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# EXISTENCE AND MULTIPLICITY RESULTS FOR PARTIAL DIFFERENTIAL INCLUSIONS VIA NONSMOOTH LOCAL LINKING

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Abstract. We consider a partial differential inclusion driven by the $p$-Laplacian and involving a nonsmooth potential, with Dirichlet boundary conditions. Under convenient assumptions on the behavior of the potential near the origin, the associated energy functional has a local linking. By means of nonsmooth Morse theory, we prove the existence of at least one or two nontrivial solutions, respectively, when the potential is $p$-superlinear or at most asymptotically $p$-linear at infinity.

## 1. Introduction

This paper is devoted to the study of the following partial differential inclusion (PDI, for short) with Dirichlet boundary conditions:

$$\begin{cases} -\Delta_p u \in \partial j(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^N$ ($N > 1$) is a bounded domain with a $C^2$ boundary, $p > 1$, $j : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function and for a.e. $x \in \Omega$, $u \mapsto j(x,u)$ is locally Lipschitz, with subcritical growth. By $\partial j(x,u)$ we denote Clarke's generalized subdifferential of the function $u \mapsto j(x,u)$ (see Section 2).

The energy functional $\varphi$ associated to problem (1.1), defined by

$$\varphi(u) = \frac{\|\nabla u\|_p^p}{p} - \int_\Omega j(x,u)\,dx \text{ for all } u \in W_0^{1,p}(\Omega),$$

is locally Lipschitz continuous in the Sobolev space $W_0^{1,p}(\Omega)$, hence the weak solutions are defined as critical points of $\varphi$ in the sense of nonsmooth critical point theory introduced by Clarke [6].

The variational study of PDI's has its roots in the work of Chang [4], with applications to elliptic equations with discontinuous nonlinearities. Since then, variational methods based on min-max theorems were developed for several types of PDI's with

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Dirichlet or Neumann boundary conditions and nonlinear differential operators, see for instance [13, 14, 15, 16, 22, 24] and the monographs [3, 12, 21] (see also [1] for a different approach based on degree theory).

Morse theory for nonsmooth functionals was started by Corvellec [8] with the definition of critical groups for isolated critical points of a continuous (or even lower semicontinuous) functional on a complete metric space, in the framework of metric critical point theory introduced by Degiovanni [10] (see also [9, 11]), and applied to elliptic equations with general growth conditions which produce continuous energy functionals (see the survey [2]). The first application of nonsmooth Morse theory to PDI's, to the best of our knowledge, was proposed in [7] along with the introduction of critical groups at infinity and a suitable nonsmooth implicit function theorem, in the case of locally Lipschitz continuous energy functionals. Speaking generally, we can say that most results of Morse theory for $C^1$-functionals on Banach spaces (based on singular homology theory and critical groups) can be extended to the nonsmooth case, while getting a nonsmooth version of Morse lemma and of the subsequent results for $C^2$ functionals on Hilbert space are still open issues, due to the lack of an effective notion of second-order derivative (though some hints may come from [5]).

Here we focus on the case when $j(x,\cdot)$ is $p$-linear near the origin, precisely we assume that for a.e. $x \in \Omega$ and all $t$ with $|t|$ small enough

$$\lambda_1 |t|^p \le p j(x,t) \le \hat{\lambda} |t|^p$$

for some $\hat{\lambda} \in (\lambda_1, \lambda_2)$, where $\lambda_1$, $\lambda_2$ with $0 < \lambda_1 < \lambda_2$ denote the first and second eigenvalue of the negative $p$-Laplacian in $W^{1,p}_0(\Omega)$, respectively. Such assumption forces for the functional $\varphi$ a *homological local linking* at 0, which produces a nontrivial sequence of critical groups at 0. The notion of homological local linking, closely related to Morse theory, was introduced by Perera in [23] for $C^1$ functionals, and applied by Liu [18] to prove existence results for $p$-Laplacian elliptic equations, and by Liu [19] and Liu and Su [20] to prove multiplicity results. Here such topological notion is first extended to the nonsmooth case and applied to PDI's.

As usual, the other required information is the behavior of $\varphi$ at infinity, which, compared with the critical groups at 0, leads to detecting nontrivial critical points. Precisely, we will prove that:

- $(a)$ if $j(x,\cdot)$ is $p$-superlinear at infinity (with a nonsmooth Ambrosetti-Rabinowitz condition), then (1.1) has at least *one* nontrivial solution;
- $(b)$ if $j(x,\cdot)$ is $p$-sublinear, or asymptotically $p$-linear with resonance, at infinity, then (1.1) has at least *two* nontrivial solutions.

Our results extend to the nonsmooth framework those of [18, 20].

The structure of the paper is the following: in Section 2 we recall the basic notions of nonsmooth critical point theory and Morse theory, and we prove existence and multiplicity results for nonsmooth functionals with a homological local linking; in Section 3 we establish a variational framework for PDI's and prove some preliminary lemmas; in Section 4 we deal with the $p$-superlinear case; and in Section 5 we deal with the $p$-sublinear case.

**Notation.** The measure of sets is always the $N$-dimensional Lebesgue measure. By $B_\rho(u)$ we denote the open ball in $W_0^{1,p}(\Omega)$, centered at $u$ with radius $\rho > 0$ (similarly $\overline{B}_\rho(u)$, $\partial B_\rho(u)$ denote the closed ball and the sphrere, respectively). By $C > 0$ we will denote several constants.

## 2. Nonsmooth Morse theory and local linking

In this section we recall some notions from nonsmooth critical point theory, focusing in particular on nonsmooth Morse theory and the notion of local linking. Our main reference is [21] (see also [6, 12]).

Let $(X, \|\cdot\|)$ be a reflexive Banach space with dual $(X^*, \|\cdot\|_*)$. A functional $\varphi : X \to \mathbb{R}$ is said to be *locally Lipschitz continuous*, if for every $u \in X$ there exist a neighborhood $U$ of $u$ and $L > 0$ such that

$$|\varphi(v) - \varphi(w)| \leq L\|v - w\| \text{ for all } v, w \in U.$$

The *generalized directional derivative* of $\varphi$ at $u$ along the direction $v \in X$ is

$$\varphi^\circ(u; v) = \limsup_{w \to u,\, t \to 0^+} \frac{\varphi(w + tv) - \varphi(w)}{t}.$$

The *generalized subdifferential* of $\varphi$ at $u$ is the set

$$\partial\varphi(u) = \big\{u^* \in X^* : \langle u^*, v\rangle \leq \varphi^\circ(u; v) \text{ for all } v \in X\big\}.$$

For easy reference, in the next lemma we recall some basic properties useful for what follows (see [21, Section 3.2]):

**Lemma 2.1.** *Let $\varphi$, $\psi : X \to \mathbb{R}$ be locally Lipschitz continuous. Then*

(i) $\varphi^\circ(u; \cdot)$ *is positively homogeneous, sub-additive, and continuous for all $u \in X$;*
(ii) $\varphi^\circ(u; -v) = (-\varphi)^\circ(u; v)$ *for all $u, v \in X$;*
(iii) *if $\varphi \in C^1(X)$, then $\varphi^\circ(u; v) = \langle\varphi'(u), v\rangle$ for all $u, v \in X$;*
(iv) $(\varphi + \psi)^\circ(u; v) \leq \varphi^\circ(u; v) + \psi^\circ(u; v)$ *for all $u, v \in X$.*

**Lemma 2.2.** *Let $\varphi, \psi : X \to \mathbb{R}$ be locally Lipschitz continuous. Then*

(i) $\partial\varphi(u)$ *is convex, closed and weakly* compact for all $u \in X$;*
(ii) *the multifunction $\partial\varphi : X \to 2^{X^*}$ is upper semicontinuous with respect to the weak* topology on $X^*$;*
(iii) *if $\varphi \in C^1(X)$, then $\partial\varphi(u) = \{\varphi'(u)\}$ for all $u \in X$;*
(iv) $\partial(\lambda\varphi)(u) = \lambda\partial\varphi(u)$ *for all $\lambda \in \mathbb{R}$, $u \in X$;*
(v) $\partial(\varphi + \psi)(u) \subseteq \partial\varphi(u) + \partial\psi(u)$ *for all $u \in X$;*
(vi) *for all $u, v \in X$ there exists $u^* \in \partial\varphi(u)$ such that $\langle u^*, v\rangle = \varphi^\circ(u; v)$;*
(vii) *if $g \in C^1(\mathbb{R}, X)$, then $\varphi \circ g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz, and for all $t \in \mathbb{R}$*

$$\partial(\varphi \circ g)(t) \subseteq \big\{\langle u^*, g'(t)\rangle : u^* \in \partial\varphi(g(t))\big\}$$

(viii) *if $u$ is a local minimizer (or maximizer) of $\varphi$, then $0 \in \partial\varphi(u)$.*

For all $u \in X$ we set

$$m_\varphi(u) := \min_{u^* \in \partial\varphi(u)} \|u^*\|_*$$

(see Lemma 2.2 (i)). We say that $u \in X$ is a *critical point* of $\varphi$, if $m_\varphi(u) = 0$ (or, equivalently, if $0 \in \partial\varphi(u)$). The set of critical points of $\varphi$ is denoted by

$$K(\varphi) = \big\{u \in X : m_\varphi(u) = 0\big\} = \big\{u \in X : 0 \in \partial\varphi(u)\big\},$$

while for all $c \in \mathbb{R}$ we set

$$K_c(\varphi) = \big\{u \in K(\varphi) : \varphi(u) = c\big\}.$$

We say that $c \in \mathbb{R}$ is a *critical level* of $\varphi$, if $K_c(\varphi) \neq \emptyset$. In most results of critical point theory, we use the following nonsmooth *Palais-Smale condition*:

($PS$) Every sequence $(u_n)$ in $X$, s.t. $(\varphi(u_n))$ is bounded in $\mathbb{R}$ and $m_\varphi(u_n) \to 0$, has a convergent subsequence.

For all $c \in \mathbb{R}$ we set

$$\varphi^c = \big\{u \in X : \varphi(u) < c\big\},\ \overline{\varphi}^c = \big\{u \in X : \varphi(u) \leqslant c\big\}.$$

We say that $u \in K(\varphi)$ is an *isolated* critical point of $\varphi$, if there exists a neighborhood $U \subset X$ of $u$ s.t.

$$K(\varphi) \cap U = \{u\}.$$

In such case, for all integer $m \geqslant 0$ we define the *$m$-th critical group of $\varphi$ at $u$* as

$$C_m(\varphi, u) = H_m(\overline{\varphi}^c \cap U, \overline{\varphi}^c \cap U \setminus \{u\}),$$

where $H_m(A, B)$ stands for the $m$-th singular homology group of a topological pair $A \supseteq B$, understood as a vector space on $\mathbb{R}$ (for a general introduction to singular homology theory we refer to [21, Section 6.1]). Due to the excision property of singular homology groups, $C_m(\varphi, u)$ is invariant with respect to $U$. Critical groups for nonsmooth functionals were introduced in [8] in the framework of metric critical point theory, and developed for locally Lipschitz continuous functionals in [7].

We shall use the following decomposition result (see [7, Lemma 4]):

**Proposition 2.3.** *Let $\varphi : X \to \mathbb{R}$ be locally Lipschitz continuous satisfying ($PS$), $a < c < b \leqslant \infty$ be s.t. $c$ is the only critical level of $\varphi$ in $[a, b]$ and $K_c(\varphi)$ is a finite set. Then, for all integer $m \geqslant 0$*

$$H_m(\overline{\varphi}^b, \overline{\varphi}^a) = \bigoplus_{u \in K_c(\varphi)} C_m(\varphi, u).$$

We extend to the nonsmooth framework the notion of *homological local linking*, originally introduced in [23] (see also [21, Definition 6.82]):

**Definition 2.4.** Let $\varphi : X \to \mathbb{R}$ be locally Lipschitz continuous, $u \in K_c(\varphi)$, $m, n \geqslant 1$ be integers. We say that $\varphi$ has a $(m, n)$-local linking at $u_0$, if there exist a neighborhood $U \subset X$ of $u_0$, and subsets $E_0 \subset E$, $D$ of $X$ s.t. $u_0 \notin E_0$, $E_0 \cap D = \emptyset$, and

(i) $K(\varphi) \cap \overline{\varphi}^c \cap U = \{u_0\}$;
(ii) $\varphi(u) \leqslant c < \varphi(v)$ for all $u \in E$, $v \in U \cap D \setminus \{u_0\}$;
(iii) $\dim \operatorname{im}(i^*_{m-1}) - \dim \operatorname{im}(j^*_{m-1}) \geqslant n$, where $i^*_{m-1} : H_{m-1}(E_0) \to H_{m-1}(X \setminus D)$, $j^*_{m-1} : H_{m-1}(E_0) \to H_{m-1}(E)$ are homomorphisms induced by the inclusion mappings $E_0 \hookrightarrow X \setminus D$, $E_0 \hookrightarrow E$.

Whenever $u_0$ is a critical point of the type above, we can have explicit information about the critical groups of $\varphi$ at $u_0$ (see [21, Theorem 6.87] for the smooth case):

**Proposition 2.5.** *Let $\varphi : X \to \mathbb{R}$ be locally Lipschitz continuous, $u_0 \in K_c(\varphi)$ be an isolated critical point, $m, n \geqslant 1$ be integers, s.t. $\varphi$ has a $(m,n)$-local linking at $u_0$. Then,*

$$\dim C_m(\varphi, u_0) \geqslant n.$$

*Proof.* Let $U$, $E_0$, $E$, $D$ be as in Definition 2.4. By [21, Definition 6.9], the following sequence is exact:

$$C_m(\varphi, u_0) \xrightarrow{\ \partial\ } H_{m-1}(\overline{\varphi}^c \cap U \setminus \{u_0\}) \xrightarrow{\ l^*\ } H_{m-1}(\overline{\varphi}^c \cap U),$$

where $l^*$ is the homomorphism induced by the inclusion mapping $\overline{\varphi}^c \cap U \setminus \{u_0\} \hookrightarrow \overline{\varphi}^c \cap U$. So we have

$$\dim\ker(l^*) = \dim\operatorname{im}(\partial) \leqslant \dim C_m(\varphi, u_0). \tag{2.1}$$

The next step consists in proving that

$$\dim\ker(l^*) \geqslant n. \tag{2.2}$$

Indeed, the following diagram is commutative:

$$\begin{tikzcd}
H_{m-1}(X \setminus D) & H_{m-1}(E_0) \arrow[l, "i^*_{m-1}"'] \arrow[r, "j^*_{m-1}"] \arrow[d, "h^*"] & H_{m-1}(E) \arrow[d, "k^*"] \\
& H_{m-1}(\overline{\varphi}^c \cap U \setminus \{u_0\}) \arrow[ul, "p^*"] \arrow[r, "l^*"] & H_{m-1}(\overline{\varphi}^c \cap U)
\end{tikzcd} ,$$

where homomorphisms $i^*_{m-1}$, $j^*_{m-1}$ are as in Definition 2.4 (iii), and $p^*$, $h^*$, $k^*$ are induced by the corresponding inclusion mappings. Looking at the diagram and applying the rank-nullity theorem, we see that

$$\begin{aligned}
\dim\operatorname{im}(i^*_{m-1}) &= \dim\operatorname{im}(p^* \circ h^*) \\
&\leqslant \dim\operatorname{im}(h^*),
\end{aligned}$$

as well as

$$\begin{aligned}
\dim\operatorname{im}(j^*_{m-1}) &\geqslant \dim\operatorname{im}(k^* \circ j^*_{m-1}) \\
&= \dim\operatorname{im}(l^* \circ h^*) \\
&= \dim\operatorname{im}(h^*) - \dim\ker(l^*|_{\operatorname{im} h^*}) \\
&\geqslant \dim\operatorname{im}(h^*) - \dim(\ker l^*).
\end{aligned}$$

So, by Definition 2.4 (iii) we have

$$\begin{aligned}
\dim\ker(l^*) &\geqslant \dim\operatorname{im}(h^*) - \dim\operatorname{im}(j^*_{m-1}) \\
&\geqslant \dim\operatorname{im}(i^*_{m-1}) - \dim\operatorname{im}(j^*_{m-1}),
\end{aligned}$$

which proves (2.2). From (2.1), (2.2) we readily conclude. $\square$

We extend to the nonsmooth framework the main result of [18] (see also [21, Proposition 6.91]), namely an existence result for critical points of a functional having a local linking:

**Theorem 2.6.** *Let $\varphi : X \to \mathbb{R}$ be locally Lipschitz continuous satisfying $(PS)$, $K(\varphi)$ be finite, $a < c < b$ be real, $m, n \geqslant 1$ be integer s.t. $K_c(\varphi) = \{u_0\}$, $\varphi$ has a $(m,n)$-local linking at $u_0$, and in addition*

$$K_a(\varphi) = K_b(\varphi) = \emptyset,\ H_m(\overline{\varphi}^b, \overline{\varphi}^a) = 0.$$

*Then, there exists $u_1 \in K(\varphi)$ s.t. one of the following holds:*

(i) $a < \varphi(u_1) < c$, $C_{m-1}(\varphi, u_1) \neq 0$;
(ii) $c < \varphi(u_1) < b$, $C_{m+1}(\varphi, u_1) \neq 0$.

*Proof.* Since $K(\varphi)$ is finite, we can fix $\varepsilon > 0$ s.t.

$$a < c - \varepsilon < c < c + \varepsilon < b,$$

and

$$K(\varphi) \cap \big\{u \in X :\ c - \varepsilon \leqslant \varphi(u) \leqslant c + \varepsilon\big\} = \{u_0\}.$$

By Propositions 2.3, 2.5 we have

$$H_m(\overline{\varphi}^{c+\varepsilon}, \overline{\varphi}^{c-\varepsilon}) = C_m(\varphi, u_0) \neq 0.$$

Consider the chain $\overline{\varphi}^a \subset \overline{\varphi}^{c-\varepsilon} \subset \overline{\varphi}^{c+\varepsilon} \subset \overline{\varphi}^b$ and apply [21, Lemma 6.90] to get

$$\dim H_m(\overline{\varphi}^{c+\varepsilon}, \overline{\varphi}^{c-\varepsilon}) \leqslant \dim H_{m-1}(\overline{\varphi}^{c-\varepsilon}, \overline{\varphi}^a) + \dim H_{m+1}(\overline{\varphi}^b, \overline{\varphi}^{c+\varepsilon}) + \dim H_m(\overline{\varphi}^b, \overline{\varphi}^a),$$

and by assumption the last addendum is 0. So, one of the groups $H_{m-1}(\overline{\varphi}^{c-\varepsilon}, \overline{\varphi}^a)$ or $H_{m+1}(\overline{\varphi}^b, \overline{\varphi}^{c+\varepsilon})$ is nontrivial. In the first case, by Proposition 2.3 again we can find $u_1 \in K(\varphi)$ satisfying (i). In the second case, similarly we can find $u_1 \in K(\varphi)$ satisfying (ii). □

The next result, namely a nonsmooth extension of the main result of [20], is a multiplicity theorem for a bounded below functional having a local linking:

**Theorem 2.7.** *Let $\varphi : X \to \mathbb{R}$ be locally Lipschitz continuous satisfying $(PS)$ and bounded below, $u_0 \in K(\varphi)$, $m, n \geqslant 1$ be integers be s.t. $\varphi$ has a $(m,n)$-local linking at $u_0$ and $u_0$ is not a global minimizer of $\varphi$. Then, $\varphi$ has at least three critical points.*

*Proof.* By hypothesis $\varphi$ satisfies $(PS)$ and

$$\inf_{u \in X} \varphi(u) = \mu > -\infty.$$

It follows from Ekeland's principle that $\varphi$ has a global minimizer $u_2 \in X$, in particular we have

$$\varphi(u_2) = \mu < c := \varphi(u_0).$$

Arguing by contradiction, assume

$$K(\varphi) = \{u_0, u_2\}. \tag{2.3}$$

Then, $u_2$ is both a strict local minimizer and an isolated critical point of $\varphi$, hence by elementary properties of singular homology (see [21, Axiom 7 and Remark 6.10, p. 143]) we have for all integer $k \geqslant 0$

$$C_k(\varphi, u_2) = \delta_{k,0}\mathbb{R}.$$

Now fix $a, b \in \mathbb{R}$ s.t. $\mu < a < c < b$. By the nonsmooth second deformation theorem (see [7, Theorem 4]), $\{u_2\}$, $\overline{\varphi}^b$ are strong deformation retracts of $\overline{\varphi}^a$, $X$ respectively, which, along with [21, Proposition 6.12, Corollary 6.15], implies for all integer $k \geqslant 0$

$$H_k(\overline{\varphi}^b, \{u_2\}) = H_k(X, \{u_2\}) = 0, \ H_k(\overline{\varphi}^a, \{u_2\}) = 0.$$

By [21, Proposition 6.21], the following sequence is exact:

$$H_m(\overline{\varphi}^a, \{u_2\}) \longrightarrow H_m(\overline{\varphi}^b, \{u_2\}) \longrightarrow H_m(\overline{\varphi}^b, \overline{\varphi}^a) \longrightarrow H_{m-1}(\overline{\varphi}^a, \{u_2\}).$$

By the computation above we deduce

$$H_m(\overline{\varphi}^b, \overline{\varphi}^a) = 0.$$

Then, Theorem 2.6 implies the existence of $u_1 \in K(\varphi)$ s.t. either $a < \varphi(u_1) < c$, or $c < \varphi(u_1) < b$, against (2.3). Thus, $\varphi$ has at least three critical points. $\square$

**Remark 2.8.** All results of this section can be extended to the case where $X$ is a complete metric space and $\varphi : X \to \mathbb{R}$ is continuous, satisfying a metric $(PS)$-type condition, see [8]. Regarding Theorem 2.7, we observe that assuming $\varphi$ bounded below and satisfying $(PS)$ is in fact equivalent to assuming $\varphi$ coercive (see [21, Proposition 5.22] for the $C^1$ case).

## 3. Variational methods for PDI's

Here we establish a variational framework for problem (1.1) and some preparatory results under the following minimal hypotheses:

$\mathbf{H}_0$ $j : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory mapping s.t. $j(x, \cdot)$ is locally Lipschitz continuous and $j(x, 0) = 0$ for a.e. $x \in \Omega$, and there exist $c_0 > 0$, $q \in (1, p^*)$ s.t. for a.e. $x \in \Omega$, all $t \in \mathbb{R}$, and all $\xi \in \partial j(x, t)$

$$|\xi| \leqslant c_0(1 + |t|^{q-1}).$$

Here $p^*$ denotes the critical Sobolev exponent, i.e.,

$$p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } N > p \\ \infty & \text{if } N \leqslant p. \end{cases}$$

We set $X = W^{1,p}_0(\Omega)$, endowed with the norm $\|u\| = \|\nabla u\|_p$, while for all $r \in [1, \infty]$ we denote by $\|\cdot\|_r$ the usual norm of $L^r(\Omega)$. We recall that the embedding $X \hookrightarrow L^r(\Omega)$ is continuous and compact for all $r \in [1, p^*)$. We rephrase the $p$-Laplacian as an operator $A : X \to X^*$ defined by

$$\langle A(u), v\rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx,$$

which is an $(S)_+$-map (see [21, Proposition 2.72]). As seen in the Introduction, we define an energy functional for (1.1) by setting for all $u \in X$

$$\varphi(u) = \frac{\|u\|^p}{p} - \int_\Omega j(x, u) \, dx.$$

We also define a set-valued Nemytskii operator

$$N(u) = \big\{ w \in L^{q'}(\Omega) : \, w(x) \in \partial j(x, u(x)) \text{ for a.e. } x \in \Omega \big\}.$$

We say that $u \in X$ is a *(weak) solution* of (1.1), if there exists $w \in N(u)$ s.t. for all $v \in X$

$$\langle A(u), v\rangle = \int_\Omega wv\,dx. \tag{3.1}$$

**Lemma 3.1.** *Let* $\mathbf{H}_0$ *hold. Then,* $\varphi : X \to \mathbb{R}$ *is locally Lipschitz continuous and for all* $u \in X$

$$\partial\varphi(u) \subseteq A(u) - N(u).$$

*Proof.* Clearly,

$$u \mapsto \frac{\|u\|^p}{p}$$

is a $C^1$-functional with gradient $A$. Now set for all $u \in L^q(\Omega)$

$$J(u) = \int_\Omega j(x, u)\,dx.$$

Then, by $\mathbf{H}_0$ and [21, Proposition 3.49], $J$ is Lipschitz continuous on any bounded subset of $L^q(\Omega)$, with $\partial J(u) \subset N(u)$. We identify $L^{q'}(\Omega)$ as a subspace of $X^*$, so the same holds for $J|_X$. By Lemma 2.2 (v) we conclude. $\square$

**Lemma 3.2.** *Let* $\mathbf{H}_0$ *hold and* $u \in K(\varphi)$*. Then,* $u \in C^1_0(\overline{\Omega})$ *is a solution of* (1.1)*.*

*Proof.* By Lemma 3.1 we have $A(u) \in N(u)$ in $X^*$, i.e., we can find $w \in N(u)$ s.t. (3.1) holds for all $v \in X$. By $\mathbf{H}_0$ we have for a.e. $x \in \Omega$

$$|w(x)| \leqslant c_0(1 + |u(x)|^{q-1}).$$

So, by (3.1) and [21, Theorem 8.4], we deduce $u \in L^\infty(\Omega)$. Then, by Lieberman's nonlinear regularity theory (see [17] or [21, Theorem 8.10]) we have $u \in C^1_0(\overline{\Omega})$. $\square$

Finally, we prove that $\varphi$ satisfies $(PS)$ along *bounded* sequences:

**Lemma 3.3.** *Let* $\mathbf{H}_0$ *hold and* $(u_n)$ *be a bounded sequence in* $X$*, s.t.* $(\varphi(u_n))$ *is bounded in* $\mathbb{R}$ *and* $m_\varphi(u_n) \to 0$*. Then,* $(u_n)$ *has a convergent subsequence.*

*Proof.* For all integer $n \geqslant 0$ we can find $u_n^* \in \partial\varphi(u_n)$ s.t.

$$\|u_n^*\|_* = m_\varphi(u_n).$$

By Lemma 3.1, in turn, we find $w_n \in N(u_n)$ s.t.

$$u_n^* = A(u_n) - w_n$$

(in $X^*$). Passing if necessary to a subsequence, we have $u_n \rightharpoonup u$ in $X$, $u_n \to u$ in $L^q(\Omega)$, and $u_n(x) \to u(x)$ for a.e. $x \in \Omega$. By $\mathbf{H}_0$, $(w_n)$ is bounded in $L^{q'}(\Omega)$. So, for all $n$ we have

$$\begin{aligned}\langle A(u_n), u_n - u\rangle &= \langle u_n^*, u_n - u\rangle + \int_\Omega w_n(u_n - u)\,dx \\ &\leqslant \|u_n^*\|_*\|u_n - u\| + \|w_n\|_{q'}\|u_n - u\|_q\end{aligned}$$

(where we have used Hölder's inequality), and the latter tends to 0 as $n \to \infty$. So

$$\limsup_n \langle A(u_n), u_n - u\rangle = 0.$$

By the $(S)_+$-property of $A$, we have $u_n \to u$ in $X$. $\square$

## 4. Existence result for the superlinear case

First we recall some spectral properties of $-\Delta_p$ in $X$. Consider the eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

It is well known (see [21, Propositions 9.47, 9.49]) that problem (4.1) admits an unbounded sequence of variational (Lyusternik-Schnirelmann) eigenvalues

$$0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \ldots \leqslant \lambda_k \ldots$$

In particular, $\lambda_1 > 0$ is simple and isolated with the variational characterization

$$\lambda_1 = \min_{u \in X \setminus \{0\}} \frac{\|u\|^p}{\|u\|_p^p}. \tag{4.2}$$

We denote by $\hat{u}_1 \in \mathrm{int}(C_0^1(\overline{\Omega}))$ the (unique) positive, $L^p(\Omega)$-normalized eigenfunction associated to $\lambda_1$. Besides, in the interval $(\lambda_1, \lambda_2)$ there lie no eigenvalues of (4.1).

The latter information is the basis for our local linking scheme. Indeed, let us denote by $Y \subset X$ the (1-dimensional) eigenspace associated to $\lambda_1$, namely,

$$Y = \big\{ u \in X : \|u\|^p = \lambda_1 \|u\|_p^p \big\} = \mathrm{span}(\hat{u}_1). \tag{4.3}$$

and by $Z$ any direct complement to $Y$ in $X$ (i.e., $Z \subset X$ is a closed subspace s.t. $X = Y \oplus Z$), then we have for all $u \in Z$

$$\|u\|^p \geqslant \lambda_2 \|u\|_p^p. \tag{4.4}$$

Indeed, the Krasnosel'skii genus of $Z \setminus \{0\}$ is $\mathrm{gen}(Z \setminus \{0\}) \geqslant 2$, hence by [21, Theorem 9.45] we have

$$\frac{1}{\lambda_2} \geqslant \min \Big\{ \frac{\|u\|_p^p}{p} : u \in Z, \ \frac{\|u\|^p}{p} = 1 \Big\}.$$

In this section we prove an existence result for problem (1.1), extending to the nonsmooth framework the result of [18]. We assume the following hypotheses:

- $\mathbf{H}_1$ $j : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory mapping s.t. $j(x, \cdot)$ is locally Lipschitz continuous and $j(x, 0) = 0$ for a.e. $x \in \Omega$, and
  - (i) $|\xi| \leqslant c_0(1 + |t|^{q-1})$ for a.e. $x \in \Omega$, all $t \in \mathbb{R}$, $\xi \in \partial j(x, t)$ ($c_0 > 0$, $q \in (p, p^*)$);
  - (ii) $\lambda_1 |t|^p \leqslant p j(x, t) \leqslant \hat{\lambda} |t|^p$ for a.e. $x \in \Omega$, all $|t| \leqslant \delta$ ($\delta > 0$, $\hat{\lambda} \in (\lambda_1, \lambda_2)$);
  - (iii) $0 < \mu j(x, t) \leqslant \xi t$ for a.e. $x \in \Omega$, all $|t| \geqslant M$, $\xi \in \partial j(x, t)$ ($M > 0$, $\mu > p$).

Clearly $\mathbf{H}_1$ incorporates $\mathbf{H}_0$. Hypothesis $\mathbf{H}_1$ (ii) implies that 0 is a local minimizer of $j(x, \cdot)$, for a.e. $x \in \Omega$, hence $0 \in N(0)$. Thus, $0 \in X$ is a critical point of $\varphi$, i.e., (1.1) admits the trivial solution. Finally, we note that $\mathbf{H}_1$ (iii) is a nonsmooth Ambrosetti-Rabinowitz condition, forcing for $j(x, \cdot)$ a $p$-superlinear growth at infinity.

**Example 4.1.** The following locally Lipschitz continuous (autonomous) potential $j : \mathbb{R} \to \mathbb{R}$ satisfies hypotheses $\mathbf{H}_1$:

$$j(t) = \begin{cases} \hat{\lambda}\dfrac{|t|^p}{p} & \text{if } |t| \leqslant 1 \\ \hat{\lambda}\dfrac{|t|^\mu}{p} & \text{if } |t| > 1, \end{cases}$$

with $\hat{\lambda} \in (\lambda_1, \lambda_2)$, $\mu \in (p, p^*)$.

We begin by establishing some properties of the energy functional $\varphi$:

**Lemma 4.2.** *Let $\mathbf{H}_1$ hold. Then, $\varphi$ satisfies $(PS)$.*

*Proof.* Let $(u_n)$ be a sequence in $X$, s.t. $|\varphi(u_n)| \leqslant C$ for all $n \in \mathbb{N}$, $m_\varphi(u_n) \to 0$. By Lemma 3.1, there exist sequences $(\varepsilon_n)$ in $\mathbb{R}$ and $(w_n)$ in $L^{q'}(\Omega)$ s.t. $\varepsilon_n \to 0^+$, $w_n \in N(u_n)$, and $\|A(u_n) - w_n\|_* \leqslant \varepsilon_n$ for all $n \in \mathbb{N}$. Then we have

$$\frac{\|u_n\|^p}{p} - \int_\Omega j(x, u_n)\, dx \leqslant C,$$
$$-\|u_n\|^p + \int_\Omega w_n u_n\, dx \leqslant \varepsilon_n \|u_n\|$$

(the latter is obtained by testing with $u_n$). Multiplying the first inequality above by $\mu$ and adding the second, we get

$$\Big(\frac{\mu}{p} - 1\Big)\|u_n\|^p \leqslant \int_\Omega \big(\mu j(x, u_n) - w_n u_n\big)\, dx + \varepsilon_n \|u_n\| + C. \tag{4.5}$$

By Rademacher's theorem (see [12, Theorem A.2.4]), $j(x, \cdot)$ is a.e. differentiable in $\mathbb{R}$ with $j'(x, t) \in \partial j(x, t)$, so integrating on $\mathbf{H}_1$ (i) we have

$$\begin{aligned} |j(x,t)| &\leqslant \Big| \int_0^t j'(x, \tau)\, d\tau \Big| \\ &\leqslant c_0\Big(|t| + \frac{|t|^q}{q}\Big). \end{aligned}$$

Plugging such estimate and $\mathbf{H}_1$ (ii), (iii) into (4.5), we get for all $n \in \mathbb{N}$

$$\begin{aligned} \Big(\frac{\mu}{p} - 1\Big)\|u_n\|^p &\leqslant \int_{\{|u_n|<M\}} 2c_0(|u_n| + |u_n|^q)\, dx + \varepsilon_n \|u_n\| + C \\ &\leqslant \varepsilon_n \|u_n\| + C \end{aligned}$$

(with a bigger $C > 0$ independent of $n$). Since $\mu > p$, we see that $(u_n)$ is bounded in $X$. The conclusion then follows from Lemma 3.3. □

In the following, we will assume that 0 is an *isolated* critical point of $\varphi$:

**Lemma 4.3.** *Let $\mathbf{H}_1$ hold. Then, $\varphi$ has a $(1,1)$-local linking at* 0.

*Proof.* Fix $\rho > 0$ s.t. $K(\varphi) \cap \overline{B}_\rho(0) = \{0\}$. Let $Y$ be defined by (4.3). Since $\hat{u}_1 \in C_0^1(\overline{\Omega})$, by reducing $\rho > 0$ if necessary, for all $u \in Y \cap \overline{B}_\rho(0)$ we have $\|u\|_\infty \leqslant \delta$ ($\delta > 0$ defined as in $\mathbf{H}_1$ (ii)), hence

$$\varphi(u) \leqslant \frac{\|u\|^p}{p} - \int_\Omega \frac{\lambda_1 |u|^p}{p}\, dx = 0. \tag{4.6}$$

Besides, set

$$D = \{u \in X : \|u\|^p \geqslant \lambda_2 \|u\|_p^p\}.$$

Clearly, $Y \cap D = \{0\}$. By $\mathbf{H}_1$ (i), (ii), (4.4), and the embedding $X \hookrightarrow L^q(\Omega)$, we have for all $u \in D$

$$\begin{aligned}
\varphi(u) &\geqslant \frac{\|u\|^p}{p} - \int_{\{|u|\leqslant\delta\}} \frac{\hat{\lambda}|u|^p}{p}\,dx - \int_{\{|u|>\delta\}} c_0\Big(\frac{|u|^q}{\delta^{q-1}} + \frac{|u|^q}{q}\Big)\,dx \\
&\geqslant \frac{\|u\|^p}{p} - \hat{\lambda}\frac{\|u\|_p^p}{p} - C\|u\|_q^q \\
&\geqslant \Big(1 - \frac{\hat{\lambda}}{\lambda_2}\Big)\frac{\|u\|^p}{p} - C\|u\|^q.
\end{aligned}$$

Since $\hat{\lambda} < \lambda_2$ and $q > p$, the mapping

$$t \mapsto \Big(1 - \frac{\hat{\lambda}}{\lambda_2}\Big)\frac{t^p}{p} - Ct^q$$

is positive in a right hand neighborhood of 0. So, by further reducing $\rho > 0$, we have for all $u \in D \cap \overline{B}_\rho(0) \setminus \{0\}$

$$\varphi(u) > 0. \tag{4.7}$$

Set $u_0 = 0$, $c = 0$, $U = \overline{B}_\rho(0)$, $E_0 = Y \cap \partial B_\rho(0)$, $E = Y \cap \overline{B}_\rho(0)$, and $D$ as above in Definition 2.4. Clearly $E_0 \subset E \subset U$, $E_0 \cap D = \emptyset$, $0 \notin E_0$. Condition (i) (of Definition 2.4) follows from the choice of $\rho > 0$. By (4.6), (4.7) we have (ii).

It remains to prove (iii) with $m = n = 1$, arguing as in [21, Proposition 6.84]. Denoting by $Z \subset X$ any direct complement of $Y$ (as before), by (4.4) we have $Z \subseteq D$. For all $u \in X \setminus D$ there exist unique $v \in Y \setminus \{0\}$, $w \in Z$ s.t. $u = v + w$. We define a continuous deformation $\eta : (X \setminus D) \times [0,1] \to (X \setminus D)$ by setting

$$\eta(v + w, t) = (1 - t)(v + w) + t\frac{\rho v}{\|v\|}.$$

So we see that $E_0$ is a strong deformation retract of $X \setminus D$. Thus, the homomorphism $i_0^* : H_0(E_0) \to H_0(X \setminus D)$ induced by the inclusion mapping is bijective, and since $E_0$ consists of two points we have by [21, Example 6.42]

$$\dim \operatorname{im}(i_0^*) = 2. \tag{4.8}$$

Besides, since $E$ is contractible (it is in fact a line segment), we have

$$H_0(E, E_0) = H_{-1}(E_0, *) = 0,$$

hence the homomorphism $j_0^* : H_0(E_0) \to H_0(E)$ induced by the inclusion mapping is surjective. So,

$$\dim \operatorname{im}(j_0^*) = \dim H_0(E) = 1. \tag{4.9}$$

From (4.8), (4.9) we get condition (iii). Thus, we conclude that $\varphi$ has a $(1,1)$-local linking at 0. □

The following lemmas define the asymptotic behavior of $\varphi$:

**Lemma 4.4.** *Let* $\mathbf{H}_1$ *hold. Then, for all* $u \in \partial B_1(0)$

$$\lim_{t\to\infty} \varphi(tu) = -\infty.$$

*Proof.* Without loss of generality, we may assume $\mu \leqslant p^*$ in $\mathbf{H}_1$ (iii). First we prove that for a.e. $x \in \Omega$, all $|t| \geqslant M$

$$j(x,t) \geqslant j(x,M)\frac{|t|^\mu}{M^\mu}. \tag{4.10}$$

Indeed, arguing as in the proof of Lemma 4.2 and integrating on $\mathbf{H}_1$ (iii), we have for all $t \geqslant M$

$$\int_M^t \frac{j'(x,\tau)}{j(x,\tau)}\,d\tau \geqslant \int_M^t \frac{\mu}{\tau}\,d\tau,$$

hence

$$\ln\Big(\frac{j(x,t)}{j(x,M)}\Big) \geqslant \ln\Big(\frac{t^\mu}{M^\mu}\Big),$$

which yields (4.10). The argument for $t \leqslant -M$ is analogous. Now fix $u \in \partial B_1(0)$. For all $t > 0$ big enough, we have $|tu| \geqslant M$ in a subset of $\Omega$ with positive measure, hence by (4.10) we have

$$\begin{aligned}
\varphi(tu) &\leqslant \frac{t^p}{p} - \int_{\{|u|\geqslant M/t\}} j(x,M)\frac{|tu|^\mu}{M^\mu}\,dx + \int_{\{|u|<M/t\}} |j(x,tu)|\,dx \\
&\leqslant \frac{t^p}{p} - \int_\Omega j(x,M)\frac{|tu|^\mu}{M^\mu}\,dx + \int_{\{|u|<M/t\}} j(x,M)\,dx + \int_\Omega c_0\Big(M + \frac{M^q}{q}\Big)\,dx \\
&\leqslant \frac{t^p}{p} - \frac{t^\mu}{M^\mu}\int_\Omega j(x,M)|u|^\mu\,dx + C,
\end{aligned}$$

and the latter tends to $-\infty$ as $t \to \infty$, since $\mu > p$ and $j(\cdot,M)|u|^\mu \geqslant 0$ and does not vanish identically in $\Omega$. So we conclude. □

**Lemma 4.5.** *Let* $\mathbf{H}_1$ *hold. Then, for all* $a < 0$ *small enough,* $u \in \overline{\varphi}^a$*,* $u^* \in \partial\varphi(u)$

$$\langle u^*, u\rangle < 0.$$

*Proof.* First we note that, by Lemma 4.4, $\varphi$ is unbounded from below in $X$. Fix

$$a < \inf_{u\in\overline{B}_\rho(0)} \varphi(u) \leqslant 0$$

(to be better determined later). Then, for all $u \in \overline{\varphi}^a$, $u^* \in \partial\varphi(u)$ we have $\|u\| > 1$, and by Lemma 3.1 we can find $w \in N(u)$ s.t.

$$u^* = A(u) - w.$$

So, using $\mathbf{H}_1$ (i), (ii) we compute

$$\begin{aligned}
\langle u^*, u\rangle &= \|u\|^p - \int_\Omega wu\,dx \\
&\leqslant \|u\|^p - \int_{\{|u|<M\}} wu\,dx - \int_{\{|u|\geqslant M\}} \mu j(x,u)\,dx \\
&\leqslant \|u\|^p + \int_\Omega c_0(M + M^q)\,dx + \int_{\{|u|<M\}} \mu j(x,u)\,dx - \int_\Omega \mu j(x,u)\,dx
\end{aligned}$$

$$\leqslant \Big(1-\frac{\mu}{p}\Big)\|u\|^p+\mu\varphi(u)+c_0(M+M^q)|\Omega|+\mu\int_\Omega c_0\Big(M+\frac{M^q}{q}\Big)\,dx$$
$$\leqslant \Big(1-\frac{\mu}{p}\Big)+\mu a+(1+\mu)c_0(M+M^q)|\Omega|,$$

and the latter is negative as soon as we choose

$$a<\min\Big\{\inf_{u\in\overline{B}_\rho(0)}\varphi(u),\ \Big(1+\frac{1}{\mu}\Big)c_0(M+M^q)|\Omega|\Big\}.$$

So we conclude. □

Now we can prove our existence result for the superlinear case:

**Theorem 4.6.** *Let* $\mathbf{H}_1$ *hold. Then,* (1.1) *has at least one solution* $u_1\in C^1_0(\overline{\Omega})\setminus\{0\}$.

*Proof.* Without loss of generality we may assume that $K(\varphi)$ is a finite set. From Lemma 4.3 we know that $0\in K(\varphi)$ and $\varphi$ has a $(1,1)$-local linking at 0. Fix $a<0$ as in Lemma 4.5, and

$$b>\max_{u\in K(\varphi)}\varphi(u)\geqslant 0.$$

Also without loss of generality we may assume that

$$K_0(\varphi)=\{0\},\ K_a(\varphi)=K_b(\varphi)=\emptyset.$$

By Lemmas 4.4, 4.5 and the nonsmooth implicit function theorem (see [7, Lemma 3]), there exists a mapping $T\in C(\partial B_1(0),(1,\infty))$ s.t. for all $u\in\partial B_1(0)$, $t\geqslant 1$

$$\varphi(tu)\begin{cases}>a & \text{if } t<T(u)\\ =a & \text{if } t=T(u)\\ <a & \text{if } t>T(u).\end{cases}$$

In particular we have

$$\overline{\varphi}^a=\big\{tu:\ u\in\partial B_1(0),\ t\geqslant T(u)\big\}.$$

Define a continuous deformation $\eta: B_1^c(0)\times[0,1]\to B_1^c(0)$ by setting for all $t\geqslant 1$, $u\in\partial B_1(0)$, $s\in[0,1]$

$$\eta(tu,s)=\begin{cases}(1-s)tu+sT(u)u & \text{if } t<T(u)\\ tu & \text{if } t\geqslant T(u).\end{cases}$$

So we see that $\overline{\varphi}^a$ is a strong deformation retract of $B_1^c(0)$, as is $\partial B_1(0)$. Besides, since there are no critical values of $\varphi$ in $[b,\infty)$, by the nonsmooth second deformation theorem (see [7, Theorem 4]) $\overline{\varphi}^b$ is a strong deformation retract of $X$.

We apply [21, Proposition 6.12, Corollary 6.15] to get

$$H_1(\overline{\varphi}^b,\overline{\varphi}^a)=H_1(X,\overline{\varphi}^a)=H_1(X,B_1^c(0))=H_1(X,\partial B_1(0))=0,$$

the last equality coming from the fact that $\partial B_1(0)$ is contractible (recall that $\dim X=\infty$). By Theorem 2.6 (with $c=0$, $m=1$), there exists $u_1\in K(\varphi)$ s.t. either $\varphi(u_1)<0$ or $\varphi(u_1)>0$, hence $u_1\neq 0$. By Lemma 3.2, $u_1\in C^1_0(\overline{\Omega})$ solves (1.1). □

## 5. Multiplicity results for the (sub) linear case

In this section we deal with a potential which is $p$-sublinear or asymptotically $p$-linear at infinity, under assumptions which ensure coercivity of $\varphi$. We distinguish between the *non-resonant* and *resonant* cases, but in both cases we prove existence of at least two non-trivial solutions of (1.1). Our results extend to the nonsmooth framework those of [20].

We consider first the non-resonant case, under the following hypotheses:

**H**$_2$ $j : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory mapping s.t. $j(x, \cdot)$ is locally Lipschitz continuous and $j(x, 0) = 0$ for a.e. $x \in \Omega$, and

- (i) $|\xi| \leqslant c_0(1 + |t|^{q-1})$ for a.e. $x \in \Omega$, all $t \in \mathbb{R}$, $\xi \in \partial j(x, t)$ ($c_0 > 0$, $q \in (p, p^*)$);
- (ii) $\lambda_1|t|^p \leqslant pj(x, t) \leqslant \hat{\lambda}|t|^p$ for a.e. $x \in \Omega$, all $|t| \leqslant \delta$ ($\delta > 0$, $\hat{\lambda} \in (\lambda_1, \lambda_2)$);
- (iii) $\displaystyle\limsup_{|t|\to\infty} \frac{pj(x, t)}{|t|^p} < \lambda_1$ uniformly for a.e. $x \in \Omega$.

Clearly **H**$_2$ incorporates **H**$_0$. As in the previous case, by **H**$_2$ (ii) we have $0 \in K(\varphi)$.

**Example 5.1.** The following locally Lipschitz continuous (autonomous) potential $j : \mathbb{R} \to \mathbb{R}$ satisfies hypotheses **H**$_2$:

$$j(t) = \begin{cases} \hat{\lambda}\dfrac{|t|^p}{p} & \text{if } |t| \leqslant 1 \\ \hat{\lambda}\dfrac{|t|^r}{p} & \text{if } |t| > 1, \end{cases}$$

with $\hat{\lambda} \in (\lambda_1, \lambda_2)$, $r \in (1, p)$.

The main difference, with respect to the previous case, is that $\varphi$ is coercive:

**Lemma 5.2.** *Let* **H**$_2$ *hold. Then,*

$$\lim_{\|u\|\to\infty} \varphi(u) = \infty.$$

*Proof.* By **H**$_2$ (iii) we can find $\theta \in (0, \lambda_1)$, $M > 0$ s.t. for a.e. $x \in \Omega$, all $|t| \geqslant M$

$$j(x, t) < \frac{\theta|t|^p}{p}.$$

So, for all $u \in X$ we have

$$\begin{aligned} \varphi(u) &\geqslant \frac{\|u\|^p}{p} - \int_{\{|u|\geqslant M\}} \frac{\theta|u|^p}{p}\,dx - \int_{\{|u|<M\}} c_0\Big(M + \frac{M^q}{q}\Big)\,dx \\ &\geqslant \frac{\|u\|^p}{p} - \frac{\theta\|u\|_p^p}{p} + \int_{\{|u|<M\}} \frac{\theta|u|^p}{p}\,dx - c_0\Big(M + \frac{M^q}{q}\Big)|\Omega| \\ &\geqslant \Big(1 - \frac{\theta}{\lambda_1}\Big)\frac{\|u\|^p}{p} - C, \end{aligned}$$

and the latter tends to $\infty$ as $\|u\| \to \infty$. $\square$

We prove now our first multiplicity result:

**Theorem 5.3.** *Let* **H**$_2$ *hold. Then,* (1.1) *has at least two solutions* $u_1, u_2 \in C_0^1(\overline{\Omega}) \setminus \{0\}$.

*Proof.* By Lemma 5.2, $\varphi$ is coercive in $X$. Besides, by $\mathbf{H}_2$ (i) it is easily seen that $\varphi$ is sequentially weakly l.s.c. in $X$. So we have

$$\min_{u \in X} \varphi(u) = m > -\infty.$$

We check that $\varphi$ satisfies $(PS)$. Indeed, let $(u_n)$ be a sequence in $X$, s.t. $|\varphi(u_n)| \leqslant C$ and $m_\varphi(u_n) \to 0$ as $n \to \infty$. By Lemma 5.2, $(u_n)$ is bounded in $X$, hence by Lemma 3.3 it has a convergent subsequence.

Arguing as in Lemma 4.3, we see that $\varphi$ has a $(1,1)$-local linking at 0. We distinguish two cases:

$(a)$ If $m = 0$, i.e., 0 is a global minimizer of $\varphi$, then just as in the proof of Lemma 4.3 we see that $\varphi(u) = 0$ for all $u \in E$, so $\varphi$ admits infinitely many critical points.

$(b)$ If $m < 0$, i.e., 0 is not a global minimizer of $\varphi$, then by Theorem 2.7 $\varphi$ admits at least three critical points.

In both cases we find $u_1, u_2 \in K(\varphi)\backslash\{0\}$, $u_1 \neq u_2$. By Lemma 3.2, $u_1, u_2 \in C_0^1(\overline{\Omega})$ solve (1.1). □

Now we turn to the resonant case, i.e., $pj(x,t) \sim \lambda_1|t|^p$ as $|t| \to \infty$, but with a tempering condition:

$\mathbf{H}_3$ $j : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory mapping s.t. $j(x,\cdot)$ is locally Lipschitz continuous and $j(x,0) = 0$ for a.e. $x \in \Omega$, and

(i) $|\xi| \leqslant c_0(1 + |t|^{q-1})$ for a.e. $x \in \Omega$, all $t \in \mathbb{R}$, $\xi \in \partial j(x,t)$ ($c_0 > 0$, $q \in (p, p^*)$);

(ii) $\lambda_1|t|^p \leqslant pj(x,t) \leqslant \hat{\lambda}|t|^p$ for a.e. $x \in \Omega$, all $|t| \leqslant \delta$ ($\delta > 0$, $\hat{\lambda} \in (\lambda_1, \lambda_2)$);

(iii) $\displaystyle\lim_{|t|\to\infty} \frac{pj(x,t)}{|t|^p} = \lambda_1$ uniformly for a.e. $x \in \Omega$;

(iv) $\displaystyle\lim_{|t|\to\infty} \Big[\min_{\xi \in \partial j(x,t)} (\xi t) - pj(x,t)\Big] = \infty$ uniformly for a.e. $x \in \Omega$.

Clearly $\mathbf{H}_3$ incorporates $\mathbf{H}_0$. As in the previous case, by $\mathbf{H}_3$ (ii) we have $0 \in K(\varphi)$. In comparison with $\mathbf{H}_2$, we see that $\mathbf{H}_3$ (iii) allows resonance at infinity, so $\mathbf{H}_3$ (iv) is required to ensure coercivity of the energy functional.

**Example 5.4.** The following locally Lipschitz continuous (autonomous) potential $j : \mathbb{R} \to \mathbb{R}$ satisfies hypotheses $\mathbf{H}_3$:

$$j(t) = \begin{cases} \hat{\lambda}\dfrac{|t|^p}{p} & \text{if } |t| \leqslant 1 \\ \hat{\lambda}\dfrac{|t|^p}{p} - \ln(|t|) & \text{if } |t| > 1, \end{cases}$$

with $\hat{\lambda} \in (\lambda_1, \lambda_2)$.

Still we have coercivity:

**Lemma 5.5.** *Let* $\mathbf{H}_3$ *hold. Then,*

$$\lim_{\|u\|\to\infty} \varphi(u) = \infty.$$

*Proof.* Set for all $(x,t)\in\Omega\times\mathbb{R}$

$$\tilde{\jmath}(x,t)=j(x,t)-\frac{\lambda_1|t|^p}{p},$$

then $\tilde{\jmath}:\Omega\times\mathbb{R}\to\mathbb{R}$ satisfies $\mathbf{H}_0$. Besides, by $\mathbf{H}_3$ (iii), (iv) we have uniformly for a.e. $x\in\Omega$

$$\lim_{|t|\to\infty}\frac{p\tilde{\jmath}(x,t)}{|t|^p}=0, \tag{5.1}$$

$$\lim_{|t|\to\infty}\Big[\min_{\xi\in\partial\tilde{\jmath}(x,t)}(\xi t)-p\tilde{\jmath}(x,t)\Big]=\infty. \tag{5.2}$$

As in the proof of Lemma 4.2 we see that $\tilde{\jmath}(x,\cdot)$ is differentiable a.e. in $\mathbb{R}$, and by (5.2) for all $K>0$ there exists $M>0$ s.t.

$$\tilde{\jmath}'(x,t)-p\tilde{\jmath}(x,t)\geqslant K$$

for a.e. $x\in\Omega$ and a.e. $|t|\geqslant M$. The product formula for derivatives holds a.e. (see [6, Propositon 2.3.13]), so we have

$$\frac{d}{dt}\Big[\frac{\tilde{\jmath}(x,t)}{|t|^p}\Big]=\frac{\tilde{\jmath}'(x,t)t-p\tilde{\jmath}(x,t)}{|t|^{p+1}}\geqslant\frac{K}{|t|^{p+1}}$$

for a.e. $x\in\Omega$ and a.e. $|t|\geqslant M$. Integrating in $[t,T]$ for any $M\leqslant t<T$, we have

$$\frac{\tilde{\jmath}(x,T)}{T^p}-\frac{\tilde{\jmath}(x,t)}{t^p}\geqslant\int_t^T\frac{K}{\tau^{p+1}}\,d\tau=\frac{K}{p}\Big(\frac{1}{t^p}-\frac{1}{T^p}\Big).$$

Letting $T\to\infty$ and using (5.1), we have for a.e. $x\in\Omega$, $t\geqslant M$

$$\tilde{\jmath}(x,t)\leqslant-\frac{K}{p}. \tag{5.3}$$

A similar argument leads to (5.3) for all $t\leqslant -M$. Arguing by contradiction, let $(u_n)$ be a sequence in $X$, s.t. $\varphi(u_n)\leqslant C$, $\|u_n\|\to\infty$ as $n\to\infty$. Set for all $n\in\mathbb{N}$

$$v_n=\frac{u_n}{\|u_n\|},$$

then passing if necessary to a subsequence we have $v_n\rightharpoonup v$ in $X$, $v_n\to v$ in $L^p(\Omega)$, and $v_n(x)\to v(x)$ for a.e. $x\in\Omega$. By (5.3) we have for all $n\in\mathbb{N}$

$$\begin{aligned}
\frac{C}{\|u_n\|^p}&\geqslant\frac{\varphi(u_n)}{\|u_n\|^p}\\
&=\frac{1}{p}\int_\Omega\big(|\nabla v_n|^p-\lambda_1|v_n|^p\big)\,dx-\int_\Omega\frac{\tilde{\jmath}(x,u_n)}{\|u_n\|^p}\,dx\\
&\geqslant\frac{1}{p}\big(1-\lambda_1\|v_n\|_p^p\big)+\int_{\{|u_n|\geqslant M\}}\frac{K}{p\|u_n\|^p}\,dx-\int_{\{|u_n|<M\}}\frac{\tilde{\jmath}(x,u_n)}{\|u_n\|^p}\,dx\\
&\geqslant\frac{1}{p}\Big(1-\frac{K|\Omega|}{\|u_n\|^p}\Big)-\frac{c_0}{\|u_n\|^p}\Big(M+\frac{M^q}{q}\Big)|\Omega|.
\end{aligned}$$

By the inequality above, choosing a bigger $C>0$ we have for all $n\in\mathbb{N}$

$$\|v_n\|_p^p\geqslant\frac{1}{\lambda_n}-\frac{C}{\|u_n\|^p}.$$

Passing to the limit as $n \to \infty$, we get

$$\|v\|_p^p \geqslant \frac{1}{\lambda_1},$$

while by construction $\|v\|^p = 1$. By (4.2) we deduce

$$\|v\|^p = \lambda_1 \|v\|_p^p = 1.$$

So, $v \in X$ is a $\lambda_1$-eigenfunction of (4.1), hence, either $v > 0$ in $\Omega$, or $v < 0$ in $\Omega$. Thus, we have $|u_n(x)| \to \infty$ for a.e. $x \in \Omega$, which by (5.3) again implies $\tilde{\jmath}(x, u_n) \to -\infty$ for a.e. $x \in \Omega$. Then, for all $n \in \mathbb{N}$ we have

$$\begin{aligned}\varphi(u_n) &\geqslant \frac{\|u_n\|^p}{p} - \frac{\lambda_1 \|u_n\|_p^p}{p} - \int_\Omega \tilde{\jmath}(x, u_n)\, dx \\ &\geqslant -\int_\Omega \tilde{\jmath}(x, u_n)\, dx,\end{aligned}$$

and the latter tends to $\infty$ as $n \to \infty$ by Fatou's lemma, against $\varphi(u_n) \leqslant C$. So we conclude. $\square$

We can now prove our second multiplicity result:

**Theorem 5.6.** *Let* $\mathbf{H}_3$ *hold. Then,* (1.1) *has at least two solutions* $u_1, u_2 \in C_0^1(\overline{\Omega}) \setminus \{0\}$.

*Proof.* By Lemma 5.5, $\varphi$ is coercive in $X$. Besides, by $\mathbf{H}_3$ (i) $\varphi$ is sequentially weakly l.s.c. in $X$, hence it is bounded from below. Then, the conclusion follows exactly as in Theorem 5.3. $\square$

**Remark 5.7.** The variational method we have employed for problem (1.1), as well as the Morse-theoretic approach, can be easily extended to a more general class of PDI's, namely

$$\begin{cases} -\Delta_p u \in F(x.u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $F : \Omega \times \mathbb{R} \to 2^{\mathbb{R}}$ is a set-valued mapping satisfying the following conditions:

(i) $F(x, \cdot) : \mathbb{R} \to 2^{\mathbb{R}}$ is u.s.c. with convex compact values for a.e. $x \in \Omega$;
(ii) $\min F$, $\max F : \Omega \times \mathbb{R} \to \mathbb{R}$ are $\mathcal{L} \otimes \mathcal{B}$-measurable;
(iii) $|\xi| \leqslant c_0(1 + |t|^{q-1})$ for a.e. $x \in \Omega$, all $t \in \mathbb{R}$, and all $x \in F(x,t)$ ($c_0 > 0$, $q \in (p, p^*)$).

The main step consists in defining a suitable potential $j$ s.t. $\partial j(x,t) \subset F(x,t)$ for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$, and then study (1.1) (see [13, 14] for details).

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