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The Gross-Saccoman Conjecture is True

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Consider a graph with perfect nodes but independent edge failures with identical probability ρ. The reliability is the connectedness probability of the random graph. A graph with n nodes and e edges is *uniformly optimally reliable* (UOR) if it has the greatest reliability among all graphs with the same number of nodes and edges, for all values of ρ. In 1997, Gross and Saccoman proved that the simple UOR graphs for $e = n$, $e = n + 1$ and $e = n + 2$ are also optimal when the classes are extended to include multigraphs [6]. The authors conjectured that the UOR simple graphs for $e = n + 3$ are optimal in multigraphs as well. A proof of the Gross-Saccoman conjecture is introduced.

KEYWORDS
Graph Theory, Uniformly optimally reliable graph, Gross-Saccoman conjecture, Network Reliability, Optimization, Multigraphs.

1 | PRELIMINARIES

In this work a graph may have multiple edges. Consider a graph $G = (V, E)$ with n nodes and e edges whose failures are independent, with identical probability ρ. The *unreliability polynomial*, $U_G(\rho)$, is the probability that the resulting random graph is not connected:

$$U_G(\rho) = \sum_{k=0}^{e} m_k(G)\rho^k(1-\rho)^{e-k}, \tag{1}$$

where $m_k(G)$ is the number of edge-cuts of size k. The *reliability* is $R_G(\rho) = 1 - U_G(\rho)$. A graph G is *uniformly optimally reliable* (UOR), if $R_G(\rho) \geq R_H(\rho)$ for all $\rho \in [0, 1]$ and all graphs H with the same number of nodes and edges. This concept has been introduced by Boesch in 1986 [2]. However, the progress in this field is slow, and there are several open conjectures. It is obvious that if $m_k(G) \leq m_k(H)$ for all $k \in \{0, 1, \ldots, e\}$ and all graphs H, then G is UOR. The converse is an enigmatic conjecture posed by Boesch in his foundational article [2].

Let us denote by $\Omega(n, e)$ the set of all graphs with n nodes and e edges, and by $\Omega_s(n, e)$ the subclass of simple graphs. All the trees share the same unreliability polynomial, and they are UOR in the class $\Omega_s(n, n-1)$. The UOR graphs in the class $\Omega_s(n, n)$ are elementary cycles. The first non-trivial UOR graphs were discovered by Boesch et al. [3]. The authors formally proved that some elementary subdivisions of a θ-graph and K_4 are UOR in the respective classes $\Omega_s(n, n+1)$ and $\Omega_s(n, n+2)$; see Figure 1 for a representation of a θ-graph. They conjectured that special subdivisions of the complete bipartite graph $K_{3,3}$ are UOR for the classes $\Omega_s(n, n+3)$, and Wang formally proved that the conjecture is true [9]. A natural question is to determine if the optimality is a hereditary property from $\Omega_s(n, e)$ to $\Omega(n, e)$. This question is intrinsically attractive from a theoretical viewpoint, and it also finds applications in communication systems, where a trade-off between redundancy (i.e., repeated links) and multi-path diversity should be met. Gross and Saccoman proved that the UOR graphs from the previous classes $e = n$, $e = n+1$ and $e = n+2$ are also UOR for multigraphs. They conjectured that the elementary subdivisions of $K_{3,3}$ that Wang proved to be UOR in the classes $\Omega_s(n, n+3)$ are also UOR for multigraphs. Here it is proved that the Gross-Saccoman conjecture is true.

This article is organized in the following manner. Section 2 presents the families of UOR graphs for the respective classes $e = n+1$, $e = n+2$ and particularly when $e = n+3$, which is our focus. A proof-strategy is presented in Section 3, where the set $\Omega(n, n+3)$ is partitioned into different sub-classes. The study of different hierarchical sub-classes is presented in Section 4, while the main result is proved in Section 5.

Without loss of generality, we will restrict our attention to connected graphs solely. For general graph-theoretic terminology, the reader is invited to consult the excellent book authored by Harary [7]. Some concepts are presented here for our specific purpose. A *cut-node* is a node v such that $G - v$ is not connected. A graph is *biconnected* if it has no cut-nodes. A *chain* is an elementary path, whose internal nodes have degree 2 and the external nodes have degree strictly greater than 2. A θ-graph consists of two nodes with degree 3 joined by three chains.

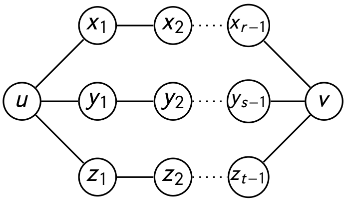

FIGURE 1 θ-graph with lengths r, s and t.

The *co-rank* of a biconnected graph G with n nodes and e edges, is $c(G) = e - n + 1$. A *bridge* is an edge e such that $G - e$ is not connected. A *cut* is an edge-set $C \subseteq E$ such that $G - C$ is not connected. The number of cuts of size k for a graph G, or k-*cuts*, is here denoted $m_k(G)$. A graph G is *stronger* than H, and it is denoted $G \succeq H$, if both graphs have the same number of nodes and edges, and $m_k(G) \leq m_k(H)$ for all $k \in \{0, \ldots, e\}$. We also say that H is *weaker* than G. The relation $\succeq$ will be used in the proof, and it is a partial order in the set $\Omega(n, e)$. A graph G is the *strongest* in an arbitrary graph class Ω, if $G \succeq H$ for all $H \in \Omega$. The strongest graph in $\Omega_s(n, e)$ is UOR, whenever it exists. There are infinite pairs of (n, e) such that a UOR does not exist in simple graphs [8], and in multigraphs [4]. The reader is invited to consult [1] for an overview of the discovered UOR graphs so far. A reliability-increasing transformation is a bijective edge-mapping $f : E(G) \to E(G')$, such that $R_{G'}(\rho) \geq R_G(\rho)$ for all $\rho \in [0, 1]$. Furthermore, a reliability-increasing transformation f is *strong* if the inverse function f^{-1} assigns cuts in G' to cuts in G. Observe that if such a function exists, then G has at least as many cuts as G', and $G' \succeq G$.

2 | UNIFORMLY OPTIMALLY RELIABLE SIMPLE GRAPHS

Boesch et al. proved that a sequence θ_n of θ-graphs are UOR graphs in the classes $\Omega_s(n, n+1)$, when the length of its chains differ at most by one. In the class $\Omega_s(n, n+2)$, the same authors proved that some special elementary subdivisions of K_4 are UOR for all $n \geq 4$. To define the corresponding family $K(n, n+2)$ of UOR graphs, first observe that K_4 accepts an edge-partition into three perfect matchings, $E(K_4) = \{e_1, e_2\} \cup \{e_3, e_4\} \cup \{e_5, e_6\}$. Consider the natural division $n-4 = 6s+r$ for some $r \leq 5$. We must insert $n-4$ nodes into the six edges such that the lengths of the resulting chains are $l_{e_i} = s+2$ if $i \leq r$, or $l_{e_i} = s+1$ otherwise; see Figure 2 for an illustrative description. Boesch et al. observed that $K_{3,3}$ also accepts a tri-partition of perfect matchings, and they conjectured that identical subdivisions lead to a novel family $G_u(n, n+3)$ of graphs, which are also UOR for the class $\Omega_s(n, n+3)$. The conjecture is true, and a full proof was introduced by Wang [9]. Later, Gross and Saccoman proved that the elementary cycles, balanced θ-graphs and $K(n, n+2)$ are UOR even when the classes are extended to multigraphs. The authors proposed the following:

Conjecture 1 (Gross-Saccoman [6]) *$G_u(n, n+3)$ is UOR in the extended set of multigraphs $\Omega(n, n+3)$, for all $n \geq 6$.*

A proof is here included. First, an explicit definition for $G_u(n, n+3)$ is given. First, consider an arbitrary partition of the edge-set of $K_{3,3}$ into three perfect matchings: $E(K_{3,3}) = \{e_1, e_2, e_3\} \cup \{e_4, e_5, e_6\} \cup \{e_7, e_8, e_9\}$. Consider the natural division $n-6 = 9s+r$, for some $r \leq 8$. The lengths of the respective chains are $l_{e_i} = s+2$, if $i \leq r$, or $l_{e_i} = s+1$ otherwise. In Figure 2, $G_u(n, n+3)$ is represented when $n = 16$. The nodes are labeled following the insertion order. In this case $l_{e_1} = 3$, and $l_{e_i} = 2$ for all $i \in \{2, \ldots, 9\}$.

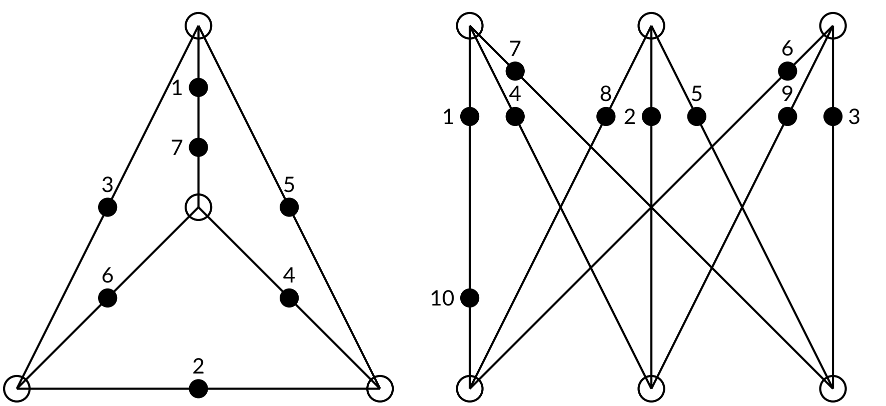

FIGURE 2 Graphs $K(n, n+2)$ for $n = 11$ (left), and $G_u(n, n+3)$ for $n = 16$ (right).

3 | PROOF STRATEGY

Lemma 1 restricts our study to biconnected graphs. It was proved by Wang [9] and rediscovered by Canale et al. [5]:

Lemma 1 *For every non-biconnected graph $G \in \Omega(n,e)$, there exists a strong reliability-increasing transformation $f : G \to G'$ such that G' is biconnected, whenever $e \geq n$.*

In particular, for every non-biconnected graph G there exists some biconnected graph G' such that $G' \geq G$. Figures 3 and 4 illustrate the idea of the proof, where the bridges are first included in cycles one-by-one (Figure 3), and then shortcuts are considered to avoid cut-nodes (Figure 4). The authors proved the result for the set $\Omega_s(n,e)$, but the same transformations work in $\Omega(n,e)$. In fact, observe that after Step 1, $e = (vw)$ is not a bridge any more in the graph $G^1 = G - (xy) + (yv)$, where the edge (xy) belongs to some elementary cycle from G. A bridgeless graph G^1_{out} is obtained after an iterative application of Step 1. The resulting graph G^1_{out} could have some cut-node w. However, in Step 2 we consider $G^2 = G^1_{out} - (wx) + (xy)$, and w is not a cut-node any more. A biconnected graph G^2_{out} is obtained after an iterative application of Step 2, and the authors in [5] proved that $G^2_{out} \geq G^1_{out}$, and $G^1_{out} \geq G$. Finally, Lemma 1 is proved choosing $G' = G^2_{out}$. The reader is invited to consult [5] for further details.

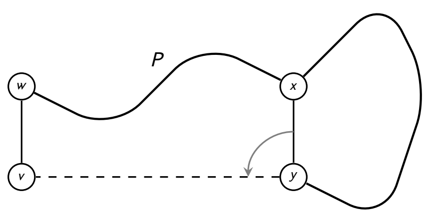

FIGURE 3 Step 1: avoiding the bridge $e = (vw)$.

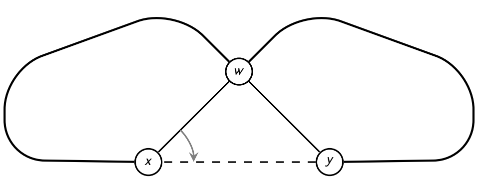

FIGURE 4 Step 2: avoiding the cut-node w.

The co-rank of a biconnected graph $G \in \Omega(n, n+3)$ is $c(G) = 4$. In biconnected graphs $m_0(G) = m_1(G) = 0$, and if we remove five edges or more, the resulting subgraph is not connected: $m_i(G) = \binom{n+3}{i}$, for all $i \geq 5$. From Expression (1), the simultaneous minimization of the cuts $m_2(G)$, $m_3(G)$ and $m_4(G)$, is enough for a biconnected graph to become the strongest. Observe that the co-rank is increased a unit when a repeated edge is included. Recall that the addition of an *open ear* in a graph G is the addition of an external elementary path $P_{u,v}$ (this is, $P_{u,v} \cap G = \{u, v\}$), that connects non-adjacent nodes u and v in the graph G. A celebrated work from Whitney states that every biconnected graph is an iterative augmentation of open ears, starting from an elementary cycle [10]. Observe that the addition of an open ear of an elementary cycle is always a θ-graph. Then the biconnected graphs in $\Omega(n, n+3)$ can be partitioned into four sub-classes:

1. Ω^1: Cycles with three repeated edges.
2. Ω^2: θ-graphs with two repeated edges.
3. Ω^3: θ-graphs with one open ear and one repeated edge.
4. Ω^4: Simple graphs: $\Omega^4 = \Omega_s(n, n+3)$.

Given two graph-sets Ω and Ω', we say that Ω' *dominates* Ω and denote $\Omega \to \Omega'$, if for every $G \in \Omega$ there exists some stronger graph $G' \in \Omega'$. In this case, Ω *is dominated by* Ω'. This relation is transitive, as well as the precedence relation $\geq$ between graphs.

The proof-strategy follows a *laddering-domination* technique, where we formally show that $\Omega^i \to \Omega^{i+1}$, that is, Ω^i is dominated by Ω^{i+1}, for $i \in \{1, 2, 3\}$. In order to prove the chain of dominations $\Omega^1 \to \Omega^2 \to \Omega^3 \to \Omega^4$, four steps are considered:

1. Prove the existence of the strongest graph $C^* \in \Omega^1$ (Subsection 4.1).
2. Show that C^* is weaker than some graph $H \in \Omega^2$ (Subsection 4.2).
3. For every graph $G \in \Omega^2$, find a stronger graph $G' \in \Omega^3$ (Subsection 4.3).
4. Prove the existence of the strongest graph $K^* \in \Omega^3$ (Subsection 4.4).
5. Find a simple graph $H_s \in \Omega^4 = \Omega_s(n, n+3)$ such that $H_s \geq K^*$ (Subsection 4.5).

The main result follows from the previous statements and the transitivity of the domination. In fact, by Steps 1-4 we immediately obtain that $\Omega^1 \to \Omega^2 \to \Omega^3 \to \Omega^4$, and $G_u(n, n+2)$ is the strongest graph in the strongest family Ω^4. The aforementioned relations will be proved using elementary combinatorics (counting cuts) in some cases, and exploiting the following property shared by some sequences of UOR graphs in some others:

Property 1 (Self-Similarity) *A graph-sequence $\{G_n\}_{n\geq 1}$ verifies the self-similarity property if there exists some natural n_0 such that $G_{n+1} * e_{n+1} = G_n$, for all $n \geq n_0$, where the symbol $*$ stands for edge-contraction, for some edge-sequence $e_{n+1} \in E(G_{n+1})$.*

Since the contraction is the inverse operation of the subdivision, the three sequences $\theta_n \in \Omega_s(n, n+1)$, $K(n, n+2) \in \Omega_s(n, n+2)$ and $G_u(n, n+3) \in \Omega_s(n, n+3)$ share the self-similarity property. In fact, these graph-sequences are derived by iterative elementary subdivisions, and the previous graph is recovered after the contraction of the subdivided edges: $G_{n+1} * e_{n+1} = G_n$.

A curious fact is that the intermediate graph K^* is derived using the self-similarity property shared by all the members from the sequence $K(n, n+2)$. Using the fact that $K(n, n+2)$ is the strongest sequence in $\Omega_s(n, n+2)$, we will duplicate the last subdivided edge e, denoted by e', and $K^* = K(n, n+2) \cup \{e'\}$ will be the strongest in the family Ω^3. Gross and Saccoman pointed out that the analysis in $\Omega(n, n+3)$ is involved, given its large number of sub-classes. This self-similarity property greatly simplifies the proof, and avoids an exhaustive classification of the sub-class Ω^3, as we will see in Subsection 4.4. Additionally, it could also be considered for future work in the study of new graph-sets $\Omega(n, e)$.

4 | HIERARCHICAL SUB-CLASSES

4.1 | Strongest graph in Ω^1

The strongest graph $C^* \in \Omega^1$ will be found by counting cuts. Figure 5 presents the three types of graphs belonging to Ω^1. The three types $C^{(3)}$, $C^{(2,1)}$ and $C^{(1,1,1)}$ can be identified with the corresponding partitions: $3 = 2+1 = 1+1+1$, where 3 means that the same edge is picked 3 times to include repeated edges, and three different edges are picked for the case $1+1+1$. The repeated edges are contiguous in Figure 5; however, the number of cuts does not depend on the relative position of the repeated edges.

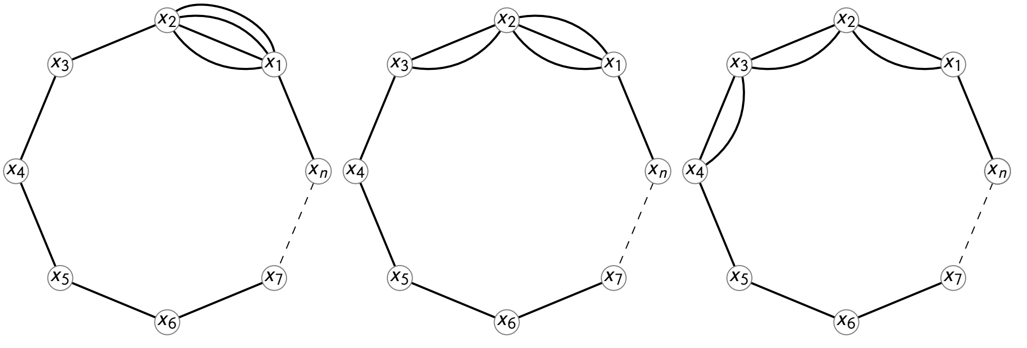

FIGURE 5 Graph-types $C^{(3)}$ (left), $C^{(2,1)}$ (middle) and $C^* = C^{(1,1,1)}$ (right).

Lemma 2 *$C^{(1,1,1)} = C^*(n)$ is the strongest graph in the class Ω^1.*

Proof We count cuts and use the identity $\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1}$:

$$
\begin{aligned}
m_2(C^{(3)}) &= \binom{n-1}{2} > \binom{n-2}{2} = m_2(C^{(2,1)}) > \binom{n-3}{2} = m_3(C^{(1,1,1)});\\
m_3(C^{(3)}) &= \binom{n-1}{3} + 4\binom{n-1}{2} = \binom{n-2}{3} + 5\binom{n-2}{2} + 4(n-2) > m_3(C^{(2,1)})\\
&= \binom{n-2}{3} + 5\binom{n-2}{2} + n - 2 = \binom{n-3}{3} + 6\binom{n-3}{2} + 5(n-3) + n - 2\\
&> \binom{n-3}{3} + 6\binom{n-3}{2} + 3(n-3) = m_3(C^{(1,1,1)});\\
m_4(C^{(3)}) &= \binom{n-1}{4} + 4\binom{n-1}{3} + 6\binom{n-1}{2} = \binom{n-2}{4} + 5\binom{n-2}{3} + 10\binom{n-2}{2} + 6(n-2) > m_4(C^{(2,1)})\\
&= \binom{n-2}{4} + 5\binom{n-2}{3} + 10\binom{n-2}{2} + 4(n-2)\\
&= \binom{n-3}{4} + 6\binom{n-3}{3} + 15\binom{n-3}{2} + 14(n-2)\\
&> \binom{n-3}{4} + 6\binom{n-3}{3} + 15\binom{n-3}{2} + 12(n-3) = m_4(C^{(1,1,1)}) \blacksquare
\end{aligned}
$$

4.2 | The Sub-Class Ω^1 is dominated by Ω^2

By Lemma 2, we know that C^* is the strongest in Ω^1. If we find a member $H \in \Omega^2$ that is stronger than C^*, the whole sub-class Ω^1 is dominated by Ω^2.

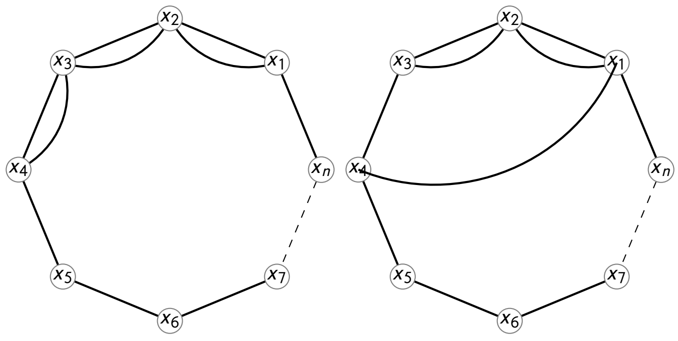

FIGURE 6 Graphs $C^* \in \Omega^1$ (left) and $H \in \Omega^2$ (right)

Lemma 3 *$H \geq C^*$ for some $H \in \Omega^2$.*

Proof Consider $H = C^* - (x_3, x_4) \cup \{(x_1, x_4)\}$ depicted in Figure 6. By direct counting, we show that $m_k(H) \leq m_k(C^*)$ for $k \in \{2, 3, 4\}$:

$$\begin{aligned}
m_2(C^*) &= \binom{n-3}{2} \geq \binom{n-4}{2} = m_2(H); \\
m_3(C^*) &= \binom{n-3}{3} + 6\binom{n-3}{2} + 3(n-3) = \binom{n-4}{3} + 7\binom{n-4}{2} + 6(n-4) + 3(n-3) \\
&\geq \binom{n-4}{3} + 6\binom{n-4}{2} + (n-4) + 2 = m_3(H); \\
m_4(C^*) &= \binom{n-3}{4} + 6\binom{n-3}{3} + 15\binom{n-3}{2} + 12(n-3) = \binom{n-4}{4} + 7\binom{n-4}{3} + 21\binom{n-4}{2} + 15(n-4) + 12(n-3) \\
&\geq \binom{n-4}{4} + 6\binom{n-4}{3} + 15\binom{n-4}{2} + 8(n-4) + 7 = m_4(H). \blacksquare
\end{aligned}$$

4.3 | The Sub-Class Ω^2 is dominated by Ω^3

Recall that Ω^2 consists of a θ-graph with two repeated edges. These repeated edges can either belong to the same chain (graph-types A and C) or not (graph-type B). Then, all the graphs belonging to Ω^2 have types A, B or C. The reader can appreciate that the number of cuts does not depend on the position of the repeated edges in the chain. Then, Figure 7 presents an exhaustive list of the graph-types G_A, G_B and G_C to be analyzed in Ω^2. We will show that for every graph $G_X \in \Omega^2$ there exists some stronger graph $G'_X \in \Omega^3$. The key is to adequately replace a repeated edge by another simple edge, for each graph-type. Figure 8 presents the respective graphs G'_A, G'_B and $G'_C \in \Omega^3$. Let r, s and t be the lengths of the left, middle and right chains of the underlying θ-graph. Intuitively, the left chain is more robust under single failures in the respective graphs belonging to Ω^3. This is the rationale behind the presented construction.

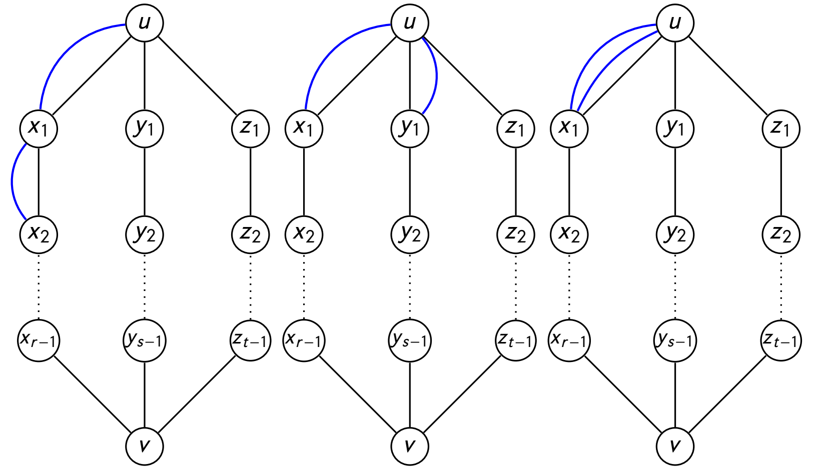

FIGURE 7 Graph-types $G_A, G_B, G_C \in \Omega^2$.

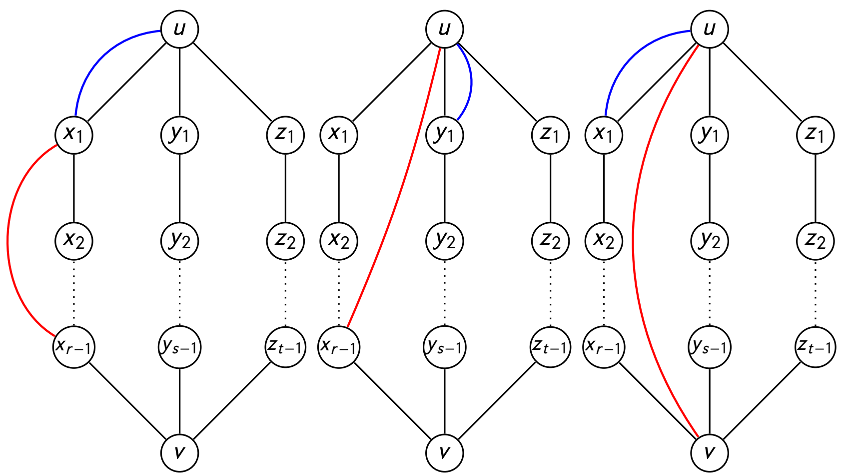

FIGURE 8 Graph-types $G'_A, G'_B, G'_C \in \Omega^3$.

Lemma 4 *The graph $G'_X \in \Omega^3$ is stronger than $G_X \in \Omega^2$ for all $X \in \{A, B, C\}$.*

Proof We will prove that $m_k(G'_X) \leq m_k(G_X)$ for all $k \in \{2, 3, 4\}$ and $X \in \{A, B, C\}$. In order to count 2-cuts, observe that we must pick two edges belonging to the same chain; otherwise, the resulting graph is connected. The graphs G'_X were constructed in such a way that the number of 2-cuts in each chain of G'_X and G_X are identical. Then, $m_2(G'_X) = m_2(G_X)$, for each $X \in \{A, B, C\}$. The differences between G_X and G'_X exist only in the left chain. In order to compare 3-cuts, the differences in counting occur when the three edges are picked from the left chain, or when a single edge is picked from each chain, and:

$$
\begin{aligned}
m_3(G_A) - m_3(G'_A) &= [\binom{r-2}{3} + 4\binom{r-2}{2} + 2(r-2) + (r-2)st] - [\binom{r-2}{3} + 4\binom{r-2}{2} + (r-2) + (1)st + 1] \\
&= (r-3)(st+1) \geq 0; \\
m_3(G_B) - m_3(G'_B) &= [\binom{r-1}{3} + 2\binom{r-1}{2} + (r-1) + (r-1)(s-1)t] - [\binom{r-1}{3} + 2\binom{r-1}{2} + (r-1) + (1)(s-1)t] \\
&= (r-2)(s-1)t \geq 0; \\
m_3(G_C) - m_3(G'_C) &= [\binom{r-1}{3} + 3\binom{r-1}{2} + (r-1)st] - [\binom{r-1}{3} + 3\binom{r-1}{2} + r - 1] = (r-1)(st-1) \geq 0.
\end{aligned}
$$

Finally, in order to compare m_4 we count the number of spanning-trees or the *tree-number*, which is the complement: $\tau(G_X) = \binom{n+3}{4} - m_4(G_X)$. Alternatively, we will show that $\tau(G'_X) \geq \tau(G_X)$ for each $X \in \{A, B, C\}$. Let us study graph-type A first. Recall that spanning-trees are maximally acyclic subgraphs. Then, we must pick precisely four edges in order to break all the cycles, and the spanning-tree cannot have multiple edges either. Let us denote for convenience $\tau(r, s, t) = rs + rt + st$, the tree-number of a θ-graph with lengths r, s and t. Closed forms for the tree-numbers can be found if we consider all the possible states of the repeated edges, breaking all the cycles (and repeated edges). For graph-type A, there are $\binom{2}{1} \times \binom{2}{1} = 4$ ways to obtain a simple subgraph, and each selection has $\tau(r-2, s, t)$ possible spanning-trees (those links cannot be picked twice to avoid repetitions). There are also $\binom{4}{3} = 4$ ways to pick precisely three elements of the repeated edges, and each selection provides $s + t$ ways to pick an edge from a different chain. A similar reasoning holds for G'_A, and:

$$
\begin{aligned}
\tau(G_A) &= 4\theta(r-2, s, t) + 4(s+t) = 4\theta(r, s, t) - 4(s+t); \\
\tau(G'_A) &= 2\theta(r-1, s, t) + 2(r-2)(st + s + t) + (r-1)(s+t) \\
&\geq 2\theta(r-1, s, t) + 2\theta(r-2, s, t) + (r-1)(s+t) = 4\theta(r, s, t) + (r-6)(s+t).
\end{aligned}
$$

Since the graph G'_A makes sense when $r \geq 4$, clearly $(r-6)(s+t) \geq -4(s+t)$, and $\tau(G'_A) \geq \tau(G_A)$.

An analogous reasoning holds for G_B and G'_B. The main difference is that we can remove the four repeated edges in G_B, adding a unit in the tree-number:

$$
\begin{aligned}
\tau(G_B) &= 4\theta(r-1, s-1, t) + 2(t+s-1) + 2(t+r-1) + 1 = 4\theta(r-1, s-1, t) + 2(t+s-1) + r + t + (r+t-1); \\
\tau(G'_B) &= 2\theta(r, s-1, t) + 2(r-1)((s-1)t + s - 1 + t) + r + t + (r-1)(1+t) \\
&\geq 4\theta(r-1, s-1, t) + 2(t+s-1) + r + t + (r-1)(1+t).
\end{aligned}
$$

Since the graph G'_B makes sense when $r \geq 3$, then $(r-1)(1+t) = r + t(r-1) - 1 \geq r + t - 1$, and $\tau(G'_B) \geq \tau(G_B)$.

Finally, to count $\tau(G_C)$ we must pick either two or three edges from the triple repetition, and:

$$
\begin{aligned}
\tau(G_C) &= 3\theta(r-1, s, t) + s + t; \\
\tau(G'_C) &= 2\theta(r-1, s, t) + s + t + 2(r-1)st + st.
\end{aligned}
$$

It suffices to prove that $2(r-1)st + st \geq \theta(r-1, s, t) = (r-1)s + (r-1)t + st$. But this inequality holds if and only if $2st \geq s + t$, which is true, since $2st \geq 2\max\{s, t\} \geq s + t$.

Observe that the constructed graphs G'_X are stronger whenever $r \geq 4$ for graph-type A, $r \geq 3$ for graph-type B and $r \geq 2$ for graph-type C. In fact, the equality $G'_X = G_X$ is obtained when $r = 3$, $r = 2$ or $r = 1$ in the respective graphs $X \in \{A, B, C\}$. Let us discuss the applicability of the previous reasoning as a function of r and s:

- Graph-type A: if $r = 3$, we get $G'_A = G_A$. However, we can adjust G'_A, replacing (x_1, x_{r-1}) by (x_1, v). The previous reasoning works with minor variations. It is clear that $r \geq 2$ in the graph G_A, so the last case is $r = 2$. In this case, we can consider $G''_A = G_A - (u, x_1) \cup \{(x_1, y_1)\} \in \Omega^2$. A direct calculation shows that G''_A is stronger than G_A.
- Graph-type B: if $r = 2$, we get $G'_B = G_B$. In this case, we can consider $G''_B = G'_B - (u, x_{r-1}) \cup \{(u, v)\}$, and this new graph is stronger than G_B. If $r = 1$ but $s \geq 2$, we consider G'_C, and by symmetry G'_C it is stronger than G_B. In fact, the roles of the left and middle chains are just exchanged. Finally, if $r = s = 1$, we get $G_B = C^{(3)} \in \Omega^1$, and it does not belong to Ω^2 (in fact, $C^{(3)}$ is weaker than C^* by Lemma 2).
- Graph-type C: if $r \geq 2$, the previous reasoning works. Finally, if $r = 1$ we obtain an elementary cycle $G_C = C_n \cup \{e_1, e_2, e_3\}$, where the e_i are three repeated edges linking non-adjacent nodes in the cycle (if such nodes were adjacent, we would get $C^{(3)} \in \Omega^1$). Consider $G''_C = G_C - e_1 \cup \{e\}$, where $e = (y_{\lfloor s/2 \rfloor}, z_{\lfloor t/2 \rfloor})$ is an edge that connects the middle points between the chains of lengths s and t. The study of this *crossing* is analogous to Lemma 3, and with a similar calculation it can be checked that G''_C is stronger than G_C. ■

4.4 | Strongest graph in Ω^3 and Self-Similarity

The subclass Ω^3 is the repetition of an edge in particular graphs belonging to the set $\Omega_s(n, n+2)$. Here, we exploit the self-similarity property 1 to build the strongest graph $K^* = K(n, n+2) \cup \{e'\}$, where e' is the repeated edge of $e : K(n, n+2) * e = K(n-1, n+1)$. First, a technical lemma that avoids loops will be used:

Lemma 5 *If $G \geq H$ in $\Omega(n, e)$, then $G \cup \{l\} \geq H \cup \{l\}$ in $\Omega(n, e+1)$, where $l = (v, v)$ is an arbitrary loop.*

Proof The loop can appear in the cuts or not. If $G \geq H$, then $m_{k+1}(G \cup \{l\}) = m_k(G) + m_{k+1}(G) \leq m_k(H) + m_{k+1}(H) = m_{k+1}(H \cup \{l\})$, for all k. Additionally, the connectedness is not modified with loops, and $m_0(G \cup \{l\}) = m_0(G) \leq m_0(H) = m_0(H \cup \{l\})$, so $G \cup \{l\} \geq H \cup \{l\}$.

Normally, loops are out of the scope in reliability analysis, since they are useless. However, Lemma 5 is considered in the following:

Proposition 6 (Self-Similarity) *$K^* = K(n, n+2) \cup \{e'\}$ is the strongest in the set Ω^3.*

Proof Consider an arbitrary graph $G \in \Omega^3$. Then, $G = H \cup \{f\}$, where f is a repeated edge in a simple graph $H \in \Omega_s(n, n+2)$. The cuts from $G = H \cup \{f\}$ and $K^* = K(n, n+2) \cup \{e'\}$ either contain the repeated edge (respectively, f and e'), or not. By sum-rule:

$$m_i(G) = m_{i-1}(H) + m_i(H * f \cup \{l\}),$$
$$m_i(K^*)(n) = m_{i-1}(K(n, n+2)) + m_i(K(n-1, n+1) \cup \{l\}),$$

where $H * f$ denotes the edge-contraction, and l represents a loop that appears after the contraction of the repeated edge f. Since $H \in \Omega_s(n, n+2)$ and $K(n, n+2)$ is the strongest in this class, $m_{i-1}(H) \leq m_{i-1}(K(n, n+2))$. It suffices to prove that $m_i(K(n-1, n+1) \cup \{l\}) \leq m_i(H * f \cup \{l\})$.

Recall that Gross and Saccoman already proved that $K^*(n-1, n+1)$ is the strongest family of graphs, even in the set $\Omega(n-1, n+1)$ of graphs with multiple edges [6]. Since $H * f \in \Omega(n, n+2)$, we have $K^*(n-1, n+1) \geq H * f$. By Lemma 5, $K^*(n-1, n+1) \cup \{l\} \geq H * f \cup \{l\}$, and in particular $m_i(K(n-1, n+1) \cup \{l\}) \leq m_i(H * f \cup \{l\})$. ■

4.5 | The Sub-Class Ω^3 is dominated by Ω^4

It suffices to prove that $G_u(n, n+2) \geq K^*(n)$. Consider H_s depicted in Figure 9:

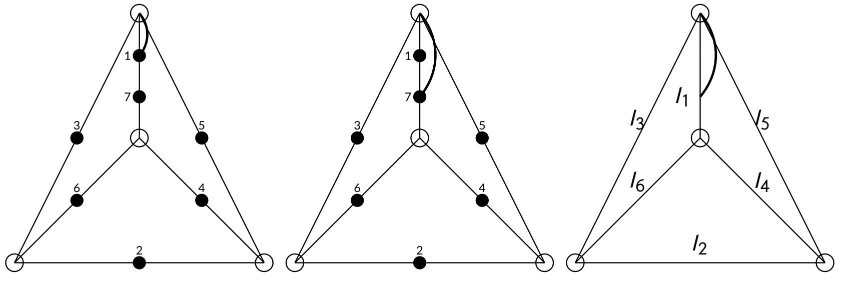

FIGURE 9 Respective graphs K^* (left), H_s (middle) and lengths of the respective chains (right)

Proposition 7 *There exists a simple graph $H_s \in \Omega_4 = \Omega_s(n, n+3)$ such that $H_s \geq K^*(n)$.*

Proof Consider the chain in K^* that contains the repeated edge, $C = \{(x_0, x_1), \ldots, (x_{l_1-1}, x_{l_1})\}$, where $e = (x_0, x_1)$ is repeated, and define $H_s = (K^* - \{e\}) \cup \{(x_0, x_{l_1-1})\}$. Both graphs are depicted in Figure 9. It is clear that $m_2(H_s) = m_2(K^*)$. In order to compare m_3 and m_4, the key is to note that the resulting chain C' induced in H_s is *stronger* than the original chain C induced in K^*. If we pick two edges or more from the same chain, the cuts appear in both cases. A difference is appreciated when three edges are picked from different chains. Further, the three chains must be adjacent in order to obtain cuts. Therefore, if we pick single edges from triads of adjacent chains, most of the terms are cancelled, and the following difference is met:

$$
\begin{aligned}
m_3(K^*) - m_3(H_s) &= [l_4 l_6(l_1-1) + l_3 l_5(l_1-1) + l_2 l_3 l_6 + l_2 l_4 l_5] - [l_4 l_6 \times 1 + l_3 l_5 \times 1 + l_2 l_3 l_6 + l_2 l_4 l_5] \\
&= l_4 l_6(l_1-2) + l_3 l_5(l_1-2) \geq 0,
\end{aligned}
$$

since the chain C, with at least one inserted node, has length $l_1 \geq 2$. Similarly, the only difference in the cuts for m_4 is found when the four edges are picked from different chains:

$$
\begin{aligned}
m_4(K^*) - m_4(H_s) &= [\sum_{i=2}^{6} \frac{\prod_{j=2}^{6} l_j}{l_i} + 2(l_2 l_3 l_6 + l_2 l_4 l_5) + (l_1 - 1) \sum_{2 \leq i < j < k \leq 6} l_i l_j l_k] \\
&- [\sum_{i=2}^{6} \frac{\prod_{j=2}^{6} l_j}{l_i} + l_1(l_2 l_3 l_6 + l_2 l_4 l_5) + 1 \times \sum_{2 \leq i < j < k \leq 6} l_i l_j l_k] \\
&= (l_1 - 2) \sum_{2 \leq i < j < k \leq 6} l_i l_j l_k - (l_1 - 2)(l_2 l_3 l_6 + l_2 l_4 l_5) \geq 0.
\end{aligned}
$$

Therefore, $m_i(K^*) \leq m_i(H_s)$ for all i, and $H_s \geq K^*(n)$. ■

5 | MAIN RESULT

The main result is just a corollary of the previous study: Conjecture 1 is true. A synthesis of the whole reasoning is included here for completeness:

Theorem 8 *$G_u(n, n+3)$ is UOR in the extended set of multigraphs $\Omega(n, n+3)$, for all $n \geq 6$.*

Proof The partition $\Omega(n, n+3) = \cup_{i=0}^{4} \Omega^i$ is considered, where $4 - i$ is the number of repeated edges in Ω^i. The set Ω^0 contains non-biconnected graphs, and they can be discarded by Lemma 1. The set Ω^4 contains simple graphs, and Wang proved that $G_u(n, n+3)$ are the strongest in this class. The strongest member from Ω^1 is C^* (see Lemma 2), and by Lemma 3, it is weaker than some member $H \in \Omega^2$. Similarly, for every graph in Ω^2, there exists a stronger member belonging to Ω^3 (Lemma 4). By the self-similarity property and Proposition 6, the graph $K^* = K(n, n+2) \cup \{e'\}$ is the strongest in the class Ω^3. By Proposition 7, H_s is stronger than K^*. Since G_u is stronger than H_s, it is also true that G_u is stronger than K^*. Therefore, the following chain of domination holds: $\Omega^1 \to \Omega^2 \to \Omega^3 \to \Omega^4$, and G_u is the strongest in Ω^4. This implies that G_u is the strongest in the set $\Omega(n, n+3)$, and the Gross-Saccoman Conjecture is true. ■

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