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Some New Results for the Kampé de Fériet Function with an Application

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Abstract: The generalized hypergeometric functions in one and several variables and their natural generalizations appear in many mathematical problems and their applications. The theory of generalized hypergeometric functions in several variables comes from the fact that the solutions of the partial differential equations appearing in a large number of applied problems of mathematical physics have been expressed in terms of such generalized hypergeometric functions. In particular, the Kampé de Fériet function (in two variables) has proved its practical utility in representing solutions to a wide range of problems in pure and applied mathematics, statistics, and mathematical physics. In this context, in a very recent paper, Progri successfully calculated the ${}_2F_2$ generalized hypergeometric function for a particular set of parameters and expressed the result in terms of the difference between two Kampé de Fériet functions. Inspired by his work, in the present paper, we obtain three results for a terminating ${}_3F_2$ series of arguments 1 and 2, together with a transformation formula of a ${}_3F_2(z)$ generalized hypergeometric function in terms of the difference between two Kampé de Fériet functions. One application of this result is also provided. The paper concludes with six reduction formulas for the Kampé de Fériet function. Of note, symmetry occurs naturally in the generalized hypergeometric functions ${}_pF_q$ and the Kampé de Fériet function involving two variables, which are the two most important functions discussed in this paper.

Keywords: Gauss hypergeometric function; generalized hypergeometric function; unit argument; Ramanujan's summations; reducibility; Kampé de Fériet function

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1. Introduction

The generalized hypergeometric function ${}_pF_q[z]$ is defined by [1], (p. 404)

$${}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \ldots (a_p)_n}{(b_1)_n (b_2)_n \ldots (b_q)_n} \frac{z^n}{n!},$$

where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1) = \Gamma(\alpha+n)/\Gamma(\alpha)$ is the Pochhammer symbol (or the shifted factorial). Here p and q are nonnegative integers, and the parameters a_j $(1 \le j \le p)$ and b_j $(1 \le j \le q)$ can have arbitrary complex values, with zero or negative integer values of b_j excluded. The sum converges for $|z| < \infty$ $(p \le q)$, $|z| < 1$ $(p = q+1)$ and $|z| = 1$ $(p = q+1$, and $\Re(s) > 0)$, where s is the parametric excess defined by $s = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j$.

Symmetry occurs in the numerator parameters $a_1, a_2, \ldots, a_p$, and symmetry also occurs in the denominator parameters $b_1, b_2, \ldots, b_q$ of the generalized hypergeometric function

$${}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}; z\right].$$

That means that every reordering of the numerator parameters $a_1, a_2, \ldots, a_p$ of the generalized hypergeometric function provides the same function, and every reordering of the denominator parameters $b_1, b_2, \ldots, b_q$ of the generalized hypergeometric function produces the same function.

In the theory and application of hypergeometric and generalized hypergeometric series, summation and transformation formulas play a very important role. In our present investigation, we require Kummer's second theorem [2], (p. 126), for the confluent hypergeometric function

$$e^{-z}\,{}_1F_1\left[\begin{matrix} a \\ 2a \end{matrix};2z\right] = {}_0F_1\left[\begin{matrix} - \\ a+\frac{1}{2} \end{matrix};\frac{z^2}{4}\right]. \tag{1}$$

Kummer [3] established this result from the theory of differential equations. Bailey [4] gave a different proof using the two Gauss summation theorems for ${}_2F_1$. From (1), it is possible to deduce the two results recorded in [2], (p. 127)

$${}_2F_1\left[\begin{matrix} -2n, a \\ 2a \end{matrix};2\right] = \frac{(\frac{1}{2})_n}{(a+\frac{1}{2})_n}, \tag{2}$$

and

$${}_2F_1\left[\begin{matrix} -2n-1, a \\ 2a \end{matrix};2\right] = 0, \tag{3}$$

for a nonnegative integer n.

The following result contiguous to (1) (written here in slightly different form), which is a special case of a general result recorded in [5], (7.11.7 for $n = 1$, p. 579) viz

$$e^{-z}\,{}_1F_1\left[\begin{matrix} a \\ 2a+1 \end{matrix};2z\right] = {}_0F_1\left[\begin{matrix} a \\ a+\frac{1}{2} \end{matrix};\frac{z^2}{4}\right] - \frac{z}{(2a+1)}\,{}_0F_1\left[\begin{matrix} - \\ a+\frac{3}{2} \end{matrix};\frac{z^2}{4}\right]. \tag{4}$$

From this last result (4), it is possible to deduce the following two contiguous evaluations recorded in [6]

$${}_2F_1\left[\begin{matrix} -2n, a \\ 2a+1 \end{matrix};2\right] = \frac{(\frac{1}{2})_n}{(a+\frac{1}{2})_n}, \tag{5}$$

$${}_2F_1\left[\begin{matrix} -2n-1, a \\ 2a+1 \end{matrix};2\right] = \frac{(\frac{3}{2})_n}{(2a+1)(a+\frac{3}{2})_n}. \tag{6}$$

In addition, from [6], we have

$${}_2F_1\left[\begin{matrix} -2n, a \\ 2a+2 \end{matrix};2\right] = \frac{(\frac{1}{2})_n(\frac{1}{2}a+\frac{3}{2})_n}{(a+\frac{3}{2})_n(\frac{1}{2}a+\frac{1}{2})_n}, \tag{7}$$

where in (5)–(7), n is also a nonnegative integer.

In addition, the well-known quadratic transformation for the Gauss hypergeometric function due to Kummer [3] is (see also [1], (15.8.13))

$$(1-z)^c\,{}_2F_1\left[\begin{matrix} a, c \\ 2a \end{matrix};2z\right] = {}_2F_1\left[\begin{matrix} \frac{1}{2}c, \frac{1}{2}c+\frac{1}{2} \\ a+\frac{1}{2} \end{matrix};\left(\frac{z}{1-z}\right)^2\right], \tag{8}$$

provided $|z| < \frac{1}{2}$ and $|z/(1-z)| < 1$. Another form of this formula can be written by setting $z/(1-z) = -x$ to obtain

$$(1-x)^{-c}\,{}_2F_1\left[\begin{matrix} a, c \\ 2a \end{matrix};\frac{-2x}{1-x}\right] = {}_2F_1\left[\begin{matrix} \frac{1}{2}c, \frac{1}{2}c+\frac{1}{2} \\ a+\frac{1}{2} \end{matrix};x^2\right]. \tag{9}$$

Employing the beta-integral method for (9), Krattenthaler and Rao [7], (Eq. (3.4)) obtained the following hypergeometric identity

$$ {}_3F_2\left[\begin{matrix} a, c, d \\ 2a, 1-e+c+d \end{matrix}; 2\right] = \frac{\Gamma(e-c)\Gamma(e-d)}{\Gamma(e)\Gamma(e-c-d)}\, {}_4F_3\left[\begin{matrix} \frac{1}{2}c, \frac{1}{2}c+\frac{1}{2}, \frac{1}{2}d, \frac{1}{2}d+\frac{1}{2} \\ a+\frac{1}{2}, \frac{1}{2}e, \frac{1}{2}e+\frac{1}{2} \end{matrix}; 1\right], \tag{10} $$

provided c or d is a nonpositive integer.

In 2011, Choi and Rathie [8] obtained two results contiguous to (8), one of which is given here:

$$ (1-z)^c\, {}_2F_1\left[\begin{matrix} a, c \\ 2a+1 \end{matrix}; 2z\right] = {}_2F_1\left[\begin{matrix} \frac{1}{2}c, \frac{1}{2}c+\frac{1}{2} \\ a+\frac{1}{2} \end{matrix}; \left(\frac{z}{1-z}\right)^2\right] - \frac{cz}{(2a+1)(1-z)}\, {}_2F_1\left[\begin{matrix} \frac{1}{2}c+\frac{1}{2}, \frac{1}{2}c+1 \\ a+\frac{3}{2} \end{matrix}; \left(\frac{z}{1-z}\right)^2\right], \tag{11} $$

provided $|z| < 1$ and $|z/(1-z)| < 1$. As discussed above, another form of this result is

$$ (1-x)^{-c}{}_2F_1\left[\begin{matrix} a, c \\ 2a+1 \end{matrix}; \frac{-2x}{1-x}\right] = {}_2F_1\left[\begin{matrix} \frac{1}{2}c, \frac{1}{2}c+\frac{1}{2} \\ a+\frac{1}{2} \end{matrix}; x^2\right] + \frac{cx}{2a+1}\, {}_2F_1\left[\begin{matrix} \frac{1}{2}c+\frac{1}{2}, \frac{1}{2}c+1 \\ a+\frac{3}{2} \end{matrix}; x^2\right]. \tag{12} $$

Employing the beta-integral method for (12), Ibraim et al. [9] from Equation (11) obtained the following identity

$$ {}_3F_2\left[\begin{matrix} a, c, d \\ 2a+1, 1-e+c+d \end{matrix}; 2\right] = \frac{\Gamma(e-c)\Gamma(e-d)}{\Gamma(e)\Gamma(e-c-d)}\Bigg\{ {}_4F_3\left[\begin{matrix} \frac{1}{2}c, \frac{1}{2}c+\frac{1}{2}, \frac{1}{2}d, \frac{1}{2}d+\frac{1}{2} \\ a+\frac{1}{2}, \frac{1}{2}e, \frac{1}{2}e+\frac{1}{2} \end{matrix}; 1\right] + \frac{cd}{e(2a+1)}\, {}_4F_3\left[\begin{matrix} \frac{1}{2}c+\frac{1}{2}, \frac{1}{2}c+1, \frac{1}{2}d+\frac{1}{2}, \frac{1}{2}d+1 \\ a+\frac{3}{2}, \frac{1}{2}e+\frac{1}{2}, \frac{1}{2}e+1 \end{matrix}; 1\right]\Bigg\}, \tag{13} $$

provided c or d is a nonpositive integer.

Very recently, by means of Kummer's second theorem (1), Progri [10] from Equation (1) expressed a certain ${}_2F_2$ hypergeometric function as the difference of two Kampé de Fériet double hypergeometric functions. The Kampé de Fériet function [11] is defined by

$$ F^{p:r,k}_{q:s,\ell}\left[\begin{matrix} (a_j) : (b_j); (b'_j) \\ (c_j) : (d_j); (d'_j) \end{matrix}; x, y\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a_1)_{m+n}\ldots(a_p)_{m+n}(b_1)_m\ldots(b_r)_m(b'_1)_n\ldots(b'_k)_n}{(c_1)_{m+n}\ldots(c_q)_{m+n}(d_1)_m\ldots(d_s)_m(d'_1)_n\ldots(d'_\ell)_n}\,\frac{x^m y^n}{m!n!}, $$

where the double sum converges

$$ \begin{aligned} &\text{(i) For } |x|, |y| < \infty \text{ when } p+r < q+s+1,\ p+k < q+\ell+1; \\ &\text{(ii) For } |x|, |y| < 1\ (p \le q),\ |x|^\alpha + |y|^\alpha < 1\ (p > q) \text{ when} \\ &\qquad p+r = q+s+1,\ p+k = q+\ell+1, \end{aligned} \tag{14} $$

where $\alpha := 1/(p-q)$. Progri's result then takes the form

$$ e^{-z}{}_2F_2\left[\begin{matrix} a, b \\ 2a, b+1 \end{matrix}; 2z\right] = F^{0:2;1}_{2:1;0}\left[\begin{matrix} - & : & \frac{1}{2}b, \frac{1}{2}b+\frac{1}{2}; & 1 \\ \frac{1}{2}b+\frac{1}{2}, \frac{1}{2}b+1 & : & a+\frac{1}{2} \quad ; & - \end{matrix}; \frac{z^2}{4}, \frac{z^2}{4}\right] - \frac{z}{b+1}\, F^{0:2;1}_{2:1;0}\left[\begin{matrix} - & : & \frac{1}{2}b, \frac{1}{2}b+\frac{1}{2}; & 1 \\ \frac{1}{2}b+1, \frac{1}{2}b+\frac{3}{2} & : & a+\frac{1}{2} \quad ; & - \end{matrix}; \frac{z^2}{4}, \frac{z^2}{4}\right]. \tag{15} $$

Symmetry occurs naturally in the Kampé de Fériet function defined by (14) for the numerator parameters a_j $(j = 1\ldots,p)$, b_j $(j = 1\ldots,r)$, and b'_j $(j = 1\ldots,k)$ and also for the denominator parameters c_j $(j = 1\ldots,q)$, d_j $(j = 1\ldots,s)$, and d'_j $(j = 1\ldots,\ell)$. Thus, the generalized hypergeometric function ${}_pF_q$ and the Kampé de Fériet function, which are the most significant functions investigated in this research, exhibit natural symmetry.

The hypergeometric functions of one and several variables occur naturally in a wide variety of problems in applied mathematics, physics (theoretical and mathematical), engineering sciences, statistics, and operations research. There have been applications of hypergeometric functions in one and several variables in such diverse fields as

(a) Statistical distribution theory related to beta, gamma, and normal distributions
(b) Communications engineering;
(c) Theory of Lie algebras and Lie groups;
(d) Integral transforms (including integral equations);
(e) Perturbation theory;
(f) Decision theory, etc.

These have been presented in detail in two texts [12,13] by Exton.

The rest of the paper is organized as follows. In Section 2, we derive three new and general results for the terminating ${}_3F_2$ series of argument 1 and 2. As special cases, we recover the fresults recorded in [6]. In Section 3, we establish a new transformation formula for the generalized hypergeometric function ${}_3F_2$ expressed as the difference of two Kampé de Fériet functions. Finally, in Section 4, we present six results for the reducibility of the Kampé de Fériet function.

2. Three Results for the Terminating ${}_3F_2$ Series

In this section, we establish three results for terminating ${}_3F_2$ series asserted in the following theorem:

Theorem 1. *Let m be a nonnegative integer. Then, the following three results for terminating ${}_3F_2$ series hold true:*

$$ {}_3F_2\left[\begin{matrix}-2m,a,b\\2a,b+1\end{matrix};2\right] = \frac{(\frac{1}{2})_m(\frac{1}{2}b)_m}{(a+\frac{1}{2})_m(\frac{1}{2}b+1)_m}\,{}_3F_2\left[\begin{matrix}-m,1,\frac{1}{2}-a-m\\\frac{1}{2}-\frac{1}{2}b-m,1-\frac{1}{2}b-m\end{matrix};1\right], \tag{16} $$

$$ \begin{aligned} &{}_3F_2\left[\begin{matrix}-2m-1,a,b\\2a,b+1\end{matrix};2\right]\\ &= \frac{(\frac{3}{2})_m(\frac{1}{2}b)_m(\frac{1}{2}b+\frac{1}{2})_m}{(b+1)(a+\frac{1}{2})_m(\frac{1}{2}b+1)_m(\frac{1}{2}b+\frac{3}{2})_m}\,{}_3F_2\left[\begin{matrix}-m,1,\frac{1}{2}-a-m\\\frac{1}{2}-\frac{1}{2}b-m,1-\frac{1}{2}b-m\end{matrix};1\right], \end{aligned} \tag{17} $$

and

$$ {}_3F_2\left[\begin{matrix}-m,1,\frac{1}{2}-a-m\\1-\frac{1}{2}a-m,\frac{3}{2}-\frac{1}{2}a-m\end{matrix};1\right] = \frac{(\frac{1}{2}a+\frac{1}{2})_m(\frac{1}{2}a+1)_m}{(\frac{1}{2}a-\frac{1}{2})_m(\frac{1}{2}a)_m}. \tag{18} $$

Proof. In order to establish (16) and (17), we proceed as follows. We write Progri's result (15) in the form

$$ S := e^{-z}{}_3F_2\left[\begin{matrix}a,b\\2a,b+1\end{matrix};2z\right] = F_1(z) - \frac{zF_2(z)}{(b+1)}, \tag{19} $$

where, for $j = 1,2$,

$$ F_j(z) := F^{0:2;1}_{2:1;0}\left[\begin{matrix}-&:&\frac{1}{2}b,\frac{1}{2}b+\frac{1}{2}&;&1\\\frac{1}{2}b+\frac{1}{2}j,\frac{1}{2}b+\frac{1}{2}j+\frac{1}{2}&:&a+\frac{1}{2}&;&-\end{matrix};\frac{z^2}{4},\frac{z^2}{4}\right]. \tag{20} $$

Expressing S in series form, we find

$$S=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m 2^n (a)_n (b)_n}{(2a)_n (b+1)_n}\frac{z^{m+n}}{m!n!}.$$

We replace m by $m-n$, using the result from [2], (p. 56)

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\mathcal{A}(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\mathcal{A}(k,n-k),$$

together with the elementary identity $(m-n)!=(-1)^n m!/(-m)_n$, yields after some simplification

$$\begin{aligned} S &= \sum_{m=0}^{\infty}\frac{(-1)^m z^m}{m!}\sum_{n=0}^{m}\frac{2^n(-m)_n(a)_n(b)_n}{(2a)_n(b+1)_n n!} \\ &= \sum_{m=0}^{\infty}\frac{(-1)^m z^m}{m!}\,{}_3F_2\left[\begin{matrix}-m,a,b\\ 2a,b+1\end{matrix};2\right]. \end{aligned}$$

Separating this last expression into even and odd powers of z, we finally obtain

$$S=\sum_{m=0}^{\infty}\frac{z^{2m}}{(2m)!}\,{}_3F_2\left[\begin{matrix}-2m,a,b\\ 2a,b+1\end{matrix};2\right]-\sum_{m=0}^{\infty}\frac{z^{2m+1}}{(2m+1)!}\,{}_3F_2\left[\begin{matrix}-2m-1,a,b\\ 2a,b+1\end{matrix};2\right]. \tag{21}$$

Proceeding as above, using $(\alpha)_{m-n}=(-1)^n(\alpha)_m/(1-\alpha-m)_n$, it is not difficult to see that

$$F_1(z)=\sum_{m=0}^{\infty}\frac{(\frac{1}{2}b)_m(z^2/4)^m}{(a+\frac{1}{2})_m(\frac{1}{2}b+1)_m m!}\,{}_3F_2\left[\begin{matrix}-m,1,\frac{1}{2}-a-m\\ \frac{1}{2}-\frac{1}{2}b-m,1-\frac{1}{2}b-m\end{matrix};1\right],$$

and

$$F_2(z)=\sum_{m=0}^{\infty}\frac{(\frac{1}{2}b)_m(\frac{1}{2}b+\frac{1}{2})_m(z^2/4)^m}{(a+\frac{1}{2})_m(\frac{1}{2}b+1)_m(\frac{1}{2}b+\frac{3}{2})_m m!}\,{}_3F_2\left[\begin{matrix}-m,1,\frac{1}{2}-a-m\\ \frac{1}{2}-\frac{1}{2}b-m,1-\frac{1}{2}b-m\end{matrix};1\right].$$

Finally, the expressions for S, $F_1(z)$, and $F_2(z)$ may be substituted in (19). Then, equating the coefficients of z^{2m} and z^{2m+1} on both sides and using the identities

$$(2m)!=2^{2m}(\tfrac{1}{2})_m m!,\qquad (2m+1)!=2^{2m}(\tfrac{3}{2})_m m!,$$

we obtain the results stated in (16) and (17).

For the derivation of (18), we put $b=a-1$ into (16). Then, making use of the summation (7), we at once obtain the result (18). This completes the proof of Theorem 1. □

Remark 1. *In (16) and (17), if we take $b=2a$ and make use of Vandermonde's theorem*

$${}_2F_1\left[\begin{matrix}-m,a\\ c\end{matrix};1\right]=\frac{(c-a)_m}{(c)_m},$$

for a nonnegative integer m, we obtain the known results (5) and (6), respectively.

Remark 2. *If we let $b\to\infty$ in (16) and (17), then we recover the results (2) and (3), respectively.*

3. A New Transformation Formula

In this section, we establish a transformation formula asserted in the following theorem:

Theorem 2. *Let $Z = z/(1-z)$. The following transformation formula holds true*

$$(1-z)^c {}_3F_2\left[\begin{matrix} a,b,c \\ 2a,b+1 \end{matrix};2z\right] = F^{2:2;1}_{2:1;0}\left[\begin{matrix} \frac{1}{2}c, \frac{1}{2}c+\frac{1}{2} & : & \frac{1}{2}b, \frac{1}{2}b+\frac{1}{2}; & 1 \\ \frac{1}{2}b+\frac{1}{2}, \frac{1}{2}b+1 : & & a+\frac{1}{2} & ; & - \end{matrix};Z^2,Z^2\right]$$

$$-\frac{cZ}{(b+1)} F^{2:2;1}_{2:1;0}\left[\begin{matrix} \frac{1}{2}c+\frac{1}{2}, \frac{1}{2}c+1 : & \frac{1}{2}b, \frac{1}{2}b+\frac{1}{2}; & 1 \\ \frac{1}{2}b+1, \frac{1}{2}b+\frac{3}{2} : & a+\frac{1}{2} \quad ; & - \end{matrix};Z^2,Z^2\right], \tag{22}$$

provided $|z| < \frac{1}{2}$ and $|Z| < 1$.

Proof. In (15), we replace z by zt, multiply both sides by $e^{-t}t^{c-1}$ (with $\Re(c) > 0$), and integrate the resulting expressions with respect to t over the interval $[0,\infty)$. We then have

$$S = \int_0^\infty e^{-t}t^{c-1} {}_2F_2\left[\begin{matrix} a,b \\ 2a,b+1 \end{matrix};2zt\right] dt = S_1 - \frac{zt}{b+1} S_2, \tag{23}$$

where

$$S_j = \int_0^\infty e^{-(1-z)t}t^{c-1}F_j(zt)\,dt \qquad (j=1,2),$$

with $F_j(zt)$ being the Kampé de Fériet functions defined in (20).

From [1], (16.11.7), the leading asymptotic behavior of the above ${}_2F_2$ function is $O(t^{-a-1}e^{2zt})$ as $t \to +\infty$ when $|\arg z| < \frac{1}{2}\pi$, and $O(t^{-\gamma})$ $\gamma = \min\{|a|,|b|\}$ as $t \to +\infty$ when $\frac{1}{2}\pi \le |\arg z| \le \pi$. Consequently, the integral S converges when $\Re(z) < \frac{1}{2}$. Now, expressing the ${}_2F_2$ function as a series, changing the order of summation and integration (justified by absolute convergence), and straightforwardly evaluating the gamma function integral, we have

$$S = \Gamma(c)\,{}_3F_2\left[\begin{matrix} a,b,c \\ 2a,b+1 \end{matrix};2z\right] \qquad (|z| < \tfrac{1}{2}). \tag{24}$$

From (A3) in Appendix A, the asymptotic behavior of the Kampé de Fériet functions appearing in S_1 and S_2 is $O(e^{zt})$ ($|\arg z| \le \frac{1}{2}\pi$) and $O(e^{\bar{z}t})$ ($\frac{1}{2}\pi < |\arg z| \le \pi$), respectively. Consequently, the integrals S_1 and S_2 also require $\Re(z) < \frac{1}{2}$ for convergence. Expanding the Kampé de Fériet function in S_1 in double series form (which converges for $|zt| < \infty$ by (14)) and proceeding as above, we obtain

$$S_1 = \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{(\frac{1}{2}b)_m(\frac{1}{2}b+\frac{1}{2})_m(z/2)^{2m+2n}}{(\frac{1}{2}b+\frac{1}{2})_{m+n}(\frac{1}{2}b+1)_{m+n}(a+\frac{1}{2})_m m!} \int_0^\infty e^{-(1-z)t}t^{2m+2n+c-1}dt$$

$$= \frac{\Gamma(c)}{(1-z)^c} \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{(\frac{1}{2}c)_{m+n}(\frac{1}{2}c+\frac{1}{2})_{m+n}(\frac{1}{2}b)_m(\frac{1}{2}b+\frac{1}{2})_m}{(\frac{1}{2}b+\frac{1}{2})_{m+n}(\frac{1}{2}b+1)_{m+n}(a+\frac{1}{2})_m m!} \left(\frac{z}{1-z}\right)^{2m+2n}$$

$$= \frac{\Gamma(c)}{(1-z)^c} \left\{ F^{2:2;1}_{2:1;0}\left[\begin{matrix} \frac{1}{2}c, \frac{1}{2}c+\frac{1}{2} & : & \frac{1}{2}b, \frac{1}{2}b+\frac{1}{2}; & 1 \\ \frac{1}{2}b+\frac{1}{2}, \frac{1}{2}b+1 : & & a+\frac{1}{2} & ; & - \end{matrix};Z^2,Z^2\right],\right.$$

where we have put $Z := z/(1-z)$ (with $|Z| < 1$ by (14)) and have made use of the result $(c)_{2m+2n} = (\frac{1}{2}c)_{m+n}(\frac{1}{2}c+\frac{1}{2})_{m+n}2^{2m+2n}$. A similar treatment of S_2 together with (24) then yields the result stated in (22). The restriction $\Re(c) > 0$, necessary for the convergence of the integrals in (23), can be removed by appeal to analytic continuation. □

Remark 3. *If we replace z by z/c in (22) and let $c \to +\infty$, we recover Progri's result (15).*

Remark 4. *If we let $b \to \infty$ in (22) and note that*

$$\lim_{b\to+\infty} F^{2:2;1}_{2:1;0}\left[\begin{matrix} \frac{1}{2}c, \frac{1}{2}c+\frac{1}{2} & : & \frac{1}{2}b, \frac{1}{2}b+\frac{1}{2}; & 1 \\ \frac{1}{2}b+\frac{1}{2}, \frac{1}{2}b+1 : & & a+\frac{1}{2} ; & — \end{matrix} ; Z^2, Z^2\right] = {}_2F_1\left[\begin{matrix} \frac{1}{2}c, \frac{1}{2}c+\frac{1}{2} \\ a+\frac{1}{2} \end{matrix} ; Z^2\right],$$

and

$$\lim_{b\to+\infty} F^{2:2;1}_{2:1;0}\left[\begin{matrix} \frac{1}{2}c+\frac{1}{2}, \frac{1}{2}c+1 : & \frac{1}{2}b, \frac{1}{2}b+\frac{1}{2}; & 1 \\ \frac{1}{2}b+1, \frac{1}{2}b+\frac{3}{2} : & a+\frac{1}{2} ; & — \end{matrix} ; Z^2, Z^2\right] = {}_2F_1\left[\begin{matrix} \frac{1}{2}c+\frac{1}{2}, \frac{1}{2}c+1 \\ a+\frac{1}{2} \end{matrix} ; Z^2\right],$$

we recover Kummer's transformation (8).

As an application of the result in Theorem 2, by employing the beta-integral method, we establish the identity asserted in the following theorem:

Theorem 3. *If c or $d = 0, -1, -2, \ldots$, then the following identity holds true:*

$$\begin{aligned} &{}_4F_3\left[\begin{matrix} a, b, c, d \\ 2a, b+1, 1-e+c+d \end{matrix} ; 2\right] \\ &= \frac{\Gamma(e-c)\Gamma(e-d)}{\Gamma(e)\Gamma(e-c-d)} \left\{ F^{4:2;1}_{4:1;0}\left[\begin{matrix} \frac{1}{2}c, \frac{1}{2}c+\frac{1}{2}, \frac{1}{2}d, \frac{1}{2}d+\frac{1}{2} & : & \frac{1}{2}b, \frac{1}{2}b+\frac{1}{2} & ; & 1 \\ \frac{1}{2}b+\frac{1}{2}, \frac{1}{2}b+1, \frac{1}{2}e, \frac{1}{2}e+\frac{1}{2} & : & a+\frac{1}{2} & ; & — \end{matrix} ; 1, 1\right] \right. \\ &\left. + \frac{cd}{e(b+1)} F^{4:2;1}_{4:1;0}\left[\begin{matrix} \frac{1}{2}c+\frac{1}{2}, \frac{1}{2}c+1, \frac{1}{2}d+\frac{1}{2}, \frac{1}{2}d+1 : & \frac{1}{2}b, \frac{1}{2}b+\frac{1}{2} & ; & 1 \\ \frac{1}{2}b+1, \frac{1}{2}b+\frac{3}{2}, \frac{1}{2}e+\frac{1}{2}, \frac{1}{2}e+1 : & a+\frac{1}{2} & ; & — \end{matrix} ; 1, 1\right] \right\}. \end{aligned} \tag{25}$$

The result (25) also holds when $a = -1, -2, \ldots$, subject to the convergence conditions $\Re(a+e-c-d) > 0$ and $\Re(b+e-c-d) > 0$.

Proof. If we substitute $Z = z/(1-z) = -x$ into the result (22), then we have

$$(1-x)^{-c}{}_3F_2\left[\begin{matrix} a, b, c \\ 2a, b+1 \end{matrix} ; \frac{-2x}{1-x}\right] = \hat{F}_1(x) + \frac{cx}{b+1}\hat{F}_2(x), \tag{26}$$

where, for $j = 1, 2$,

$$\hat{F}_j(x) := F^{2:2;1}_{2:1;0}\left[\begin{matrix} \frac{1}{2}c+\frac{1}{2}j-\frac{1}{2}, \frac{1}{2}c+\frac{1}{2}j : & \frac{1}{2}b, \frac{1}{2}b+\frac{1}{2} ; & 1 \\ \frac{1}{2}b+\frac{1}{2}j, \frac{1}{2}b+\frac{1}{2}j+\frac{1}{2} : & a+\frac{1}{2} ; & — \end{matrix} ; x^2, x^2\right].$$

Now, we multiply both sides of (26) by $x^{d-1}(1-x)^{e-d-1}$ (where we temporarily assume $\Re(e) > \Re(d) > 0$) and integrate with respect to x over the interval $[0,1]$ to produce

$$\begin{aligned} &\int_0^1 x^{d-1}(1-x)^{e-c-d-1}{}_3F_2\left[\begin{matrix} a, b, c \\ 2a, b+1 \end{matrix} ; \frac{-2x}{1-x}\right] dx \\ &= \int_0^1 x^{d-1}(1-x)^{e-d-1}\hat{F}_1(x)\,dx + \frac{c}{b+1}\int_0^1 x^{d}(1-x)^{e-d-1}\hat{F}_2(x)\,dx. \end{aligned} \tag{27}$$

When $c = -m$, $m = 0, 1, 2, \ldots$, both the ${}_3F_2(x)$ and the Kampé de Fériet functions $\hat{F}_j(x)$ terminate. Then, the integral on the left-hand side of (27) becomes

$$\begin{aligned} &\sum_{n=0}^{m} \frac{(-m)_n(a)_n(b)_n(-2)^n}{(2a)_n(b+1)_n n!} \int_0^1 x^{d+n-1}(1-x)^{e-d+m-n-1}dx \\ &= \frac{\Gamma(d)\Gamma(e-c-d)}{\Gamma(e-c)} \sum_{n=0}^{m} \frac{(-m)_n(a)_n(b)_n(d)_n 2^n}{(2a)_n(b+1)_n(1-e+c+d)_n}, \end{aligned}$$

upon evaluation of the beta integral (subject to $\Re(e) > \Re(d) > 0$) followed by some routine simplification. The sum in the above expression can then be identified with the ${}_4F_3$ series in (25). Expanding $\hat{F}_1(x)$ in series form and carrying out the integration, we find

$$\int_0^1 x^{d-1}(1-x)^{e-d-1}\hat{F}_1(x)\,dx = \frac{\Gamma(d)\Gamma(c-d)}{\Gamma(e)}$$
$$\times F^{4:2;1}_{4:1;0}\left[\begin{matrix} \frac{1}{2}c, \frac{1}{2}c+\frac{1}{2}, \frac{1}{2}d, \frac{1}{2}d+\frac{1}{2} & : & \frac{1}{2}b, \frac{1}{2}b+\frac{1}{2} & ; & 1 \\ \frac{1}{2}b+\frac{1}{2}, \frac{1}{2}b+1, \frac{1}{2}e, \frac{1}{2}e+\frac{1}{2} & : & a+\frac{1}{2} & ; & - \end{matrix} ;1,1\right],$$

with a similar result for the integral involving $\hat{F}_2(x)$. This then establishes (25) when c is a nonpositive integer.

When $a = -m$, $m = 1, 2, \ldots$ we may proceed in a similar manner by interchanging the order of summation and integration to find that the integral on the left-hand side of (27) now results in the beta integral containing the factor $(1-x)^{e-c-d-n-1}$, thus requiring the condition $\Re(e-c-d) > m$ or equivalently $\Re(a+e-c-d) > 0$. The treatment of the integrals involving $\hat{F}_j(x)$ is the same, except that as the resulting Kampé de Fériet functions now no longer terminate, we need the conditions [14], Theorem 1

$$\Re(a+e-c-d) > 0, \qquad \Re(b+e-c-d) > 0, \tag{28}$$

for convergence. The condition $\Re(e) > \Re(d) > 0$ may be removed by analytic continuation. As the parameters c and d are interchangeable, the result (25) will also hold for $d = -m$. This completes the proof of Theorem 3. □

Remark 5. *If we set $b \to +\infty$ in (25), we recover the result (10).*

4. Six Results for the Reducibility of the Kampé de Fériet Function

The six results for the reducibility of the Kampé de Fériet function to be established in this section are given in the following theorem.

Theorem 4. *The following results hold true:*

$$F^{0:1;1}_{2:0;0}\left[\begin{matrix} - & : & a & ; & 1 \\ a+\frac{1}{2}, a+1 & : & - & ; & - \end{matrix} ;x,x\right] = {}_0F_1\left[\begin{matrix} - \\ a+\frac{1}{2} \end{matrix} ;x\right], \tag{29}$$

$$F^{0:1;1}_{2:0;0}\left[\begin{matrix} - & : & a & ; & 1 \\ a+1, a+\frac{3}{2} & : & - & ; & - \end{matrix} ;x,x\right] = {}_0F_1\left[\begin{matrix} - \\ a+\frac{3}{2} \end{matrix} ;x\right], \tag{30}$$

$$F^{2:1;1}_{2:0;0}\left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2} & : & b & ; & 1 \\ b+\frac{1}{2}, b+1 & : & - & ; & - \end{matrix} ;x,x\right] = {}_2F_1\left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2} \\ b+\frac{1}{2} \end{matrix} ;x\right], \tag{31}$$

$$F^{2:1;1}_{2:0;0}\left[\begin{matrix} \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}a+1 & : & b & ; & 1 \\ b+1, b+\frac{3}{2} & : & - & ; & - \end{matrix} ;x,x\right] = {}_2F_1\left[\begin{matrix} \frac{1}{2}a+\frac{1}{2}, \frac{1}{2}a+1 \\ b+\frac{3}{2} \end{matrix} ;x\right]. \tag{32}$$

In addition, we have

$$F^{4:1;1}_{4:0;0}\left[\begin{matrix} \frac{1}{2}c, \frac{1}{2}c+\frac{1}{2}, \frac{1}{2}d, \frac{1}{2}d+\frac{1}{2} & : & a & ; & 1 \\ a+\frac{1}{2}, a+1, \frac{1}{2}e, \frac{1}{2}e+\frac{1}{2} & : & - & ; & - \end{matrix} ;1,1\right] = {}_4F_3\left[\begin{matrix} \frac{1}{2}c, \frac{1}{2}c+\frac{1}{2}, \frac{1}{2}d, \frac{1}{2}d+\frac{1}{2} \\ a+\frac{1}{2}, \frac{1}{2}e, \frac{1}{2}e+\frac{1}{2} \end{matrix} ;1\right], \tag{33}$$

and

$$F^{4:1;1}_{4:0;0}\left[\begin{matrix} \frac{1}{2}c+\frac{1}{2}, \frac{1}{2}c+1, \frac{1}{2}d+\frac{1}{2}, \frac{1}{2}d+1 & : & a & ; & 1 \\ a+1, a+\frac{3}{2}, \frac{1}{2}e+\frac{1}{2}, \frac{1}{2}e+1 & : & - & ; & - \end{matrix} ;1,1\right]$$
$$= {}_4F_3\left[\begin{matrix} \frac{1}{2}c+\frac{1}{2}, \frac{1}{2}c+1, \frac{1}{2}d+\frac{1}{2}, \frac{1}{2}d+1 \\ a+\frac{3}{2}, \frac{1}{2}e+\frac{1}{2}, \frac{1}{2}e+1 \end{matrix} ;1\right], \tag{34}$$

subject to the convergence conditions $\Re(a+e-c-d) > 0$ and $\Re(2a+e-c-d) > 0$ for general parameter values.

Proof. To establish (29) and (30), we take $b = 2a$ in Progri's result (15) to find

$$e^{-z}{}_1F_1\left[\begin{matrix} a \\ 2a+1 \end{matrix};2z\right] = F^{0:1;1}_{2:0;0}\left[\begin{matrix} - & : & a & ; & 1 \\ a+\frac{1}{2}, a+1 & : & - & ; & - \end{matrix};\frac{z^2}{4},\frac{z^2}{4}\right]$$

$$-\frac{z}{2a+1}F^{0:1;1}_{2:0;0}\left[\begin{matrix} - & : & a & ; & 1 \\ a+1, a+\frac{3}{2} & : & - & ; & - \end{matrix};\frac{z^2}{4},\frac{z^2}{4}\right]. \tag{35}$$

Comparison of (35) with (4) immediately establishes (29) and (30) upon replacing $z^2/4$ by x. In the same manner, (31) and (32) can be established by comparison of (22) when $b = 2a$ with (11).

The results (33) and (34) can be established by comparison of (25) when $b = 2a$ with (13). Although derived for c or $d = -m$, $m = 0, 1, 2, \ldots$, they can be extended by analytic continuation to the general values of a ($\neq -m$), c, and d provided the stated convergence conditions hold; see also, (28). □

The results from (29) to (34) can also be deduced from a general result mentioned in Equation (32) from [11], (p. 28).

Remark 6. *If we set $b = 2a$ in (25) and make use of (33) and (34), we obtain the result (13).*

5. Conclusions

In this paper, using a recent result of Progri, we obtained:

(a) Three new results for the terminating ${}_3F_2$ generalized hypergeometric function of arguments 1 and 2;
(b) A new transformation formula for a ${}_3F_2(z)$ generalized hypergeometric function. The result has been expressed in terms of the difference between two Kampé de Fériet functions;
(c) As an application, by employing the well-known beta integral method, an identity;
(d) Six reduction formulas for the Kampé de Fériet function.

We believe that the results established in this paper have not appeared in the literature and represent a contribution to the theory of generalized hypergeometric functions of one and two variables. It is hoped that the results could be of potential use in the area of applied mathematics, statistics, and mathematical physics. All the results mentioned in the paper were verified numerically using *Mathematica*.

Finally, since the results presented in the paper are general in character and generalize and unify several results available in the existing literature, by employing three results for the terminating ${}_3F_2$ with arguments 1 and 2 presented in Theorem 1, we can further obtain some new results. Similarly, we can use the results given in Theorems 2–4 to obtain several new results. These results are under investigations and will form a subsequent paper in this direction.

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Appendix A. Order Estimate for the Kampé de Fériet Functions in (23)

In this appendix, we estimate the order of the Kampé de Fériet functions appearing in (23) as $t \to +\infty$. For simplicity of presentation, we suppose that the parameters a and b are real with $b > 0$. We first present integral representations for these two functions.

Lemma A1. *The following integral representations when $b > 0$ hold true:*

$$F^{0:2;1}_{2:1;0}\left[\begin{matrix} - & : & \frac{1}{2}b, \frac{1}{2}b+\frac{1}{2} \ ; & 1 \\ \frac{1}{2}b+\frac{1}{2}, \frac{1}{2}b+1 & : & a+\frac{1}{2} \ ; & - \end{matrix} ; x, y\right]$$
$$= b\int_0^1 v^{b-1} {}_0F_1\left(\begin{matrix} - \\ a+\frac{1}{2} \end{matrix}; v^2 x\right) {}_0F_1\left(\begin{matrix} - \\ \frac{1}{2} \end{matrix}; (1-v)^2 y\right) dv,$$

and

$$F^{0:2;1}_{2:1;0}\left[\begin{matrix} - & : & \frac{1}{2}b, \frac{1}{2}b+\frac{1}{2} \ ; & 1 \\ \frac{1}{2}b+1, \frac{1}{2}b+\frac{3}{2} & : & a+\frac{1}{2} \ ; & - \end{matrix} ; x, y\right]$$
$$= b(b+1)\int_0^1 v^{b-1}(1-v)\, {}_0F_1\left(\begin{matrix} - \\ a+\frac{1}{2} \end{matrix}; v^2 x\right) {}_0F_1\left(\begin{matrix} - \\ \frac{3}{2} \end{matrix}; (1-v)^2 y\right) dv.$$

Proof. The results follow by series expansion of the two hypergeometric functions with the resulting integral evaluated as a beta function and use of the duplication formula for the Pochhammer symbol. □

Now,

$${}_0F_1\left(\begin{matrix} - \\ a+\frac{1}{2} \end{matrix}; v^2 x\right) = \Gamma(a+\tfrac{1}{2})(v\sqrt{x})^{\frac{1}{2}-a} I_{a-\frac{1}{2}}(2v\sqrt{x}),$$

where I is the modified Bessel function, and

$${}_0F_1\left(\begin{matrix} - \\ \frac{1}{2} \end{matrix}; (1-v)^2 y\right) = \cosh[2(1-v)\sqrt{y}], \quad {}_0F_1\left(\begin{matrix} - \\ \frac{3}{2} \end{matrix}; (1-v)^2 y\right) = \frac{\sinh[2(1-v)\sqrt{y}]}{2(1-v)\sqrt{y}}.$$

Then, with $x = y = \tau^2/4$ and $\mu := a + \frac{1}{2}$, we obtain

$$F_1(\tau) = F^{0:2;1}_{2:1;0}\left[\begin{matrix} - & : & \frac{1}{2}b, \frac{1}{2}b+\frac{1}{2} \ ; & 1 \\ \frac{1}{2}b+\frac{1}{2}, \frac{1}{2}b+1 & : & a+\frac{1}{2} \ ; & - \end{matrix} ; \frac{\tau^2}{4}, \frac{\tau^2}{4}\right]$$
$$= b\Gamma(a+\tfrac{1}{2})\int_0^1 v^{b-1}(\tfrac{1}{2}v\tau)^{-\mu} I_\mu(v\tau)\cosh[(1-v)\tau]\, dv, \tag{A1}$$

and

$$\frac{\tau}{b+1}F_2(\tau) = \frac{\tau}{(b+1)}F^{0:2;1}_{2:1;0}\left[\begin{matrix} - & : & \frac{1}{2}b, \frac{1}{2}b+\frac{1}{2} \ ; & 1 \\ \frac{1}{2}b+1, \frac{1}{2}b+\frac{3}{2} & : & a+\frac{1}{2} \ ; & - \end{matrix} ; \frac{\tau^2}{4}, \frac{\tau^2}{4}\right]$$
$$= b\Gamma(a+\tfrac{1}{2})\int_0^1 v^{b-1}(\tfrac{1}{2}v\tau)^{-\mu} I_\mu(v\tau)\sinh[(1-v)\tau]\, dv. \tag{A2}$$

Since both Kampé de Fériet functions in (A1) and (A2) are functions of τ^2, it follows that it is sufficient to consider their behavior only in $|\arg\tau| \leq \frac{1}{2}\pi$.

We are therefore led to consideration the of the integrals $e^\tau J_\pm(\tau)$, where

$$J_\pm(\tau) = \int_0^1 v^{b-1} f(v\tau)\,(1 \pm e^{-2(1-v)\tau})\, dv, \quad f(\xi) := \frac{\Gamma(\mu+1)}{(\frac{1}{2}\xi)^\mu} I_\mu(\xi) e^{-\xi}.$$

From the well-known asymptotic behavior $I_\mu(\xi) \sim e^\xi/\sqrt{2\pi\xi}$ as $|\xi| \to \infty$ in $|\arg\,\xi| < \frac{1}{2}\pi$ (and $I_\mu(\xi) = O(\xi^{-1/2})$ when $\arg\,\xi = \pm\frac{1}{2}\pi$), it is found that

$$|f(v\tau)| < f(0) = 1 \;\; (a > 0), \qquad |f(v\tau)| < 2^\mu \Gamma(\mu+1)|\tau|^{-a} \;\; (a < 0),$$

as $|\tau| \to \infty$ with $v \in [0,1]$. Hence, $|J_\pm(\tau)| = O(1)$ $(a > 0)$, $O(\tau^{-a})$ $(a < 0)$ for $|\tau| \to \infty$ in $|\arg\,\tau| \leq \frac{1}{2}\pi$.

Consequently, the functions $F_1(\tau)$ and $\tau F_2(\tau)$ in (A1) and (A2) satisfy the order estimates

$$O(e^\tau) \;\; (a > 0), \qquad O(\tau^{-a}e^\tau) \;\; (a < 0), \tag{A3}$$

as $|\tau| \to \infty$ in $|\arg\,\tau| \leq \frac{1}{2}\pi$. In the sectors $\frac{1}{2}\pi < |\arg\,\tau| \leq \pi$, the exponential factor in (A3) is replaced by $\exp\overline{\tau}$, where the bar denotes the complex conjugate.

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