ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫЕ УРАВНЕНИЯ И ФУНКЦИОНАЛЬНЫЙ АНАЛИЗ

INTEGRO-DIFFERENTIAL EQUATIONS AND FUNCTIONAL ANALYSIS

Серия «Математика»
2022. Т. 42. С. 43—58

Онлайн-доступ к журналу:
http://mathizv.isu.ru

ИЗВЕСТИЯ
Иркутского
государственного
университета

Research article

УДК 517.957
MSC 35K40, 35K57
DOI https://doi.org/10.26516/1997-7670.2022.42.43

Solutions of the Second-order Nonlinear Parabolic System Modeling the Diffusion Wave Motion

Alexander L. Kazakov[1,2✉], **Anna A. Lempert**[1,2]

[1] V. M. Matrosov Institute for System Dynamics and Control Theory SB RAS, Irkutsk, Russian Federation
[2] Irkutsk National Research Technical University, Irkutsk, Russian Federation
✉ kazakov@icc.ru

Abstract. The paper continues a long series of our research and considers a second-order nonlinear evolutionary parabolic system. The system can be a model of various convective and diffusion processes in continuum mechanics, including mass transfer in a binary mixture. In hydrology, ecology, and mathematical biology, it describes the propagation of pollutants in water and air, as well as population dynamics, including the interaction of two different biological species. We construct solutions that have the type of diffusion (heat) wave propagating over a zero background with a finite velocity. Note that the system degenerates on the line where the perturbed and zero (unperturbed) solutions are continuously joined. A new existence and uniqueness theorem is proved in the class of analytical functions. In this case, the solution has the desired type and is constructed in the form of characteristic series, the convergence of which is proved by the majorant method. We also present two new classes of exact solutions, the construction of which, due to ansatzes of a specific form, reduces to integrating systems of ordinary differential equations that inherit a singularity from the original formulation. The obtained results are expected to be helpful in modeling the evolution of the Baikal biota and the propagation of pollutants in the water of Lake Baikal near settlements.

Keywords: parabolic partial differential equations, analytical solution, diffusion wave, existence theorem, exact solution, mathematical modeling.

Acknowledgements: The study is funded by RFBR (project No. 20-07-00407); RFBR and the Ministry of Science and Technology, Taiwan (project No. 20-51-S52003).

For citation: Kazakov A. L., Lempert A. A. Solutions of the Second-order Nonlinear Parabolic System Modeling the Diffusion Wave Motion. *The Bulletin of Irkutsk State University. Series Mathematics*, 2022, vol. 42, pp. 43–58.
https://doi.org/10.26516/1997-7670.2022.42.43

Научная статья

Решения нелинейной параболической системы второго порядка, моделирующие движение диффузионной волны

А. Л. Казаков[1,2]✉, **А. А. Лемперт**[1,2]

[1] Институт динамики систем и теории управления имени В. М. Матросова СО РАН, Иркутск, Российская Федерация

[2] Иркутский национальный исследовательский технический университет, Иркутск, Российская Федерация

✉ kazakov@icc.ru

Аннотация. В статье, продолжающей большой цикл исследований авторов, рассмотрена нелинейная эволюционная параболическая система второго порядка, которая моделирует различные конвективные и диффузионные процессы в механике сплошных сред, включая массообмен в бинарной смеси. В гидрологии, экологии и математической биологии система применяется для описания процессов распространения загрязняющих примесей в воде и воздухе и популяционной динамики, включая взаимодействие двух различных биологических видов. Для указанной системы рассматриваются решения, имеющие тип диффузионной (тепловой) волны, распространяющейся по нулевому фону с конечной скоростью. При этом на линии, вдоль которой непрерывно стыкуются возмущенное и нулевое (невозмущенное) решения, система испытывает вырождение. Доказывается новая теорема существования и единственности в классе аналитических функций. Решение при этом имеет искомый тип и строится в виде характеристических рядов, сходимость которых доказывается методом мажорант. Также найдены два новых класса точных решений, построение которых за счет использования анзацев специального вида сводится к интегрированию систем обыкновенных дифференциальных уравнений, наследующей особенность от исходной постановки. Полученные результаты предполагается использовать для моделирования эволюции байкальской биоты и распространения загрязняющих примесей в воде Байкала вблизи населенных пунктов.

Ключевые слова: параболические уравнения с частными производными, аналитическое решение, диффузионная волна, теорема существования, точное решение, математическое моделирование

Благодарности: Исследование выполнено при финансовой поддержке РФФИ (проект № 20-07-00407 А); РФФИ и Министерства Науки и Технологии, Тайвань (проект № 20-51-S52003).

Ссылка для цитирования: Kazakov A. L., Lempert A. A. Solutions of the Second-order Nonlinear Parabolic System Modeling the Diffusion Wave Motion // Известия Иркутского государственного университета. Серия Математика. 2022. Т. 42. С. 43–58.
https://doi.org/10.26516/1997-7670.2022.42.43

1. Introduction

Let us consider the system of second-order nonlinear evolutionary parabolic equations

$$\begin{aligned} T_t &= [\Phi_2(T)]_{xx} + [\Phi_1(T)]_x + \Phi_0(T,S), \\ S_t &= [\Psi_2(S)]_{xx} + [\Psi_1(S)]_x + \Psi_0(T,S). \end{aligned} \tag{1.1}$$

Here t, x are independent variables: time and a spatial coordinate, respectively; $T(t,z)$ and $S(t,x)$ are unknown functions; $\Phi_i, \Psi_i,\ i = 0,1,2$ are sufficiently smooth specified functions.

System (1.1) and its generalizations are often used as models of various convective [1; 2] and reaction-diffusion [6; 24] processes. Also, such systems arise in mathematical biology [16] when describing population dynamics, in particular, the interaction of two different biological species [16; 21], as well as when modeling pollution propagation [17]. A particular mention deserves the fact that the well-known porous medium equation [19; 23] is a particular case of equations (1.1). Special attention should be paid to the case when system (1.1) degenerates [4]. For example, if the equalities hold

$$\Phi_2'(0) = \Phi_1'(0) = \Phi_0(0,0) = \Psi_2'(0) = \Psi_1'(0) = \Phi_0(0,0) = 0. \tag{1.2}$$

If (1.2) holds, system (1.1) may have solutions having the type of heat [19] (filtration [20], diffusion [10]) waves that describe disturbances propagating along the zero background with a finite velocity. Such a situation, as you know, is not typical for equations and systems of parabolic type [15] and arises due to singularity. The boundary between the unperturbed zero background and the perturbed positive solution is called a wavefront. The joined solution is called a HDW-type solution [10].

We have been studying such solutions for the long time. At the beginning we considered the porous medium equation and its generalizations in one-dimensional [7; 12] and non-one-dimensional [9; 14] cases. Then the results were expanded to reaction–diffusion systems [10]. Our approach includes three directions. The first is a proving of new existence and uniqueness theorems for HDW-type solutions in the class of analytical functions. The proof is carried out in the traditions of the scientific school of Academician A.F. Sidorov [5; 20]: the solution is constructed as a characteristic series, its convergence is proved by the majorant method. Since these theorems, as all the similar, starting with the Cauchy-Kovalevskaya theorem [3], are local, the questions of the domain of existence of the solution and its properties outside a small neighborhood of the diffusion wavefront are relevant. We can answer this question using exact solutions [18], the construction of which forms the second direction. Such solutions are also helpful for verifying the results of numerical algorithms operating [22], the development of which is the third direction.

Previously [8;10;11;13], we considered systems having form (1.1) without convective terms, i.e., $[\Phi_1(T)]_x = [\Psi_1(S)]_x \equiv 0$. At the same time, each of the papers is devoted to the specific formulation. In [10; 13], the plane symmetric case for power functions Φ_2, Ψ_2 was studied in detail. In [11], we considered the cases of cylindrical and spherical symmetry. Finally, in the article [8], the functions Φ_2, Ψ_2 are analytical and have a general form.

This paper deals with the formulation where $[\Phi_1(T)]_x \not\equiv 0$, $[\Psi_1(S)]_x \not\equiv 0$ for the first time. A new existence and uniqueness theorem for the HDW-type solution in the class of analytical functions is proved. In two different cases, the problem is reduced to the Cauchy problem for a system of ordinary differential equations (ODEs) that inherits all the specific features of the original one. At the same time, the theorem and form of the ODEs differ significantly from those obtained earlier and generalize the corresponding results from [8; 10; 13].

2. Problem Formulation

If the functions $\Phi_1(T), \Phi_2(T), \Psi_1(S), \Psi_2(S)$ are differentiable, system (1.1) takes the form

$$\begin{aligned} T_t &= [\Phi_2'(T)T_x]_x + \Phi_1'(T)T_x + \Phi_0(T,S), \\ S_t &= [\Psi_2'(S)S_x]_x + \Psi_1'(S)S_x + \Psi_0(T,S). \end{aligned} \tag{2.1}$$

Let us consider the case of power nonlinearities, which is most often found in the literature. Let

$$\Phi_1'(T) = \lambda_1 T^{\sigma_1}, \Phi_2'(T) = \lambda_2 T^{\sigma_2}, \Psi_1'(S) = \mu_1 S^{\delta_1}, \Psi_2'(S) = \mu_2 S^{\delta_2}, \tag{2.2}$$

where $\sigma_i, \delta_i, \lambda_i, \mu_1, i = 1, 2$ are positive constants.

The substitution $u = \lambda_2 T^{\sigma_2}$, $v = \mu_2 S^{\delta_2}$ brings system (2.1), (2.2) to the form

$$\begin{aligned} u_t &= u u_{xx} + u_x^2/\sigma + A u^\alpha u_x + f(u,v), \\ v_t &= v v_{xx} + v_x^2/\delta + B v^\beta v_x + g(v,u). \end{aligned} \tag{2.3}$$

Here $A, B, \alpha, \beta, \delta, \sigma$ are positive constants, $f(u,v), g(u,v)$ are specified functions,

$$\sigma = \sigma_2, \delta = \delta_2, \alpha = \frac{\sigma_1}{\sigma_2}, A = \frac{\lambda_1}{\lambda_2^\alpha}, \beta = \frac{\delta_1}{\delta_2}, B = \frac{\mu_1}{\mu_2^\beta},$$

$$f(u,v) = \frac{\sigma \lambda_2^{1/\sigma}}{u^{1/\sigma - 1}} \Phi_0\left(\frac{u^{1/\sigma}}{\lambda_2^{1/\sigma}}, \frac{v^{1/\delta}}{\mu_2^{1/\delta}}\right), \; g(v,u) = \frac{\delta \mu_2^{1/\delta}}{v^{1/\delta - 1}} \Psi_0\left(\frac{u^{1/\sigma}}{\lambda_2^{1/\sigma}}, \frac{v^{1/\delta}}{\mu_2^{1/\delta}}\right).$$

For system (2.3), we consider the following boundary conditions:

$$u(t,x)|_{x=a(t)} = 0, \; v(t,x)|_{x=a(t)} = 0. \tag{2.4}$$

The function $x = a(t)$ is sufficiently smooth. Due to the specifics of the system, we can assume that $a(0) = 0$ without losing the generality.

A function is called analytical at a point (on a set) if it coincides in some neighborhood with its Taylor expansion. If $f(0,0) = g(0,0)$, it is easy to see that problem (2.3), (2.4) has the trivial solution $u \equiv 0, v \equiv 0$, which is analytical at any point in the Oxy plain.

The next section is devoted to an existence theorem for a nontrivial analytical solution to problem (2.3), (2.4), in which the functions u and v have the same sign on both sides of the line $x = a(t)$. This property makes it possible to construct a piecewise analytical function of variables t, x, composed of the trivial and positive parts of the non-trivial solutions, continuously joined along the line $a(t)$. The function is a HDW-type solution, the curve $x = a(t)$ is a diffusion wavefront.

3. Existence Theorem

Theorem 1. *Let the functions $f(u, v)$, $g(v, u)$, and $a(t)$ are analytical at the points $u = 0, v = 0$, and $t = 0$, respectively; $\alpha, \beta \in \mathbb{N}$; $f(0,0) = 0$, $g(0,0) = 0$, $\sigma > 0$, $\delta > 0$, $a'(0) \neq 0$. Assume also $u_x(0,0)$ and $v_x(0,0)$ simultaneously vanish or not vanish.*

Then problem (2.3), (2.4) *has the unique analytical solution $u = u(t, x)$, $v = v(t, x)$ in some neighborhood of $t = 0, x = 0$, and $uv|_{x \neq a(t)} > 0$.*

Proof. The case $\alpha = \beta = 0$ leads to the problem from [10], where we proved the corresponding theorem. It thus becomes a particular case of this one. The proof of the theorem is carried out in three stages. At the first stage, a formal solution is constructed in the form of a characteristic series. The second stage proves its local convergence. At the end, we show that the functions u and v outside the line $x = a(t)$ have the same sign and do not vanish. In order not to repeat the reasonings already repeatedly published, the justification of the second stage is given briefly.

Let us change the independent spatial variable as $z = x - a(t)$. Then problem (2.3), (2.4) takes the form

$$\begin{aligned} u_t &= uu_{zz} + u_z^2/\sigma + [a'(t) + Au^\alpha]u_z + f(u, v), \\ v_t &= vv_{zz} + v_z^2/\delta + [a'(t) + Bv^\beta]v_z + g(v, u). \end{aligned} \tag{3.1}$$

$$u(t, z)|_{z=0} = 0, v(t, z)|_{z=0} = 0. \tag{3.2}$$

The solution to problem (3.1), (3.2) is constructed as the Taylor series

$$u(t, z) = \sum_{n=0}^{\infty} \frac{u_n(t) z^n}{n!}, u_n = \left.\frac{d^n u}{dz^n}\right|_{z=0}; \; v(t, z) = \sum_{n=0}^{\infty} \frac{v_n(t) z^n}{n!}, v_n = \left.\frac{d^n v}{dz^n}\right|_{z=0}. \tag{3.3}$$

One can easily verify that series (3.3) are characteristic [3]. Let us determine their coefficients by a recurrent procedure.

It follows from (3.2) that $u_0 = v_0 = 0$. Assuming in 3.1) $z = 0$, we obtain the system of two quadratic equations to find $u_1(t)$ and $v_1(t)$:

$$u_1^2/\sigma + a'(t)u_1 = 0,\ v_1^2/\delta + a'(t)v_1 = 0, \tag{3.4}$$

which has four solutions. Two of them, in which one of the desired variables vanishes, do not satisfy the condition of the Theorem.

From the remaining two solutions, let us consider the case $u_1 = 0, v_1 = 0$. Then $u_2 = v_2 = \ldots = u_k = v_k = \ldots = 0$, therefore, we obtain the unique trivial solution $u \equiv 0, v \equiv 0$.

The last solution to (3.4) is

$$u_1(t) = -\sigma a'(t), \quad v_1(t) = -\delta a'(t). \tag{3.5}$$

This case corresponds to a nontrivial analytical solution to problem (3.1), (3.2). To determine the coefficients u_2, v_2, we differentiate (3.1) with respect to z and set $z = 0$. It brings us to

$$u_2 = \frac{1}{(1+\sigma)a'(t)}\left[\sigma a''(t) + \alpha A u_0^{\alpha-1}u_1^2 + f_{1,0}u_1 + f_{0,1}v_1\right],$$

$$v_2 = \frac{1}{(1+\delta)a'(t)}\left[\delta a''(t) + \beta B v_0^{\beta-1}v_1^2 + g_{1,0}u_1 + g_{0,1}v_1\right].$$

Here

$$f_{1,0} = \frac{\partial f(u,v)}{\partial u}\Big|_{u=0,v=0}, f_{0,1} = \frac{\partial f(u,v)}{\partial v}\Big|_{u=0,v=0},$$

$$g_{1,0} = \frac{\partial g(u,v)}{\partial u}\Big|_{u=0,v=0}, g_{0,1} = \frac{\partial g(u,v)}{\partial v}\Big|_{u=0,v=0}.$$

Assume the coefficients of series (3.3) up to n are found. To determine u_{n+1} and v_{n+1}, we differentiate (3.1) n times with respect to z and set $z = 0$. After collecting terms and resolving with respect to the desired variables, we obtain

$$u_{n+1} = \frac{1}{a'(1+n\sigma)}\left[\sum_{k=2}^{n}\left(C_n^k + \frac{1}{\sigma}C_n^{k-1}\right)u_k u_{n+2-k} + \right.$$

$$\left. + A\sum_{k=1}^{n} C_n^k u_{n+1-k}\left(\sum_{\substack{i_1,\ldots,i_\alpha=0,\\ i_1+\ldots+i_\alpha=k}}^{k} C_k^{i_1,\ldots,i_\alpha} u_{i_1}u_{i_2}\ldots u_{i_\alpha}\right) + f_n - u_n'\right], \quad n \geq 2. \tag{3.6}$$

$$v_{n+1} = \frac{1}{a'(1+n\delta)}\left[\sum_{k=2}^{n}\left(C_n^k + \frac{1}{\sigma}C_n^{k-1}\right)v_k v_{n+2-k} + \right.$$

$$+B\sum_{k=1}^{n} C_n^k v_{n+1-k}\left(\sum_{\substack{i_1,\ldots,i_\beta=0,\\ i_1+\ldots+i_\beta=k}}^{k} C_k^{i_1,\ldots,i_\beta} v_{i_1}v_{i_2}\ldots v_{i_\beta}\right)+g_n-v_n'\Bigg],\quad n\geq 2. \tag{3.7}$$

Here

$$f_n=\frac{\partial^n f(u,v)}{\partial z^n}\Big|_{z=0},\quad g_n=\frac{\partial^n g(v,u)}{\partial z^n}\Big|_{z=0},\quad n=1,2,\ldots,$$

where the derivatives of $f(u,v)$ and $g(u,v)$ are found according to the rule of differentiation of complex functions.

Thus, the coefficients of series (3.3) are uniquely determined by formulas (3.6), (3.7). This completes the first stage of the proof.

The proof of convergence for the trivial case is not required, but for the nontrivial case it is carried out by the classical majorant method. Let us introduce the auxiliary unknown functions U and V by the formulas

$$u(t,z)=-\sigma a'(t)z+U(t,z)z^2,\ v(t,z)=-\delta a'(t)z+V(t,z)z^2.$$

The correctness of such a replacement follows from conditions (3.2) and equalities (3.5).

The replacement brings the first equation of system (3.2) to the form

$$-\sigma a''z+z^2U_t=$$

$$=(-\sigma a'z+z^2U)(2U+4zU_z+z^2U_{zz})+\frac{1}{\sigma}(-\sigma a'+2zU+z^2U_z)^2+$$

$$+A[a'+(-\sigma a'z+z^2U)^\alpha](-\sigma a'+2zU+z^2U_z)+f(-\sigma a'z+z^2U,-\delta a'z+z^2V). \tag{3.8}$$

After carrying out standard transformations, including collecting terms (see [8]), equation (3.8) can be rewritten as

$$\begin{gathered}2(1+\sigma)U+(4\sigma+1)\,zU_z+\sigma z^2U_{zz}=f_0(t,z)+\\ +zf_1(t,z,U,U_t,V)+z^2f_2(t,z,U,V,U_z)+z^3f_3(t,z,U,V,U_z,U_{zz}).\end{gathered} \tag{3.9}$$

The second equation of system (3.2) is similarly transformed to

$$2(1+\delta)V+(4\delta+1)\,zV_z+\delta z^2V_{zz}=$$

$$=g_0(t,z)+zg_1(t,z,V,V_t,U)+z^2g_2(t,z,V,U,V_z)+z^3g_3(t,z,V,U,V_z,V_{zz}). \tag{3.10}$$

Here $f_i,g_i,\ i=1,\ldots,3$ are already known analytical functions of their arguments, whose explicit forms are not given due to cumbersomeness.

The boundary conditions for U and V follow from the compatibility conditions for equations (3.9), (3.10) and have the form

$$U|_{z=0}=\frac{f_0(t,0)}{2(1+\sigma)},\ V|_{z=0}=\frac{g_0(t,0)}{2(1+\delta)}. \tag{3.11}$$

Problem (3.9)–(3.11) is equivalent to problem (3.1), (3.2) in the class of analytical functions. Thus, these problems are solvable (and unsolvable) in the specified class simultaneously.

The solution to (3.9)–(3.10) is constructed as the Taylor series

$$U(t,z)=\sum_{n=0}^{\infty}U_n(t)\frac{z^n}{n!},\ V(t,z)=\sum_{n=0}^{\infty}V_n(t)\frac{z^n}{n!}, \tag{3.12}$$

whose coefficients are determined according to the described above recursive procedure. We don't give their exact form since it is not important in this case. For a simpler case, the corresponding formulas are presented in [8].

Let us construct a uniform majorant for both equations of system (3.9), (3.10). If the majorant estimates

$$U_0(t),V_0(t)\ll W_0(t);\ U_1(t),V_1(t)\ll W_1(t);$$
$$f_1(t,z,U,U_t,V),g_1(t,z,V,V_t,U)\ll h_1(t,z,W,W_t,W);$$
$$f_2(t,z,U,V,U_z),g_2(t,z,U,V,U_z)\ll h_2(t,z,W,W,W_z);$$
$$f_3(t,z,U,V,U_z,U_{zz}),g_3(t,z,U,V,U_z,U_{zz})\ll h_3(t,z,W,W,W_z,W_{zz})$$

hold, then the solution to the problem

$$\frac{\partial^2 W}{\partial z^2}=\frac{\partial h_1(t,W,W_t,W)}{\partial z}+h_2(t,W,W,W_z)+zh_3(t,z,W,W,W_z,W_{zz}), \tag{3.13}$$

$$W|_{z=0}=W_0(t),\quad W_z|_{z=0}=W_1(t) \tag{3.14}$$

is a majorant for series (3.12). This fact is easy to verify by constructing the solution to (3.13), (3.14) in the form of the Taylor series.

Next, we prove that problem (3.13), (3.14) has a unique analytical solution majorizing zero. To do this, let us differentiate (3.13) by z, resolve the resulting expression with respect to W_{zzz} and add the third boundary condition $W_{zz}(t,0)=W_2(t)$. Then we arrive to the problem of the Cauchy-Kovalevskaya type

$$\frac{\partial^3 W}{\partial z^3}=H(t,z,W,W_t,W_z,W_{tz},W_{zz}), \tag{3.15}$$

$$W|_{z=0}=W_0(t),W_z|_{z=0}=W_1(t),W_{zz}|_{z=0}=W_2(t), \tag{3.16}$$

where H is an analytical function that majorizes zero. According to the Cauchy-Kovalevskaya theorem, problem (3.15), (3.16) has a unique analytical solution majorizing zero. In turn, this solution is a majorant for series (3.13) by constructing procedure. Thus, we finally obtain that series (3.3) are convergent. The second stage of the proof is completed.

It follows from (3.5) that the functions $u(t,x)$ and $v(t,x)$ have the same sign on both sides of the line $x=a(t)$ in some neighborhood of $x=0,t=0$.

In fact, $u_1(t)[x-a(t)]$ and $v_1(t)[x-a(t)]$ are the first terms of the characteristic series, i.e., in some small neighborhood of the diffusion wavefront, they determine the signs of the unknown functions.

Thus, the third stage of the proof is completed. □

Remark 1. The counterexample constructed in [10] shows that if the conditions of Theorem 1 are not satisfied, when $u_x(0,0)v_x(0,0)=0$, $u_x^2(0,0)+v_x^2(0,0)>0$, problem (2.3), (2.4) may or may not have an analytical solution. In the second case, the convergence radii of series (3.3) are zero.

4. Exact solutions

This section pertains to finding such nontrivial exact solutions to problem (2.3), (2.4), which constructing reduces to integrating the Cauchy problems for systems of ordinary differential equations (ODEs). Previously, this problem was studied in the particular case when $A=B=0$ [10] (see also [8; 13]).

We specify the type of the functions f and g. Let

$$f(u,v)=A_1v^{\gamma-\xi}u^{\xi}, g(v,u)=B_1u^{\lambda-\eta}v^{\eta},$$

where $A_1, B_1, \gamma, \xi, \eta \in \mathbb{R}$, $\gamma>0, \lambda>0, 0\le \xi, \eta \le \gamma$. Problem (3.1), (3.2) takes the form

$$\begin{aligned} u_t &= uu_{zz}+u_z^2/\sigma+[a'(t)+Au^{\alpha}]u_z+A_1v^{\gamma-\xi}u^{\xi},\\ v_t &= vv_{zz}+v_z^2/\delta+[a'(t)+Bv^{\beta}]v_z+B_1u^{\lambda-\eta}v^{\eta}, \end{aligned} \tag{4.1}$$

$$u(t,0)=0,\ v(t,0)=0. \tag{4.2}$$

As we showed in [7; 12], the main ansatzes that allow us to perform the mentioned reduction have the form $x-a(t)$ and $x/a(t)$. Let us consider these cases separately. Note that if $\xi, \eta, \in \mathbb{N}\cup\{0\}, \gamma, \lambda \in \mathbb{N}$, then with proper selection of the function $a(t)$, problem (4.1) has an analytical solution represented by the Taylor series (3.3) according to Theorem 1.

Generalized traveling wave. Let us construct a solution to system (4.1) in the form

$$u=\psi(t)p(z), v=\psi(t)q(z),\ z=x-a(t). \tag{4.3}$$

Substituting (4.3) into (4.1) and dividing by $\psi^2(t)$, we obtain

$$pp''+\frac{(p')^2}{\sigma}+\left[\frac{a'(t)}{\psi(t)}+A\psi^{\alpha-1}(t)p^{\alpha}\right]p'+A_1\psi^{\gamma-2}(t)q^{\gamma-\xi}p^{\xi}-\frac{\psi'(t)}{\psi^2(t)}p=0,$$

$$qq'' + \frac{(q')^2}{\delta} + \left[\frac{a'(t)}{\psi(t)} + B\psi^{\beta-1}(t)q^{\beta}\right]q' + B_1\psi^{\lambda-2}(t)p^{\lambda-\eta}q^{\eta} - \frac{\psi'(t)}{\psi^2(t)}q = 0. \tag{4.4}$$

To turn (4.4) into ODEs with respect to $p(z), q(z)$, it is necessary and sufficient that the following conditions hold:

$$\begin{gathered} a'(t)/\psi(t) = \text{const}, \psi'(t)/\psi^2(t) = \text{const}, \psi^{\alpha-1}(t) = \text{const}, \\ \psi^{\beta-2}(t) = \text{const}, \psi^{\gamma-2}(t) = \text{const}, \psi^{\lambda-2}(t) = \text{const}. \end{gathered} \tag{4.5}$$

Here, in turn, two sub-cases are possible: 1) $\psi(t) = \text{const}$; 2) $\psi(t) \neq \text{const}$.

1. To begin with, assume $\psi(t) = \text{const}$. Then from the first equality of (4.4), it follows that $a(t) = \mu t$, where $\mu \neq 0$. Without losing the generality of consideration, we can set $\psi = 1$, and then the system of partial differential equations (PDEs) is reduced to the ODEs:

$$\begin{gathered} pp'' + (p')^2/\sigma + (\mu + Ap^{\alpha})p' + A_1q^{\gamma-\xi}p^{\xi} = 0, \\ qq'' + (q')^2/\delta + (\mu + Bq^{\beta})q' + B_1p^{\lambda-\eta}q^{\eta} = 0. \end{gathered} \tag{4.6}$$

2. Let now $\psi(t) \neq \text{const}$. It follows from (4.4) that $\alpha = \beta = 1$, $\gamma = \lambda = 2$. Solving the first two equalities of (4.4) as an ODEs, we get that $\psi(t) = \omega/(\mu t+\nu), a(t) = \omega\ln(\mu t+\nu)$, where $\omega, \mu, \nu \in \mathbb{R}, \omega > 0, \mu > 0, \nu > 0$. Then the PDEs is reduced to the ODEs as

$$\begin{gathered} pp'' + (p')^2/\sigma + (\mu + Ap)p' + A_1q^{2-\xi}p^{\xi} + \mu p/\omega = 0, \\ qq'' + (q')^2/\delta + (\mu + Bq)q' + B_1p^{2-\eta}q^{\eta} + \mu q/\omega = 0. \end{gathered} \tag{4.7}$$

The initial conditions for (4.6) and (4.7) are

$$p(0) = 0, \; q(0) = 0. \tag{4.8}$$

The conditions (4.4) are a direct consequence of (4.2).

Generalized self-similar solutions. Here we construct the solution to system (4.1) in the form

$$u = \phi(t)r(y), v = \phi(t)s(y), \; y = -z/a(t) = 1 - x/a(t). \tag{4.9}$$

Substituting (4.9) into (4.1) and multiplying by $a^2(t)/\psi^2(t)$, we obtain

$$\begin{gathered} rr'' + \tfrac{(r')^2}{\sigma} + \tfrac{a(t)}{\phi(t)}\left[A\phi^{\alpha}(t)r^{\alpha} + a'(t)(1-y)\right]r' + \\ \tfrac{a^2(t)}{\phi^2(t)}[A_1\phi^{\gamma}(t)s^{\gamma-\xi}r^{\xi} - \phi'(t)r] = 0, \\ ss'' + \tfrac{(s')^2}{\delta} + \tfrac{a(t)}{\phi(t)}\left[B\phi^{\beta}(t)s^{\beta} + a'(t)(1-y)\right]s' + \\ \tfrac{a^2(t)}{\phi^2(t)}[B_1\phi^{\lambda}(t)r^{\lambda-\eta}s^{\eta} - \phi'(t)s] = 0. \end{gathered} \tag{4.10}$$

To reduce (4.4) to ODEs with respect to $r(y), s(y)$, it is necessary and sufficient that the following conditions hold:

$$a(t)a'(t)/\phi(t) = \text{const}, \ a^2(t)\phi'(t)/\phi^2(t) = \text{const}, \ \phi^{\alpha-1}(t)a(t) = \text{const},$$
$$\phi^{\gamma-2}(t)a^2(t) = \text{const}, \ \phi^{\beta-1}(t)a(t) = \text{const}, \ \phi^{\lambda-2}(t)a^2(t) = \text{const}. \tag{4.11}$$

If $\psi(t) = \text{const}$, we have sub-case 1 from the previous subsection. So, let $\phi'(t) \neq 0$. Substituting the first equality of (4.11) to the second one, we get

$$a(t)a''(t)/[a'(t)]^2 = \text{const} = C. \tag{4.12}$$

Here, as above, two sub-cases are possible: 1) $C = 1$; 2) $C \neq 1$.

1. Let $C = 1$. Then the solution to (4.12) have the form $a(t) = \eta \exp(\mu t)$, where μ, η are non-zero constants. You can easily make sure that the necessary and sufficient conditions for the remaining relations from (4.11) to be fulfilled are the equalities $\alpha = \beta = 1/2$, $\gamma = \lambda = 1$, and then $\psi(t) = \eta^2 \exp(2\mu t)$. The system of equations for determining $r(y), s(y)$ takes the form

$$\begin{aligned} &rr'' + (r')^2/\sigma + [A\sqrt{r} + \mu(1-y)]r' + A_1 s^{1-\xi} r^{\xi} - 2\mu r = 0, \\ &ss'' + (r')^2/\delta + [B\sqrt{s} + \mu(1-y)]s' + B_1 r^{1-\eta} s^{\eta} - 2\mu s = 0. \end{aligned} \tag{4.13}$$

2. Let $C \neq 1$. Then the solution to (4.12) have the form $a(t) = (\mu t + \eta)^{\omega}$, where $\mu \neq 0, \eta > 0, \omega > 0$ are constants. The necessary and sufficient conditions for the remaining relations from (4.11) to be fulfilled are the equalities $\alpha = \beta = (\omega - 1)/(2\omega - 1), \gamma = 2\alpha, \lambda = 2\beta$, and then $\psi(t) = \omega(\mu t + \eta)^{2\omega-1}$. This raises the additional restrictions $\alpha \neq 1/2, \alpha \neq 1$, $\gamma \neq 1, \gamma \neq 2$. The system of equations for determining $r(y), s(y)$ takes the form

$$\begin{aligned} &rr'' + (r')^2/\sigma + [Ar^{\alpha} + \mu(1-y)]r' + A_1 s^{\gamma-\xi} r^{\xi} + 2\mu r/(2-\gamma) = 0, \\ &ss'' + (s')^2/\delta + [Bs^{\alpha} + \mu(1-y)]s' + B_1 r^{\gamma-\eta} s^{\eta} + 2\mu s/(2-\gamma) = 0, \end{aligned} \tag{4.14}$$

The condition $\alpha > 0$ raises the extra restriction $(\omega - 1)/(2\omega - 1) > 0$, therefore, $\omega \in (-\infty, 1/2), (1, +\infty)$.

The initial conditions for (4.13) and (4.14) directly follow from (4.2) and have the form

$$r(0) = 0, \ s(0) = 0. \tag{4.15}$$

Discussion. Problems (4.6), (4.8); (4.7), (4.8); (4.13), (4.15), and (4.14), (4.15) inherit a singularity from the original formulation, so the issue of the existence and uniqueness of the solution requires additional study.

First, it is necessary to extend the initial data to the Cauchy conditions. Due to the degeneracy of the systems, the conditions for derivatives at zero cannot be chosen arbitrarily. Setting in both parts of the systems (4.6), (4.7) $z = p = q = 0$, and in (4.13), (4.14) $y = s = r = 0$, we get the same type of systems of quadratic equations

$$[P'(0)]^2/\sigma + \mu[P'(0)] = 0, \ [Q'(0)]^2/\delta + \mu[Q'(0)] = 0, \tag{4.16}$$

where symbol P can be either p or r, and symbol Q either q or s. System (4.16) has 4 real roots: $P_1(0) = 0, Q_1(0) = 0$; $P_2(0) = -\sigma\mu, Q_2(0) = 0$; $P_3(0) = 0, Q_1(0) = -\delta\mu$; $P_4(0) = -\sigma\mu, Q_4(0) = -\delta\mu$. Each of them allows us to set conditions on the first derivatives and obtain consistent Cauchy conditions at zero.

The case $P(0) = Q(0) = P'(0) = Q'(0)$ leads to the trivial solution $P \equiv 0, Q \equiv 0$. The second and third cases are also of no interest, since for $\xi > 0, \eta > 0$, one of the unknown functions is identically zero, and the second is determined from the Cauchy problem for a single ODE [7]. The most significant is the fourth case, when the Cauchy conditions for systems (4.6), (4.8) have the form

$$p(z)|_{z=0} = 0, p'(z)|_{z=0} = -\sigma\mu, \ q(z)|_{z=0} = 0, q'(z)|_{z=0} = -\delta\mu, \tag{4.17}$$

and for systems (4.13), (4.14) are

$$r(y)|_{y=0} = 0, r'(y)|_{y=0} = -\sigma\mu, \ s(y)|_{y=0} = 0, s'(y)|_{y=0} = -\delta\mu. \tag{4.18}$$

The issue of the solvability of problems (4.6), (4.17); (4.7), (4.17); (4.13), (4.18); and (4.14), (4.18) has not yet been solved in the general case, although for integer values of the powers, their analytical solvability follows from Theorem 1. Based on the results of previous studies, we make the assumption that a solution will exist for $\alpha > 0, \beta > 0$, $\gamma \geq 1, \lambda \geq 1$.

5. Conclusion

Summarizing the study, we note that convective terms first have been taken into account for the problem of solution constructing to nonlinear parabolic systems with singularity. This fact has entailed a significant complication of the problem statement. In addition to proving the new existence theorem that generalizes some known similar ones (see [10]), we have found the exact solutions. Meaningful results have been obtained for the first time for the case when the powers γ and λ in system (4.1) are different (see [8; 11]). Besides the theoretical significance, the results obtained may be helpful from the applied point of view, in particular, in frameworks of modeling the Baikal biota and the propagation of pollutants in the water of Lake Baikal near settlements.

Further research in this direction may be related to applying the results obtained for developing and testing an efficient numerical algorithm for constructing solutions to the considered problems [11], which can be based on the boundary element approach or the collocation method. From a theoretical point of view, the proof or refutation of the hypothesis expressed at the end of the previous section of the paper is significant. Another important and promising area of research is obtaining a desired (optimal) boundary regime by controlling the diffusion front.

References

1. Andreev V.K., Gaponenko Yu.A., Goncharova O.N., Pukhnachev V.V. *Mathematical models of convection*. Berlin, Walter De Gruyter, 2012, 417 p. https://doi.org/10.1515/9783110258592
2. Bergman T.L., Lavine A.S., Incopera F.P., Dewitt D.P. *Fundamentals of heat and mass transfer*. NY, John Wiley & Sons, 2011, 992 p.
3. Courant R., Hilbert D. *Methods of Mathematical Physics. Vol. II: Partial Differential Equations*. NY, John Wiley & Sons, 2008, 852 p.
4. DiBenedetto E. *Degenerate Parabolic Equations*. NY, Springer-Verlag, 1993, 388 p. https://doi.org/10.1007/978-1-4612-0895-2.
5. Filimonov M.Yu. Representation of solutions of boundary value problems for nonlinear evolution equations by special series with recurrently calculated coefficients. *Journal of Physics: Conference Series*, 2019, pp. 012071. https://doi.org/10.1088/1742-6596/1268/1/012071
6. Gambino G., Lombardo M.C., Sammartino M., Sciacca V. Turing pattern formation in the Brusselator system with nonlinear diffusion. *Physical Review E*, 2013, vol. 88, pp. 042925. https://doi.org/10.1103/PhysRevE.88.042925
7. Kazakov A.L. On exact solutions to a heat wave propagation boundary-value problem for a nonlinear heat equation. *Siberian Electronic Mathematical Reports*, 2019, vol. 16, pp. 1057–1068. https://doi.org/10.33048/semi.2019.16.073 (in Russian)
8. Kazakov A.L., Kuznetsov P.A. Analytical Diffusion Wave-type Solutions to a Nonlinear Parabolic System with Cylindrical and Spherical Symmetry. *Bulletin of Irkutsk State University, Series Mathematics*, 2021, vol. 37, pp. 31–46. https://doi.org/10.26516/1997-7670.2021.37.31
9. Kazakov A.L., Kuznetsov P.A. On the analytic solutions of a special boundary value problem for a nonlinear heat equation in polar coordinates. *Journal of Applied and Industrial Mathematics*, 2018, vol. 12, no. 2, pp. 255–263. https://doi.org/10.1134/S1990478918020060
10. Kazakov A.L., Kuznetsov P.A., Lempert A.A. Analytical solutions to the singular problem for a system of nonlinear parabolic equations of the reaction-diffusion type. *Symmetry*, 2020, vol. 12, no. 6, pp. 999. https://doi.org/10.3390/SYM12060999
11. Kazakov A. L., Kuznetsov P.A., Spevak L.F. Construction of Solutions to a Boundary Value Problem with Singularity for a Nonlinear Parabolic System. *Journal of Applied and Industrial Mathematics*, 2021, vol. 15, no. 4, pp. 616–626. https://doi.org/10.1134/S1990478921040050
12. Kazakov A.L., Orlov Sv.S., Orlov S.S. Construction and study of exact solutions to a nonlinear heat equation. *Siberian Mathematical Journal*, 2018, vol. 59, no. 3, pp. 427–441. https://doi.org/10.1134/S0037446618030060

13. Kazakov A.L., Spevak L.F. Exact and approximate solutions of a degenerate reaction–diffusion system. *Journal of Applied Mechanics and Technical Physics*, 2021, vol. 62, no. 4, pp. 673–683. https://doi.org/10.1134/S0021894421040179
14. Kazakov A.L., Nefedova O.A., Spevak L.F. Solution of the problem of initiating the heat wave for a nonlinear heat conduction equation using the boundary element method. *Computational Mathematics and Mathematical Physics*, 2019, vol. 59, no. 6, pp. 1015–1029. https://doi.org/10.1134/S0965542519060083
15. Ladyzenskaja O.A., Solonnikov V.A, Ural'ceva N.N. *Linear and Quasi-linear Equations of Parabolic Type. Translations of mathematical monographs.* Providence, American Mathematical Society Publ., 1968, 648 p.
16. Murray J. *Mathematical biology: I. An introduction.* Interdisciplinary applied mathematics, vol. 17. NY, Springer, 2002, 575 p.
17. Perthame B. *Parabolic equations in biology. Growth, reaction, movement and diffusion.* NY, Springer, 2015, 211 p. https://doi.org/10.1007/978-3-319-19500-1.
18. Polyanin A.D., Zaitsev V.F. *Handbook of nonlinear partial differential equations.* NY, Chapman and Hall/CRC, 2003, 840 p. https://doi.org/10.1201/9780203489659
19. Samarskii A.A., Galaktionov V.A., Kurdyumov S.P., Mikhailov A.P. *Blow-up in quasilinear parabolic equations.* Berlin, New-York, Walter de Gruyte, 1995, 554 p. https://doi.org/10.1515/9783110889864
20. Sidorov A.F. *Izbrannye trudy: Matematika. Mekhanika* [Selected Works: Mathematics. Mechanics]. Moscow, Fizmatlit Publ., 2001, 576 p. (in Russian)
21. Telch B. Global boundedness in a chemotaxis quasilinear parabolic predator–prey system with pursuit-evasion. *Nonlinear Analysis: Real World Applications*, 2021, vol. 59, pp. 103269. https://doi.org/10.1016/j.nonrwa.2020.103269
22. Vabishchevich P.N. Monotone Schemes for Convection–Diffusion Problems with Convective Transport in Different Forms. *Computational Mathematics and Mathematical Physics*, 2021, vol. 61, no. 1, pp. 90–102. DOI: 10.1134/S0965542520120155.
23. Vazquez J.L. *The porous medium equation: mathematical theory.* Oxford, Clarendon Press, 2007, 648 p. https://doi.org/10.1093/acprof:oso/9780198569039.001.0001
24. Zemskov E.P. Turing instability in reaction–diffusion systems with nonlinear diffusion. *Journal of Experimental and Theoretical Physics*, 2013, vol. 117, no. 4, pp. 764–769. https://doi.org/10.1134/S1063776113120194

Список источников

1. Современные математические модели конвекции / В. К. Андреев, Ю. А. Гапоненко, О. Н. Гончарова, В. В. Пухначев. М. : Физматлит, 2008. 368 с.
2. Fundamentals of heat and mass transfer / T. L. Bergman, A. S. Lavine, F. P. Incopera, D. P. Dewitt. New-York : John Wiley & Sons, 2011. 992 p.
3. Курант Р. Уравнения с частными производными. М. : Мир, 1964. 830 с.
4. DiBenedetto E. Degenerate Parabolic Equations. New-York : Springer-Verlag, 1993. 388 p. https://doi.org/10.1007/978-1-4612-0895-2.
5. Filimonov M. Yu. Representation of solutions of boundary value problems for nonlinear evolution equations by special series with recurrently calculated coefficients // Journal of Physics: Conference Series. 2019. P. 012071. https://doi.org/10.1088/1742-6596/1268/1/012071

6. Gambino G., Lombardo M. C., Sammartino M., Sciacca V. Turing pattern formation in the Brusselator system with nonlinear diffusion // Physical Review E. 2013. Vol. 88. P. 042925. https://doi.org/10.1103/PhysRevE.88.042925
7. Казаков А. Л. О точных решениях краевой задачи о движении тепловой волны для уравнения нелинейной теплопроводности // Сибирские электронные математические известия. 2019. Т. 16. С. 1057–1068. https://doi.org/10.33048/semi.2019.16.073
8. Kazakov A. L., Kuznetsov P. A. Analytical Diffusion Wave-type Solutions to a Nonlinear Parabolic System with Cylindrical and Spherical Symmetry // Известия Иркутского государственного университета. Серия Математика. 2021. Т. 37. С. 31–46. https://doi.org/10.26516/1997-7670.2021.37.31
9. Казаков А. Л., Кузнецов П. А. Об аналитических решениях одной специальной краевой задачи для нелинейного уравнения теплопроводности в полярных координатах // Сибирский журнал индустриальной математики. 2018. Т. 21. № 2(74). С. 56–65. https://doi.org/10.17377/SIBJIM.2018.21.205
10. Kazakov A. L., Kuznetsov P. A., Lempert A. A. Analytical solutions to the singular problem for a system of nonlinear parabolic equations of the reaction-diffusion type // Symmetry. 2020. Vol. 12, N. 6. P. 999. https://doi.org/10.3390/SYM12060999
11. Казаков А. Л., Кузнецов П. А., Спевак Л. Ф. Построение решений краевой задачи с вырождением для нелинейной параболической системы // Сибирский журнал индустриальной математики. 2021. Т. 24, № 4(88). С. 64–78. https://doi.org/10.33048/SIBJIM.2021.24.405
12. Казаков А. Л., Орлов Св. С., Орлов С. С. Построение и исследование некоторых точных решений нелинейного уравнения теплопроводности // Сибирский математический журнал. 2018. Т. 59, № 3. С. 544–560. https://doi.org/10.17377/smzh.2018.59.306
13. Казаков А. Л., Спевак Л. Ф. Точные и приближенные решения вырождающейся системы реакция – диффузия // Прикладная механика и техническая физика. 2021. Т. 62, № 4. С. 169–180. https://doi.org/10.15372/PMTF20210417
14. Казаков А. Л., Нефедова О. А., Спевак Л. Ф. Решение задач об инициировании тепловой волны для нелинейного уравнения теплопроводности методом граничных элементов // Журнал вычислительной математики и математической физики. 2019. Т. 59, № 6. С. 1047–1062. https://doi.org/10.1134/S0044466919060085
15. Ладыженская О. А., Солонников В. А., Уральцева Н. Н. Линейные и квазилинейные уравнения параболического типа. М. : Наука, 1967. 736 с.
16. Murray J. Mathematical biology: I. An introduction, third edition. Interdisciplinary applied mathematics. Vol. 17. New York : Springer, 2002. 575 p.
17. Perthame B. Parabolic equations in biology. Growth, reaction, movement and diffusion. New York : Springer, 2015. 211 p. https://doi.org/10.1007/978-3-319-19500-1.
18. Полянин А. Д., Зайцев В. Ф. Справочник по нелинейным уравнениям математической физики. Точные решения. М. : Физматлит, 2002. 432 с.
19. Режимы с обострением в задачах для квазилинейных параболических уравнений / А. А. Самарский, В. А. Галактионов, С. П. Курдюмов, А. П. Михайлов. М. : Наука, 1987. 534 с.
20. Сидоров А. Ф. Избранные труды: Математика. Механика. М. : Физматлит, 2001. 576 с.

21. Telch B. Global boundedness in a chemotaxis quasilinear parabolic predator–prey system with pursuit-evasion // Nonlinear Analysis: Real World Applications, 2021. Vol. 59. Art. no. 103269. https://doi.org/10.1016/j.nonrwa.2020.103269
22. Вабищевич П. Н. Монотонные схемы для задач конвекции-диффузии с конвективным переносом в различной форме // Журнал вычислительной математики и математической физики, 2021. Т. 61, № 1. С. 95-–107. https://doi.org/10.31857/S0044466920120157.
23. Vazquez J. L. The porous medium equation: Mathematical theory. Oxford : Clarendon Press, 2007. 648 p. https://doi.org/10.1093/acprof:oso/9780198569039.001.0001
24. Земсков Е. П. Неустойчивость Тьюринга в реакционно-диффузионных системах с нелинейной диффузией // Журнал экспериментальной и теоретической физики. 2013. Т. 144, № 4. С. 878–884.

Об авторах

Казаков Александр Леонидович, д-р физ.-мат. наук, проф., Институт динамики систем и теории управления им. В. М. Матросова СО РАН, Российская Федерация, 664003, г. Иркутск, kazakov@icc.ru, https://orcid.org/0000-0002-3047-1650.

Лемперт Анна Ананьевна, канд. физ.-мат. наук, Институт динамики систем и теории управления им. В. М. Матросова СО РАН, Российская Федерация, 664003, г. Иркутск, lempert@icc.ru, https://orcid.org/0000-0001-9562-7903.

About the authors

Alexander L. Kazakov, Doct Sci. (Phys.Mathematics), Prof., V. M. Matrosov Institute for System Dynamics and Control Theory SB RAS, Irkutsk, 664003, Russian Federation, kazakov@icc.ru, https://orcid.org/0000-0002-3047-1650

Anna A. Lempert, Cand. Sci. (Phys.Math.), V. M. Matrosov Institute for System Dynamics and Control Theory SB RAS, Irkutsk, 664003, Russian Federation, lempert@icc.ru, https://orcid.org/0000-0001-9562-7903.

Поступила в редакцию / Received 26.07.2022
Поступила после рецензирования / Revised 29.08.2022
Принята к публикации / Accepted 07.09.2022