentum amplituhedron"? We leave these and other questions to future work.

A Mathematica package `orthitroid`

While working on this project we have developed a Mathematica package **`orthitroids`** which contains many useful functions for studying the positive orthogonal Grassmannian and the orthogonal momentum amplituhedron. It parallels some of the functionality of the **`positroids`** package [20] and the **`amplituhedronBoundaries`** package [16]. In this appendix we provide a list of some useful functions from the **`orthitroids`** package. We use the following three name spaces for distinguishing functions:

- `opos` — for functions related to the othogonal positive Grassmannian $OG_+(k)$;
- `omom` — for functions related to the orthogonal momentum amplituhedron $\mathcal{O}_k$;
- `mod` — for functions related to the worldsheet associahedron $\mathcal{M}'_{0,2k}(\mathbb{R})$, the positive part of the moduli space of $2k$ points on the Riemann sphere.

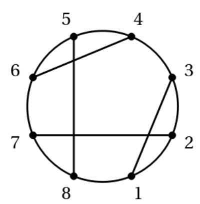
Our package is included as supplementary material together with **`examples.nb`**, a Mathematica notebook that demonstrates the package's functionality.

A.1 Positive orthogonal Grassmannian

- `oposTopCell[`*k_*`]:` returns the permutation for the top-dimensional orthitroid cell of the positive orthogonal Grassmannian.
- `oposPermToStrand[`*perm_*`]:` returns a *lensless strand diagram* for the permutation *perm*.

```
In[1]:= oposPermToStrand[{{1,3},{2,7},{4,6},{5,8}}]
```

Out[1]=



- `oposPermToYoungNice[`*perm_*`]:` returns the Young diagram for the orthitroid cell labelled by the permutation *perm*.

```
In[2]:= oposPermToYoungNice[{{1,3},{2,7},{4,6},{5,8}}]
```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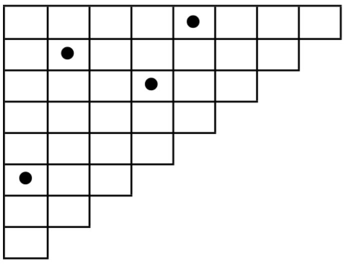
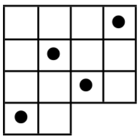
Out[2]= 

- `oposPermToYoungReducedNice[`*perm_*`]`: returns the reduced (folded) Young diagram for the orthitroid cell labelled by the permutation *perm*.

In[3]:= `oposPermToYoungReducedNice[{{1,3},{2,7},{4,6},{5,8}}]`

Out[3]= 

- `oposDimension[`*perm_*`]`: returns the positive orthogonal Grassmannian dimension of the orthitroid cell labelled by the permutation *perm*.

In[4]:= `oposDimension[{{1,3},{2,7},{4,6},{5,8}}]`

Out[4]= 3

- `oposBoundary[`*perm_*`]`: returns the list of codimension-one boundaries of the orthitroid cell labelled by the permutation *perm*.

In[5]:= `oposBoundary[{{1,3},{2,7},{4,6},{5,8}}]`

Out[5]=
```
{{{1,2},{3,7},{4,6},{5,8}},{{1,3},{2,5},{4,6},{7,8}},
{{1,3},{2,7},{4,5},{6,8}},{{1,7},{2,3},{4,6},{5,8}},
{{1,3},{2,8},{4,6},{5,7}},{{1,3},{2,7},{4,8},{5,6}}}
```

- `oposInverseBoundary[`*perm_*`]`: returns the list of orthitroid cells which have the orthitroid cell labelled by the permutation *perm* as a codimension-one boundary.

In[6]:= `oposInverseBoundary[{{1,3},{2,7},{4,6},{5,8}}]`

Out[6]=
```
{{{1,4},{2,7},{3,6},{5,8}},{{1,5},{2,7},{3,8},{4,6}},
{{1,3},{2,6},{4,7},{5,8}}}
```

- `oposStratification[`*perm_*`]`: returns all boundaries (of all codimensions) of the orthitroid cell labelled by the permutation *perm*.

- `oposInverseStratification[`*perm_*`]`: returns the list of orthitroid cells which have the orthitroid cell labelled by the permutation *perm* in its boundary stratification.

- `oposPermToMat` [*perm_*] : returns a matrix parametrizing the orthitroid cell labelled by the permutation *perm* with $c_i = \cosh(\theta_i)$ and $s_i = \sinh(\theta_i)$.

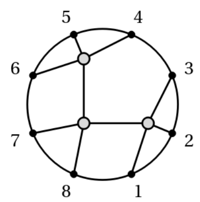
In[7]:= `MatrixForm@oposPermToMat[{{1,3},{2,7},{4,6},{5,8}}]`

Out[7]= $$\begin{pmatrix} 1 & 0 & -s_3 & 0 & 0 & 0 & -c_3 s_1 & -c_1 c_3 \\ 0 & 1 & c_3 & 0 & 0 & 0 & s_1 s_3 & c_1 s_3 \\ 0 & 0 & 0 & 1 & 0 & -s_2 & -c_1 c_2 & -c_2 s_1 \\ 0 & 0 & 0 & 0 & 1 & c_2 & c_1 s_2 & s_1 s_2 \end{pmatrix}$$

A.2 Orthogonal momentum amplituhedron

- `omomPermToForest` [*perm_*] : returns the OG forest for the permutation *perm*.

In[8]:= `omomPermToForest[{{1,3},{2,7},{4,6},{5,8}}]`

Out[8]= 

- `omomDimension` [*perm_*] : returns the orthogonal momentum amplituhedron dimension of the cell labelled by the permutation *perm*.

In[9]:= `omomDimension[{{1,3},{2,7},{4,6},{5,8}}]`

Out[9]= 3

- `omomBoundary` [*perm_*] : returns the list of codimension-one boundaries of the orthogonal momentum amplituhedron cell labelled by the permutation *perm*.

- `omomInverseBoundary` [*perm_*] : returns the list of orthogonal momentum amplituhedron cells which have the cell labelled by the permutation *perm* as a codimension-one boundary.

In[10]:= `omomInverseBoundary[{{1,3},{2,7},{4,6},{5,8}}]`

Out[10]= `{{{1,3},{2,6},{4,7},{5,8}},{{1,5},{2,7},{3,8},{4,6}}}`

- `omomStratification` [*perm_*] : returns all boundaries (of all codimensions) of the orthogonal momentum amplituhedron cell labelled by the permutation *perm*.

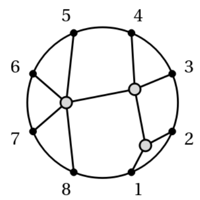
- `omomInverseStratification` [*perm_*] : returns the list of orthogonal momentum amplituhedron cells which have the cell labelled by the permutation *perm* in its boundary stratification.

A.3 Moduli space

- `modDiagonalsToPlanarTree[` *k_* `][` *listOfDiags_* `]`: returns a planar tree on $2k$ leaves which is dual to the dissection of a regular $2k$-gon specified by the diagonals in *listOfDiags*.

In[11]:= `modDiagonalsToPlanarTree[4][{{1,3},{1,5}}]`

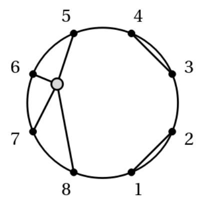
Out[11]=



- `modDiagonalsToForest[` *k_* `][` *listOfDiags_* `]`: returns the corresponding OG forest of type $\underline{k}$ obtained from `modDiagonalsToPlanarTree[` *k* `][` *listOfDiags* `]` via the map described in this paper.

In[12]:= `modDiagonalsToForest[4][{{1,3},{1,5}}]`

Out[12]=



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