On the Equivalence between Kalman Filter at Steady State and DPLL

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***Abstract*—Fundamental results in the literature previously showed that a class of Kalman filter converges to a digital phase lock loop (DPLL) structure at the second and third order. We generalize these results at any order and give the closed-form linear relation, and its inverse, between the steady-state Kalman gains and the loop filter constants. Both relations are simple and only involve Stirling numbers of the first and second kind. This new result may help in a deeper understanding of the equivalence between Kalman filter and DPLL and be of practical interest in high dynamic scenarios.**

***Index Terms*—Carrier synchronization, digital phase lock loop (DPLL), Kalman filter, Kalman gain, loop filter constant, phase tracking, Stirling numbers of the second kind, unsigned Stirling numbers of the first kind.**

I. Introduction

Phase lock loops (PLL) are traditionally used in receivers to track the carrier phase of a signal. They are found in many domains of application including communications, localization, navigation, sonar, radar, etc. Initially designed in the analog domain, it is common today to implement them as digital PLL (DPLL) [1]–[6]. An abundant literature exists regarding their performance; a glimpse of them can be found, *e.g.*, in [5]–[7].

Despite their popularity, DPLLs are not robust in challenging conditions (*e.g.*, complex dynamics), partly since their constant loop bandwidth prevents them from adaptivity in changing scenarios. Alternate techniques have thus been conceived; many of them are surveyed in [8]. Among them Kalman filter (KF) based techniques have been considered for their ability to optimally—in a mean-squared error sense (MSE)—address time-varying scenarios owing to their adaptive loop bandwidth, *e.g.*, [9]–[11].

Important results in the communications field actually showed a strong relation between DPLL and KF-based techniques. In [12]–[14], the authors firstly proved that applying nonlinear state estimation theory to angle-modulated communication signals results in a quasi-optimum DPLL structure at any order. Twenty years later, the comparison went one step further at the second order [15], [16]. In [15], KF was shown to have the structure of a second-order DPLL with time-varying loop filter gains equal to the Kalman gains. In [16], KF was studied at steady state, *i.e.*, once the Kalman gains have converged. The resulting closed-loop transfer function was shown to be equivalent to that of a second-order DPLL with a simple relation between the steady-state Kalman gains and the loop filter constants. Twenty more years later, the authors in [17] extended this result to the third order. Nonetheless, slow dynamics was considered and the equivalence was established *via* an analog PLL formulation.

Studying KF at convergence is of utmost importance. As underlined in [18], KF behaves at steady state as a time-invariant filter so that it is particularly relevant to compare it with traditional tracking techniques. *Pros* and *cons* of time-varying bandwidth loop filters can then be convincingly demonstrated; deep insight can also be gained to design a fair tuning of both approaches. As an illustration, in [15] the authors compare the performance of KF and DPLL at the second order. They set the DPLL loop filter constants equal to the steady-state Kalman gains—so as to obtain the same steady-state performance—and showed the improved acquisition performance of the KF compared to the DPLL. Further studies can be found on the performance of steady-state KF about the convergence time [19], [20], the asymptotic MSE [17], [21] and lower bounds [22].

Nonetheless, the *op. cited* results (aside that in [19]) have only been established at the second order or at the third order with approximation. In challenging applications, high-order dynamics are at play, for instance in missile tracking systems, in deep space communications [23]–[25], or in global navigation satellite system (GNSS) for highly maneuvering airborne or space-borne platforms [26]–[28]. Higher-order loops have the potential then to track high dynamics without increasing the loop bandwidth and thus may address weaker signals [4], [24]. Studying steady-state KF performance at high-order is thus of practical interest.

In this paper, we attempt to contribute in this direction. To this end, we study the equivalence *at any order* between a class of KF at steady state and a class of DPLL. Two main theorems are enunciated to interpret the KF as a DPLL, and reciprocally, while given the closed-form relations between steady-state Kalman gains and loop filter constants. These relations are exemplified till the fifth or sixth order so as to be directly used in practical high dynamic scenarios. Interestingly, our approach assumes an ideal phase discriminator as in [4],

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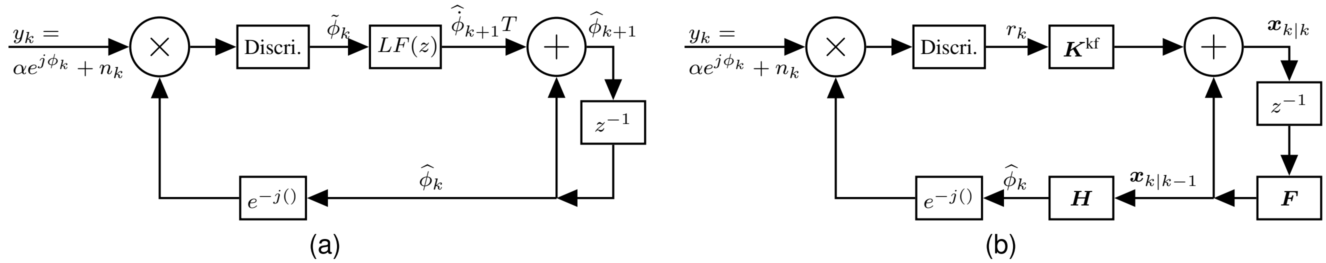

Fig. 1. (a) DPLL architecture. (b) KF-based tracking architecture (KFPLL).

hence it addresses also other locked loops like delay locked loop [29].

The class of DPLL considered is detailed in Section II. Similarly, the class of KF considered for phase tracking is detailed in Section III where an emphasis is made on the transfer function at steady state. The two main results about the equivalence between DPLL and steady-state KF are then enunciated in Section IV. Demonstrations of these results are gathered in Section V while Section VI includes some concluding remarks.

Notations. $\mathcal{I}_K \triangleq \{0, \ldots, K-1\}$ is a set of finite integers. $\mathcal{Z}\{x_k\}(z)$ is the z-transform of the discrete-time signal x_k. $\binom{n}{k} = n!/[(n-k)!k!]$ is the binomial coefficient for $n \geq k \geq 0$; the definition is extended to $\binom{n}{k} = 0$ if $k > n$ or $k < 0$. $\delta_{n,k}$ is the Kronecker delta.

II. The Conventional DPLL with Phase and Phase-Rate Feedback

In this Section we recall the architecture of the DPLL of interest as depicted in Fig. 1a. Particularly, we assume a so-called phase and phase-rate feedback without any computation delay as defined in [4], [30]. We further give the equivalent linear model.

A. Architecture

A standard DPLL is designed to receive a noisy input signal as

$$y_k = \alpha e^{j\phi_k} + n_k \tag{1}$$

with α the amplitude of the component that carries the phase of interest ϕ_k at instant k and n_k the noise input.

The received signal (1) is then correlated with a local replica that carries the estimated phase $\widehat{\phi}_k$ at the same instant. In practice, this correlation can stem from the integration of several complex input samples so as to increase the signal-to-noise ratio (SNR) or decrease the computational complexity [30, p.2-1]. In any event, we denote by T the loop update rate.

The residual phase, namely the error between the actual phase and that just estimated, is extracted via a so-called phase discriminator [29, Sec. 8.6.1], [8]. This predetection stage ideally outputs, in absence of noise, the phase error

$$\tilde{\phi}_k = \phi_k - \widehat{\phi}_k. \tag{2}$$

The model (2) can be inaccurate partly due to the phase discriminator that intrinsically is nonlinear outside a given operating range. We discard any possible imperfections to develop our linear model in what follows.

In practice, the phase residual (2) contains noise that is removed (at least partially) by a loop filter. The latter is usually low-pass and based on proportional-integral gains. The filter output can be seen as the estimated phase rate at time $k+1$ multiplied by the update interval T such as

$$\begin{aligned}\widehat{\dot{\phi}}_{k+1}T = K_1^{\mathrm{d}}\tilde{\phi}_k + K_2^{\mathrm{d}}\sum_{i_1=1}^{k}\tilde{\phi}_{i_1} + K_2^{\mathrm{d}}\sum_{i_1=1}^{k}\sum_{i_2=1}^{i_1}\tilde{\phi}_{i_2} \\ + \ldots + K_N^{\mathrm{d}}\sum_{i_1=1}^{k}\sum_{i_2=1}^{i_1}\cdots\sum_{i_{N-1}=1}^{i_{N-2}}\tilde{\phi}_{i_{N-1}}\end{aligned} \tag{3}$$

where K_n^{d} is the nth filter constants assuming a Nth order loop ($N \geq 1$). Note that we have assumed here integration *via* the rectangular method [31, Sec. 1.5]. Alternate techniques like the trapezoidal or Simpson's rule would result in a different relation than in (3) [32, Ch. 5].

Finally, the estimate phase at the next instant $k+1$ is built assuming a loop with phase and phase-rate feedback [4], [30], namely

$$\widehat{\phi}_{k+1} = \widehat{\phi}_k + \widehat{\dot{\phi}}_{k+1}T. \tag{4}$$

This updated value is used to generate the next replica to be correlated with the new received signal samples. The loop is hence closed.

B. Linear Model

In linear regime, the DPLL is tantamount to a filter with transfer function in the z-domain such as

$$H^{\mathrm{d}}(z) \triangleq \frac{\widehat{\Phi}(z)}{\Phi(z)} \tag{5}$$

with $\Phi(z) \triangleq \mathcal{Z}\{\phi_k\}(z)$ and $\widehat{\Phi}(z) \triangleq \mathcal{Z}\left\{\widehat{\phi}_k\right\}(z)$. For a Nth order DPLL, it is shown in [4, eqs. (36)-(37)] that (5) has the form

$$H^{\mathrm{d}}(z) = \frac{D^{\mathrm{d}}(z) - (z-1)^N}{D^{\mathrm{d}}(z)} \tag{6}$$

with

$$\begin{aligned}D^{\mathrm{d}}(z) = (z-1)^N + K_1^{\mathrm{d}}(z-1)^{N-1} + K_2^{\mathrm{d}}z(z-1)^{N-2} \\ + K_3^{\mathrm{d}}z^2(z-1)^{N-3} + \ldots + K_N^{\mathrm{d}}z^{N-1}.\end{aligned} \tag{7}$$

For a comprehensive view, the transfer function of the loop filter only is finally

$$LF(z) \triangleq \mathcal{Z}\left\{\widehat{\phi}_{k+1}T\right\}(z)/\mathcal{Z}\left\{\tilde{\phi}_k\right\}(z)$$
$$= K_1^{\mathrm{d}} + K_2^{\mathrm{d}}\frac{1}{1-z^{-1}} + \ldots + K_N^{\mathrm{d}}\frac{1}{(1-z^{-1})^{N-1}}.$$

III. The Conventional KFPLL with Newtonian Transition Matrix

In this Section we describe the architecture of the KF of interest as depicted in Fig. 1b. We particularly choose a transition matrix stemming from a so-called kinematic or Newtonian model [31], [33]. We derive from it the equivalent loop transfer function at steady state and *at any order*. Similar calculations can be found in [16] and [17] but were restricted to the second and third order, respectively.

A. State-Space Model

In a phase tracking problem, the KF algorithm is usually based on the following state-space model, assuming a perfect linearization of the phase error by the discriminator, [15]–[17], [34]

$$\boldsymbol{x}_k = \boldsymbol{F}\boldsymbol{x}_{k-1} + \boldsymbol{w}_{k-1} \quad (8a)$$
$$\tilde{y}_k = \boldsymbol{H}\boldsymbol{x}_k + \tilde{n}_k \quad (8b)$$

where

$$\boldsymbol{x}_k = \begin{bmatrix}\phi_k & \dot{\phi}_k & \ldots & \overset{N-1}{\dot{\phi}_k}\end{bmatrix}^\top \quad (9a)$$
$$\boldsymbol{H} = \begin{bmatrix}1 & 0\ldots 0\end{bmatrix} \quad (9b)$$

with

- $\boldsymbol{x}_k$ the $N \times 1$ state-space vector whose elements are the phase and its derivatives till the $(N-1)$th order discretized at the time instant kT;
- $\boldsymbol{F}$ the $N \times N$ state transition matrix;
- $\boldsymbol{w}_k$ the Gaussian process noise;
- $\tilde{y}_k$ the noisy phase on receive;
- $\boldsymbol{H}$ the $1 \times N$ measurement matrix;
- $\tilde{n}_k$ the phase noise assumed Gaussian distributed with known power [35]–[37].

In practice, $\tilde{y}_k$ is not directly measured instead the discriminator output is used as a measurement residual as seen later in (11) and discussed in [38].

In this work we focus on a particular class of transition matrix that we are referring to as Newtonian transition matrix (adopting a similar terminology as in [31, p. 132]), *i.e.*,

$$\boldsymbol{F} = \begin{pmatrix} 1 & \frac{T}{1!} & \frac{T^2}{2!} & \ldots & \frac{T^{N-1}}{(N-1)!} \\ 0 & 1 & \frac{T}{1!} & \ldots & \frac{T^{N-2}}{(N-2)!} \\ 0 & 0 & 1 & \ldots & \frac{T^{N-3}}{(N-3)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}. \quad (10)$$

This choice is important to establish later the equivalence with the previously described DPLL. (As will appear from the derivations in Section V, (10) intrinsically confers to the KF an architecture of DPLL with *rectangular* integration.) No specific assumptions is made about the covariance matrices of the measurement and process noises. In that regards, our state-space model (8)-(9)-(10) addresses conventional phase dynamics such as described, *e.g.*, in [17], [33], but may cover a broader scope. In the remainder of the paper, this state-space approach is dubbed KFPLL.

B. Architecture at Steady State

Kalman filtering aims at obtaining at each instant k the posterior distribution of $\boldsymbol{x}_k|\tilde{\boldsymbol{Y}}_k$ from (8) where $\tilde{\boldsymbol{Y}}_k = \begin{bmatrix}\tilde{y}_1 & \ldots & \tilde{y}_k\end{bmatrix}$ gathers all measurements till instant k. To that end, a two-stage scheme updates the moments of the prediction and posterior distributions defined as [39]

$$\boldsymbol{x}_k|\tilde{\boldsymbol{Y}}_{k-1} \sim \mathcal{N}\left(\boldsymbol{x}_{k|k-1}, \boldsymbol{P}_{k|k-1}\right)$$
$$\boldsymbol{x}_k|\tilde{\boldsymbol{Y}}_k \sim \mathcal{N}\left(\boldsymbol{x}_{k|k}, \boldsymbol{P}_{k|k}\right)$$

where $\mathcal{N}(\boldsymbol{m}, \boldsymbol{P})$ is the Gaussian distribution with mean $\boldsymbol{m}$ and covariance matrix $\boldsymbol{P}$.

The pair $(\boldsymbol{F}, \boldsymbol{H})$ in (8) is completely observable[1], hence the KF converges here necessarily to *a* steady state [33, Sec. 5.25]. The filtering architecture is then driven by [40, Sec. 7.3]

$$\boldsymbol{x}_{k|k} = \boldsymbol{F}\boldsymbol{x}_{k-1|k-1} + \boldsymbol{K}^{\mathrm{kf}} r_k \quad (11a)$$
$$r_k = \phi_k - \widehat{\phi}_k \quad (11b)$$
$$\widehat{\phi}_k = \boldsymbol{H}\boldsymbol{F}\boldsymbol{x}_{k-1|k-1}. \quad (11c)$$

where $\boldsymbol{K}^{\mathrm{kf}} \triangleq \begin{bmatrix}K_1^{\mathrm{kf}} & \ldots & K_N^{\mathrm{kf}}\end{bmatrix}^\top$ is the asymptotic Kalman gain vector and where, though not explicitly stated, the predicted state estimate is

$$\boldsymbol{x}_{k|k-1} = \boldsymbol{F}\boldsymbol{x}_{k-1|k-1}. \quad (12)$$

Similarly to the DPLL and as in [16], the phase estimate $\widehat{\phi}_k$ in (11c) is chosen as that carried by the replica and thus directly depends on the predicted state estimate (12), *viz*

$$\widehat{\phi}_k = [\boldsymbol{x}_{k|k-1}]_1. \quad (13)$$

Interestingly, the output of a KF is conventionally the state estimate $\boldsymbol{x}_{k|k}$ so that another phase of interest could be from (11a) [41, Fig. 1]

$$[\boldsymbol{x}_{k|k}]_1 = \widehat{\phi}_k + K_1^{\mathrm{kf}} \times r_k. \quad (14)$$

Finally, r_k in (11b) is the conventional KF innovation term and corresponds to the phase error at the discriminator output. Note that to develop next the equivalent linear filter, r_k is written for a noise-free input $\tilde{y}_k = \phi_k$.

[1] Noting that $\boldsymbol{H}\boldsymbol{F}^n = \boldsymbol{H}\boldsymbol{F}^{n-1}\boldsymbol{F}$, it can be shown by induction that $[\boldsymbol{H}\boldsymbol{F}^n]_{n,p} = n^p T^p/p!$ for $n, p \in \mathcal{I}_N$ so that the observability matrix is full rank [33, eq. (1.3.8-5)].

C. Linear Model at Steady State

Herein, we transpose the KF equations (11) in the z-domain to obtain the equivalent transfer function at convergence. Let us note $[\boldsymbol{x}_{k|k}]_n \triangleq \overset{n}{\dot{x}}_{k|k}$. Then using (10), the state estimate update (11a) can be expressed in the z-domain as N equations, *i.e.*, for $n \in \mathcal{I}_N$,

$$(1 - z^{-1})\overset{n}{\dot{X}}(z) = F_n(z)R(z) \tag{15}$$

with $\overset{n}{\dot{X}}(z) \triangleq \mathcal{Z}\left\{\overset{n}{\dot{x}}_{k|k}\right\}(z)$, $R(z) \triangleq \mathcal{Z}\{r_k\}(z)$ and

$$F_n(z) \triangleq \frac{1}{z-1}\sum_{p=1}^{N-1-n}\frac{T^p}{p!}F_{n+p}(z) + K^{\mathrm{kf}}_{n+1} \tag{16}$$

with $F_{N-1}(z) \triangleq K^{\mathrm{kf}}_N$ when the sum in (16) is empty. The innovation term (11b) and the phase estimate (11c) are also transformed in the z-domain, using (9b), as

$$R(z) = \Phi(z) - \widehat{\Phi}(z) \tag{17a}$$

$$\widehat{\Phi}(z) = z^{-1}\sum_{n=0}^{N-1}\frac{T^n}{n!}\overset{n}{\dot{X}}(z) \tag{17b}$$

Injecting (15) in (17b) while using (17a), the phase estimate (17b) becomes

$$\widehat{\Phi}(z) = G(z)\left(\Phi(z) - \widehat{\Phi}(z)\right) \tag{18}$$

where we have defined

$$G(z) \triangleq \frac{z^{-1}}{1-z^{-1}}\sum_{n=0}^{N-1}\frac{T^n}{n!}F_n(z). \tag{19}$$

Using (18), the transfer function of the KFPLL can be expressed as

$$H^{\mathrm{kf}}(z) \triangleq \frac{\widehat{\Phi}(z)}{\Phi(z)} = \frac{G(z)}{1+G(z)} \tag{20a}$$

$$= \frac{D^{\mathrm{kf}}(z) - (z-1)^N}{D^{\mathrm{kf}}(z)} \tag{20b}$$

where, to ease our upcoming demonstrations, we have set in the last line

$$D^{\mathrm{kf}}(z) \triangleq (z-1)^N\left(1 + G(z)\right). \tag{21}$$

In what follows, we show that $D^{\mathrm{kf}}(z)$ has a similar functional form as $D^{\mathrm{d}}(z)$ in (7) and explicit the induced relation of the K^{d}_n's wrt the K^{kf}_n's and reciprocally.

IV. Results: Equivalence between KFPLL and DPLL

The main results of this work are summarized in the two following Theorems and demonstrated in Section V. They are compared with that of the literature at the second and third order.

A. Main Results

Theorem IV.1 (Convergence of KFPLL to DPLL and closed–form linear relation between K^{d}_n and $\{K^{\mathrm{kf}}_p\}_{p=n,\ldots,N}$)**.** *Any KFPLL as described in Section III is at convergence tantamount to a DPLL as described in Section II with filter constants given by, for $n = 1, \ldots, N$,*

$$K^d_n = \sum_{p=n}^{N}(-1)^{p+n}S_{p-1,n-1}T^{p-1}K^{kf}_p \tag{22}$$

with $S_{p,n}$ defined in (23)*. For $n = 1$,* (22) *boils down to $K^d_1 = K^{kf}_1$.*

Lemma IV.2 (Closed-form expression of $S_{p,n}$)**.** *The coefficients $S_{p,n}$ are defined by*

$$S_{0,0} = 1 \tag{23a}$$

$$S_{p,0} = 0 \qquad p \geq 1 \tag{23b}$$

$$S_{p,n} = \frac{1}{p!}\sum_{q=0}^{n-1}(-1)^q\binom{n}{q}(n-q)^p \qquad 1 \leq t \leq p \tag{23c}$$

which is summarized by, for $p \geq n \geq 0$,

$$S_{p,n} = \frac{n!}{p!}S(p,n) \tag{24}$$

where $S(p,n)$ are the Stirling numbers of the second kind [42, p. 26], [43, Ch. 6.1].

Theorem IV.3 (Interpretation of a DPLL as a steady-state KFPLL and closed-form relation between K^{kf}_n and $\{K^{\mathrm{d}}_p\}_{p=n,\ldots,N}$)**.** *Any DPLL as described in Section II can be seen as a steady-state KFPLL as described in Section III; providing that values for the two covariance matrices involved in the state-space model* (8) *can be found to result in*[2]*, for $n = 1, \ldots, N$*

$$T^{n-1}K^{kf}_n = \sum_{p=n}^{N}c_{p-1,n-1}K^d_p \tag{25}$$

with $c_{p,n}$ defined in (26)*. For $n = 1$,* (25) *boils down to $K^{kf}_1 = K^d_1$.*

Lemma IV.4 (Closed-form expression of $c_{p,n}$)**.** *The coefficients $c_{p,n}$ are defined by*

$$c_{p,n} = \frac{n!}{p!}c(p,n) \tag{26}$$

where $c(p,n)$ are the unsigned Stirling numbers of the first kind [42, p. 29], [43, Ch. 6.1]. They can be numerically obtained by the recurrence formula

$$c(0,0) = 1$$

$$c(n,0) = c(0,n) = 0 \qquad n > 0$$

$$c(p+1,n) = p\,c(p,n) + c(p,n-1) \qquad n > 0.$$

Formula (22)-(26) are exemplified in Tables I-V, respectively.

[2] The answer to this question is out of the scope of this paper. It would require to study solutions to the discrete time algebraic Riccati equation [40, Sec. 7.3]. Elements of answer are given in [34, eqs. (7),(9)] at the second order where the K^{kf}_n's are expressed in closed-form wrt the noise parameters and in [17] at the third order where Riccati equations are solved under some approximations.

TABLE I
EQUIVALENT LOOP FILTER CONSTANTS FOR A STEADY-STATE KFPLL

$N=1$	$N=2$	$N=3$	$N=4$	$N=5$
$K_1^{\mathrm{d}}=K_1^{\mathrm{kf}}$	$K_1^{\mathrm{d}}=K_1^{\mathrm{kf}}$	$K_1^{\mathrm{d}}=K_1^{\mathrm{kf}}$	$K_1^{\mathrm{d}}=K_1^{\mathrm{kf}}$	$K_1^{\mathrm{d}}=K_1^{\mathrm{kf}}$
	$K_2^{\mathrm{d}}=TK_2^{\mathrm{kf}}$	$K_2^{\mathrm{d}}=TK_2^{\mathrm{kf}}-\frac{T^2}{2!}K_3^{\mathrm{kf}}$	$K_2^{\mathrm{d}}=TK_2^{\mathrm{kf}}-\frac{T^2}{2!}K_3^{\mathrm{kf}}+\frac{T^3}{3!}K_4^{\mathrm{kf}}$	$K_2^{\mathrm{d}}=TK_2^{\mathrm{kf}}-\frac{T^2}{2!}K_3^{\mathrm{kf}}+\frac{T^3}{3!}K_4^{\mathrm{kf}}-\frac{T^4}{4!}K_5^{\mathrm{kf}}$
		$K_3^{\mathrm{d}}=T^2K_3^{\mathrm{kf}}$	$K_3^{\mathrm{d}}=T^2K_3^{\mathrm{kf}}-T^3K_4^{\mathrm{kf}}$	$K_3^{\mathrm{d}}=T^2K_3^{\mathrm{kf}}-T^3K_4^{\mathrm{kf}}+\frac{7}{12}T^4K_5^{\mathrm{kf}}$
			$K_4^{\mathrm{d}}=T^3K_4^{\mathrm{kf}}$	$K_4^{\mathrm{d}}=T^3K_4^{\mathrm{kf}}-\frac{3}{2}T^4K_5^{\mathrm{kf}}$
				$K_5^{\mathrm{d}}=T^4K_5^{\mathrm{kf}}$

TABLE II
EQUIVALENT STEADY-STATE KF GAINS FOR A DPLL

$N=1$	$N=2$	$N=3$	$N=4$	$N=5$
$K_1^{\mathrm{kf}}=K_1^{\mathrm{d}}$	$K_1^{\mathrm{kf}}=K_1^{\mathrm{d}}$	$K_1^{\mathrm{kf}}=K_1^{\mathrm{d}}$	$K_1^{\mathrm{kf}}=K_1^{\mathrm{d}}$	$K_1^{\mathrm{kf}}=K_1^{\mathrm{d}}$
	$TK_2^{\mathrm{kf}}=K_2^{\mathrm{d}}$	$TK_2^{\mathrm{kf}}=K_2^{\mathrm{d}}+\frac{1}{2}K_3^{\mathrm{d}}$	$TK_2^{\mathrm{kf}}=K_2^{\mathrm{d}}+\frac{1}{2}K_3^{\mathrm{d}}+\frac{1}{3}K_4^{\mathrm{d}}$	$TK_2^{\mathrm{kf}}=K_2^{\mathrm{d}}+\frac{1}{2}K_3^{\mathrm{d}}+\frac{1}{3}K_4^{\mathrm{d}}+\frac{1}{4}K_5^{\mathrm{d}}$
		$T^2K_3^{\mathrm{kf}}=K_3^{\mathrm{d}}$	$T^2K_3^{\mathrm{kf}}=K_3^{\mathrm{d}}+K_4^{\mathrm{d}}$	$T^2K_3^{\mathrm{kf}}=K_3^{\mathrm{d}}+K_4^{\mathrm{d}}+\frac{11}{12}K_5^{\mathrm{d}}$
			$T^3K_4^{\mathrm{kf}}=K_4^{\mathrm{d}}$	$T^3K_4^{\mathrm{kf}}=K_4^{\mathrm{d}}+\frac{3}{2}K_5^{\mathrm{d}}$
				$T^4K_5^{\mathrm{kf}}=K_5^{\mathrm{d}}$

TABLE III
COEFFICIENTS $(-1)^{n+p}S_{p,n}$ FOR EQUIVALENT DPLL FILTER CONSTANTS IN (22)

n \ p	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	0	1	$-\frac{1}{2!}$	$\frac{1}{3!}$	$-\frac{1}{4!}$	$\frac{1}{5!}$	$-\frac{1}{6!}$
2	0	0	1	-1	$\frac{7}{12}$	$-\frac{1}{4}$	$\frac{31}{360}$
3	0	0	0	1	$-\frac{3}{2}$	$\frac{5}{4}$	$-\frac{3}{4}$
4	0	0	0	0	1	-2	$\frac{13}{6}$
5	0	0	0	0	0	1	$-\frac{5}{2}$
6	0	0	0	0	0	0	1

B. Comparison to Previous Results

Results of Theorem IV.1 agree with that presented in [16, eq. (32)][3], and [34, eq. (19)] at the second order, and, implicitly in [17, eq. (22)] at the third order. Indeed, for the latter, if we had derived the closed-loop transfer function (20a) with input equal to the phase estimate (14) instead of (13) we would have obtained the same expression as in [17, eq. (22)]. Interestingly, the authors of [17] do not reach the exact same conclusion. Instead, they showed that their KFPLL is similar to a DPLL but only under the approximation of a slow dynamics and while considering the structure of an analog PLL. To further ease the comparison between results found in the literature and Theorems IV.1-IV.3, the expression of $D^{\mathrm{kf}}(z)$ in (21) is specialized later in Lemma V.4 at the second and third order.

[3]To be exact, one needs to consider in [16] the clock delay equal to 1 and the probability to make a correction equal to 1. In addition, Kalman gains absorb the update interval T^{n-1}. Doing so, the transfer function [16, eq. (32)] matches that in (20) with Lemma (V.4).

TABLE IV
CLOSED-FORM EXPRESSIONS OF SOME $S_{p,t}$

$S_{p,1}=\frac{1}{p!}$	$p\geq 1$
$S_{p,2}=\frac{2^p-2}{p!}$	$p\geq 2$
$S_{p,3}=\frac{3^p-3\times 2^p+3}{p!}$	$p\geq 3$
$S_{p,4}=\frac{4^p-4\times 3^p+6\times 2^p-4}{p!}$	$p\geq 4$
$S_{p,p-1}=\frac{p-1}{2!}$	$p\geq 2$
$S_{p,p}=1$	$p\geq 1$

TABLE V
COEFFICIENTS $c_{p,n}$ FOR EQUIVALENT KFPLL GAINS IN (25)

n \ p	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	0	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$
2	0	0	1	1	$\frac{11}{12}$	$\frac{5}{6}$	$\frac{137}{180}$
3	0	0	0	1	$\frac{3}{2}$	$\frac{7}{4}$	$\frac{15}{8}$
4	0	0	0	0	1	2	$\frac{17}{6}$
5	0	0	0	0	0	1	$\frac{5}{2}$
6	0	0	0	0	0	0	1

V. DERIVATIONS

Herein we give the intermediate results followed by their proofs that finally lead to that of Theorems IV.1 and IV.3.

A. Intermediate Elements of Demonstrations

First, $F_n(z)$ defined by the recursion (16) is expressed in closed-form via Lemmas V.1 and V.2. Injecting this explicit expression in (19) and (21), $D^{\mathrm{kf}}(z)$ can be rewritten as proposed in Lemma V.3. We then identify the polynomial coefficients of $D^{\text{kf}}(z)$ with that of $D^{\text{d}}(z)$ in Lemma V.5 and rewrite the obtained relation in matrix representation in Lemma V.6. It relates *via* a linear relation the K_n^{kf}'s and the K_n^{d}'s. One of the two matrices involved can easily be seen as invertible in Lemma V.7. Using two additional relations established in Lemmas V.8 and V.9, we obtain a closed-form expression of the latter matrix inverse times the second matrix in Lemma V.10. This matrix product can be inverted then in closed-form in Lemma V.11. Using all these previous Lemmas, the main results in Theorems IV.1 and IV.3 can finally be demonstrated.

Lemma V.1 (Closed-form expression of $F_n(z)$)**.** *The functions $F_n(z)$ in* (16) *can be expressed for $n \in \mathcal{I}_N$ as*

$$F_n(z) = \sum_{p=0}^{N-1-n} \alpha_p(z) T^p K_{n+1+p}^{kf} \tag{27}$$

where $\alpha_p(z)$ are defined by the recursion

$$\alpha_0(z) \triangleq 1 \tag{28a}$$

$$\alpha_p(z) \triangleq \frac{1}{z-1} \sum_{r=1}^{p} \frac{\alpha_{p-r}(z)}{r!} \quad p \geq 1 \tag{28b}$$

that is noticeably independent of the order N.

Lemma V.2 (Closed-form expression of $\alpha_p(z)$)**.** *The term $\alpha_p(z)$ defined in* (28) *can be further developed for $p \geq 1$ as*

$$\alpha_p(z) = \sum_{t=1}^{p} \frac{S_{p,t}}{(z-1)^t} \tag{29}$$

where for $1 \leq t \leq p$

$$S_{p,t} \triangleq \sum_{i_j \geq 1 / \sum_{j=1}^{t} i_j = p} \frac{1}{\prod_{j=1}^{t} i_j!} \tag{30}$$

with the special case $S_{p,1} = 1/p!$ for $t = 1$. According to [44, Th. 2.2] and [45, eq. (27)], $S_{p,t}$ can equivalently written as in (23c) *and* (24)*, respectively.*

Lemma V.3 (Closed-form expression of $D^{\text{kf}}(z)$)**.** *$D^{kf}(z)$ in* (21) *can be rewritten as*

$$D^{kf}(z) = (z-1)^N + K_1^{kf}(z-1)^{N-1} + \sum_{t=0}^{N-2} \left(\sum_{p=t+1}^{N-1} S_{p,t+1} T^p K_{p+1}^{kf} \right) z(z-1)^{N-t-2}. \tag{31}$$

Lemma V.4 (Expressions of $D^{\text{kf}}(z)$ for $N = 2$ and $N = 3$)**.** *To ease the comparison with previous results in the literature in Section IV-B, we specialize* (31) *as follows*

$$D^{kf}(z) = (z-1)^2 + (z-1)K_1^{kf} + zTK_2^{kf} \qquad (N=2)$$

$$D^{kf}(z) = (z-1)^3 + (z-1)^2 K_1^{kf} + z(z-1)\left(TK_2^{kf} - \frac{T^2}{2}K_3^{kf}\right) + z^2T^2K_3^{kf} \quad (N=3).$$

Lemma V.5 (Formulation of $D^{\text{kf}}(z)$ as $D^{\text{d}}(z)$)**.** *$D^{kf}(z)$ in* (31) *can be formulated as $D^d(z)$ in* (7) *iff*

$$K_1^{kf} = K_1^d \tag{32a}$$

and for $k = 1, \ldots, N-1$,

$$\sum_{t=0}^{N-1-k} (-1)^t \binom{N-2-t}{k-1} \left(\sum_{p=t+1}^{N-1} S_{p,t+1} T^p K_{p+1}^{kf} \right) = \sum_{p=1}^{k} \binom{N-1-p}{k-p} K_{p+1}^d. \tag{32b}$$

Lemma V.6 (Matrix representation of the linear relation between $\{K_n^{\text{kf}}\}_{n=2,\ldots,N}$ and $\{K_n^{\text{d}}\}_{n=2,\ldots,N}$ matrices)**.** *Without a dedicated proof, we can simply rewrite the $N-1$ equations in* (32b) *as*

$$\boldsymbol{M}^{kf} \begin{pmatrix} TK_2^{kf} \\ \vdots \\ T^{N-1}K_N^{kf} \end{pmatrix} = \boldsymbol{M}^d \begin{pmatrix} K_2^d \\ \vdots \\ K_N^d \end{pmatrix} \tag{33}$$

where for $n, p = 1, \ldots, N-1$

$$[\boldsymbol{M}^{kf}]_{n,p} \triangleq \sum_{t=1}^{p} (-1)^{t+1} \binom{N-1-t}{n-1} \frac{t!}{p!} S(p,t) \tag{34a}$$

$$[\boldsymbol{M}^{d}]_{n,p} \triangleq \binom{N-1-p}{n-p} \tag{34b}$$

with the usual convention $S(p,t) = \binom{p}{t} = 0$ if $t > p$. Elements of the first column of $\boldsymbol{M}^{kf}$ are simply $\binom{N-2}{n-1}$ while that of the last column are $(-1)^{N-1-n}E(N-1, n-1)/(N-1)!$ with $E(n,k)$ the Eulerian numbers [43, eq. (6.40)].
$\boldsymbol{M}^d$ is a lower triangular matrix with diagonal elements $\binom{N-1-p}{0} = 1$ and thus invertible; it is actually the Pascal's triangle expressed as a matrix.

Lemma V.7 (Inverse of $\boldsymbol{M}^{\text{d}}$)**.** *The inverse of $\boldsymbol{M}^d$ is a lower triangular matrix such that for $n, p = 1, \ldots, N-1$*

$$[\boldsymbol{M}^{d^{-1}}]_{n,p} = (-1)^{p-n} \binom{N-1-p}{n-p}. \tag{35}$$

It is the same as $\boldsymbol{M}^d$ but with alternate elements negated starting with the first subdiagonal.

As become clear after, the two following Lemmas are useful to prove the two last Lemmas.

Lemma V.8 (Sum over a product of binomial coefficients)**.** *We have for $n = 1, \ldots, N-1$, $t = 1, \ldots, n$,*

$$\sum_{q=1}^{n} (-1)^q \binom{N-1-q}{n-q} \binom{N-1-t}{q-1} = -\binom{t-1}{n-1}. \tag{36}$$

Lemma V.9 (Sum of Lah numbers and Stirling numbers of the second kind)**.** *We have for $p \geq n \geq 1$*

$$\sum_{t=n}^{p} (-1)^{t+p} S(p,t) L(t,n) = S(p,n) \tag{37}$$

where $L(t,n) \triangleq t!/n!\binom{t-1}{n-1}$ is the unsigned Lah numbers [46].

Lemma V.10 (Closed-form expression of $\boldsymbol{M}^{\text{d}^{-1}}\boldsymbol{M}^{\text{kf}}$)**.** *For $n, p = 1, \ldots, N-1$*

$$[\boldsymbol{M}^{d^{-1}}\boldsymbol{M}^{kf}]_{n,p} = (-1)^{n+p} S_{p,n} \tag{38}$$

Lemma V.11 (Closed-form expression of $\boldsymbol{M}^{\text{kf}^{-1}}\boldsymbol{M}^{\text{d}}$)**.** *For $n, p = 1, \ldots, N-1$*

$$[\boldsymbol{M}^{kf^{-1}}\boldsymbol{M}^{kf}]_{n,p} = c_{p,n} \tag{39}$$

B. Proofs of Intermediate Elements of Demonstration

Proof of Lemma V.1. We use strong induction [47, Sec. 1.2.1]. Using (16), we have $F_{N-1}(z) = K_N^{\text{kf}}$ which proves the Lemma for $n = N-1$. Assuming the Lemma true for $n, \ldots, N-1$ with $n \leq 1$ and using (16) yields

$$\begin{aligned} F_{n-1}(z) &= \frac{1}{z-1}\sum_{p=1}^{N-n}\frac{T^p}{p!}F_{n-1+p}(z) + K_n^{\text{kf}} \\ &= \frac{1}{z-1}\sum_{p=1}^{N-n}\frac{T^p}{p!}\sum_{r=0}^{N-n-p}\alpha_r(z)T^r K_{n+p+r}^{\text{kf}} + K_n^{\text{kf}} \\ &= \sum_{s=1}^{N-n}\left\{\frac{1}{z-1}\sum_{p=1}^{s}\frac{\alpha_{s-p}(z)}{p!}\right\}T^s K_{n+s}^{\text{kf}} + K_n^{\text{kf}} \end{aligned}$$

where a simple change of variables and an inversion of the indices of summation have been made to obtain the last line. We thus recognize the same functional form as in (27)-(28) for the index $n-1$ which proves the Lemma for $n-1$ and ends our demonstration. □

Proof of Lemma V.2. We use strong induction. Using (28), we have $\alpha_1(z) = (z-1)^{-1}$ which proves the Lemma for $p = 1$. Assuming the Lemma true for $1, \ldots, p$ and using (28) yields

$$\alpha_{p+1}(z) = \frac{1}{z-1}\sum_{r=1}^{p}\frac{\alpha_{p+1-r}(z)}{r!} + \frac{1}{z-1}\frac{\alpha_0(z)}{(p+1)!} \quad (40)$$

$$= \sum_{r=1}^{p}\frac{1}{r!}\sum_{t=1}^{p+1-r}\frac{S_{p+1-r,t}}{(z-1)^{t+1}} + \frac{1}{(p+1)!}\frac{1}{z-1} \quad (41)$$

$$= \sum_{t=2}^{p+1}\sum_{r=1}^{p+2-t}\frac{S_{p+1-r,t-1}}{r!}\frac{1}{(z-1)^t} + \frac{1}{(p+1)!}\frac{1}{z-1} \quad (42)$$

where a simple change of variables and an inversion of the indices of summation have been made to obtain the last line. Using (30), we can further write

$$\begin{aligned} \sum_{r=1}^{p+2-t}\frac{S_{p+1-r,t-1}}{r!} &= \sum_{r=1}^{p+1-(t-1)}\frac{1}{r!}\sum_{i_j\geq 1/\sum_{j=1}^{t-1} i_j = p+1-r}\frac{1}{\prod_{j=1}^{t-1} i_j!} \\ &= \sum_{i_j\geq 1/\sum_{j=1}^{t} i_j = p+1}\frac{1}{\prod_{j=1}^{t} i_j!} \\ &= S_{p+1,t}. \end{aligned} \quad (43)$$

Injecting (43) in (42), we get $\alpha_{p+1}(z) = \sum_{t=1}^{p+1} S_{p+1,t}$ which ends our demonstration. □

Proof of Lemma V.3. Injecting (27) and (29) in (19) yields

$$\begin{aligned} G(z)(z-1)^N &= \sum_{n=0}^{N-1} K_{n+1}^{\text{kf}}\frac{T^n}{n!}(z-1)^{N-1} \\ &+ \sum_{n=0}^{N-1}\sum_{t=1}^{N-1-n}\sum_{p=t}^{N-1-n} S_{p,t}\frac{T^{n+p}}{n!}K_{n+p+1}^{\text{kf}}(z-1)^{N-t-1}. \end{aligned}$$

We further isolate terms related to the index $n = 0$ and for $n \geq 1$ we inverse the indices of summation n and t in the last sum. Then, we regroup terms indexed by (t, n) from $n = 1, \ldots, N-1$ with the term indexed by $(t+1, n = 0)$ and obtain

$$\begin{aligned} G^{(N)}(z)(z-1)^N &= K_1^{\text{kf}} N(z-1)^{N-1} \\ &+ \sum_{t=1}^{N-3}\left\{\sum_{p=t+1}^{N-1} S_{p,t+1}T^p K_{p+1}^{\text{kf}}(z-1)^{N-t-2}\right. \\ &+ \left.\sum_{n=1}^{N-1-t}\sum_{p=t}^{N-1-n} S_{p,t}\frac{T^{n+p}}{n!}K_{n+p+1}^{\text{kf}}(z-1)^{N-t-1}\right\} \\ &+ \sum_{n=1}^{N-1} K_{n+1}^{\text{kf}}\frac{T^n}{n!}\left((z-1)^{N-1} + (z-1)^{N-2}\right) \\ &+ T^{N-1}K_N^{\text{kf}}\ (1 + z - 1) \end{aligned}$$

Applying a change of variable at the third line (*i.e.*, summation on the sub-diagonals $n+p$) and using (43) (*i.e.*, $S_{p,t+1} = \sum_{r=t}^{p-1} S_{r,t}/(p-r)!$), we obtain

$$\begin{aligned} G(z)(z-1)^N &= K_1^{\text{kf}}(z-1)^{N-1} \\ &+ \sum_{t=0}^{N-2}\left(\sum_{p=t+1}^{N-1} S_{p,t+1}T^p K_{p+1}^{\text{kf}}\right) z(z-1)^{N-t-2}. \end{aligned} \quad (44)$$

Injecting (44) in (21), we obtain $D^{\text{kf}}(z)$ as in (31). □

Proof of Lemma V.5. Applying binomial theorem in (31), we obtain after some rearrangement on the indices of summation

$$\begin{aligned} D^{\text{kf}}(z) - (z-1)^N &= K_1^{\text{kf}}\binom{N-1}{0}(-1)^{N-1} \\ &+ \sum_{k=1}^{N-1}(-1)^{N-1-k}\left[K_1^{\text{kf}}\binom{N-1}{k}\right. \\ &+ \left.\sum_{t=0}^{N-1-k}\left(\sum_{p=t+1}^{N-1} S_{p,t+1}T^p K_{p+1}^{\text{kf}}\right)\binom{N-t-2}{k-1}(-1)^{-t}\right] z^k. \end{aligned}$$

Using the same arguments in (7), we obtain

$$D^{\text{d}}(z) - (z-1)^N = \sum_{k=0}^{N-1}(-1)^{N-1-k}\sum_{t=0}^{k} K_{t+1}^{\text{d}}\binom{N-1-t}{k-t} z^k.$$

$D^{\text{kf}}(z) - (z-1)^N$ and $D^{\text{d}}(z) - (z-1)^N$ are thus both $N-1$th order polynomials. They are identical iff there coefficients are equal leading to (32). □

Proof of Lemma V.7. We have for $n, p = 1, \ldots, N-1$

$$\begin{aligned} &\sum_{q=1}^{N-1}\binom{N-1-q}{n-q}\binom{N-1-p}{q-p}(-1)^{q-p} \\ &= \sum_{q=1}^{N-1}\binom{N-1-q}{N-1-n}\binom{N-1-p}{N-1-q}(-1)^{q-p} \\ &= (-1)^{-p+(N-1)}\sum_{q=0}^{N-2}\binom{q}{N-1-n}\binom{N-1-p}{q}(-1)^q \\ &= \binom{0}{p-n} = \delta_{n,p} \end{aligned}$$

where, to obtain the last line, we have used a classical identity about the sum of products of binomial coefficients, *i.e.*, [43, eq. (5.24)]. □

Proof of Lemma V.8. We first notice that negating the upper index leads to [43, (5.14)]

$$\binom{[N-2]-q}{[n-1]-q} = (-1)^{n-1-q}\binom{[n-1]-[N-2]-1}{[n-1]-q}.$$

Then using Vandermonde's convolution [43, eq. (5.27)] and negating again the upper index successively yields

$$\begin{aligned}
&\sum_{q=0}^{n-1}(-1)^{q+1}\binom{[N-2]-q}{[n-1]-q}\binom{[N-2]-[t-1]}{q}\\
&=\sum_{q=0}^{n-1}(-1)^{q+n-q}\binom{[n-1]-[N-2]-1}{[n-1]-q}\binom{[N-2]-[t-1]}{q}\\
&=(-1)^{n-1+1}\binom{[n-1]-[t-1]-1}{n-1}\\
&=-\binom{t-1}{n-1}.
\end{aligned}$$

□

Proof of Lemma V.9. The matrix representation in [48, p. 423] states that $L(t,n) = \sum_{j=0}^{t} c(t,j)S(j,n)$. Observing that $L(t,n) = 0$ if $t < n$, we have thus for $p \geq n \geq 1$

$$\begin{aligned}
\sum_{t=n}^{p}(-1)^{t+p}S(p,t)L(t,n) &= \sum_{j=0}^{p}\sum_{t=0}^{p}(-1)^{t+p}S(p,t)c(t,j)S(j,n)\\
&=\sum_{j=0}^{p}(-1)^{j-p}\delta_{pj}S(j,n)\\
&=S(p,n)
\end{aligned}$$

where we have used in the line before last the inversion formula proposed in [49, Prop. 1.9.1], [43, Tab. 264]. To obtain (37), it remains only to use the definition of $S_{p,n}$ in (24). □

Proof of Lemma V.10. Using (34a) and (35), we obtain for $n, p = 1, \ldots, N-1$

$$\begin{aligned}
&[\boldsymbol{M}^{\mathrm{d}^{-1}}\boldsymbol{M}^{\mathrm{kf}}]_{n,p}\\
&=\sum_{q=1}^{n}\sum_{t=1}^{p}(-1)^{n+q+t-1}\binom{N-1-q}{n-q}\binom{N-1-t}{q-1}S_{p,t}\\
&=\sum_{t=1}^{p}\sum_{q=1}^{n}(-1)^{q}\binom{N-1-q}{n-q}\binom{N-1-t}{q-1}(-1)^{n+t-1}\frac{t!}{p!}S(p,t)\\
&=\sum_{t=1}^{p}(-1)^{n+t}\binom{t-1}{n-1}\frac{t!}{p!}S(p,t) \qquad \text{[Lemma V.8]}\\
&=(-1)^{n-p}\frac{n!}{p!}\sum_{t=n}^{p}(-1)^{t+p}S(p,t)L(t,n) \quad \left[\binom{t}{n-1}=0 \text{ if } t\leq n\right]\\
&=(-1)^{n-p}\frac{n!}{p!}S(p,n) \quad \text{[Lemma V.9]}.
\end{aligned}$$

Using (24), we finally get (38). □

Proof of Lemma V.11. Using (24)-(26), we have for $n, p = 1, \ldots, N-1$

$$\begin{aligned}
&\sum_{k=1}^{N-1}(-1)^{n-k}S_{k-1,n-1}c_{p-1,k-1}\\
&=\sum_{k=0}^{N-2}(-1)^{n-k+1}S_{k,n-1}c_{p-1,k}\\
&=\frac{(n-1)!}{(p-1)!}(-1)^{n-p}\sum_{k=0}^{N-2}(-1)^{p-1-k}c(p-1,k)S(k,n-1)\\
&=\frac{(n-1)!}{(p-1)!}(-1)^{n-p}\delta_{n,p}=\delta_{n,p}
\end{aligned}$$

where, to obtain the last line, we have used the inversion formula proposed in [49, Prop. 1.9.1], [43, Tab. 264], which ends our demonstration. □

C. Proof of Main Theorems

Proof of Theorem IV.1. From Lemma V.7, we conclude first that the system (33) is invertible by a left-multiplication with $\boldsymbol{M}^{\mathrm{d}^{-1}}$. Considering also (32a) we conclude that the K_n^{d}'s are determined by a unique linear relation to the K_n^{kf}'s. A closed-form expression of the latter can be obtained using Lemma V.10. □

Proof of Theorem IV.3. Using Lemma V.11 jointly with (32a), we directly obtain that the K_n^{kf}'s are determined by a unique linear relation to the K_n^{d}'s. Nonetheless, for any values of K_n^{d}'s nothing ensures—at least to our knowledge—that the values obtained for K_n^{kf}'s can be explained by a state-space model as described in Section 8. Indeed, covariance matrices of the process and measurement noises still need to be found. □

VI. Conclusion

The equivalence between KF and DPLL has been considered for the problem of phase estimation. The study focused on two specific architectures. For the DPLL, the feedback stage is performed by a phase and phase-rate feedback without computation delay while numerical integrators are implemented *via* the rectangular method. For the KF, the state-space model assumes a Newtonian transition matrix. Both architectures were examined under the assumption of a perfect linear regime. Two main results were then demonstrated *at any order*. Firstly, the KF converges to a DPLL with filter constants equal to a weighted sum of the KF gains; the latter weights involve only the Stirling numbers of the second kind. Secondly, the DPLL is a steady-state KF with Kalman gains equal to a weighted sum of the filter constants; the latter weights involve only the unsigned Stirling numbers of the first kind. Nonetheless to be valid, the steady-state KF gains still need to be explained by appropriate covariance matrices involved in the state-space model.

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