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Caputo Fractional Evolution Equations in Discrete Sequences Spaces

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Abstract: In this paper, we treat some fractional differential equations on the sequence Lebesgue spaces $\ell^p(\mathbb{N}_0)$ with $p \geq 1$. The Caputo fractional calculus extends the usual derivation. The operator, associated to the Cauchy problem, is defined by a convolution with a sequence of compact support and belongs to the Banach algebra $\ell^1(\mathbb{Z})$. We treat in detail some of these compact support sequences. We use techniques from Banach algebras and a Functional Analysis to explicity check the solution of the problem.

Keywords: Caputo fractional derivation; convolution product; Banach algebras

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1. Introduction

The main objective of this paper is to study the following semidiscrete Cauchy differential equation

$$\begin{cases} \partial_t u(n,t) = Bu(n,t) + g(n,t), & n \in \mathbb{N}_0,\ t > 0, \\ u(n,0) = \varphi(n), & n \in \mathbb{N}_0, \end{cases} \tag{1}$$

where B is a convolution operator in the discrete variable, i.e.,

$$Bu(n,t) = \sum_{j \geq 0} b(n-j)u(j,t), \tag{2}$$

and the sequence b belongs to the Banach algebra $\ell^1(\mathbb{Z})$. A first example is the one-dimensional discrete Laplacian, Δ_d, which is defined by $b = \delta_{-1} - 2\delta_0 + \delta_1$, where δ_i denotes the discrete Dirac measure given by the Kronecker delta, i.e., $\delta_i(n) = 1$ if $i = n$ and 0 in other case. Equation (1) is usually called the lattice diffusion equation or the semidiscrete heat equation.

These classes of equations have received a wide interest in the mathematical literature in the last years. They appear in diverse areas of knowledge. For example, in probability theory, the function $u(n,t)$ of (1) with $B = \Delta_d$, expresses the probability that a continuous-time symmetric random walk arrives at point n at time t; ([1], [Section 4]). In chemical physics, (1) describes the flow of a liquid in an infinite row of tanks where two neighbors are always connected [2], [Section 3]. Another amazing application takes place in transport theory. Equation (1) expresses the dynamics of an infinite chain of cars, each of them being coupled to its two neighbors. The function $u(n;t)$ is the displacement of car n at time t from its equilibrium point ([3], [Example 1]). Quite recently, Slavik [4] studied the asymptotic behavior of solutions of (1) when $B = \Delta_d$, showing that a bounded solution approaches the average of the initial values if the average exists. In the case that $b = \delta_{-1} - \delta_0$ in (2), we obtain the forward difference operator $B = \Delta$ and then the Equation (2) describes the semidiscrete transport system, treated recently by Abadias et al. in [5].

Other interesting references, such as [6,7], present fundamental solutions for (1) and the second-order semidiscrete equation

$$\begin{cases} \partial_{tt}u(n,t) = Bu(n,t) + g(n,t), & n \in \mathbb{N}_0,\ t > 0, \\ u(n,0) = \varphi(n),\ \ u_t(n,0) = \phi(n), & n \in \mathbb{N}_0, \end{cases} \tag{3}$$

when $B = -(-\Delta_d)^\alpha$ is the fractional power of discrete fractional Laplacian. In the particular case of [7], the authors apply operator theory techniques and some of the properties of the Bessel functions to obtain a theory of uni-parametric operators (C_0-semigroups and cosine operators) generated by Δ_d and $-(-\Delta_d)^\alpha$ on the Lebesgue space $\ell^p(\mathbb{Z})$. Moreover, note that the fractional forward difference operator $B = -(-\Delta)^\alpha$ has been treated in [5] where the maximum and comparison principles in the context of Fourier Analysis are shown.

However, there is no attempt (to the best of our knowledge) to present explicitly fundamental solutions of the general Equation (1) on the sequence space $\ell^p(\mathbb{N}_0)$ for $p \geq 1$ instead of on the sequence space $\ell^p(\mathbb{Z})$ for $p \geq 1$ ([8]).

The main technique in this paper is that we apply our knowledge from Banach algebras and Functional Analysis to fractional differential systems. This useful approach that we follow in this paper, allows us to obtain a completely new point of view. We prove results by introducing this new method and describing both the qualitative and quantitative behavior of the fundamental solutions of (1) in a unified way.

More generally, and to present simultaneously our studies of the subdiffusive and superdiffusive cases connected to Equations (1) and (3), in this article, we deal with the representation of the fundamental solutions for the following semidiscrete system:

$$\begin{cases} \mathbb{D}_t^\beta u(n,t) = Bu(n,t) + g(n,t), & n \in \mathbb{N}_0,\ t > 0, \\ u(n,0) = \varphi(n), & n \in \mathbb{N}_0, \end{cases} \tag{4}$$

in case $0 < \beta \leq 1$ and

$$\begin{cases} \mathbb{D}_t^\beta u(n,t) = Bu(n,t) + g(n,t), & n \in \mathbb{N}_0,\ t > 0, \\ u(n,0) = \varphi(n),\ \ u_t(n,0) = \phi(n), & n \in \mathbb{N}_0, \end{cases} \tag{5}$$

in case $1 < \beta \leq 2$. In both cases, B is the convolution operator $Bf(n) := (b * f)(n)$ defined for $f \in \ell^p(\mathbb{N}_0)$, $p \in [1,\infty]$, $b \in \ell^1(\mathbb{Z})$ and $\beta \in (0,2]$ is a real number. The symbol $\mathbb{D}_t^\beta$ denotes the Caputo fractional derivative of order $\beta > 0$.

The paper is organized as follows. In the first section, we introduce the main results about the Banach algebras and, in particular, about the spaces $\ell^p(\mathbb{N}_0)$ and $\ell^1(\mathbb{Z})$. In the second section, we consider some particular finite difference operators in $\ell^p(\mathbb{N}_0)$, mainly

$$\begin{aligned} \mathcal{D}f(n) &= f(n+1) - f(n-1) = ((\delta_{-1} - \delta_1) * f)(n), \\ \Delta_2 f(n) &= f(n) - 2f(n+1) + f(n+2) = ((\delta_0 - 2\delta_{-1} + \delta_{-2}) * f)(n), \\ \nabla_2 f(n) &= f(n) - 2f(n-1) + f(n-2) = ((\delta_0 - 2\delta_1 + \delta_2) * f)(n), \end{aligned}$$

for $f \in \ell^p(\mathbb{N}_0)$. Finally, we present Theorem 6 where we include the representation of the fundamental solutions for semidiscrete Caputo fractional differential equations.

This paper contains part of the results included in the Master Thesis of the first author, entitled "Semigrupos y operadores coseno generados por operadores de diferencias finitas en espacios de sucesiones de Lebesgue", Universidad de La Rioja, (2021).

Notation The usual set numbers $\mathbb{N}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ are used. We write as $\mathbb{T} = \{e^{i\theta} : \theta \in [-\pi,\pi)\}$ the unit circumference (or also called torus) and $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. The Dirac measures δ_0 and δ_n are $\delta_n(j) = 0$ if $n \neq j$ and $\delta_n(n) = 1$ for $n, j \in \mathbb{Z}$. We denote by χ_I the indicator function on set I (i.e., $\chi_I(n) = 1$ if $n \in I$ and $\chi_I(n) = 0$ if $n \notin I$).

Furthermore, I_n and J_n are the Bessel functions and H_n the usual Hermite polynomials; Γ is the Gamma function, Φ_β the Wright function and $E_{\alpha,\beta}$ the Mittag-Leffler function.

Given X Banach space, X' is the dual of Banach space and $\mathcal{B}(X)$ the set of linear bounded operators on X; given $A \in \mathcal{B}(X)$, we write $A' \in \mathcal{B}(X')$ is the adjoint of the operator A.

2. A Banach Algebra Framework

Given $1 \leq p \leq \infty$, we recall that the Banach spaces $(\ell^p(\mathbb{N}_0), \| \quad \|_p)$ are formed by infinite sequences $f = (f(n))_{n\in\mathbb{N}_0} \subset \mathbb{C}$ such that

$$\begin{aligned} ||f||_p : &= \left(\sum_{n=0}^{\infty} |f(n)|^p\right)^{\frac{1}{p}} < \infty, \qquad 1 \leq p < \infty; \\ ||f||_\infty : &= \sup_{n\geq 0} |f(n)| < \infty. \end{aligned}$$

We remind that the natural embeddings $\ell^1(\mathbb{N}_0) \hookrightarrow \ell^p(\mathbb{N}_0) \hookrightarrow \ell^\infty(\mathbb{N}_0)$, for $1 \leq p \leq \infty$, and that the dual of $\ell^p(\mathbb{N}_0)$ is identified with $\ell^{p'}(\mathbb{N}_0)$ where $\frac{1}{p} + \frac{1}{p'} = 1$ for $1 < p < \infty$ and $p = 1$ if $p' = \infty$.

In the case that $f \in \ell^1(\mathbb{N}_0)$ and $g \in \ell^p(\mathbb{N}_0)$, we define

$$(f * g)(n) := \sum_{j=0}^{n} f(n-j)g(j), \qquad n \in \mathbb{Z}.$$

From Young's Inequality, it follows that $f * g \in \ell^p(\mathbb{N}_0)$, and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$. We denote by $T_g \in \mathcal{B}(\ell^p(\mathbb{N}_0))$ defined by $T_g(f) = f * g$ for $g \in \ell^p(\mathbb{N}_0)$. The element g is called the symbol of the convolution operator T_g.

Note that $(\ell^1(\mathbb{N}_0), *)$ is a commutative Banach algebra with identity that we denote by $\delta_0 := \chi_{\{0\}}$. We observe that $\delta_1 * \delta_1 = \delta_2$ and, in general, $\delta_n * \delta_m = \delta_{n+m}$ for $n, m \in \mathbb{N}_0$. As usual, we write $f^2 = f * f$ and $f^n = f^{n-1} * f$ for $n \geq 2$.

The Gelfand transform associated to $(\ell^1(\mathbb{N}_0), *)$, is the $\mathcal{Z}$-transform, $\mathcal{Z} : \ell^1(\mathbb{N}_0) \to C(\mathbb{D})$, (or Taylor series), where

$$\mathcal{Z}(f)(z) := \sum_{n\geq 0} f(n)z^n, \quad z \in \mathbb{D}.$$

We recall that the resolvent set of f, denoted as

$$\rho_{\ell^1(\mathbb{N}_0)}(f) := \{\lambda \in \mathbb{C} \ : \ (\lambda\delta_0 - f)^{-1} \in \ell^1(\mathbb{N}_0)\}$$

and the spectrum of f, $\sigma_{\ell^1(\mathbb{N}_0)}(f) = \mathbb{C}\backslash\rho_{\ell^1(\mathbb{N}_0)}(f)$.

In what follows, we apply the general theory of Functional Analysis and commutative Banach Algebra as framework. In the following theorem, we collect some results that will be of our interest, see [9].

Theorem 1. *The following properties hold:*

(i) *The spectrum* $Spec(\ell^1(\mathbb{N}_0))$ *is compact and, consequently, homeomorphic to the unit complex circle,* $\mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$.

(ii) $\sigma_{\ell^1(\mathbb{N}_0)}(f) \subset \{z \in \mathbb{C} \ ; \ |z| < \|f\|_1\}$ *and*

$$(\lambda\delta_0 - f)^{-1} = \sum_{n\geq 0} \lambda^{-n-1} f^n, \qquad \|f\|_1 < |\lambda|. \tag{6}$$

(iii) *The algebra* $\ell^1(\mathbb{N}_0)$ *is a semi-simple regular Banach algebra and the* $\mathcal{Z}$*-transform is injective.*

(iv) $\mathcal{Z}(f * g) = \mathcal{Z}(f)\mathcal{Z}(g)$ *and*

$$\sigma_{\ell^1(\mathbb{N}_0)}(f) = \mathcal{Z}(f)(\mathbb{D}), \qquad f \in \ell^1(\mathbb{N}_0). \tag{7}$$

(v) *Given* $b \in \ell^1(\mathbb{N}_0)$ *and the linear convolution operator* $T_b(f) := b * f$ *for* $f \in \ell^p(\mathbb{N}_0)$ *for* $p \geq 1$*. Then,*

$$\sigma_{\mathcal{B}(\ell^p(\mathbb{N}_0))}(T_b) \subset \sigma_{\ell^1(\mathbb{N}_0)}(f) = \mathcal{Z}(f)(\mathbb{D}).$$

We recall that the Banach algebra $(\ell^1(\mathbb{Z}), \| \quad \|_1)$ is formed by bi-infinite sequences $f = (f(n))_{n\in\mathbb{Z}} \subset \mathbb{C}$ such that

$$\|f\|_1 := \sum_{n=-\infty}^{\infty} |f(n)| < \infty.$$

Given $f, g \in \ell^1(\mathbb{Z})$, the product in the algebra $\ell^1(\mathbb{Z})$ is the usual convolution product given by

$$(f * g)(n) := \sum_{j=-\infty}^{\infty} f(n-j)g(j), \qquad n \in \mathbb{Z}.$$

Note that $(\ell^1(\mathbb{Z}), *)$ is also a commutative Banach algebra with identity and $\delta_n * \delta_m = \delta_{n+m}$ for $n, m \in \mathbb{Z}$. The Gelfand transform associated to $(\ell^1(\mathbb{Z}), *)$, is the Fourier series (or discrete Fourier transform), $\mathcal{F} : \ell^1(\mathbb{Z}) \to C(\mathbb{T})$, where

$$\hat{f}(\theta) := \mathcal{F}(f)(e^{i\theta}) := \sum_{n\in\mathbb{Z}} f(n)e^{in\theta}, \quad \theta \in \mathbb{T}.$$

The spectrum $\mathrm{Spec}(\ell^1(\mathbb{Z}))$ is homeomorphic to the unit torus, $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and $\sigma_{\ell^1(\mathbb{Z})}(b) = \mathcal{F}(b)(\mathbb{T})$ ([8], (Theorem 2.1)).

Definition 1. *Given $\alpha, \beta > 0$, we define the vector-valued Mittag-Leffler function, $E_{\alpha,\beta} : \ell^1(\mathbb{Z}) \to \ell^1(\mathbb{Z})$, by*

$$E_{\alpha,\beta}(a) := \sum_{j=0}^{\infty} \frac{a^j}{\Gamma(\alpha j + \beta)}, \qquad a \in \ell^1(\mathbb{Z}).$$

Note that

$$E_{1,1}(a) = \sum_{j=0}^{\infty} \frac{a^j}{j!} = e^a; \qquad E_{2,1}(a) = \sum_{j=0}^{\infty} \frac{a^j}{(2j)!}.$$

The set $\exp(\ell^1(\mathbb{Z})) := \{e^a \; ; a \in \ell^1(\mathbb{Z})\}$ is the connected component of δ_0 in the set of regular elements in $\ell^1(\mathbb{Z})$ ([9], Theorem 6.4.1).

Now, we remind the usual terminology in semigroup theory: an element $a \in \ell^1(\mathbb{Z})$ is called the generator of the entire group given by the exponential function $(e^{za})_{z\in\mathbb{C}}$. The cosine function is expressed by its generator in terms of Mittag-Leffer function $Cos(z, a) := E_{2,1}(z^2 a)$, see [10], (Sections 3.1 and 3.14). Moreover, the Laplace transform of an entire group or a cosine function is connected with the resolvent of its generator as follows:

$$\begin{aligned} (\lambda - a)^{-1} &= \int_0^\infty e^{-\lambda s} e^{as} ds, \qquad \lambda > \|a\|_1, \\ \lambda(\lambda^2 - a)^{-1} &= \int_0^\infty e^{-\lambda s} Cos(s, a) ds, \qquad \lambda > \sqrt{\|a\|_1}, \end{aligned} \tag{8}$$

see, for example, ([10], p. 213).

Example 1. For $\alpha, \beta > 0$, we have that

$$E_{\alpha,\beta}(z\delta_0) = E_{\alpha,\beta}(z)\delta_0; \qquad E_{\alpha,\beta}(z\delta_1) = \sum_{j=0}^{\infty} \frac{z^j \delta_j}{\Gamma(\alpha j + \beta)}.$$

In particular, $e^{z\delta_1} = \sum_{j=0}^{\infty} \frac{z^j \delta_j}{j!}$ and $Cos(z, \delta_1) = \sum_{j=0}^{\infty} \frac{z^{2j}\delta_j}{(2j)!}$ are generated by δ_1.

In the next proposition, we present some technical properties of these Mittag-Leffler functions in the Banach algebra $\ell^1(\mathbb{Z})$. As usual, we consider vector-valued integration (in the sense of Bochner) in the Banach space $\ell^1(\mathbb{Z})$, see, for example, ([11], Section 1.2).

Proposition 1. *For $\alpha, \beta > 0$ and $a \in \ell^1(\mathbb{Z})$, we have that*

(i) $\|E_{\alpha,\beta}(a)\|_1 \le E_{\alpha,\beta}(\|a\|_1)$.
(ii) $\mathcal{F}(E_{\alpha,\beta}(a)) = E_{\alpha,\beta}(\mathcal{F}(a))$; *in particular,* $\mathcal{F}(e^{az}) = e^{z\mathcal{F}(a)}$ *and* $\mathcal{F}(Cos(z,a)) = Cos(\mathcal{F}(z), a)$ *for* $z \in \mathbb{C}$.
(iii) $\sigma_{\ell^1(\mathbb{Z})}(E_{\alpha,\beta}(a)) = E_{\alpha,\beta}(\sigma_{\ell^1(\mathbb{Z})}(a))$.
(iv) *The following Laplace transform formula holds*

$$\int_0^\infty e^{-\lambda t} t^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(t^\alpha a)dt = k!\lambda^{\alpha-\beta}\Big((\lambda^\alpha - a)^{-1}\Big)^{(k+1)}, \quad \Re(\lambda) > \|a\|_1^{1/\alpha}, \tag{9}$$

for $k \in \mathbb{N} \cup \{0\}$.

Theorem 2. *Given $b \in \ell^1(\mathbb{Z})$ and $p \ge 1$, we define the operator $T_b : \ell^p(\mathbb{N}_0) \to \ell^p(\mathbb{N}_0)$ by*

$$T_b(f)(n) := b * f(n) = \sum_{j\ge 0} b(n-j)f(j), \qquad n \in \mathbb{N}_0.$$

(i) $T_b \in \mathcal{B}(\ell^p(\mathbb{N}_0))$ *and* $\|b\|_p \le \|T_b\|_{\mathcal{B}(\ell^p(\mathbb{N}_0))} \le \|b\|_1$.
(ii) $\sigma_{\mathcal{B}(\ell^p(\mathbb{N}_0))}(T_b) \subset \sigma_{\ell^1(\mathbb{Z})}(b) = \mathcal{F}(b)(\mathbb{T})$.

Proof. (i) It is clear that $T_b \in \mathcal{B}(\ell^p(\mathbb{N}_0))$ and $\|T_b\| \le \|b\|_1$. Now, take $f = \delta_n$, and $T_b(\delta_n)(j) = b(j-n)$ for $n, j \ge 0$. Then,

$$\|T_b(\delta_n)\|_p^p = \sum_{j=0}^{\infty} |b(j-n)|^p = \sum_{l=-n}^{\infty} |b(l)|^p$$

for $n \ge 0$. We conclude that $\|b\|_p \le \|T_b\|_{\mathcal{B}(\ell^p(\mathbb{N}_0))}$.
(ii) Now, take $\lambda \in \rho_{\ell^1(\mathbb{Z})}(b)$. Then, $(\lambda - b)^{-1} \in \ell^1(\mathbb{Z}) \subset \mathcal{B}(\ell^p(\mathbb{N}_0))$ and $\lambda \in \rho_{\mathcal{B}(\ell^p(\mathbb{N}_0))}(T_b)$. We conclude that $\sigma_{\mathcal{B}(\ell^p(\mathbb{N}_0))}(T_b) \subset \sigma_{\ell^1(\mathbb{Z})}(b)$. □

3. Some Finite Difference Operators in $\ell^1(\mathbb{Z})$

Sequences of compact support, i.e., elements in the set

$$c_c(\mathbb{Z}) := \{a \in \ell^1(\mathbb{Z}) : \exists\, m \in \mathbb{Z}_+ : a(n) = 0, \forall\, |n| > m)\}$$

are an important case of finite difference operators. In such a case, the discrete Fourier Transform of $a \in c_c(\mathbb{Z})$ is a trigonometric polynomial

$$\mathcal{F}(a)(e^{i\theta}) = \sum_{j=-m}^{m} a(j)e^{ij\theta}. \tag{10}$$

It is interesting to observe that if $\sum_{j=-m}^{m} a(j) = 0$, then $0 \in \sigma_{\ell^1(\mathbb{Z})}(a)$. This follows immediately from $\sigma_{\ell^1(\mathbb{Z})}(b) = \mathcal{F}(b)(\mathbb{T})$.

Definition 2. *For $f \in \ell^p(\mathbb{N}_0)$, with $1 \le p \le \infty$, we define the following operators*

1. $-\Delta f(n) = f(n) - f(n+1) = ((\delta_0 - \delta_{-1}) * f)(n)$.

2. $\nabla f(n) = f(n) - f(n-1) = ((\delta_0 - \delta_1) * f)(n)$.
3. $\mathcal{D}f(n) = f(n+1) - f(n-1) = ((\delta_{-1} - \delta_1) * f)(n)$.
4. $\Delta_d f(n) = f(n+1) - 2f(n) + f(n-1) = ((\delta_{-1} - 2\delta_0 + \delta_1) * f)(n)$.
5. $\Delta_2 f(n) = f(n) - 2f(n+1) + f(n+2) = ((\delta_0 - 2\delta_{-1} + \delta_{-2}) * f)(n)$.
6. $\nabla_2 f(n) = f(n) - 2f(n-1) + f(n-2) = ((\delta_0 - 2\delta_1 + \delta_2) * f)(n)$.
7. $\Delta_{dd} f(n) = f(n+2) - 2f(n) + f(n-2) = ((\delta_{-2} - 2\delta_0 + \delta_2) * f)(n)$.

for $n \in \mathbb{N}_0$ *and* $f(n) = 0$ *for* $n < 0$.

The above operators are often used in the context of a numerical analysis. The operators $-\Delta$ and ∇ are connected with the Euler scheme of approximation. The discrete Laplacian Δ_d is the second-order central difference approximation for the second-order derivative. The double Laplacian, the operator Δ_{dd}, is introduced in Bateman's seminar paper ([12], Page 506), to treat the equations of Born and Karman on crystal lattices in vibration. Other operators Δ_2, $\mathcal{D}$ and ∇_2 are also considered in [12].

To consider the action of these operators in $\mathcal{B}(\ell^p(\mathbb{N}_0))$, we study these operators as elements in the Banach algebra $\ell^1(\mathbb{Z})$ as Theorem 2 shows. Operators $-\Delta$, ∇, Δ_d and Δ_{dd} have been studied in detail in ([8], Theorem 3.2, 3.3, 3.4 and 3.5). In the following subsections, we treat $\mathcal{D}$, Δ_2 and ∇_2.

In Table 1, we collect some basic results of the finite difference operators given in Definition 2. In Figure 1, we also plot the spectrum $\sigma_{\ell^1(\mathbb{Z})}(a)$ for these finite difference operators.

Table 1. Finite difference operators in $\ell^1(\mathbb{Z})$.

A	$A(f) = a * f$	a	$\|a\|_1$	$\mathcal{F}(a)(z)$
$-\Delta$	$f(n) - f(n+1)$	$\delta_0 - \delta_{-1}$	2	$1 - z$
∇	$f(n) - f(n-1)$	$\delta_0 - \delta_1$	2	$1 - \frac{1}{z}$
$\mathcal{D}$	$f(n+1) - f(n-1)$	$\delta_{-1} - \delta_1$	2	$z - \frac{1}{z}$
Δ_d	$f(n+1) - 2f(n) + f(n-1)$	$\delta_{-1} - 2\delta_0 + \delta_1$	4	$z + \frac{1}{z} - 2$
Δ_2	$f(n+2) - 2f(n+1) + f(n)$	$\delta_0 - 2\delta_{-1} + \delta_{-2}$	4	$(z-1)^2$
∇_2	$f(n) - 2f(n-1) + f(n-2)$	$\delta_0 - 2\delta_1 + \delta_2$	4	$(\frac{1}{z} - 1)^2$
Δ_{dd}	$f(n+2) - 2f(n) + f(n-2)$	$\delta_{-2} - 2\delta_0 + \delta_2$	4	$(z - \frac{1}{z})^2$

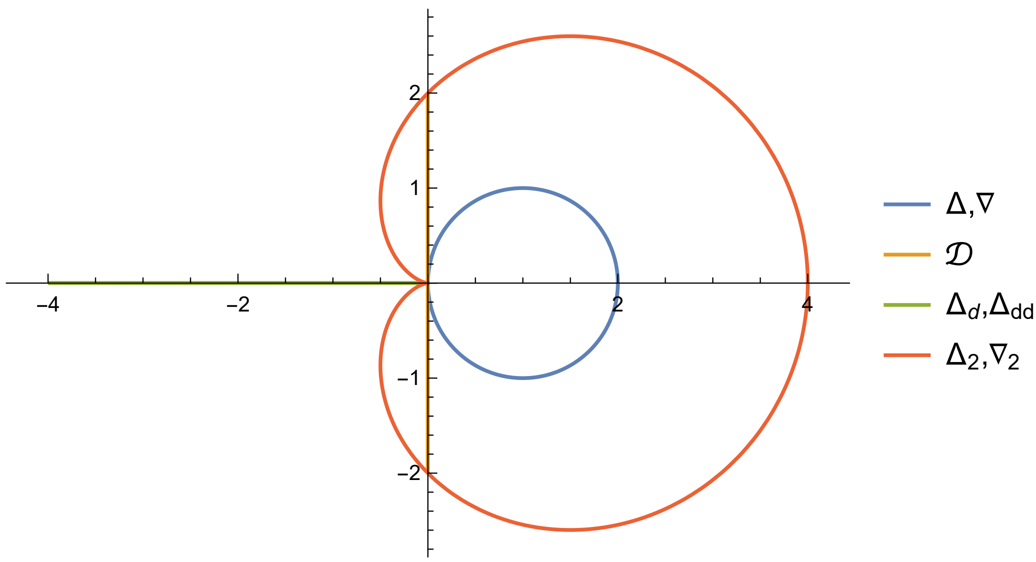

Figure 1. Spectrum of finite difference operators in $\ell^1(\mathbb{Z})$.

3.1. The Operator $\mathcal{D}$

This operator is a finite difference operator of order 1 defined by $\mathcal{D}f(n) = f(n+1) - f(n-1)$. We present some of these properties in the next theorem.

Theorem 3. *The operator $\mathcal{D}f = a * f$ with $a = \delta_{-1} - \delta_1$, has the following properties.*

(i) *The norm of a is equal to 2, $\|a\|_1 = 2$.*
(ii) *The discrete Fourier transform of a is given by $\mathcal{F}(a)(z) = z - \frac{1}{z}$ with $z \in \mathbb{T}$.*
(iii) *The spectrum of a is $\sigma_{\ell^1(\mathbb{Z})}(\mathcal{D}) = [-2i, 2i]$.*
(iv) *The group generated by $-a$ is $e^{-za}(n) = J_n(2z)$, $z \in \mathbb{C}$, $n \in \mathbb{Z}$ and $\|e^{-ta}\|_1 \le e^{2t}$, with $t \in \mathbb{R}$ and $t > 0$.*
(v) *For $\lambda \in \mathbb{C} \setminus [-2i, 2i]$,*

$$(\lambda\delta_0 + a)^{-1} = 2^{-n}\frac{(\sqrt{\lambda^2+4} - \lambda)^n}{\sqrt{\lambda^2+4}}, \qquad n \in \mathbb{Z}.$$

Proof. Items (i) and (ii) are straightforward. To show (iii), we have that

$$\sigma_{\ell^1(\mathbb{Z})}(\mathcal{D}) = \mathcal{F}(a)(\mathbb{T}) = \{z \in \mathbb{C} : z = w - \frac{1}{w}, |w| = 1\} = \{z \in \mathbb{C} : z = 2i\Im(w), |w| = 1\} = [-2i, 2i].$$

We define for a while $e_z \in \ell^1(\mathbb{Z})$ with $z \in \mathbb{C}$ by $e_z(n) = J_n(2z)$. Note that

$$\|e_z\|_1 = \sum_{n\in\mathbb{Z}} |J_n(2z)| \le \sum_{n\in\mathbb{Z}} I_n(2|z|) = e^{2|z|}.$$

where we have applied the generating function of modified Bessel function of the first kind I_n with $n \in \mathbb{Z}$, see, for example, [13], [Appendix].

We apply the discrete Fourier transform to obtain

$$\mathcal{F}(e_z)(w) = \sum_{n\in\mathbb{Z}} J_n(2z)w^{-n} = e^{z(\frac{1}{w} - w)} = e^{-z\mathcal{F}(a)(w)} = \mathcal{F}(e^{-za})(w),$$

with $w \in \mathbb{T}$. As the discrete Fourier transform is injective, we conclude that $e_z(n) = e^{-za}(n)$ and the item (iv) is proved.

Finally, to show item (v), we have that

$$(\lambda\delta_0 + a)^{-1}(n) = \int_0^\infty e^{-\lambda t}e^{-ta}(n)\,dt = \int_0^\infty e^{-\lambda t}J_n(2t)\,dt = 2^{-n}\frac{(\sqrt{\lambda^2+4} - \lambda)^n}{\sqrt{\lambda^2+4}},$$

where we have applied ([14], Formula 6.623) for $\Re(\lambda) > 0$ and $n \in \mathbb{Z}$. By analytic prolongation, we extend the equality for $\lambda \in \mathbb{C} \setminus [-2i, 2i]$. □

3.2. The Operator Δ_2

The operator $\Delta_2 f(n) = f(n) - 2f(n+1) + f(n+2)$ is a finite difference operator of order 2. Note that $\Delta_2 = (-\Delta)^2$. In the next theorem, we present some properties of Δ_2.

Theorem 4. *The operator $\Delta_2 f = a * f$ with $a = \delta_0 - 2\delta_{-1} + \delta_{-2}$ verifies the following properties.*

(i) *The norm of a is equal to 4, $\|a\|_1 = 4$.*
(ii) *The discrete Fourier transform $\mathcal{F}(a)(z) = (z-1)^2$ with $z \in \mathbb{T}$.*
(iii) *The spectrum of a in $\ell^1(\mathbb{Z})$ is equal to $\sigma_{\ell^1(\mathbb{Z})}(\Delta_2) = \{z \in \mathbb{C} : z = (w-1)^2, w \in \mathbb{T}\}$.*

(iv) *The group generated by a is*

$$e^{za}(n)=\frac{(i\sqrt{z})^{-n}H_{-n}(i\sqrt{z})}{(-n)!}e^{z}\chi_{-\mathbb{N}_0}(n),\qquad z\in\mathbb{C},$$

$n\in\mathbb{Z}$ and $\|e^{ta}\|_1\le e^{4t}$ for $t>0$.

(v) *The cosine function $Cos(z,a)=\frac{z^{-n}}{(-n)!}\frac{e^{-z}+(-1)^n e^{z}}{2}\chi_{-\mathbb{N}_0}(n)$.*

Proof. Items (i), (ii) and (iii) are similar to items (i), (ii) and (iii) in Theorem 3. Now, we define $e_z\in\ell^1(\mathbb{Z})$ con $z\in\mathbb{C}$ by $e_z(n)\frac{(i\sqrt{z})^{-n}H_{-n}(i\sqrt{z})}{(-n)!}e^{z}\chi_{-\mathbb{N}_0}(n)$ where H_n is the Hermite polynomial. First, we check that $e_z\in\ell^1(\mathbb{Z})$,

$$\begin{aligned}||e_z||_1&=e^{\Re(z)}\sum_{j=0}^{\infty}|\frac{(i\sqrt{z})^jH_j(i\sqrt{z})}{j!}|\le e^{|z|}\sum_{j=0}^{\infty}\frac{|\sqrt{z}|^j}{j!}|H_j(i\sqrt{z})|=e^{|z|}\sum_{j=0}^{\infty}\frac{|\sqrt{z}|^j}{j!}|\frac{2^j}{\sqrt{\pi}|\int_{-\infty}^{\infty}i^n(\sqrt{z}+t)^ne^{-t^2}\,dt}\\&\le e^{|z|}\sum_{j=0}^{\infty}\frac{|\sqrt{z}|^j}{j!}\frac{2^j}{\sqrt{\pi}}\int_{-\infty}^{\infty}(|\sqrt{z}|+|t|)^je^{-t^2}\,dt=\frac{2e^{|z|}}{\sqrt{\pi}}\sum_{j=0}^{\infty}\frac{(2|\sqrt{z}|)^j}{j!}\int_0^{\infty}(|\sqrt{z}|+t)^je^{-t^2}\,dt\\&=\frac{2e^{|z|}}{\sqrt{\pi}}\int_0^{\infty}e^{-t^2}\sum_{j=0}^{\infty}\frac{(2|\sqrt{z}|(|\sqrt{z}|+t))^j}{j!}\,dt=\frac{2e^{|z|}}{\sqrt{\pi}}\int_0^{\infty}e^{-t^2}e^{2|\sqrt{z}|(|\sqrt{z}|+t)}\,dt=\frac{2e^{4|z|}}{\sqrt{\pi}}\int_0^{\infty}e^{-(t-|\sqrt{z}|)^2}\,dt\\&=\frac{2e^{4|z|}}{\sqrt{\pi}}\int_{-|\sqrt{z}|}^{\infty}e^{-u^2}\,du<\frac{2e^{4|z|}}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-u^2}\,du=2e^{4|z|}<\infty.\end{aligned}$$

We calculate the discrete Fourier transform of e_z

$$\begin{aligned}\mathcal{F}(e_z)(w)&=\sum_{n\in\mathbb{Z}}\frac{(i\sqrt{z})^{-n}H_{-n}(i\sqrt{z})}{(-n)!}e^{z}\chi_{-\mathbb{N}_0}(n)w^{-n}=e^{z}\sum_{j=0}^{\infty}\frac{H_j(i\sqrt{z})}{j!}(wi\sqrt{z})^j\\&=e^{z}e^{-(wi\sqrt{z})^2+2(wi\sqrt{z})(i\sqrt{z})}=e^{z(w-1)^2}=e^{z\mathcal{F}(a)(w)}=\mathcal{F}(e^{za})(w),\end{aligned}$$

where we have applied ([14], Formula 8.951), for $w\in\mathbb{T}$. As the discrete Fourier transform is injective, we conclude that $e_z(n)=e^{za}(n)$ for $n\in\mathbb{Z}$.

To calculate the cosine function generated by a, we apply ([10], Example 3.14.15)

$$\begin{aligned}Cos(z,a)&=\frac{1}{2}(e^{-z(\delta_0-\delta_{-1})}+e^{z(\delta_0-\delta_{-1})})=\frac{1}{2}\left(e^{-z}\frac{z^{-n}}{(-n)!}\chi_{-\mathbb{N}_0(n)}+e^{z}\frac{(-z)^{-n}}{(-n)!}\chi_{-\mathbb{N}_0(n)}\right)\\&=\frac{z^{-n}}{(-n)!}\left(\frac{e^{-z}+(-1)^ne^{z}}{2}\right)\chi_{-\mathbb{N}_0}(n),\end{aligned}$$

and we conclude the proof. □

3.3. The Operator ∇_2

In this subsection, we treat the finite difference operator of order 2, defined by $\nabla_2 f(n)=f(n)-2f(n-1)+f(n-2)$. Note that e $\nabla_2=\nabla^2$. We present some basic properties of operator ∇_2.

Theorem 5. *The operator $\nabla_2 f=a*f$ with $a=\delta_0-2\delta_1+\delta_2$ has the following properties.*

(i) *The norm of a in $\ell^1(\mathbb{Z})$ is equal to 4, $\|a\|_1=4$.*

(ii) *The discrete Fourier transform $\mathcal{F}(a)(z)=(\frac{1}{z}-1)^2$ for $z\in\mathbb{T}$.*

(iii) *The spectrum of a in $\ell^1(\mathbb{Z})$ is equal to $\sigma_{\ell^1(\mathbb{Z})}(\nabla_2)=\{z\in\mathbb{C}:z=(\frac{1}{w}-1)^2,w\in\mathbb{T}\}$.*

(iv) *The group generated by a is given by*

$$e^{za}(n)=\frac{(i\sqrt{z})^{n}H_{n}(i\sqrt{z})}{(n)!}e^{z}\chi_{\mathbb{N}_0}(n),\qquad z\in\mathbb{C},$$

$n \in \mathbb{Z}$ and $\|e^{ta}\|_1 \le e^{4t}$, for $t > 0$.

(v) *The cosine function $Cos(z,a)(n) = \frac{z^n}{n!}\frac{e^{-z}+(-1)^n e^z}{2}\chi_{\mathbb{N}_0}(n)$.*

Proof. We skip the proof of items (i), (ii) and (iii). To show (iv), we define $e_z(n) = \frac{(i\sqrt{z})^n H_n(i\sqrt{z})}{n!} e^z \chi_{\mathbb{N}_0}(n)$ for $z \in \mathbb{C}$. As in Theorem 4, $e_z \in \ell^1(\mathbb{Z})$ and

$$\mathcal{F}(e_z)(w) = \sum_{n\in\mathbb{Z}} \frac{(i\sqrt{z})^n H_n(i\sqrt{z})}{n!} e^z \chi_{\mathbb{N}_0}(n) w^{-n} = e^z \sum_{n=0}^{\infty} \frac{H_n(i\sqrt{z})}{n!}\left(\frac{i\sqrt{z}}{w}\right)^n$$
$$= e^z e^{-(\frac{i\sqrt{z}}{w})^2 + 2(\frac{i\sqrt{z}}{w})(i\sqrt{z})} = e^{z(\frac{1}{w}-1)^2} = e^{z\mathcal{F}(a)(w)} = \mathcal{F}(e^{za})(w),$$

for $w \in \mathbb{T}$. We conclude that $e_z(n) = e^{za}(n)$ for $z \in \mathbb{C}$. As $\nabla_2 = (\nabla)^2$, we apply again ([10], Example 3.14.15), to obtain the cosine function generated by a. □

Remark 1. The connection between semigroups and cosine functions is given by the so-called Weierstrass formula,

$$e^{at} = \frac{1}{\sqrt{\pi t}} \int_0^{\infty} e^{-\frac{s^2}{4t}} Cos(s) ds, \qquad t > 0, \tag{11}$$

for $a \in \ell^1(\mathbb{Z})$, see, for example, ([10], Theorem 3.14.17). In the case that we apply the Weierstrass formula in the conditions of Theorems 4 and 5 to obtain the well-known formula

$$(2i)^{-n}\sqrt{\pi} H_n(i\omega) = \int_{-\infty}^{\infty} x^n e^{-(x-\omega)^2} dx, \qquad \omega \in \mathbb{C},$$

see, for example, ([14], Formula 3.462(4)).

4. Fundamental Solution for Semidiscrete Evolution Equations

In this section, we consider the operator $Bf(n) := (b * f)(n)$, with $b \in \ell^1(\mathbb{Z})$, $f \in \ell^p(\mathbb{N}_0)$, $p \in [1,\infty]$ and $n \in \mathbb{N}_0$. We obtain an explicit representation of solutions for the following time/space fractional evolution equation:

$$\begin{cases} \mathbb{D}_t^{\beta} u(n,t) = Bu(n,t) + g(n,t), & n \in \mathbb{N}_0,\ t > 0. \\ u(n,0) = \varphi(n), \quad u_t(n,0) = \phi(n) & n \in \mathbb{N}_0, \end{cases}$$

Here, $\beta \in (0,2]$ is real number. We recall that $\mathbb{D}_t^{\beta}$ denotes the Caputo fractional derivative given by

$$\mathbb{D}_t^{\beta} v(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} v'(s) ds = (g_{1-\beta} * v')(t), \qquad t > 0,$$

for $0 < \beta < 1$ and

$$\mathbb{D}_t^{\beta} v(t) = \frac{1}{\Gamma(2-\beta)} \int_0^t (t-s)^{1-\beta} v''(s) ds = (g_{2-\beta} * v'')(t), \qquad t > 0,$$

for $1 < \beta < 2$. For $\beta = 1$ and $\beta = 2$, we consider the usual first- and second-order derivation. Note that

$$\lim_{\beta\to 1^-} \mathbb{D}_t^{\beta} v(t) = v'(t), \qquad \lim_{\beta\to 2^-} \mathbb{D}_t^{\beta} v(t) = v''(t), \qquad t > 0,$$

however,

$$\lim_{\beta\to 0^+} \mathbb{D}_t^{\beta} v(t) = v(t) - v(0) \qquad \lim_{\beta\to 1^+} \mathbb{D}_t^{\beta} v(t) = v'(t) - v'(0), \qquad t > 0, \tag{12}$$

see, for example, [15,16].

The main result in this section is the following Theorem. The function $E_{\alpha,\beta}(b)$ (with $b \in \ell^1(\mathbb{Z})$) is the vector-valued Mittag-Leffler function given in Definition 1. A similar result was stated in the Banach space $\ell^p(\mathbb{Z})$ in ([8], Theorem 5.1).

Theorem 6. *Let $\varphi, \phi \in \ell^p(\mathbb{N}_0)$, and $g : \mathbb{N}_0 \times \mathbb{R}_+ \to \mathbb{C}$ be such that, for each $t \in \mathbb{R}_+$, $g(\cdot,t) \in \ell^p(\mathbb{N}_0)$ and $\sup_{s\in[0,t]} ||g(\cdot,s)||_p < \infty$ with $1 \leq p \leq \infty$.*

(i) *For $0 < \beta < 1$, the function*

$$u(n,t) =(E_{\beta,1}(t^\beta b) * \varphi)(n) + \int_0^t (t-s)^{\beta-1}\Big(E_{\beta,\beta}((t-s)^\beta b) * g(\cdot,s)\Big)(n)ds,$$

is the unique solution of the initial value problem (4) with $u(\cdot,t)$ belonging to $\ell^p(\mathbb{N}_0)$ for $t > 0$ and $n \in \mathbb{N}_0$.

(ii) *For $1 < \beta < 2$, the function*

$$\begin{aligned} u(n,t) =&(E_{\beta,1}(t^\beta b) * \varphi)(n) + t(E_{\beta,2}(t^\beta b) * \phi)(n) \\ &+ \int_0^t (t-s)^{\beta-1}\Big(E_{\beta,\beta}((t-s)^\beta b) * g(\cdot,s)\Big)(n)ds, \end{aligned}$$

is the unique solution of the initial value problem (5) with $u(\cdot,t)$ belonging to $\ell^p(\mathbb{N}_0)$ for $t > 0$ and $n \in \mathbb{N}_0$.

Proof. We show part (ii) in the case $p = \infty$. Part (i) or $1 \leq p < \infty$ are proved in a similar way. We prove the result in several steps.

Step 1. First, we show the explicit solution for (5). Taking the $\mathcal{Z}$-transform of (5), we obtain that

$$\begin{cases} \mathbb{D}_t^\beta u(z,t) = Bu(z,t) + g(z,t) \\ u(z,0) = \varphi(z);\ u_t(z,0) = \phi(z). \end{cases} \tag{13}$$

Now, taking Laplace's transformation to (13), we have:

$$\hat{u}(z,\lambda) = \frac{\lambda^{\beta-1}}{\lambda^\beta - \mathcal{F}(b)(\theta)}\varphi(z) + \frac{\lambda^{\beta-2}}{\lambda^\beta - \mathcal{F}(b)(\theta)}\phi(z) + \frac{1}{\lambda^\beta - \mathcal{F}(b)(\theta)}\hat{g}(z,\lambda). \tag{14}$$

By inverse Laplace transform, see identity (9), we obtain

$$\begin{aligned} u(z,t) = &E_{\beta,1}(\mathcal{F}(b)(\theta)t^\beta)\varphi(z) + tE_{\beta,2}(\mathcal{F}(b)(\theta)t^\beta)\phi(z) \\ &+ \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}((t-s)^\beta \mathcal{F}(b)(\theta))(z)g(z,s)ds. \end{aligned}$$

We apply Proposition 1 (ii) to obtain

$$\begin{aligned} u(n,t) =&(E_{\beta,1}(t^\beta b) * \varphi)(n) + t(E_{\beta,2}(t^\beta b) * \phi)(n) \\ &+ \int_0^t (t-s)^{\beta-1}\Big(E_{\beta,\beta}((t-s)^\beta b) * g(\cdot,s)\Big)(n)ds, \end{aligned}$$

for $n \in \mathbb{N}_0$ and $t > 0$.

Step 2. Now, we prove uniqueness. Suppose that the system (5) has two solutions u_1 and u_2 with the same initial values φ, ϕ, and write $v := u_1 - u_2$. Then, v is a solution of the following ODE

$$\mathbb{D}_t^\beta v(z,t) = Bv(z,t), \quad v(z,0) = 0, \quad v_t(z,0) = 0.$$

Due to the above ordinary differential equation having its unique solution and that the function zero is a solution, we conclude that $v(z,t) = 0$. As the $\mathcal{Z}$-transform is injective, we conclude that $v(n,t) = 0$ for every $n \in \mathbb{N}_0$ and $t \geq 0$. Hence, $u_1 = u_2$.

Step 3. By Proposition 1 (i), we obtain that

$$\begin{aligned} \|E_{\beta,1}(t^\beta b)\|_1 &\leq E_{\beta,1}(t^\beta \|b\|_1) \\ \|E_{\beta,2}(t^\beta b)\|_1 &\leq E_{\beta,2}(t^\beta \|b\|_1) \\ \|E_{\beta,\beta}(t-s)^\beta b)\|_1 &\leq E_{\beta,2}((t-s)^\beta \|b\|_1) \end{aligned}$$

for $0 < s < t$ and $b \in \ell^1(\mathbb{Z})$. Because $\sup_{s\in[0,t]} ||g(\cdot,s)||_\infty < \infty$, we conclude that

$$\int_0^t (t-s)^{\beta-1}\Big(E_{\beta,\beta}((t-s)^\beta b) * g(\cdot,s)\Big)(n)ds \in \ell^\infty(\mathbb{N}_0),$$

and the solution $u(\cdot,t) \in \ell^\infty(\mathbb{N}_0)$ for $t > 0$. □

Remark 2. Now, we may shortly treat the behavior of the solution when β tends to natural parameter, i.e., $\beta = 1,2$. For simplicity, we only present the homogeneous case, $g = 0$. In the case that $\beta \to 1^-$, the solution of Equation (4) tends to semigroup family operators $E_{1,1}(tb)$, and when $\beta \to 2^-$, the solution of Equation (5),

$$u(\cdot,t) = E_{\beta,1}(t^\beta b) * \varphi + tE_{\beta,2}(t^\beta b) * \phi, \qquad t > 0$$

tends to the well-known solution of second-order Cauchy problem, expressed in terms of cosine function and sine function generated by b, see ([10], Corollary 3.14.8).

However, in the case that $\beta \to 1^+$, the solution of Equation (5) converges to

$$u(\cdot,t) = E_{1,1}(bt) + tE_{1,2}(tb), \qquad t > 0,$$

as in the scalar case. We remark that this function is the solution of the following first-order modified Cauchy problem

$$\begin{cases} v'(n,t) = Bv(n,t) + \phi(n), & n \in \mathbb{N}_0,\ t > 0, \\ v(n,0) = \varphi(n), & n \in \mathbb{N}_0, \end{cases}$$

for $\phi, \varphi \in \ell^p(\mathbb{N}_0)$. This natural fact is connected with the interpolation property of the Caputo fractional differentiation, see (12).

The fundamental solution $u_{\beta,1}$ for Equations (4) and (5) is given by taking as initial values $\psi = \delta_0$ and $\phi = 0$. In the case $1 < \beta \leq 2$ ($\beta = 2$, is the wave equation), a second fundamental solution $u_{\beta,2}$ is given by $\psi = 0$ and $\phi = \delta_0$, see ([17], Remark 3.2).

As a corollary of Theorems 2 and 6 is the following subordination theorem for fundamental solutions. This result extends ([17], Corollary 3.5) in the space $\ell^p(\mathbb{N}_0)$.

We denote by Φ_α the entire Wright function defined by

$$\Phi_\alpha(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\alpha n + 1 - \alpha)} = \frac{1}{2\pi i}\int_\gamma \mu^{\alpha-1} e^{\mu - z\mu^\alpha} d\mu, \quad 0 < \alpha < 1, \tag{15}$$

where γ is a complex path which starts and ends at $-\infty$ and rounds the origin once counterclockwise. The Wright function is a known special function which has appeared in a wide variety of different contexts, for example, it is used for models in stochastic processes, see [18]. The proof of the next corollary is similar to ([8], Corollary 5.3), and we leave for the reader.

Corollary 1. *Let $u_{\beta,1}$ and $u_{\beta,2}$ be the fundamental solutions of problems (4) and (5) and Ψ_α the Wright function defined by (15).*

(i) *Let* $0 < \beta < 1$. *Then,*

$$u_{\beta,1}(n,t) = \int_0^\infty \Phi_\beta(\tau) u_{1,1}(n, \tau t^\beta) d\tau, \qquad n \in \mathbb{N}_0, \ t > 0.$$

(ii) *Let* $1 < \beta < 2$. *Then,*

$$\begin{aligned} u_{\beta,1}(n,t) &= \int_0^\infty \Phi_{\frac{\beta}{2}}(\tau) u_{2,1}(n, \tau t^{\frac{\beta}{2}}) d\tau, \\ u_{\beta,2}(n,t) &= \int_0^t \frac{(t-u)^{\frac{-\beta}{2}}}{\Gamma(1-\frac{\beta}{2})} \int_0^\infty \Phi_{\frac{\beta}{2}}(\tau) u_{2,2}(n, \tau u^{\frac{\beta}{2}}) d\tau \, du, \end{aligned}$$

for $n \in \mathbb{N}_0$ *and* $t > 0$.

5. Conclusions and Future Work

In this paper, we have considered Caputo fractional differential equations in the sequence Lebesgue spaces $\ell^p(\mathbb{N}_0)$ with $p \geq 1$. The associate operator is given by a convolution with a sequence in the Banach Algebra which belongs to $\ell^1(\mathbb{Z})$. We use techniques from the Functional Analysis, such as the Guelfand theory in Banach algebra, to obtain more information about the problem. We calculate the explicit solution in terms of Mittag-Leffer functions. Some particular cases (sequences of compact support) of finite difference operators are treated in detail.

An interesting problem to address in the future is to consider these techniques in the continuous case. We may study these Caputo fractional differential equations in $L^p(\mathbb{R}^+)$ and $L^p(\mathbb{R})$.

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