学位論文要旨

非圧縮粘性流れに対する圧力境界条件を含む圧力 Poisson 法と射影法の数学解析

Mathematical analysis of pressure Poisson methods and projection methods involving pressure boundary conditions for incompressible viscous flows

金沢大学大学院 自然科学研究科

数物科学専攻

学籍番号：　1924012003
氏名：　松井 一徳
主任指導教員名：　木村 正人
提出年月：　2022年1月

Summary

We consider pressure Poisson equations for stationary incompressible Stokes problems and time-dependent incompressible Navier–Stokes problems. The pressure Poisson equation is an elliptic partial differential equation of second order and is used in various numerical methods for incompressible viscous flows. Since there are many mechanisms that generate flow by creating pressure differences, one often sets a Dirichlet boundary condition for the pressure Poisson equation. However, in general, the pressure of the boundary condition for the numerical methods differs from the exact pressure solution of the original problem.

The thesis aims to provide a mathematical analysis for the pressure Poisson equation from the viewpoint of additional boundary conditions. We establish error estimates in suitable norms between solutions to a stationary Stokes problem and the corresponding pressure Poisson problem in terms of the additional boundary condition. In addition, for a pseudo-compressibility problem that interpolates the Stokes and pressure Poisson problems, we also give error estimates in suitable norms between the solutions to the pseudo-compressibility problem, the pressure Poisson problem, and the Stokes problem for several additional boundary condition cases. Moreover, we propose a new additional boundary condition for the projection method for the time-dependent Navier–Stokes problem with a Dirichlet-type pressure boundary condition and no tangent flow.

1 Pressure Poisson problem

Let Ω be a bounded domain of $\mathbb{R}^d$ ($d = 2$ or $d = 3$) with Lipschitz continuous boundary Γ. For the boundary Γ, we assume that there exist two relatively open subsets Γ_1, Γ_2 of Γ satisfying the following conditions:

$$|\Gamma \setminus (\Gamma_1 \cup \Gamma_2)| = 0, \quad |\Gamma_1|, |\Gamma_2| > 0, \quad \Gamma_1 \cap \Gamma_2 = \emptyset, \quad \mathring{\overline{\Gamma_1}} = \Gamma_1, \quad \mathring{\overline{\Gamma_2}} = \Gamma_2,$$

where $\overline{A}$ is the closure of $A \subset \Gamma$ with respect to Γ, $\mathring{A}$ is the interior of A with respect to Γ, and $|A|$ is the $(d-1)$-dimensional Hausdorff measure of A.

We consider the following Stokes problem: Find $u^S : \Omega \to \mathbb{R}^d$ and $p^S : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u^S + \nabla p^S = F & \text{in } \Omega, \\ \operatorname{div} u^S = 0 & \text{in } \Omega, \\ u^S = 0 & \text{on } \Gamma_1, \\ T_n(u^S, p^S) = t^b & \text{on } \Gamma_2, \end{cases} \tag{ST}$$

holds, where $F : \Omega \to \mathbb{R}^d$, $t^b : \Gamma_2 \to \mathbb{R}^d$, n is the unit outward normal vector for Γ,

$$S(u^S)_{ij} := \frac{\partial u_i^S}{\partial x_j} + \frac{\partial u_j^S}{\partial x_i}, \qquad T_n(u^S, p^S)_i := \sum_{k=1}^{d} S(u^S)_{ik} n_k - p^S n_i,$$

for all $i, j = 1, \ldots, d$. The functions u^S and p^S are the velocity and the pressure of the flow governed by (ST), respectively. Here, $T_n(u^S, p^S)$ is often called the normal stress on Γ. Let the fourth equation of (ST) be called the traction boundary condition.

By taking the divergence of the first equation, we obtain

$$\operatorname{div} F = \operatorname{div}(-\Delta u^S + \nabla p^S) = -\Delta(\operatorname{div} u^S) + \Delta p^S = \Delta p^S, \tag{1.1}$$

which is often called the pressure Poisson equation (McKee et al., 2004).

We need an additional boundary condition for solving equation (1.1). In real-world applications, the additional boundary condition is usually given by using experimental or plausible values. We consider the following boundary value problem for the pressure Poisson equation: Find $u^{PP} : \Omega \to \mathbb{R}^d$ and $p^{PP} : \Omega \to \mathbb{R}$ satisfying

$$\begin{cases} -\Delta u^{PP} - \nabla(\operatorname{div} u^{PP}) + \nabla p^{PP} = F & \text{in } \Omega, \\ -\Delta p^{PP} = -\operatorname{div} F & \text{in } \Omega, \\ u^{PP} = 0, \dfrac{\partial p^{PP}}{\partial n} = g^b & \text{on } \Gamma_1, \\ T_n(u^{PP}, p^{PP}) = t^b, p^{PP} = p^b & \text{on } \Gamma_2, \end{cases} \tag{PPT}$$

where $g^b : \Gamma_1 \to \mathbb{R}$ and $p^b : \Gamma_2 \to \mathbb{R}$ are the data for the additional boundary conditions. We call this problem the pressure Poisson problem. The second term $-\nabla(\operatorname{div} u^{PP})$ in the first equation of (PPT) is usually omitted since $\operatorname{div} u^S = 0$, but this term is necessary to treat the traction boundary condition in a weak formulation. The idea of using (1.1) instead of $\operatorname{div} u^S = 0$ is useful for calculating the pressure numerically in the Navier–Stokes problem. For example, this idea is used in the marker and cell (MAC) method (Harlow and Welch, 1965) and the projection method (Chorin, 1968; Temam, 1969).

As the boundary condition for the Stokes problem, we also consider the boundary condition introduced by Begue et al. (1987);

$$\begin{cases} u^S = 0 & \text{on } \Gamma_1, \\ u^S \times n = 0 & \text{on } \Gamma_2, \\ p^S = p^b & \text{on } \Gamma_2, \end{cases} \tag{1.2}$$

where "×" is the cross product in $\mathbb{R}^d$. On boundary Γ_2, the boundary value of the pressure is prescribed, and the velocity is parallel to the normal direction on Γ. Such a situation occurs at the end of the pipe, such as blood vessels or pipelines (Fig. 1).

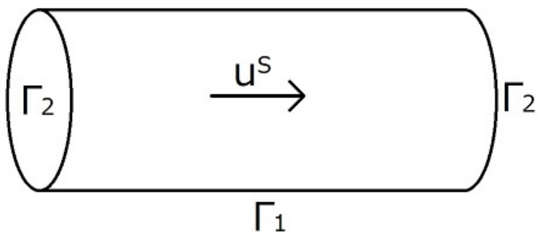

Figure 1: Image of a flow in a pipe

To define weak formulations of the problems, for $\tilde{\Gamma} \in \{\Gamma_1, \Gamma_2\}$, we set

$$H^1_{\tilde{\Gamma}}(\Omega) := \{\psi \in H^1(\Omega) \mid \psi = 0 \text{ on } \tilde{\Gamma}\}, \quad H := \{\varphi \in H^1(\Omega)^d \mid \varphi = 0 \text{ on } \Gamma_1, \varphi \times n = 0 \text{ on } \Gamma_2\}.$$

We use the same notation $(\cdot,\cdot)$ to represent the $L^2(\Omega)$ inner product for scalar-, vector- and matrix-valued functions. For the open subset $\tilde{\Gamma} \in \{\Gamma, \Gamma_1, \Gamma_2\}$ of the boundary Γ, let $H^{1/2}(\tilde{\Gamma})$ be the set of all functions $\eta \in L^2(\tilde{\Gamma})$ such that the norm

$$\|\eta\|_{H^{1/2}(\tilde{\Gamma})} := \left(\|\eta\|^2_{L^2(\tilde{\Gamma})} + \int_{\tilde{\Gamma}} \int_{\tilde{\Gamma}} \frac{|\eta(s_1) - \eta(s_2)|^2}{|s_1 - s_2|^d} ds_1 ds_2 \right)^{1/2}$$

exists and is finite, which is a Banach space with respect to $\|\cdot\|_{H^{1/2}(\tilde{\Gamma})}$, and let $\gamma_0 : H^1(\Omega) \to H^{1/2}(\Gamma)$ be the standard trace operator. For Γ_1 and Γ_2, we define the following subspaces of $H^{1/2}(\Gamma)$,

$$H^{1/2}_{\gamma_0}(\Gamma_1) \coloneqq \gamma_0(H^1_{\Gamma_2}(\Omega)), \quad H^{1/2}_{\gamma_0}(\Gamma_2) \coloneqq \gamma_0(H^1_{\Gamma_1}(\Omega)).$$

We assume the following conditions;

$$F \in L^2(\Omega)^d, \quad \operatorname{div} F \in L^2(\Omega), \quad t^b \in (H^{1/2}_{\gamma_0}(\Gamma_2)^d)^*, \quad g^b \in (H^{1/2}_{\gamma_0}(\Gamma_1))^*, \quad p^b \in H^1(\Omega).$$

The weak form of the Stokes problem (ST) is as follows: Find $(u^{S1}, p^{S1}) \in H^1_{\Gamma_1}(\Omega)^d \times L^2(\Omega)$ such that

$$\begin{cases} \dfrac{1}{2}(S(u^{S1}), S(\varphi)) - (p^{S1}, \operatorname{div}\varphi) = (F, \varphi) - \langle t^b, \varphi\rangle_{H^{1/2}_{\gamma_0}(\Gamma_2)^d} & \text{for all } \varphi \in H^1_{\Gamma_1}(\Omega)^d, \\ (\operatorname{div} u^{S1}, \psi) = 0 & \text{for all } \psi \in L^2(\Omega). \end{cases} \tag{ST'}$$

The weak form of (PPT) is as follows: Find $u^{PP} \in H^1_{\Gamma_1}(\Omega)^d$ and $p^{PP} \in H^1(\Omega)$ such that

$$\begin{cases} \dfrac{1}{2}(S(u^{PP}), S(\varphi)) - (p^{PP}, \operatorname{div}\varphi) = (F, \varphi) - \langle t^b, \varphi\rangle_{H^{1/2}_{\gamma_0}(\Gamma_2)^d} & \text{for all } \varphi \in H^1_{\Gamma_1}(\Omega)^d, \\ (\nabla p^{PP}, \nabla\psi) = -(\operatorname{div} F, \psi) + \langle g^b, \psi\rangle_{H^{1/2}_{\gamma_0}(\Gamma_1)} & \text{for all } \psi \in H^1_{\Gamma_2}(\Omega), \\ p^{PP} = p^b & \text{on } \Gamma_2. \end{cases} \tag{PPT'}$$

Here, (ST') and (PPT') have a unique solution. The weak form of the Stokes problem with the boundary condition (1.2) is as follows: Find $(u^{S2}, p^{S2}) \in H \times L^2(\Omega)$ such that

$$\begin{cases} (\nabla \times u^{S2}, \nabla \times v) - (p^{S2}, \operatorname{div} v) = (F, v) - \displaystyle\int_{\Gamma_2} p^b v \cdot n ds & \text{for all } v \in H, \\ (\operatorname{div} u^{S2}, \psi) = 0 & \text{for all } \psi \in L^2(\Omega). \end{cases} \tag{SP'}$$

There exists a unique solution to (SP'), e.g., if the boundary Γ is $C^{1,1}$-class or Ω is a polygon.

Our main results for (ST'), (PPT'), and (SP') are the following theorems.

Theorem 1.1. *If $p^{S1} \in H^1(\Omega)$ and $\Delta p^{S1} \in L^2(\Omega)$, there exists a constant $c > 0$ such that*

$$\|u^{S1} - u^{PP}\|_{H^1(\Omega)^d} + \|p^{S1} - p^{PP}\|_{H^1(\Omega)} \le c\left(\left\|\frac{\partial p^{S1}}{\partial n} - g^b\right\|_{(H^{1/2}_{\gamma_0}(\Gamma_1))^*} + \|p^{S1} - p^b\|_{H^{1/2}(\Gamma_2)}\right).$$

Theorem 1.2. *If $\Delta u^{S2} + \nabla(\operatorname{div} u^{S2}) \in L^2(\Omega)^d$, $p^{S2} \in H^1(\Omega)$ and $\Delta p^{S2} \in L^2(\Omega)$, then there exists a constant $c > 0$ such that*

$$\|p^{S2} - p^{PP}\|_{H^1(\Omega)} \le c\left\|\frac{\partial p^{S2}}{\partial n} - g^b\right\|_{(H^{1/2}_{\gamma_0}(\Gamma_1))^*},$$

$$\|u^{S2} - u^{PP}\|_{H^1(\Omega)^d} \le c\left(\left\|\frac{\partial p^{S2}}{\partial n} - g^b\right\|_{(H^{1/2}_{\gamma_0}(\Gamma_1))^*} + \|t^{S2} - t^b\|_{(H^{1/2}_{\gamma_0}(\Gamma_2)^d)^*}\right),$$

where $t^{S2} = T_n(u^{S2}, p^{S2})$.

Theorems 1.1 and 1.2 state that if we have a good prediction for the boundary data (g^b and p^b), then the pressure Poisson problem is a good approximation for the Stokes problem. In particular, we propose a new viewpoint of the pressure Poisson problem and the boundary condition (1.2). The numerical solution to the Stokes problem with the boundary condition (1.2) requires delicate choices of the weak formulation and special finite element techniques (Bertoluzza et al., 2017). On the other hand, the pressure Poisson problem has been used as a simple numerical scheme for a long time. From our results, we can confirm that the pressure Poisson problem is also available for the Stokes problem with the boundary condition (1.2).

2 ε-Stokes problem

Next, we consider the full-Dirichlet boundary condition for the Stokes problem: Find $u^S : \Omega \to \mathbb{R}^d$ and $p^S : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u^S + \nabla p^S = F & \text{in } \Omega, \\ \operatorname{div} u^S = 0 & \text{in } \Omega, \\ u^S = u^b & \text{on } \Gamma, \end{cases} \tag{S}$$

where $u^b : \Gamma \to \mathbb{R}^d$ is a given Dirichlet boundary data satisfying $\int_\Gamma u^b \cdot n ds = 0$. The corresponding pressure Poisson problem is as follows: Find $u^{PP} : \Omega \to \mathbb{R}^d$ and $p^{PP} : \Omega \to \mathbb{R}$ satisfying

$$\begin{cases} -\Delta u^{PP} + \nabla p^{PP} = F & \text{in } \Omega, \\ -\Delta p^{PP} = -\operatorname{div} F & \text{in } \Omega, \\ u^{PP} = u^b & \text{on } \Gamma, \\ +\text{boundary condition for } p^{PP}. \end{cases} \tag{PP}$$

We introduce an "interpolation" between problems (S) and (PP). For $\varepsilon > 0$, find $u^\varepsilon : \Omega \to \mathbb{R}^d$ and $p^\varepsilon : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u^\varepsilon + \nabla p^\varepsilon = F & \text{in } \Omega, \\ -\varepsilon \Delta p^\varepsilon + \operatorname{div} u^\varepsilon = -\varepsilon \operatorname{div} F & \text{in } \Omega, \\ u^\varepsilon = u^b & \text{on } \Gamma, \\ +\text{boundary condition for } p^\varepsilon. \end{cases} \tag{ES}$$

We call this problem the ε-Stokes problem (ES). The ε-Stokes problem is treated as an approximation of the Stokes problem to avoid numerical instabilities (e.g., Brezzi and Pitkäranta, 1984). The ε-Stokes problem approximates the Stokes problem (S) as $\varepsilon \to 0$ and the pressure Poisson problem (PP) as $\varepsilon \to \infty$ (Fig. 2).

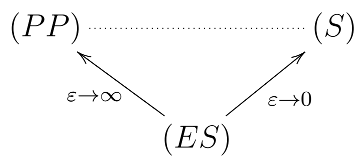

Figure 2: Sketch of the connections between problems (S), (PP) and (ES).

We specify the boundary conditions for p^{PP} and p^ε. We consider a Neumann boundary condition (2.3) and a mixed boundary condition (2.4),

$$\frac{\partial p^{PP}}{\partial n} = g^b \text{ on } \Gamma, \quad \frac{\partial p^\varepsilon}{\partial n} = g^b \text{ on } \Gamma, \tag{2.3}$$

$$\begin{cases} \dfrac{\partial p^{PP}}{\partial n} = g^b & \text{on } \Gamma_1, \\ p^{PP} = p^b & \text{on } \Gamma_2, \end{cases} \quad \begin{cases} \dfrac{\partial p^\varepsilon}{\partial n} = g^b & \text{on } \Gamma_1, \\ p^\varepsilon = p^b & \text{on } \Gamma_2, \end{cases} \tag{2.4}$$

$$p^{PP} = p^b \text{ on } \Gamma, \quad p^\varepsilon = p^b \text{ on } \Gamma, \tag{2.5}$$

where $p^b : \Gamma \to \mathbb{R}$ and $g^b = \Gamma \to \mathbb{R}$ satisfying $\int_\Gamma g^b ds = \int_\Gamma \operatorname{div} F dx$ are the given boundary data. The boundary condition (2.4) corresponds to (2.3) when $\Gamma_1 = \Gamma, \Gamma_2 = \emptyset$ and to (2.3) when $\Gamma_1 = \emptyset, \Gamma_2 = \Gamma$.

The weak form of the Stokes problem becomes as follows: Find $u^S \in H^1(\Omega)^d$ and $p^S \in L^2(\Omega)/\mathbb{R}$ such that

$$\begin{cases} (\nabla u^S, \nabla \varphi) + \langle \nabla p^S, \varphi \rangle_{H_0^1(\Omega)^d} = (F, \varphi) & \text{for all } \varphi \in H_0^1(\Omega)^d, \\ (\operatorname{div} u^S, \psi) = 0 & \text{for all } \psi \in L^2(\Omega)/\mathbb{R}, \\ u^S = u^b & \text{in } H^{1/2}(\Gamma)^n, \end{cases} \tag{S'}$$

where $L^2(\Omega)/\mathbb{R}$ is the space of all functions $f \in L^2(\Omega)$ with the average being 0 and $\langle \nabla p^S, \varphi \rangle_{H_0^1(\Omega)^d} := -(p^S, \operatorname{div} \varphi)$. We consider the following equations, which is a generalization of weak formulations of the pressure Poisson problem with the boundary conditions (2.3), (2.4), or (2.5): Find $u^{PP} \in H^1(\Omega)^d$ and $p^{PP} \in H^1(\Omega)$ such that

$$\begin{cases} (\nabla u^{PP}, \nabla \varphi) + (\nabla p^{PP}, \varphi) = (F, \varphi) & \text{for all } \varphi \in H_0^1(\Omega)^d, \\ (\nabla p^{PP}, \nabla \psi) = \langle G, \psi \rangle_Q & \text{for all } \psi \in Q, \\ u^{PP} - u^b \in H_0^1(\Omega)^d, \\ p^{PP} - p^b \in Q, \end{cases} \tag{PP'}$$

where $Q \subset H^1(\Omega)$ is a closed subspace such that there exists a constant $c > 0$ for which $\|q\|_{L^2(\Omega)} \leq c\|\nabla q\|_{L^2(\Omega)^d}$ for all $q \in Q$ (e.g., $Q = H_0^1(\Omega), H_{\Gamma_2}^1(\Omega), H^1(\Omega)/\mathbb{R}$) and $G \in Q^*$. We also consider the following equations, which is a generalization of weak formulations of the ε-Stokes problem with the boundary conditions (2.3), (2.4), or (2.5): Find $u^\varepsilon \in H^1(\Omega)^d$ and $p^\varepsilon \in H^1(\Omega)$ such that

$$\begin{cases} (\nabla u^\varepsilon, \nabla \varphi) + (\nabla p^\varepsilon, \varphi) = (F, \varphi) & \text{for all } \varphi \in H_0^1(\Omega)^d, \\ \varepsilon(\nabla p^\varepsilon, \nabla \psi) + (\operatorname{div} u^\varepsilon, \psi) = \varepsilon \langle G, \psi \rangle_Q & \text{for all } \psi \in Q, \\ u^\varepsilon - u^b \in H_0^1(\Omega)^d, \\ p^\varepsilon - p^b \in Q. \end{cases} \tag{ES'}$$

As in Section 1, we can show that there exists a constant $c > 0$ independent of ε such that

$$\|u^S - u^{PP}\|_{H^1(\Omega)^d} \leq c\|p^S - p^{PP}\|_{H^{1/2}(\Gamma)},$$
$$\|u^S - u^\varepsilon\|_{H^1(\Omega)^d} \leq c\|p^S - p^{PP}\|_{H^{1/2}(\Gamma)}.$$

From the first inequality, if we have a good predictive value for pressure on Γ, then u^{PP} is a good approximation of u^S. Moreover, u^ε is also a good approximation of u^S from the second inequality.

Our main result for (ES') and (PP') is the following theorem:

Theorem 2.1. *There exists a constant $c > 0$ independent of $\varepsilon > 0$ such that*

$$\|u^\varepsilon - u^{PP}\|_{H^1(\Omega)^d} + \|p^\varepsilon - p^{PP}\|_{H^1(\Omega)} \leq \frac{c}{\varepsilon}\|\operatorname{div} u^{PP}\|_{Q^*}.$$

for all $\varepsilon > 0$. In particular, we have

$$u^\varepsilon \to u^{PP} \text{ strongly in } H^1(\Omega)^d, \; p^\varepsilon \to p^{PP} \text{ strongly in } H^1(\Omega) \quad \text{as } \varepsilon \to \infty.$$

Furthermore, the solution to (ES') has the following asymptotic structure:

Theorem 2.2. *Let $k \in \mathbb{N}$ be arbitrary ($k \geq 1$) and let $v^{(0)} := u^{PP}$. If functions $v^{(1)}, v^{(2)}, \cdots, v^{(k)} \in H_0^1(\Omega)^d$ and $q^{(1)}, q^{(2)}, \cdots, q^{(k)} \in Q$ satisfy*

$$\begin{cases} (\nabla v^{(i)}, \nabla \varphi) + (\nabla q^{(i)}, \varphi) = 0 & \text{for all } \varphi \in H_0^1(\Omega)^d, \\ (\nabla q^{(i)}, \nabla \psi) = -(\operatorname{div} v^{(i-1)}, \psi) & \text{for all } \psi \in Q, \end{cases} \tag{2.6}$$

for all $1 \leq i \leq k$, then there exists a constant $c > 0$ independent of ε satisfying

$$\left\| u^\varepsilon - \left(u^{PP} + \frac{1}{\varepsilon} v^{(1)} + \cdots + \left(\frac{1}{\varepsilon}\right)^k v^{(k)} \right) \right\|_{H^1(\Omega)^d} \leq \frac{c}{\varepsilon^{k+1}} \| \operatorname{div} v^{(k)} \|_{Q^*},$$

$$\left\| p^\varepsilon - \left(p^{PP} + \frac{1}{\varepsilon} q^{(1)} + \cdots + \left(\frac{1}{\varepsilon}\right)^k q^{(k)} \right) \right\|_{H^1(\Omega)} \leq \frac{c}{\varepsilon^{k+1}} \| \operatorname{div} v^{(k)} \|_{Q^*}.$$

On the other hand, our main result for (ES') and (S') is the following theorem:

Theorem 2.3. *Let the map $L^2(\Omega) \ni f \mapsto [f] := f - \frac{1}{|\Omega|}\int_\Omega f \, dx \in L^2(\Omega)/\mathbb{R}$. There exists a constant $c > 0$ independent of ε such that*

$$\|u^\varepsilon\|_{H^1(\Omega)^d} + \|[p^\varepsilon]\|_{L^2(\Omega)} \leq c \quad \text{for all } \varepsilon > 0.$$

Furthermore, if the range of Q under the map $[\cdot]$ is dense in $L^2(\Omega)/\mathbb{R}$, then we obtain

$$u^\varepsilon \to u^S \text{ strongly in } H^1(\Omega)^d, \; [p^\varepsilon] \to p^S \text{ strongly in } L^2(\Omega)/\mathbb{R} \quad \text{as } \varepsilon \to 0.$$

Theorem 2.3 does not give the convergence rate. If $Q = H^1(\Omega)/\mathbb{R}$ (corresponding to the Neumann boundary condition (2.3)), then the convergence rate becomes $\sqrt{\varepsilon}$.

Theorem 2.4. *Suppose that $Q = H^1(\Omega)/\mathbb{R}$ and $p^S \in H^1(\Omega)$. Then, there exists a constant $c > 0$ independent of ε such that*

$$\|u^\varepsilon - u^S\|_{H^1(\Omega)^d} + \|p^\varepsilon - p^S\|_{L^2(\Omega)} \leq c\sqrt{\varepsilon}.$$

3 Projection method

We assume that the boundary Γ is $C^{1,1}$-class or Ω is a polygon. For fixed $T > 0$, we consider the following Navier–Stokes problem: Find $u : \Omega \times [0,T] \to \mathbb{R}^d$ and $p : \Omega \times [0,T] \to \mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial u}{\partial t} + D(u,u) - \nu \Delta u + \dfrac{1}{\rho}\nabla P = f & \text{in } \Omega \times (0,T), \\ \operatorname{div} u = 0 & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \Gamma_1 \times (0,T), \\ u \times n = 0 & \text{on } \Gamma_2 \times (0,T), \\ P = p^b & \text{on } \Gamma_2 \times (0,T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{NS}$$

where $D(v,w) := (\nabla \times v) \times w$, $P := p + \frac{\rho}{2}|u|^2$, $\nu, \rho > 0$, $f : \Omega \times (0,T) \to \mathbb{R}^d$, $p^b : \Gamma_2 \times (0,T) \to \mathbb{R}$, and $u_0 : \Omega \to \mathbb{R}^d$. The first equation of (NS) is based on

$$(u \cdot \nabla)u = (\nabla \times u) \times u + \frac{1}{2}\nabla |u|^2.$$

For Γ_2, we assume a boundary condition including a pressure value $p+\frac{\varrho}{2}|u|^2$, which is called the total pressure, stagnation pressure, or Bernoulli pressure. Usual pressure is often called static pressure to distinguish it from the total pressure. In an experimental measurement of the total and static pressure using a Pitot tube, the boss measurement is dependent on the yaw angle of the Pitot tube. Then, the effect on the total pressure $p+\frac{\varrho}{2}|u|^2$ is smaller than the effect on the usual pressure p (Holman, 2001). The boundary condition on Γ_2 in (NS) is introduced by Begue et al. (1987), and the existence of a weak velocity solution is proven by Bernard (2003) and Kim and Cao (2015).

We introduce a projection method for (NS). The projection method is one of the numerical schemes for Navier–Stokes equations (Chorin, 1968; Temam, 1969). Let $\tau(:= T/N < 1, N \in \mathbb{N})$ be a time increment and let $t_k := k\tau$ $(k = 0, 1, \ldots, N)$. We set $u_0^* := u_0$ and calculate u_k^*, u_k, p_k $(k = 1, 2, \ldots, N)$ by repeatedly solving the following problems (Step 1) and (Step 2).

(Step 1) Find $u_k^* : \Omega \to \mathbb{R}^d$ such that

$$\begin{cases} \dfrac{u_k^* - u_{k-1}}{\tau} + D(u_{k-1}^*, u_k^*) - \nu \Delta u_k^* = f(t_k) & \text{in } \Omega, \\ u_k^* = 0 & \text{on } \Gamma_1, \\ u_k^* \times n = 0 & \text{on } \Gamma_2, \\ \operatorname{div} u_k^* = 0 & \text{on } \Gamma_2. \end{cases} \tag{3.7}$$

(Step 2) Find $P_k :\to \mathbb{R}$ and $u_k :\to \mathbb{R}^d$ such that

$$\begin{cases} -\dfrac{\tau}{\rho}\Delta P_k = -\operatorname{div} u_k^* & \text{in } \Omega, \\ \dfrac{\partial P_k}{\partial n} = 0 & \text{on } \Gamma_1, \\ P_k = p^b(t_k) & \text{on } \Gamma_2, \end{cases} \tag{3.8}$$

$$u_k = u_k^* - \frac{\tau}{\rho}\nabla P_k \quad \text{in } \Omega. \tag{3.9}$$

Remark 3.1. *For the velocity boundary condition on Γ_2, we can rewrite the third and fourth equations of (Step 1) by using $\kappa := \operatorname{div} n = (d-1)\times$(mean curvature) as follows:*

$$u_k^* \times n = 0, \quad \frac{\partial u_k^*}{\partial n} \cdot n + \kappa u_k^* \cdot n = 0 \qquad \text{on } \Gamma_2.$$

In particular, if Γ_2 is flat, then it holds that

$$u_k^* \times n = 0, \quad \frac{\partial u_k^*}{\partial n} \cdot n = 0 \qquad \text{on } \Gamma_2.$$

3.1 Weak formulations

Let p_d be

$$p_d := \begin{cases} 2+\varepsilon & \text{if } d = 2, \\ 3 & \text{if } d = 3, \end{cases}$$

where $\varepsilon > 0$ is arbitrarily small. We assume $\nu = \rho = 1$ and the following conditions for f, p^b, and u_0:

$$f \in L^2(0,T;H^*), \quad p^b \in L^2(0,T;H^1(\Omega)), \quad u_0 \in L^{p_d}(\Omega)^d. \tag{3.10}$$

To define weak formulations of the Navier–Stokes equations (NS) and the projection method (Step 1) and (Step 2), we define the bilinear form $a_0 : H\times H \to \mathbb{R}$ and trilinear form $a_1 : L^{p_d}(\Omega)^d \times H\times H \to \mathbb{R}$ ($p_2 > 2, p_3 = 3$) by

$$\begin{aligned} a_0(u,v) &\coloneqq (\operatorname{div} u, \operatorname{div} v) + (\nabla\times u, \nabla\times v) && \text{for all } u,v\in H,\\ a_1(u,v,w) &\coloneqq \int_\Omega u\cdot(\nabla\times(v\times w))dx && \text{for all } u\in L^{p_d}(\Omega)^d, v,w\in H,\end{aligned}$$

We set weak formulation of (NS) as follows: Find $u\in L^2(0,T;H^1(\Omega)^d)$ and $P\in L^1(0,T;L^2(\Omega))$ such that $\frac{\partial u}{\partial t}\in L^1(0,T;H^*)$, $u(0)=u_0$, and for a.e. $t\in(0,T)$,

$$\begin{cases} \left\langle \dfrac{\partial u}{\partial t},\varphi\right\rangle_H + a_0(u,\varphi) + a_1(u,u,\varphi) - (P,\operatorname{div}\varphi) = \langle f,\varphi\rangle_H - \displaystyle\int_{\Gamma_2} p^b\varphi\cdot n ds & \text{for all } \varphi\in H,\\ \operatorname{div} u = 0 & \text{in } L^2(\Omega). \end{cases} \tag{3.11}$$

On the other hand, a weak formulation of the projection method (Step 1) and (Step 2) with the initial datum $u_0(=: u_0^*)$ is as follows:

Problem 3.2. *Let $(f_k)_{k=1}^N \subset H^*$ and $(p_k^b)_{k=1}^N\subset H^1(\Omega)$. For all $k=1,2,\dots,N$, find $(u_k^*, P_k, u_k)\in H\times H^1(\Omega)\times L^2(\Omega)^d$ such that $P_k - p_k^b \in H^1_{\Gamma_2}(\Omega)$ and*

$$\begin{cases} \dfrac{1}{\tau}(u_k^* - u_{k-1},\varphi) + a_0(u_k^*,\varphi) + a_1(u_{k-1}^*, u_k^*,\varphi) = \langle f_k,\varphi\rangle_H & \textit{for all } \varphi\in H,\\ \tau(\nabla P_k,\nabla\psi) = -(\operatorname{div} u_k^*,\psi) & \textit{for all } \psi\in H^1_{\Gamma_2}(\Omega),\\ u_k = u_k^* - \tau\nabla P_k & \textit{in } L^2(\Omega)^d. \end{cases} \tag{3.12}$$

By the Lax–Milgram theorem, the problem 3.2 has a unique solution.

Remark 3.3. *For $f\in L^2(0,T;H^*)$ and $p^b\in L^2(0,T;H^1(\Omega))$, we set for all $k=1,2,\dots,N$,*

$$f_k \coloneqq \frac{1}{\tau}\int_{t_{k-1}}^{t_k} f(t)dt, \qquad p_k^b \coloneqq \frac{1}{\tau}\int_{t_{k-1}}^{t_k} p^b(t)dt. \tag{3.13}$$

In Theorems 3.7 and 3.11, we assume $f\in C([0,T];H^*), p^b\in C([0,T];H^1(\Omega))$ and set for all $k=1,2,\dots,N$,

$$f_k \coloneqq f(t_k), \qquad p_k^b \coloneqq p^b(t_k),$$

Remark 3.4. *The function space $L^2(\Omega)^d$ has the following orthogonal decomposition:*

$$L^2(\Omega)^d = U\oplus\nabla(H^1_{\Gamma_2}(\Omega)),$$

where $U\coloneqq\{\varphi\in L^2(\Omega)^d \mid \operatorname{div}\varphi = 0$ in $L^2(\Omega), \langle\varphi\cdot n,\psi\rangle_{H^{1/2}(\Gamma)} = 0$ for all $\psi\in H^1_{\Gamma_2}(\Omega)\}$ (Guermond and Quartapelle, 1998). By the second and third equation of (3.12) and the Gauss divergence formula, it holds that for all $k=1,2,\dots,N$ and $\psi\in H^1_{\Gamma_2}(\Omega)$,

$$(u_k,\nabla\psi) = (u_k^*,\nabla\psi) - \tau(\nabla P_k,\nabla\psi) = -(\operatorname{div} u_k^*,\psi) - \tau(\nabla P_k,\nabla\psi) = 0,$$

which implies that $u_k\in U$. Since the third equation of (3.12) is equivalent to

$$u_k^* - \tau\nabla p^b(t_k) = u_k + \tau\nabla(P_k - p^b(t_k)) \qquad \textit{in } L^2(\Omega)^d,$$

Step 2 is the projection of $u_k^* - \tau\nabla p^b(t_k)$ to the divergence-free space U.

3.2 Main results: stability and convergence

For two sequences $(x_k)_{k=0}^N$ and $(y_k)_{k=1}^N$ in a Banach space E, we define a piecewise linear interpolant $\hat{x}_\tau \in W^{1,\infty}(0,T;E)$ of $(x_k)_{k=0}^N$ and a piecewise constant interpolant $\bar{y}_\tau \in L^\infty(0,T;E)$ of $(y_k)_{k=1}^N$, respectively, by

$$\begin{aligned}
\hat{x}_\tau(t) &\coloneqq x_{k-1} + \frac{t-t_{k-1}}{\tau}(x_k - x_{k-1}) && \text{for } t \in [t_{k-1}, t_k] \text{ and } k = 1, 2, \dots, N, \\
\bar{y}_\tau(t) &\coloneqq y_k && \text{for } t \in (t_{k-1}, t_k] \text{ and } k = 1, 2, \dots, N.
\end{aligned}$$

We show the stability of the projection method (3.12) and establish error estimates in suitable norms between the solutions to the Navier–Stokes equations (3.11) and the projection method (3.12).

Theorem 3.5. *Under the condition (3.10), we set $f_k \in H^*$ and $p_k^b \in H^1(\Omega)$ as (3.13) for all $k = 1, 2, \dots, N$. Then, there exists a constant $c > 0$ independent of τ such that*

$$\begin{aligned}
&\|\bar{u}_\tau\|_{L^\infty(0,T;L^2(\Omega)^d)} + \|\bar{u}_\tau^*\|_{L^\infty(0,T;L^2(\Omega)^d)} + \|\bar{u}_\tau^*\|_{L^2(0,T;H^1(\Omega)^d)} + \frac{1}{\sqrt{\tau}}\|\bar{u}_\tau - \bar{u}_\tau^*\|_{L^2(0,T;L^2(\Omega)^d)} \\
&\leq c\left(\|u_0\|_{L^2(\Omega)^d} + \|f\|_{L^2(0,T;H^*)} + \|p^b\|_{L^2(0,T;H^1(\Omega))}\right).
\end{aligned}$$

For a convergence theorem, we assume:

Hypothesis 3.6. *The solution (u, P) to (3.11) satisfies*

$$u \in C([0,T]; H \cap H^2(\Omega)^d) \cap H^1(0,T; L^2(\Omega)^d) \cap H^2(0,T;H^*), \qquad P \in C([0,T]; H^1(\Omega)).$$

We also assume $f \in C([0,T]; H^*)$ and $p^b \in C([0,T]; H^1(\Omega))$ and set in Problem 3.2 for all $k = 1, 2, \dots, N$,

$$f_k \coloneqq f(t_k), \qquad p_k^b \coloneqq p^b(t_k).$$

Theorem 3.7. *Under Hypothesis 3.6, there exist two constants $c, \tau_0 > 0$ independent of τ such that for all $0 < \tau < \tau_0$,*

$$\begin{aligned}
\|u - \bar{u}_\tau\|_{L^\infty(0,T;L^2(\Omega)^d)} + \|u - \bar{u}_\tau^*\|_{L^\infty(0,T;L^2(\Omega)^d)} + \|u - \bar{u}_\tau^*\|_{L^2(0,T;H^1(\Omega)^d)} &\leq c\sqrt{\tau}, \\
\|\bar{u}_\tau - \bar{u}_\tau^*\|_{L^2(0,T;L^2(\Omega)^d)} &\leq c\tau.
\end{aligned}$$

Remark 3.8. *For regularity of the solution (u, P) to (3.11), see Bernard (2003) and Kim (2015). In the case of the homogeneous Dirichlet boundary condition on the whole boundary Γ, high regularity properties of the solution to the Navier–Stokes equations are well-known (Boyer and Fabrie, 2013).*

Furthermore, we assume the following regularity assumptions:

Hypothesis 3.9 (Regularity of the Stokes problem)**.** *There exists a constant $c = c(\Omega, \Gamma_1, \Gamma_2) > 0$ such that for all $e \in L^2(\Omega)^d$*

$$\|w\|_{H^2(\Omega)^d} + \|r\|_{H^1(\Omega)} \leq c\|e\|_{L^2(\Omega)^d},$$

where $(w, r) \in H \times L^2(\Omega)$ is the solution to

$$\begin{cases} a_0(w, \varphi) - (r, \operatorname{div}\varphi) = (e, \varphi) & \text{for all } \varphi \in H, \\ \operatorname{div} w = 0 & \text{in } L^2(\Omega). \end{cases}$$

Hypothesis 3.10. *The solution* (u, P) *to (3.11) satisfies*

$$u \in H^1(0,T;H^1(\Omega)^d) \cap H^2(0,T;L^2(\Omega)^d) \cap H^3(0,T;H^*), \qquad P \in H^1(0,T;H^1(\Omega)).$$

Then, we can improve the convergence rate:

Theorem 3.11. *Under Hypotheses 3.6 and 3.9, there exist two constants* $\tau_1, c > 0$ *independent of* τ *such that for all* $0 < \tau < \tau_1$,

$$\|u - \bar{u}_\tau\|_{L^2(0,T;L^2(\Omega)^d)} + \|u - \bar{u}^*_\tau\|_{L^2(0,T;L^2(\Omega)^d)} \leq c\tau.$$

Furthermore, if we also assume Hypothesis 3.10, then there exist two constants $\tau_2, c > 0$ *independent of* τ *such that for all* $0 < \tau < \tau_2 (\leq \tau_1)$,

$$\|P - \bar{P}_\tau\|_{L^2(0,T;L^2(\Omega))} \leq c\sqrt{\tau}.$$

Remark 3.12. *Hypothesis 3.9 holds, e.g., if* Ω *is of class* $C^{2,1}$ *(Bernard, 2002).*

3.3 Main result: existence of a weak solution to (NS)

Using Theorem 3.5, we prove that there exists a solution to a weak formulation of (NS) weaker than (3.11). Putting $\varphi := v \in V$ in the first equation of (3.11), we obtain the following equation: for all $v \in V$,

$$\left\langle \frac{\partial u}{\partial t}, v \right\rangle_V + a_0(u,v) + a_1(u,u,v) = \langle f, v\rangle_H - \int_{\Gamma_2} p^b v \cdot n ds \tag{3.14}$$

in $L^1(0,T)$.

Corollary 3.13. *Under the condition (3.10), there exists a solution* $u \in L^2(0,T;V) \cap L^\infty(0,T; L^2(\Omega)^d) \cap C([0,T];V^*)$ *to (3.14) with* $u(0) = u_0$ *such that* $\frac{\partial u}{\partial t} \in L^{4/p_d}(0,T;V^*)$.

Remark 3.14. *For* $f \in L^2(0,T;L^2(\Omega)^d)$ *and* $p^b \in L^2(0,T;H^{1/2}(\Gamma_2))$, *local existence and uniqueness of a weak solution* u *to (3.14) with* $u_0 \in H$ *are proven by Bernard (2003). Since it holds that*

$$a_0(u,v) = \sum_{i,j=1}^{d} \int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j} dx + \int_{\Gamma_2} \kappa u \cdot v ds \quad \text{for all } u, v \in H,$$

where $\kappa := \operatorname{div} n = (d-1)\times$*(mean curvature) (cf. Remark 3.1), (3.14) is equivalent to*

$$\begin{aligned} &\left\langle \frac{\partial u}{\partial t}, v \right\rangle_V + \sum_{i,j=1}^{d} \int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j} dx + \int_{\Gamma_2} \kappa u \cdot v ds + a_1(u,u,v) \\ =& \langle f, v\rangle_H - \int_{\Gamma_2} p^b v \cdot n ds \quad \text{for all } v \in V \end{aligned} \tag{3.15}$$

in $L^1(0,T)$. *Kim and Cao (2015) prove that there exists a weak solution* u *to (3.15) with* $f \in L^2(0,T;V^*)$, $p^b \in L^2(0,T;H^{-1/2}(\Gamma_2))$, *and* $u_0 \in U$, *where* U *is defined in Remark 3.4. We demonstrate the existence of a weak solution* u *to (3.14) with a different approach than Bernard (2003) and Kim and Cao (2015).*

令和４年２月４日

学位論文審査報告書（甲）

１．学位論文題目（外国語の場合は和訳を付けること。）

Mathematical analysis of pressure Poisson methods and projection methods involving pressure boundary conditions for incompressible viscous flows

（非圧縮粘性流れに対する圧力境界条件を含む圧力 Poisson 法と射影法の数学解析）

２．論文提出者 (1) 所　　属　数物科学　専攻

(2) 氏　　名　松井　一徳（まつい　かずのり）

３．審査結果の要旨（600～650 字）

松井一徳君は、2019 年３月に数物科学専攻博士前期課程修了後、同年４月に同専攻博士後期課程に進学した。研究テーマは、圧力境界条件を含む非圧縮粘性流れ問題およびその数値解法の数学解析である。流体方程式の数学解析およびその数値解析は、偏微分方程式論・数値解析学における中心的なテーマの１つであり、応用上も非常に重要である。松井君が取り組んだ、境界条件として圧力が与えられている問題は、具体的な工学問題や射影法などの数値解法においてしばしば現れる。しかしこれまで、非圧縮粘性流れに対し圧力境界条件を数学的に厳密に取り扱った研究は多くなく、松井君の本研究はその数学的な基礎付けを与えるものとして価値の高いものである。博士論文は３つのパートからなり、第１部は、射影法または圧力ポアソン方程式法と呼ばれる定常非圧縮ストークス方程式の数学的誤差評価を証明したもので、同君の単著として DCDS-S 誌 (2021 年３月)に掲載済みである。第２部は、定常非圧縮ストークス方程式と圧力ポアソン方程式を同時に解析する数学的枠組みを与えたもので、国際共同研究の結果として２編の論文にまとめられている。第３部は、非定常非圧縮ナビエ・ストークス方程式において圧力境界条件と、射影法による数値解法の誤差評価を論じたものので、独自性と有用性に加え数学的に完成度の高い結果である (単著論文投稿中)。以上により本論文は、博士(理学)を授与するに値すると判断した。

４．審査結果 (1) 判　　定（いずれかに○印）（合　格）・　不合格

(2) 授与学位　博　士（　理学　）