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Explicit upper bounds for the Stirling numbers of the first kind

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ABSTRACT

We give explicit upper bounds for the Stirling numbers of the first kind $s(n, m)$ which are asymptotically sharp. The form of such bounds varies according to m lying in the central or non-central regions of $\{1, \ldots, n\}$. In each case, we use a different probabilistic representation of $s(n, m)$ in terms of well known random variables to show the corresponding upper bounds. Some applications concerning the Riemann zeta function and a certain subset of the Comtet numbers of the first kind are also provided.

1. Introduction

The Stirling numbers of both kinds are among the most important sequences in mathematics and have numerous applications in combinatorics, number theory, numerical analysis, probability theory, and other fields. Starting from the seminal work by Moser and Wyman [14], there are many papers devoted to obtaining asymptotic expansions for the Stirling numbers of the first kind $s(n, m)$ (see, for instance, Wilf [21], Temme [18], Hwang [11], Tsylova [19], Chelluri et al. [6], Louchard [13], and the references therein).

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As far as we know, less attention has been paid to obtain explicit upper bounds for such numbers. In this regard, denote by

$$H_n^{(a)} = \sum_{j=1}^{n} \frac{1}{j^a}, \quad a > 0, \quad n = 1, 2, \dots . \tag{1}$$

Moser and Wyman [14] gave the estimate

$$s(n+1, m+1) = \frac{(-1)^{n-m} n! \left(H_n^{(1)}\right)^m}{m!} \left(1 - \frac{m(m-1)H_n^{(2)}}{2\left(H_n^{(1)}\right)^2} + E_{n,m}\right), \tag{2}$$

where

$$|E_{n,m}| \le 2 \left(\frac{em}{H_n^{(1)}}\right)^3 .$$

Of course, formula (2) works in the range $m = o(H_n^{(1)}) = o(\log n)$. The same authors showed that more exact terms can be added in formula (2).

Using Stein-Chen Poisson approximation, Arratia and DeSalvo [3] obtained the upper and lower bounds

$$\binom{N}{m} e^{-\mu}(1 - e^{\mu} D_{n,m}) \le |s(n, n-m)| \le \binom{N}{m} e^{-\mu}(1 + e^{\mu} D_{n,m}), \tag{3}$$

for $n \ge 3$ and $n \ge m \ge 2$, where

$$N = \binom{n}{2}, \qquad \mu = \frac{\binom{m}{2}\binom{n}{3}}{\binom{N}{2}},$$

and $D_{n,m}$ is an error term uniformly bounded by 1. The inequalities in (3) are useful in the range $m = O(\sqrt{n})$, i.e., in the case in which the error term $D_{n,m}$ goes to 0, as $n \to \infty$.

The purpose of this paper is twofold. On the one hand, to give explicit upper bounds for $s(n, m)$ easy to handle and asymptotically sharp. In this respect, many functions and analytic constants are expressible by means of series involving the numbers $s(n, m)$ (see, for instance, Blagouchine [4]). Having at our disposal explicit upper bounds for $s(n, m)$, we can compute such functions and analytic constants by finite sums, providing at the same time a specific upper bound for the remainder (see the illustrative example for the Riemann zeta function given in Proposition 7.1 in the final section).

On the other hand, to open the door to possible extensions of our results to more general numbers, such as the Comtet numbers of the first kind (cf. Comtet [7] and El-Desouky et al. [10]). Indeed, we give different probabilistic representations for $s(n, m)$

according to m lying in the central or non-central regions of $\{1, \ldots, n\}$ in order to obtain our results. Certainly, these probabilistic representations are closely connected to the saddle-point method based on Cauchy's integral representations. However, we believe that such representations, written in terms of sums of independent random variables involving the Bernoulli, the uniform, and the exponential distributions, may be of independent interest. As an illustration, we give in Theorem 7.2 in the final section some estimates for a certain subset of the Comtet numbers of the first kind containing the r-Stirling numbers of the first kind introduced by Broder [5].

The paper is organized as follows. In the following section, we state our main results and compare them with other known estimates. Section 3 contains the aforementioned probabilistic representations. In Sections 4 and 5, we give the estimates of such probabilistic representations used to establish our main results, the proof of them being postponed to Section 6. The final section contains the illustration of our results and methods mentioned above.

2. Main results

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For any $n \in \mathbb{N}_0$, the Stirling numbers of the first kind $s(n, m)$ can be defined in various equivalent ways (cf. Abramowitz and Stegun [1, p. 824] and Comtet [8, Chap. 5]). For instance, they can be defined as

$$(x)_n = \sum_{m=0}^{n} s(n,m) x^m, \quad x \in \mathbb{R}, \tag{4}$$

where $(x)_n$ is the descending factorial, i.e., $(x)_n = x(x-1)\cdots(x-n+1)$, $n \in \mathbb{N}$, $(x)_0 = 1$, or in terms of their generating function as

$$\frac{\log^m(1+u)}{m!} = \sum_{n=m}^{\infty} s(n,m) \frac{u^n}{n!}. \tag{5}$$

Throughout this paper, we assume that $n = 2, 3, \ldots$. We define the functions

$$\mu_n(t) = \sum_{j=1}^{n} \frac{t}{t+j}, \qquad \sigma_n^2(t) = \sum_{j=1}^{n} \frac{t}{t+j}\left(1 - \frac{t}{t+j}\right), \qquad t > 0. \tag{6}$$

Such functions were introduced by Moser and Wyman [14] (see also Temme [18], Chelluri et al. [6], and Louchard [13]). Their respective probabilistic meaning, as the mean and the variance of a certain random variable having the Poisson-binomial distribution, will be established in Section 3.

Let $m = 1, \ldots, n-1$. Since $\mu_n(t)$ is strictly increasing and satisfies

$$\lim_{t \to 0} \mu_n(t) = 0, \qquad \lim_{t \to \infty} \mu_n(t) = n,$$

we consider the unique solution $\tau = \tau(n, m) > 0$ to the equation

$$\mu_n(\tau) = m. \tag{7}$$

Note that if $m = n$ there is no real solution τ to equation (7). On the other hand, it follows from (4) that $s(n, n) = 1$. For these reasons, we assume that $m = 1, \dots, n-1$ in our following main result.

Theorem 2.1. *Let $m = 1, \dots, n-1$ and let τ be as in (7). Then,*

$$|s(n+1, m+1)| \leq \frac{n!(\log n)^m}{m!}\left(1 + \frac{m}{\log n}\right) \tag{8}$$

and

$$|s(n+1, m+1)| \leq \frac{\Gamma(\tau+n+1)}{\Gamma(\tau)\tau^{m+1}} \min\left(1, \frac{1}{\sigma_n(\tau)\sqrt{2\pi\,(1 - \log\sigma_n(\tau)/\sigma_n^2(\tau))}} + \frac{1}{\sigma_n^2(\tau)}\right). \tag{9}$$

In addition, whenever $n - m \leq \sqrt{m+1}/4$, we have

$$|s(n+1, m+1)| \leq \binom{n+1}{m+1}\left(\frac{m+1}{2}\right)^{n-m}\left(1 + 32e^{1/6}\frac{(n-m)^2}{m+1}\right). \tag{10}$$

To see the sharpness of estimate (10), we give the following result.

Theorem 2.2. *Let $m = 1, \dots, n-1$. If $n - m \leq \sqrt{m+1}/4$, then*

$$\left|(-1)^{n-m}s(n+1, m+1) - \binom{n+1}{m+1}\left(\frac{m+1}{2}\right)^{n-m}\left(1 + \frac{5(n-m)(n-m-1)}{6(m+1)}\right)\right| \leq 2^7e^{1/6}\binom{n+1}{m+1}\left(\frac{m+1}{2}\right)^{n-m}\left(\frac{n-m}{\sqrt{m+1}}\right)^3.$$

In contrast with (10), estimates (8) and (9) are valid for any $n \geq 2$ and $m = 1, \dots, n-1$. However, their accuracy depends on the range of m. Actually, it is well known that

$$H_n^{(1)} = \log n + \gamma + O(n^{-1}),$$

where γ is Euler's constant. Thus, whenever $m = o(\log n)$, the main terms in (2) and (8) have the same order of magnitude, as $n \to \infty$. In other words, the upper bound in (8) is asymptotically sharp, provided that $m = o(\log n)$. We point out that Hwang [11] extended formula (2) into an asymptotic series valid for $m = O(\log n)$. Also, it is worthwhile to mention that Wilf [21] obtained the following expansion for each fixed $m \geq 1$

$$\frac{|s(n+1,m+1|}{n!} = \gamma_1 \frac{(\log(n+1))^m}{m!} + \gamma_2 \frac{(\log(n+1))^{m-1}}{(m-1)!} + \cdots + \gamma_{m+1} + O\left(\frac{(\log(n+1))^{m-1}}{n+1}\right),$$

where γ_j are the coefficients in the expansion

$$\frac{1}{\Gamma(z)} = \sum_{j=1}^{\infty} \gamma_j z^j,$$

with $\gamma_1 = 1$ and $\gamma_2 = \gamma = 0.57721\cdots$.

On the other hand, Chelluri et al. [6] obtained the asymptotic estimate

$$|s(n+1,m+1)| = \frac{\Gamma(\tau+n+1)}{\Gamma(\tau)\tau^{m+1}\sigma_n(\tau)\sqrt{2\pi}}(1+O(1/m)), \tag{11}$$

in the range $\sqrt{\log n} \leq m \leq n - n^{1/3}$. In this range, Moser and Wyman [14, Lemma 4.1] showed that $\sigma_n(\tau) \to \infty$, as $n \to \infty$. Therefore, the upper bound in (9) is asymptotically sharp, provided that $\sqrt{\log n} \leq m \leq n - n^{1/3}$.

From Theorem 2.2, estimate (10) is asymptotically sharp too, whenever $n - m = o(\sqrt{m})$. We mention that the accuracy of the upper bound in (10) can also be derived from the following asymptotic result by Moser and Wyman [14] (see also Tsylova [19])

$$\begin{aligned}
&(-1)^{n-m}s(n,m) \\
&= \binom{n}{m}\left(\frac{m}{2}\right)^{n-m}\left(1 + \frac{5(n-m)_2}{6m} + \frac{1}{m^2}\left((n-m)_3 + \frac{25(n-m)_4}{72}\right) + \cdots\right) \\
&= \binom{n}{m}\left(\frac{m}{2}\right)^{n-m}(1+o(1)), \quad n-m = o(\sqrt{n}).
\end{aligned} \tag{12}$$

Apart from giving specific bounds, we have included Theorem 2.2 because its proof does not require an extra effort to that in proving (10).

Finally, note that the aforementioned ranges are overlapping. As a practical guide, the upper bounds in (8), (9), and (10) should be used, respectively, in the ranges $m \leq (\log n)^r$, $(\log n)^r \leq m \leq n - n^s$, and $n - n^s \leq m \leq n$, for any $1/2 \leq r < 1$ and $1/3 \leq s < 1/2$.

3. Probabilistic representations

We consider the following sequences of random variables used to give suitable probabilistic representations of $s(n,m)$. Let $X(p)$ be a random variable having the Bernoulli distribution with success probability p, i.e.,

$$P(X(p)=1)=p=1-P(X(p)=0), \qquad 0\le p\le 1. \tag{13}$$

If $(X(p_j))_{1\le j\le n}$ is a finite sequence of independent random variables such that $X(p_j)$ has the Bernoulli distribution with success probability p_j, and denoting by $\mathbf{p}_n=(p_1,\dots,p_n)$, we define

$$W(\mathbf{p}_n)=X(p_1)+\ldots+X(p_n), \quad 0\le p_j\le 1, \quad j=1,\dots,n. \tag{14}$$

The probability distribution of $W(\mathbf{p}_n)$ is rather involved and is known in the literature as the Poisson-binomial distribution (see, for instance, Shorgin [16] and Roos [15]). Observe that the mean and the variance of $W(\mathbf{p}_n)$ are respectively given by

$$\mathbb{E}W(\mathbf{p}_n)=\sum_{j=1}^n p_j, \qquad Var(W(\mathbf{p}_n))=\sum_{j=1}^n p_j(1-p_j), \tag{15}$$

where $\mathbb{E}$ stands for mathematical expectation. Using the independence of the random variables involved, we see that

$$\mathbb{E}(1+z)^{W(\mathbf{p}_n)}=\prod_{j=1}^n \mathbb{E}(1+z)^{X(p_j)}=\prod_{j=1}^n (1+p_jz)\,, \qquad z\in\mathbb{C}. \tag{16}$$

We will use the following particular cases of (14), namely,

$$W_n(t):=W(\mathbf{p}_n), \quad \text{if } p_j=t/(t+j), \quad j=1,\dots,n, \quad t>0, \tag{17}$$

and

$$W_n:=W(\mathbf{p}_n), \quad \text{if } p_j=1/j, \quad j=1,\dots,n. \tag{18}$$

As follows from (15) and (17), the mean and the variance of $W_n(t)$ are, respectively, the quantities $\mu_n(t)$ and $\sigma_n^2(t)$ defined in (6). The connection between the random variables W_n and $W_n(t)$ is given by

$$\mathbb{E}(1+tz)^{W_n}=\mathbb{E}(1+t)^{W_n}\mathbb{E}z^{W_n(t)}, \qquad t>0, \qquad z\in\mathbb{C}, \tag{19}$$

as follows from (16) and some simple computations.

Finally, let U and T be two independent random variables such that U is uniformly distributed on $[0,1]$ and T has the exponential density with unit mean

$$\rho(x) = e^{-x}, \qquad x \geq 0. \tag{20}$$

Let $(U_j)_{j\geq 1}$ and $(T_j)_{j\geq 1}$ be two sequences of independent copies of U and T, respectively. We assume that both sequences are mutually independent and define

$$S_m = U_1T_1 + \ldots + U_mT_m. \qquad m \in \mathbb{N}. \tag{21}$$

It follows from (20) that

$$\mu := \mathbb{E}(UT) = \frac{1}{2}, \qquad \sigma^2 := Var(UT) = \mathbb{E}(UT)^2 - \mu^2 = \frac{5}{12}, \tag{22}$$

as well as

$$\mathbb{E}e^{uUT} = \mathbb{E}\left(\frac{1}{1-uU}\right) = -\frac{\log(1-u)}{u}, \qquad |u| < 1, \tag{23}$$

thus implying that

$$\mathbb{E}e^{uS_m} = \left(\frac{-\log(1-u)}{u}\right)^m, \qquad m \in \mathbb{N}, \qquad |u| < 1. \tag{24}$$

Various probabilistic representations of $s(n,m)$ in terms of the random variables considered above are provided in the following result.

Theorem 3.1. *Let $t > 0$. For any $m = 0, 1, \ldots, n$, we have*

$$(-1)^{n-m}s(n+1, m+1) = \frac{\Gamma(t+n+1)}{\Gamma(t)t^{m+1}} \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbb{E}e^{i\theta(W_n(t)-m)}\, d\theta, \tag{25}$$

as well as

$$(-1)^{n-m}s(n+1, m+1) = n!\mathbb{E}\binom{W_n}{m} = \binom{n+1}{m+1}\mathbb{E}S_{m+1}^{n-m}. \tag{26}$$

Proof. Let $x \in \mathbb{R}$. Since $s(n+1, 0) = 0$, we can rewrite (4) as

$$(x)_{n+1} = \sum_{m=0}^{n} s(n+1, m+1)x^{m+1}. \tag{27}$$

By (16), (18), and the binomial expansion, we have

$$(x)_{n+1} = (-1)^n n! x \prod_{j=1}^{n} (1 - x/j) = (-1)^n n! x \mathbb{E}(1-x)^{W_n}$$
$$= (-1)^n n! x \sum_{m=0}^{n} \mathbb{E}\binom{W_n}{m}(-x)^m.$$

Comparing this identity with (27), we obtain the first equality in (26). On the other hand, note that

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbb{E}(1+te^{i\theta})^{W_n} e^{-im\theta}\, d\theta \\ &= \sum_{j=0}^{n} \mathbb{E}\binom{W_n}{j} \frac{t^j}{2\pi}\int_{-\pi}^{\pi} e^{i\theta(j-m)}\, d\theta = t^m \mathbb{E}\binom{W_n}{m}. \end{aligned} \tag{28}$$

Choosing $z = e^{i\theta}$ in (19) and recalling (16) and (17), the first term in (28) is equal to

$$\mathbb{E}(1+t)^{W_n} \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbb{E}e^{i\theta(W_n(t)-m)}\, d\theta = \frac{\Gamma(t+n+1)}{\Gamma(t+1)n!}\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbb{E}e^{i\theta(W_n(t)-m)}\, d\theta.$$

Hence, (25) follows from the first equality in (26) and (28). The second equality in (26) was shown by Sun and Wang [17] (see also [2]). We give here a short proof of it for the sake of completeness. By (24), we have

$$\frac{\log^m(1+u)}{m!} = \frac{u^m}{m!}\mathbb{E}e^{-uS_m} = \frac{u^m}{m!}\sum_{k=0}^{\infty}\frac{(-u)^k \mathbb{E}S_m^k}{k!}$$
$$= \sum_{n=m}^{\infty}\binom{n}{m}(-1)^{n-m}\mathbb{E}S_m^{n-m}\frac{u^n}{n!}, \qquad |u| < 1.$$

This, together with (5), shows the second equality in (26) and completes the proof. □

4. Estimates concerning Poisson-binomial distributions

In this section, we give some auxiliary results referring to the random variables W_n and $W_n(t)$, which are used to prove estimates (8) and (9) in Theorem 2.1. The proof of the following lemma is similar in spirit to that of Lemma 6 in Shorgin [16].

Lemma 4.1. *Let $\mu_n(t)$ and $\sigma_n^2(t)$ be as in (6) and let $W_n(t)$ be as in (17), $t > 0$. Then,*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\mathbb{E}e^{i\theta(W_n(t)-\mu_n(t))}\right| d\theta \le \frac{1}{\sigma_n(t)\sqrt{2\pi\left(1-\log\sigma_n(t)/\sigma_n^2(t)\right)}} + \frac{1}{\sigma_n^2(t)}.$$

Proof. Let $j = 1, \dots, n$ and $-\pi \le \theta \le \pi$. Denote by $p_j = t/(t+j)$. From (17), we have

$$\mathbb{E} e^{i\theta(X_j(t)-p_j)} = e^{-ip_j\theta}(p_j e^{i\theta} + 1 - p_j),$$

thus implying that

$$\left| \mathbb{E} e^{i\theta(X_j(t)-p_j)} \right|^2 = 1 - 4p_j(1-p_j)\sin^2(\theta/2) \le e^{-4p_j(1-p_j)\sin^2(\theta/2)},$$

which, by virtue of (6) and the independence of the random variables $(X_j(t))_{1 \le j \le n}$, entails

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\pi}^{\pi} \left| \mathbb{E} e^{i\theta(W_n(t)-\mu_n(t))} \right| d\theta \le \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-2\sigma_n^2(t)\sin^2(\theta/2)}\, d\theta \\ &= \frac{2}{\pi}\int_0^{\pi/2} e^{-2\sigma_n^2(t)\sin^2\theta}\, d\theta. \end{aligned} \tag{29}$$

Let $0 < a < \pi/2$ to be chosen later on. Observe that

$$\frac{2}{\pi}\int_a^{\pi/2} e^{-2\sigma_n^2(t)\sin^2\theta}\, d\theta \le e^{-2\sigma_n^2(t)\sin^2 a}. \tag{30}$$

On the other hand, making the change $2\sigma_n(t)\sin\theta = u$, we have

$$\begin{aligned} &\frac{2}{\pi}\int_0^{a} e^{-2\sigma_n^2(t)\sin^2\theta}\, d\theta = \frac{1}{\pi\sigma_n(t)}\int_0^{2\sigma_n(t)\sin a} \frac{e^{-u^2/2}}{\sqrt{1-u^2/4\sigma_n^2(t)}}\, du \\ &\le \frac{1}{\pi\sigma_n(t)\cos a}\int_0^{\infty} e^{-u^2/2}\, du = \frac{1}{\sigma_n(t)\sqrt{2\pi}\cos a}. \end{aligned} \tag{31}$$

If $\sigma_n(t) \le 1$, the result is obviously true. Thus assume that $\sigma_n(t) > 1$. Choosing $\sigma_n^2(t)\sin^2 a = \log \sigma_n(t)$, the result follows from (29)–(31). □

Let N_λ be a random variable having the Poisson law with mean $\lambda \ge 0$, i.e.,

$$P(N_\lambda = j) = \frac{\lambda^j}{j!}e^{-\lambda}, \qquad \lambda \in \mathbb{N}_0.$$

The binomial moments of N_λ are easy to compute, since

$$\mathbb{E}\binom{N_\lambda}{m} = \sum_{j=m}^{\infty} \binom{j}{m} \frac{\lambda^j}{j!} e^{-\lambda} = \frac{\lambda^m}{m!}, \qquad m \in \mathbb{N}_0. \tag{32}$$

On the other hand, it is well known (see, for instance, Shorgin [16], Roos [15], Zacharovas and Hwang [22], and the references therein) that the random variable W_n is close to N_λ, whenever both random variables have similar means and variances. In this respect, we consider the following coupling construction. Let $(N_{\lambda_j})_{2\le j\le n}$ be a finite sequence of independent random variables such that N_{λ_j} has the Poisson distribution with mean

$$\lambda_j = -\log(1-1/j), \qquad j = 2, \dots, n. \tag{33}$$

For fixed $m \ge 1$, denote by $N_{\lambda_j}(m)$ the truncated Poisson random variable

$$N_{\lambda_j}(m) = N_{\lambda_j} \mathbf{1}_{\{N_{\lambda_j} \le m-1\}} + m\mathbf{1}_{\{N_{\lambda_j} > m\}},$$

where $\mathbf{1}_{\{\cdot\}}$ is the indicator function. It is clear that $N_{\lambda_j}(m) \le N_{\lambda_j}$ for any $m \ge 1$. Furthermore, (14) and (18) allow us to write

$$W_n = X_1 + \cdots + X_n = 1 + X_2 + \cdots + X_n, \tag{34}$$

where $(X_j)_{2\le j\le n}$ is a finite sequence of independent random variables such that X_j has the Bernoulli distribution with success probability $1/j$, $2 \le j \le n$.

Observe that $X_j \stackrel{d}{=} N_{\lambda_j}(1)$ for $2 \le j \le n$, where $\stackrel{d}{=}$ stands for equality in distribution. Then, $X_j \le N_{\lambda_j}$ for $2 \le j \le n$, and we have from (34)

$$W_n \le 1 + N_{\lambda_2} + \cdots + N_{\lambda_n} \stackrel{d}{=} 1 + N_{\lambda_2+\cdots+\lambda_n}. \tag{35}$$

Lemma 4.2. *Let W_n be as in (34). Then,*

$$\mathbb{E}\binom{W_n}{m} \le \frac{(\log n)^m}{m!} + \frac{(\log n)^{m-1}}{(m-1)!}, \qquad m = 1, ..., n.$$

Proof. By (32) and (35), we have

$$\begin{aligned}\mathbb{E}\binom{W_n}{m} &\le \mathbb{E}\binom{1+N_{\lambda_2+\cdots+\lambda_n}}{m} = \mathbb{E}\binom{N_{\lambda_2+\cdots+\lambda_n}}{m} + \mathbb{E}\binom{N_{\lambda_2+\cdots+\lambda_n}}{m-1} \\ &= \frac{(\lambda_2+\cdots+\lambda_n)^m}{m!} + \frac{(\lambda_2+\cdots+\lambda_n)^{m-1}}{(m-1)!},\end{aligned}$$

which, together with (33), shows the result. □

5. Moment estimates

The estimate given in (10) for relatively large values of m will be derived from the second equality in (26) written in terms of the moments of the random variable S_m.

General moment estimates are established in Lemma 5.1 below. To this end, let Y be a nonnegative random variable with $\mathbb{E}Y = \mu > 0$ and $Var(Y) = \sigma^2 < \infty$. Assume further that

$$\varphi(u) := \mathbb{E}e^{u(Y-\mu)} < \infty, \qquad |u| \leq u_0, \tag{36}$$

for some $u_0 > 0$. Let $(Y_j)_{1\leq j\leq m}$ be a finite sequence of independent copies of Y and denote

$$\widetilde{S}_m = Y_1 + \cdots + Y_m, \qquad m \in \mathbb{N}.$$

It turns out that the kth moment of $\widetilde{S}_m$ has the order of magnitude of that of $(\mu m)^k$, whenever $k = o(\sqrt{m})$, as shown in the following result.

Lemma 5.1. *Let $k, m \in \mathbb{N}$ be such that*

$$k \leq u_0 \mu \sqrt{m}, \tag{37}$$

where u_0 is defined in (36). Then,

$$\mathbb{E}(\widetilde{S}_m)^k \leq (\mu m)^k \left(1 + \left(\frac{k}{u_0 \mu \sqrt{m}}\right)^2 \left(\varphi^m\left(u_0/\sqrt{m}\right) + \varphi^m\left(-u_0/\sqrt{m}\right)\right)\right). \tag{38}$$

Moreover,

$$\begin{aligned} &\left|\mathbb{E}(\widetilde{S}_m)^k - (\mu m)^k \left(1 + \frac{\sigma^2}{2\mu^2} \frac{k(k-1)}{m}\right)\right| \\ &\leq (\mu m)^k \left(\frac{k}{u_0 \mu \sqrt{m}}\right)^3 \left(\varphi^m\left(u_0/\sqrt{m}\right) + \varphi^m\left(-u_0/\sqrt{m}\right)\right). \end{aligned} \tag{39}$$

Proof. If $k = 1$, estimates (38) and (39) are true, since $\mathbb{E}\widetilde{S}_m = \mu m$. For the rest of the proof, assume that $k \geq 2$. Consider the standardized random variable

$$Z_m = \frac{\widetilde{S}_m - \mu m}{\sigma \sqrt{m}}, \tag{40}$$

which obviously satisfies $\mathbb{E}Z_m = 0$. We write

$$\begin{aligned}\mathbb{E}(\widetilde{S}_m)^k &= \mathbb{E}(\sigma\sqrt{m}Z_m + \mu m)^k = \sum_{l=0}^{k}\binom{k}{l}(\mu m)^{k-l}(\sigma\sqrt{m})^l\mathbb{E}Z_m^l \\ &= (\mu m)^k\left(1+\sum_{l=2}^{k}\binom{k}{l}\left(\frac{\sigma}{\mu\sqrt{m}}\right)^l\mathbb{E}Z_m^l\right).\end{aligned} \tag{41}$$

By (37), the modulus of the sum in (41) is bounded above by

$$\begin{aligned}&\sum_{l=2}^{k}\left(\frac{\sigma k}{\mu\sqrt{m}}\right)^l\frac{\mathbb{E}|Z_m|^l}{l!} \le \left(\frac{k}{u_0\mu\sqrt{m}}\right)^2\sum_{l=2}^{\infty}\frac{(u_0\sigma)^l\mathbb{E}|Z_m|^l}{l!} \\ &\le \left(\frac{k}{u_0\mu\sqrt{m}}\right)^2\mathbb{E}e^{u_0\sigma|Z_m|} \le \left(\frac{k}{u_0\mu\sqrt{m}}\right)^2\left(\mathbb{E}e^{u_0\sigma Z_m}+\mathbb{E}e^{-u_0\sigma Z_m}\right).\end{aligned} \tag{42}$$

This, together with (36), (40), and (41), shows inequality (38). The proof of (39) follows the same pattern by noting that the term corresponding to $l = 2$ on the right-hand side in (41) is equal to

$$\frac{k(k-1)}{2\mu^2m^2}Var(\widetilde{S}_m) = \frac{\sigma^2}{2\mu^2}\frac{k(k-1)}{m}.$$

This completes the proof. □

Applying Lemma 5.1 to the case in which $Y = UT$ and, therefore, $\widetilde{S}_m = S_m$, as defined in (21), we obtain the following result.

Lemma 5.2. *Let $k, m \in \mathbb{N}$ with $k \le \sqrt{m}/4$ and let S_m be as in (21). Then,*

$$\mathbb{E}S_m^k \le \left(\frac{m}{2}\right)^k\left(1 + 32e^{1/6}\frac{k^2}{m}\right). \tag{43}$$

Moreover,

$$\left|\mathbb{E}S_m^k - \left(\frac{m}{2}\right)^k\left(1+\frac{5k(k-1)}{6m}\right)\right| \le 2^7e^{1/6}\left(\frac{m}{2}\right)^k\left(\frac{k}{\sqrt{m}}\right)^3. \tag{44}$$

Proof. Suppose that $|u| \le u_0 = 1/2$. By (23) and (36), we have in this case

$$\varphi(u) = \mathbb{E}e^{u(UT-1/2)} = -\frac{e^{-u/2}\log(1-u)}{u}.$$

Observe that

$$-\log(1-u) = u + \frac{u^2}{2} + \frac{2}{3}u^3g(u), \qquad |g(u)| \le 1.$$

This implies that

$$\varphi(u) = e^{-u/2}\left(1 + \frac{u}{2} + \frac{2}{3}u^2 g(u)\right) \le e^{2u^2/3},$$

and therefore

$$\varphi^m\left(|u_0|/\sqrt{m}\right) \le e^{1/6}. \tag{45}$$

Since $\mu = u_0 = 1/2$, inequality (43) follows from (38) and (45). Similarly, inequality (44) follows from (39) and the fact that $\sigma^2 = 5/12$, as seen in (22). This completes the proof. □

6. Proofs of Theorem 2.1 and Theorem 2.2

The upper bound in (8) follows from the first equality in (26) and Lemma 4.2. Inequality (9) follows by choosing $t = \tau$, as defined in (7), in representation (25) and then applying Lemma 4.1. Estimate (10) is an immediate consequence of the second equality in (26) and inequality (43). This completes the proof of Theorem 2.1.

Theorem 2.2 readily follows from the second equality in (26) and (44).

We finally observe that the second equality in (26) and formula (41) with $\widetilde{S}_m$ replaced by S_m give us the exact formula

$$\begin{aligned}(-1)^{n-m}s(n+1,m+1) &= \binom{n+1}{m+1}\mathbb{E}S_{m+1}^{n-m} \\ &= \binom{n+1}{m+1}\left(\frac{m+1}{2}\right)^{n-m}\left(1 + \sum_{l=2}^{n-m}\binom{n-m}{l}\left(\frac{5}{3m}\right)^{l/2}\mathbb{E}Z_m^l\right),\end{aligned} \tag{46}$$

for $0 \le m \le n$, as follows from (22). For $n - m = o(\sqrt{n})$, (46) is nothing else but formula (12) shown by Moser and Wyman [14] written in terms of the moments of the standardized random variable

$$Z_m = \frac{S_m - m/2}{\sqrt{5m/12}}.$$

7. Final considerations

In this section, we briefly illustrate some statements made in the Introduction concerning the Riemann zeta function and a certain subset of the Comtet numbers of the first kind.

In the first place, let $m \in \mathbb{N}_0$ be fixed. A nice formula relating the Riemann zeta function $\zeta(m+2)$ with the Stirling numbers of the first kind is the following

$$\zeta(m+2) = \sum_{n=m}^{\infty}\frac{|s(n+1,m+1)|}{(n+1)(n+1)!}. \tag{47}$$

This formula can be found in Jordan [12, pp. 194–195]. A very short proof of it was given by Blagouchine [4, formula (34)]. As an application of formula (8) in Theorem 2.1 (recall the comments following Theorem 2.2), we give the following explicit estimate.

Proposition 7.1. *Let $m \in \mathbb{N}_0$. For any $k \in \mathbb{N}$ with $m \le \log k$, we have*

$$\zeta(m+2) - \sum_{n=m}^{k} \frac{|s(n+1,m+1)|}{(n+1)(n+1)!} \le \frac{2}{k} \sum_{l=0}^{m} \frac{(m)_l}{m!} (\log k)^{m-l}. \tag{48}$$

Proof. Since $m \le \log k$, we have from (8)

$$\sum_{n=k+1}^{\infty} \frac{|s(n+1,m+1)|}{(n+1)(n+1)!} \le \frac{2}{m!} \sum_{n=k+1}^{\infty} \frac{(\log n)^m}{n^2} \le \frac{2}{m!} \int_k^{\infty} \frac{(\log x)^m}{x^2}\, dx,$$

since the function $h(x) = x^{-2}(\log x)^m$ is decreasing in $[k, \infty)$. Thus, the result follows by applying successively integration by parts in the last integral. □

Note that the order of magnitude of the right-hand side in (48) is that of $(\log k)^m/k$ and, therefore, the series in (47) slowly converges to $\zeta(m+2)$.

In the second place, let $\boldsymbol{\alpha} = (\alpha_j)_{j\ge 0}$ be an arbitrary sequence of real numbers. The Comtet numbers of the first kind associated to $\boldsymbol{\alpha}$, denoted by $s_{\boldsymbol{\alpha}}(n,m)$, are defined by

$$\prod_{j=0}^{n-1} (z - \alpha_j) = \sum_{m=0}^{n} s_{\boldsymbol{\alpha}}(n,m) z^m. \tag{49}$$

These numbers were introduced by Comtet [7] (see also El-Desouky et al. [10]). We will extend estimate (8) to the subset of the Comtet numbers of the first kind satisfying

$$\alpha_j = 0, \quad j = 0, 1, \dots, r-1, \quad 0 < \alpha_r < \alpha_{r+1} \le \alpha_{r+2} \le \cdots, \tag{50}$$

for some fixed $r \in \mathbb{N}$. The r-Stirling numbers of the first kind, introduced by Broder [5], are given by

$$\begin{bmatrix} n \\ m \end{bmatrix}_r = (-1)^{n-m} s_{\boldsymbol{\alpha}}(n,m), \tag{51}$$

when we choose $\alpha_j = j$, $j \ge r$, in (50). Under assumption (50), we can rewrite (49) as

$$\prod_{j=0}^{n-1} (z - \alpha_{r+j}) = \sum_{m=0}^{n} s_{\boldsymbol{\alpha}}(n+r, m+r) z^m. \tag{52}$$

Let $(\widetilde{X}_j)_{0\le j\le n-1}$ be a finite sequence of independent random variables such that $\widetilde{X}_j$ has the Bernoulli distribution with success probability

$$p_j = \frac{\alpha_r}{\alpha_{r+j}}, \quad j = 0, 1, \ldots, n-1, \tag{53}$$

and consider the random variable

$$\widetilde{W}_n = \widetilde{X}_0 + \widetilde{X}_1 + \cdots + \widetilde{X}_{n-1} = 1 + \widetilde{X}_1 + \cdots + \widetilde{X}_{n-1}. \tag{54}$$

With these ingredients, we state the following result.

Theorem 7.2. *Assume* (50). *For any* $m = 0, 1, \ldots, n$, *we have*

$$(-1)^{n-m} s_{\boldsymbol{\alpha}}(n+r, m+r) = \frac{Q_n}{\alpha_r^m} \mathbb{E}\binom{\widetilde{W}_n}{m}, \tag{55}$$

where

$$Q_n = \prod_{j=0}^{n-1} \alpha_{r+j}. \tag{56}$$

As a consequence, we have for $n \geq 2$ *and* $m = 1, \ldots, n-1$,

$$|s_{\boldsymbol{\alpha}}(n+r, m+r)| \leq \frac{Q_n}{\alpha_r^m} \left(\frac{(\log R_n)^m}{m!} + \frac{(\log R_n)^{m-1}}{(m-1)!} \right), \tag{57}$$

where

$$R_n = \prod_{j=1}^{n-1} \frac{\alpha_{r+j}}{\alpha_{r+j} - \alpha_r}. \tag{58}$$

In particular, we have for the r*-Stirling numbers of the first kind*

$$\begin{aligned} &\begin{bmatrix} n+r \\ m+r \end{bmatrix}_r \\ &\leq \frac{r(r+1)\cdots(r+n-1)}{r^m} \left(\frac{\left(\log \binom{n+r-1}{r}\right)^m}{m!} + \frac{\left(\log \binom{n+r-1}{r}\right)^{m-1}}{(m-1)!} \right). \end{aligned} \tag{59}$$

Proof. Starting from (52), we have by (53), (54), and (56)

$$\begin{aligned} &\sum_{m=0}^{n} s_{\boldsymbol{\alpha}}(n+r, m+r)(\alpha_r z)^m = \prod_{j=0}^{n-1} (\alpha_r z - \alpha_{r+j}) \\ &= (-1)^n Q_n \prod_{j=0}^{n-1} (1 - p_j z) = (-1)^n Q_n \prod_{j=0}^{n-1} \mathbb{E}(1-z)^{\widetilde{X}_j} \\ &= (-1)^n Q_n \mathbb{E}(1-z)^{\widetilde{W}_n} = (-1)^n Q_n \sum_{m=0}^{n} \mathbb{E}\binom{\widetilde{W}_n}{m} (-z)^m, \end{aligned}$$

which shows (55).

On the other hand, let $(N_{t_j})_{1\leq j\leq n-1}$ be a finite sequence of independent random variables such that N_{t_j} has the Poisson distribution with mean

$$t_j = -\log(1-p_j), \quad j = 1, \dots, n-1, \tag{60}$$

where p_j is defined in (53). Proceeding as in (32)–(35), we have $\widetilde{X}_j \leq N_{t_j}$, $1 \leq j \leq n-1$, thus implying, as in the proof of Lemma 4.2, that

$$\mathbb{E}\binom{\widetilde{W}_n}{m} \leq \frac{(t_1+\cdots+t_{n-1})^m}{m!} + \frac{(t_1+\cdots+t_{n-1})^{m-1}}{(m-1)!}.$$

This, together with (55), implies (57), since $t_1+\cdots+t_{n-1} = \log R_n$, as follows from (53) and (60). Finally, (59) follows from (57) and (58), by noting that $\alpha_{r+j} = r+j$, $j \in \mathbb{N}_0$. □

For $r = 1$, estimates (8) and (59) coincide. We finally point out that an asymptotic formula for the r-Stirling numbers of the first kind, similar to that in (11), was obtained by Corcino et al. [9] (see also Vega and Corcino [20]).

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