# INFINITELY MANY NODAL SOLUTIONS FOR ANISOTROPIC $(p,q)$-EQUATIONS

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Abstract. We consider an anisotropic $(p.q)$-Neumann problem with an indefinite potential term and a reaction which is only locally defined and odd. Using a variant of the symmetric mountain pass theorem, we show that the problem has a whole sequence of smooth nodal solutions which converge to zero in $C^1\left(\overline{\Omega}\right)$.

## 1. Introduction

Let $\Omega\subseteq\mathbb{R}^N$ be a bounded domain with a $C^2-$ boundary $\partial\Omega$. In this paper we study the following anisotropic $(p,q)$-equation)

$$\begin{cases} -\Delta_{p(z)}u(z)-\Delta_{q(z)}u(z)+\xi(z)\left|u(z)\right|^{p(z)-2}u(z) \\ \qquad\qquad\qquad\qquad\qquad\qquad = f(z,u(z)) \text{ in } \Omega, \\ \frac{\partial u}{\partial n}=0 \text{ on } \partial\Omega \end{cases} \tag{1.1}$$

Given $s\in L^{\infty}(\Omega)$ with $1<s_{-}\leq s(z)\leq s_{+}<\infty$ for a.a. $z\in\Omega$ , by $\Delta_{s(z)}$ we denote the $s(z)-$Laplacian defined by

$$\Delta_{s(z)}u=div\left(\left|Du\right|^{s(z)-2}Du\right) \text{ for all } u\in W_0^{1,s(z)}(\Omega),$$

Here we assume that the variable exponents $p(\cdot)$ and $q(\cdot)$ belong to $C^1\left(\overline{\Omega}\right)$. This allows us to use the existing regularity theory for anisotropic problems. The potential function $\xi(\cdot)\in L^{\infty}(\Omega)$ and in general it is sign-changing. The reaction function $f(z,x)$ is only locally defined, that is, $f(\cdot,\cdot)$ is defined on $\Omega\times[-\theta,\theta]$, $\theta>0$ and it is a Carathéodory function (that is, for all $x\in[-\theta,\theta]$, $z\mapsto f(z,x)$ is measurable and for a.a. $z\in\Omega$, $x\mapsto f(z,x)$ is continuous). We assume that for a.a. $z\in\Omega$, $f(z,\cdot)\mid_{[-\theta,\theta]}$ is odd. In the boundary condition $n(\cdot)$ denotes the outward unit normal on $\partial\Omega$.

Using a version of the symmetric mountain pass theorem due to Kajikiya [8] (see Proposition 3), together with suitable truncations and comparison techniques, we show that there exists a whole sequence $\{u_n\}_{n\geq1}\subseteq C^1\left(\overline{\Omega}\right)$ of nodal solutions of $(1.1)$, such that $u_n\to0$ in $C^1\left(\overline{\Omega}\right)$.

Elliptic equations with locally defined reaction, were first considered by Wang [18], who considered semilinear Dirichlet equations driven by the Laplacian and with a reaction of the form $\lambda\left|x\right|^{q-2}x+g(x,z)$ with $1<q<2$. So, the reaction has a parametric concave term (the function $\lambda\left|x\right|^{q-2}x$) and

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a perturbation $g \in C(\Omega \times \mathbb{R})$ which is odd in $x \in \mathbb{R}$ for $|x|$ small and

$$\lim_{x \to 0} \frac{g(z,x)}{|x|^{q-2} x} = 0 \text{ uniformly for a.a. } z \in \Omega.$$

No conditions are imposed on $g(z,\cdot)$ for $|x|$ big. So, the hypotheses on $g(z,\cdot)$ are local near zero. In this setting the author proves the existence of a sequence $\{u_n\}_{n\geq 1} \subseteq H_0^1(\Omega)$ of weak solutions. Later, Li-Wang [11] extended the work of [18] to Schroedinger equations and also showed that the solutions $\{u_n\}_{n\geq 1}$ are nodal.

Recently the aforementioned works were extended to Robin problems by Papageorgiou-Veltro-Veltro [15] (semilinear problems driven by the Laplacian plus an indefinite and unbounded potential) and by Papageorgiou-Radulescu [12] (nonlinear nonhomogeneous equations).

For anisotropic equations there are no such results. Infinitely many solutions for $p(x)$ −Laplacian-type equations were proved by Andrei [1], Fan-Zhang [5] (Dirichlet problems with strictly positive potential terms) and by Boureanu-Preda [2], Liang-Zhang [9] (Neumnann problems with strictly positive potential terms). All these works have reactions which are globally defined and impose conditions on $f(z,\cdot)$ as $x \to \pm\infty$. Moreover, none of these works provide sign information for the solutions produced. We mention that Boureanu-Preda [2] use the fountain theorem to obtain weak solutions $\{u_n\}_{n\geq 1}$ such that $\|u_n\|_{W^{1,p(z)}(\Omega)} \to +\infty$ while Liang-Zhang [9] use the symmetric mountain pass theorem of Kajikiya [8] to show that $\|u_n\|_{W^{1,p(z)}(\Omega)} \to 0$.

## 2. Mathematical background

Let $M(\Omega)$ be the space of all measurable functions $u : \Omega \to \mathbb{R}$. As usual we identify two such functions which differ only on a set of zero measure. Also let

$$L_1^\infty(\Omega) = \left\{ p \in L^\infty(\Omega) : 1 \leq \underset{\Omega}{\operatorname{essinf}} p \right\}.$$

Given $p \in L_1^\infty(\Omega)$, we define

$$p_- = \underset{\Omega}{\operatorname{essinf}}\, p \text{ and } p_+ = \underset{\Omega}{\operatorname{esssup}}\, p$$

and the variable exponent Lebesgue space

$$L^{p(z)}(\Omega) = \left\{ u \in M(\Omega) : \int_\Omega |u|^{p(z)} dz < \infty \right\}.$$

This space is furnished with the so called *Luxemburg norm* defined by

$$\|u\|_{p(z)} = \inf \left\{ \lambda > 0 : \int_\Omega \left( \frac{|u|}{\lambda} \right)^{p(z)} dz \leq 1 \right\}.$$

These spaces resemble the classical Lebesgue spaces. They are separable Banach spaces and they are reflexive if and only if $1 < p_- \leq p_+ < \infty$ (in fact they are uniformly convex) and simple and continuous functions with compact support are dense in $L^{p(z)}(\Omega)$. If $p,\ q\in L_1^{\infty}(\Omega)$, then $L^{p(z)}(\Omega)\hookrightarrow L^{q(z)}(\Omega)$ continuously if and only if $q(z)\le p(z)$ for a.a. $z\in\Omega$.

Let $p,\ p'\in L_1^{\infty}(\Omega)$ satisfy $\frac{1}{p(z)}+\frac{1}{p'(z)}=1$ for a.a. $z\in\Omega$. Then $L^{p(z)}(\Omega)^*=L^{p'(z)}(\Omega)$ and we have the following Holder type inequality

$$\int_{\Omega}|uv|\,dz\le\left[\frac{1}{p_-}+\frac{1}{p'_-}\right]\|u\|_{p(z)}\|v\|_{p'(z)}$$

for all $u\in L^{p(z)}(\Omega)$,all $v\in L^{p(z)}(\Omega)$ $\left(\frac{1}{p_-}+\frac{1}{p'_-}=1\right)$. We can also define variable exponent Sobolev spaces by

$$W^{1,p(z)}(\Omega)=\left\{u\in L^{p(z)}(\Omega):|Du|\in L^{p(z)}(\Omega)\right\}.$$

We equip $W^{1,p(z)}(\Omega)$ with the following norm

$$\|u\|=\|u\|_{p(z)}+\||Du|\|_{p(z)}.$$

An equivalent norm of $W^{1,p(z)}(\Omega)$ is given by

$$\|u\|'=\inf\left\{\lambda>0:\int_{\Omega}\left[\left(\frac{|Du|}{\lambda}\right)^{p(z)}+\left(\frac{|u|}{\lambda}\right)^{p(z)}\right]dz\le 1\right\}.$$

The space $W^{1,p(z)}(\Omega)$ is a separable Banach space and it is reflexive (in fact uniformly convex), if $1<p_-\le p_+<\infty$. Also, we have $W^{1,p(z)}(\Omega)\hookrightarrow W^{1,p_-}(\Omega)$ continuously.

We set

$$p^*(z)=\begin{cases}\frac{Np(z)}{N-p(z)} & \text{if}\quad p(z)\le N\\ +\infty & \text{if}\quad N<p(z).\end{cases}$$

If $p,\ q\in C\left(\overline{\Omega}\right)\cap L_1^{\infty}(\Omega)$, $p_+<N$ and $1\le q(z)\le p^*(z)$ (resp. $1\le q(z)<p^*(z)$) for all $z\in\overline{\Omega}$, then $W^{1,p(z)}(\Omega)$ is embedded continuously (resp. compactly) into $L^{q(z)}(\Omega)$.

Very useful for the analysis of these spaces and for the study of anisotropic boundary value problems, is the modular function of the space $L^{r(z)}(\Omega)$ with $r\in L_1^{\infty}(\Omega)$ defined by

$$\rho_{r(z)}(u)=\int_{\Omega}|u|^{r(z)}\,dz.$$

There is a close relation between this function and the norm $\|\cdot\|_{p(z)}$.

**Proposition 2.1.** *(a) For $u\in L^{r(z)}(\Omega)$, $u\neq 0$, we have*

$$\|u\|_{r(z)}\le\lambda\Longleftrightarrow\rho_{r(z)}\left(\frac{u}{\lambda}\right)\le 1;$$

(b) $\|u\|_{r(z)}<1$ *(resp.* $=1,\ >1$*)* $\Longleftrightarrow\rho_{r(z)}(u)<1$ *(resp.* $=1,\ >1$*)*;
(c) $\|u\|_{r(z)}<1\Longrightarrow\|u\|_{r(z)}^{r_+}\le\rho_{r(z)}(u)\le\|u\|_{r(z)}^{r_-}$ *and*
$\|u\|_{r(z)}>1\Longrightarrow\|u\|_{r(z)}^{r_-}\le\rho_{r(z)}(u)\le\|u\|_{r(z)}^{r_+}$;
(d) $\|u_n\|_{r(z)}\to 0\Longleftrightarrow\rho_{r(z)}(u_n)\to 0$;
(e) $\|u_n\|_{r(z)}\to+\infty\Longleftrightarrow\rho_{r(z)}(u_n)\to+\infty$.

A comprehensive treatment of variable exponent spaces can be found in the book of Diening-Harjulehto-Hästo-Rujicka [3]. Their use in the study of various anisotropic boundary value problems can be found in the book of Radulescu-Repovs [16].

Let $A_{r(z)} : W^{1,r(z)}(\Omega) \to W^{1,r(z)}(\Omega)^*$ be the nonlinear map defined by

$$\left\langle A_{r(z)}(u), h\right\rangle = \int_{\Omega} |Du|^{r(z)-2} (Du, Dh)_{\mathbb{R}^N}\, dz \text{ for all } u,\, h \in W^{1,r(z)}(\Omega).$$

The next proposition summarizes the main properties of this map (see, Gasinski-Papageorgiou ([6]).

**Proposition 2.2.** *The map $A_{r(z)} : W^{1,r(z)}(\Omega) \to W^{1,r(z)}(\Omega)^*$ defined above is bounded (that is, maps bounded sets to bounded sets), continuous, monotone (hence maximal monotone, too), and of type $(S)_+$, that is, if for every sequence $\{u_n\}_{n\geq 1} \subseteq W^{1,r(z)}(\Omega)$ such that $u_n \xrightarrow{w} u$ and*

$$\limsup_{n\to\infty} \left\langle A_{r(z)}(u_n), u_n - u\right\rangle \leq 0,$$

*one has*

$$u_n \to x \text{ in } X \text{ as } n \to \infty.$$

The anisotropic regularity theory (see Fan [4]) will lead us to the Banach space $C^1(\overline{\Omega})$. This is an ordered Banach space with a positive cone given by

$$C_+ = \left\{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$int\ C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\right\},$$

If $X$ is a Banach space and $\varphi \in C^1(X, \mathbb{R})$, then by $K_\varphi$ we denote the critical set of $\varphi$, that is,

$$K_\varphi = \left\{u \in X : \varphi'(u) = 0\right\}$$

We say that $\varphi(\cdot)$ satisfies the the *Palais-Smale condition* (the *PS-condition*, for short) if: *every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\in\mathbb{N}}$ is bounded and* $\varphi'(u_n) \to 0$ in $X^*$as $n \to \infty$ *admits a strongly convergent subsequence.*

As we already mentioned in the Introduction, we will use a version of the symmetric mountain pass theorem due to Kajikiya [8]. More precisely, we will use the next proposition, which is a special case of Theorem 1 of Kajikiya [8].

**Proposition 2.3.** *If $X$ is a Banach space, $\varphi \in C^1(X, \mathbb{R})$ satisfies the satisfies the $PS-$condition, it is even, bounded below, $\varphi(0) = 0$ and for every $n \in \mathbb{N}$ there exists an $n-$dimensional subspace $V_n \subseteq X$ and an $\rho_n > 0$ such that*

$$\sup\{\varphi(u) : u \in V_n, \|u\| = \rho_n\} < 0,$$

*then there exists a sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that*

$$\varphi(u_n) \leq 0 \text{ for all } n \in \mathbb{N},\ u_n \to 0 \text{ in } X.$$

The hypotheses on the data of (1.1) are the following:

$(\mathbf{H_0})$: $\xi \in L^\infty(\Omega)$, $p, q \in C^1(\overline{\Omega})$ and $1 < q_- \leq q(z) \leq q_+ < p_- \leq p(z) \leq p_+$.

$(\mathbf{H_1})$: $f:\Omega\times[-\theta,\theta]\to\mathbb{R}$ is a Carathéodory function such that $f(z,0)=0$ for a.a. $z\in\Omega$ and

(*i*) for a.a. $z\in\Omega$, $f(z,\cdot)\mid_{[-\theta,\theta]}$ is odd and $f(z,x)x\geq 0$ for $a.a.z\in\Omega$, all $x\in[-\theta,\theta]$;

(*ii*) there exists $a_\theta\in L^\infty(\Omega)$ such that

$$|f(z,x)|\leq a_\theta(z) \text{ for } a.a.z\in\Omega, \text{ all } x\in\mathbb{R} \text{ with } |x|\leq\theta;$$

(*iii*)

$$\lim_{x\to 0}\frac{f(z,x)}{|x|^{q_--2}x}=+\infty \text{ for a.a. } z\in\Omega;$$

Consider an even function $\beta\in C(\mathbb{R})$ such that:

$$0\leq\beta(x)\leq 1 \text{ for all } x\in\mathbb{R},\ \beta\mid_{[-\mu,\mu]}\equiv 1 \text{ with } 0<\mu<\theta,\quad \operatorname{supp}\beta\subseteq[-\theta,\theta].$$

We introduce the following Carathéodory function defined on $\Omega\times\mathbb{R}$:

$$f_0(z,x):=\beta(x)f(z,x)+(1-\beta(x))|x|^{p(z)-2}x. \tag{2.1}$$

For a.a. $z\in\Omega$, the function $f_0(z,.)$ has the following properties

$$\begin{gathered} f_0(z,.) \text{ is odd, } f_0(z,\cdot)\mid_{[-\mu,\mu]}=f(z,\cdot)\mid_{[-\mu,\mu]} \text{ and} \\ f_0(z,x):=\xi(z)|x|^{p(z)-2}x \text{ if } |x|\geq\theta. \end{gathered} \tag{2.2}$$

In particular we have

$$f_0(z,\theta)=\xi(z)\theta^{p(z)-2} \text{ and } f_0(z,-\theta)=-\xi(z)\theta^{p(z)-1} \text{ for } a.a.z\in\Omega. \tag{2.3}$$

We can find $r\in C(\overline{\Omega})$ such that $p_+<r(z)<p^*(z)$ for all $z\in\overline{\Omega}$ and, for some $C_1>0$:

$$|f_0(z,x)|\leq C_1\left[1+|x|^{r(z)-1}\right] \text{ for a.a. } z\in\Omega, \text{ all } x\in\mathbb{R} \tag{2.4}$$

(see hypothesis $(\mathbf{H_1})(ii)$ and (2.1)). On account of hypothesis $(\mathbf{H_1})(iii)$ and (2.4)) we see that given $\eta>0$, we can find $C_2=C_2(\eta)>0$ such that

$$f_0(z,x)x\geq\eta|x|^{q_-}-C_2|x|^{r(z)} \text{ for a.a. } z\in\Omega, \text{ all } x\in\mathbb{R}. \tag{2.5}$$

We set

$$k(z,x)=\eta|x|^{q_--2}x-C_2|x|^{r(z)-2}x \text{ for all } (z,x)\in\overline{\Omega}\times\mathbb{R}.$$

Evidently $k\in C(\overline{\Omega}\times\mathbb{R})$.

## 3. Nodal solutions

Motivated by the unilateral growth estimate (2.5), we consider the following anisotropic Neumann problem

$$\begin{cases} -\Delta_{p(z)}u(z)-\Delta_{q(z)}u(z)+|\xi(z)||u(z)|^{p(z)-2}u(z) \\ \qquad\qquad = k(z,u(z)) \text{ in } \Omega, \\ \frac{\partial u}{\partial n}=0 \text{ on } \partial\Omega \end{cases} \tag{3.1}$$

**Proposition 3.1.** *If hypotheses* $(\mathbf{H_0})$ *hold, then problem* (3.1) *has a unique positive solution* $\overline{u}\in int\ C_+$ *and since* (3.1) *is odd,* $\overline{v}=-\overline{u}\in -intC_+$ *is the unique negative solution of* (3.1).

*Proof.* We introduce the function $\widehat{k}(z,x)$ defined by

$$\widehat{k}(z,x)=\begin{cases} k(z,-\theta)-\theta^{p(z)-1} & \text{if } x<-\theta \\ k(z,x)+|x|^{p(z)-2}x & \text{if } -\theta<x<\theta \\ k(z,\theta)+\theta^{p(z)-1} & \text{if } \theta<x. \end{cases} \tag{3.2}$$

Evidently $\widehat{k}\in C\left(\overline{\Omega}\times\mathbb{R}\right)$. We set $\widehat{K}(z,x)=\int_0^x \widehat{k}(z,s)\,ds$ and consider the $C^1-$functional $\widehat{\varphi}_+ : W^{1,p(z)}(\Omega)\to\mathbb{R}$ defined by

$$\begin{aligned}\widehat{\varphi}_+(u)&=\int_\Omega \frac{1}{p(z)}|Du|^{p(z)}dz+\int_\Omega \frac{1}{q(z)}|Du|^{q(z)}dz\\ &+\int_\Omega \frac{|\xi(z)|+1}{p(z)}|u(z)|^{p(z)}dz-\int_\Omega \widehat{K}\left(z,u^+\right)dz\end{aligned}$$

for all $u\in W^{1,p(z)}(\Omega)$. From (2.5) it is clear that $\widehat{\varphi}_+(\cdot)$ is coercive. Also, using the compact embedding of $W^{1,p(z)}(\Omega)$ into $L^{p(z)}(\Omega)$, we see that $\widehat{\varphi}_+(\cdot)$ is sequentially weakly lower semicontinuous. So, by Weierstrass-Tonelli theorem, we can find $\overline{u}\in W^{1,p(z)}(\Omega)$ such that

$$\widehat{\varphi}_+(\overline{u})=\inf\left\{\widehat{\varphi}_+(u):u\in W^{1,p(z)}(\Omega)\right\}. \tag{3.3}$$

Let $u\in intC_+$ and $t\in(0,1)$ small such that $tu(z)\le\theta$ for all $z\in\overline{\Omega}$. Using (3.2) we have

$$\begin{aligned}&\widehat{\varphi}_+(tu)\le \tfrac{t^{p_-}}{p_-}\rho_{p(z)}(Du)+\tfrac{t^{q_-}}{q_-}\rho_{q(z)}(Du)\\ &+\tfrac{t^{p_-}}{p_-}\left[\|\xi\|_\infty+1\right]\rho_{p(z)}(u)-\tfrac{\eta}{q_-}t^{q_-}\|u\|_{q_-}^{q_-}+\tfrac{C_2t^{r_-}}{r_-}\rho_{r(z)}(u)\\ &\le \tfrac{t^{p_-}}{p_-}\max\left\{\|u\|^{p_-},\|u\|^{p_+}\right\}+\tfrac{t^{q_-}}{q_-}\max\left\{\|u\|^{q_-},\|u\|^{q_+}\right\}\\ &+\tfrac{t^{p_-}}{p_-}C_3\max\left\{\|u\|^{p_-},\|u\|^{p_+}\right\}-\tfrac{\eta t^{q_-}}{q_-}\|u\|_{q_-}^{q_-}\\ &+\tfrac{t^{p_-}}{p_-}C_3\max\left\{\|u\|^{p_-},\|u\|^{p_+}\right\}-\tfrac{\eta t^{q_-}}{q_-}\|u\|_{q_-}^{q_-}\\ &\qquad\qquad\qquad\qquad \text{for some } C_3,\ C_4>0\end{aligned} \tag{3.4}$$

(see Proposition 2.1). In deriving the previous estimate, we have used also the continuous (in fact compact) embeddings $W^{1,p(z)}(\Omega)\hookrightarrow L^{p(z)}(\Omega)$ and $W^{1,p(z)}(\Omega)\hookrightarrow L^{r(z)}(\Omega)$.

Since $q_-<p_-<r_-$, choosing $\eta>0$ big (see hypothesis $(\mathbf{H_1})(iii)$) it follows from (3.4) that by taking $t\in(0,1)$ even smaller if necessary, we have

$$\widehat{\varphi}_+(tu)<0,$$

hence

$$\widehat{\varphi}_+(\overline{u})<0=\widehat{\varphi}_+(0)\ \ (\text{see } (3.3)),$$

therefore

$$\overline{u}\neq 0.$$

From (3.3) we have

$$\widehat{\varphi}_+'(\overline{u})=0,$$

hence

$$\begin{aligned}\left\langle A_{p(z)}\left(\overline{u}\right),h\right\rangle+\left\langle A_{q(z)}\left(\overline{u}\right),h\right\rangle+\int_{\Omega}\left[\left|\xi\left(z\right)\right|+1\right]\left|\overline{u}\right|^{p(z)-2}\overline{u}hdz\\=\int_{\Omega}\widehat{k}\left(z,\overline{u}^{+}\right)hdz\text{ for all }h\in W^{1,p(z)}\left(\Omega\right)\end{aligned}\tag{3.5}$$

In $(3.5)$ we choose $h=-\overline{u}^{-}\in W^{1,p(z)}\left(\Omega\right)$. Then

$$\rho_{p(z)}\left(Du^{-}\right)+\rho_{q(z)}\left(Du^{-}\right)+\int_{\Omega}\left[\left|\xi\left(z\right)\right|+1\right]\left(\overline{u}^{-}\right)^{p(z)}dz=0$$

(see $(3.2)$), hence

$$\rho_{p(z)}\left(Du^{-}\right)\leq 0,$$

therefore

$$\overline{u}\geq 0,\ \overline{u}\neq 0.$$

Also in $(3.5)$ we choose $h=\left[\overline{u}-\theta\right]^{+}\in W^{1,p(z)}\left(\Omega\right)$. We obtain

$$\begin{aligned}&\left\langle A_{p(z)}\left(\overline{u}\right),\left[\overline{u}-\theta\right]^{+}\right\rangle+\left\langle A_{q(z)}\left(\overline{u}\right),\left[\overline{u}-\theta\right]^{+}\right\rangle\\&+\int_{\Omega}\left[\left|\xi\left(z\right)\right|+1\right]\left|\overline{u}\right|^{p(z)-1}\left(\overline{u}-\theta\right)^{+}dz\\&=\int_{\Omega}\left[k\left(z,\theta\right)+\theta^{p(z)-1}\right]\left[\overline{u}-\theta\right]^{+}dz\text{ (see (3.2))}\\&=\int_{\Omega}\left[\eta\theta^{q_{-}-1}-C_{2}\theta^{r(z)-1}+\theta^{p(z)-1}\right]\left[\overline{u}-\theta\right]^{+}dz\\&\leq\int_{\Omega}\left[f_{0}\left(z,\theta\right)+\theta^{p(z)-1}\right]\left[\overline{u}-\theta\right]^{+}dz\text{ (see (2.5))}\\&\leq\int_{\Omega}\left[\left|\xi\left(z\right)\right|+1\right]\theta^{p(z)-1}\left(\overline{u}-\theta\right)^{+}dz\text{ (see (2.3))}\\&=\left\langle A_{p(z)}\left(\theta\right),\left[\overline{u}-\theta\right]^{+}\right\rangle+\left\langle A\theta,\left[\overline{u}-\theta\right]^{+}\right\rangle\\&+\int_{\Omega}\left[\left|\xi\left(z\right)\right|+1\right]\theta^{p(z)-1}\left(\overline{u}-\theta\right)^{+}dz\text{ (see (2.3))},\end{aligned}$$

hence

$$\overline{u}\leq\theta.$$

So, we have proved that

$$\overline{u}\in\left[0,\theta\right],\ \overline{u}\neq 0.\tag{3.6}$$

From $(3.6)$, $(3.2)$ and $(3.5)$ it follows

$$\overline{u}\text{ is a positive solution of }(3.1).$$

From Proposition 3.1 of Gasinski-Papageorgiou [6], we have that $\overline{u}\in L^{\infty}\left(\Omega\right)$. Then, Theorem 1.3 of Fan [4] (see also Lieberman [10]), we have that $\overline{u}\in C_{+}\backslash\left\{0\right\}$. Finally the anisotropic maximum principle of Zhang [19], implies that

$$\overline{u}\in intC_{+}.$$

Next we show that this positive solution is unique. To this end we consider the integral functional $j : L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \displaystyle\int_\Omega \frac{1}{p(z)} \left| Du^{\frac{1}{q_-}} \right| dz + \int_\Omega \frac{1}{q(z)} \left| Du^{\frac{1}{q_-}} \right| dz \\ \qquad + \displaystyle\int_\Omega \frac{|\xi(z)|}{p(z)} \left( u^{\frac{1}{q_-}} \right) dz & \text{if } u \geq 0 \text{ and } u^{\frac{1}{q_-}} \in W^{1,p(z)}(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

From Theorem 2.2 of Takac-Giacomoni [17], we know that the functional $j(\cdot)$ is convex.

Suppose that $\widetilde{u} \in W^{1,p(z)}(\Omega)$ is another solution of $(3.1)$. Again we have $\widetilde{u} \in intC_+$.

On account of Proposition 4.1.22, p.274, of Papageorgiou-Radulescu-Repovs [14], we have

$$\frac{\overline{u}}{\widetilde{u}}, \ \frac{\widetilde{u}}{\overline{u}} \in L^\infty(\Omega).$$

From Theorem 2.5 (and its proofs) of Takac-Giacomoni [17], we have

$$j'(\overline{u}^{q_-})(\overline{u}^{q_-} - \widetilde{u}^{q_-}) = \int_\Omega \frac{-\Delta_{p(z)}\overline{u}(z) - \Delta_{q(z)}\overline{u}(z) + |\xi(z)| |\overline{u}(z)|^{p(z)-1}}{\overline{u}^{q_- -1}} (\overline{u}^{q_-} - \widetilde{u}^{q_-}) dz,$$

$$j'(\widetilde{u}^{q_-})(\overline{u}^{q_-} - \widetilde{u}^{q_-}) = \int_\Omega \frac{-\Delta_{p(z)}\widetilde{u}(z) - \Delta_{q(z)}\widetilde{u}(z) + |\xi(z)| |\widetilde{u}(z)|^{p(z)-1}}{\widetilde{u}^{q_-} - 1} (\overline{u}^{q_-} - \widetilde{u}^{q_-}) dz.$$

The convexity of $j(\cdot)$ implies the monotonicity of $j'(\cdot)$ and so we have

$$0 \leq \int_\Omega - C_2 \left[ \overline{u}^{r(z)-q_-} - \widetilde{u}^{r(z)-q_-} \right] (\overline{u}^{q_-} - \widetilde{u}^{q_-}) dz \leq 0,$$

(since $q_- < p_- < r_-$), hence

$$\overline{u} = \widetilde{u}.$$

This proves the uniqueness of positive solution $\overline{u} \in intC_+$.

Since problem $(3.1)$ is odd, $\overline{v} = -\overline{u} \in -intC_+$ is the unique negative solution of $(3.1)$. □

We define

$$\mathcal{S}_+ = \{u : u \text{ is a positive solutions of } (1.1) \text{ in } [0,\theta]\}$$
$$\mathcal{S}_- = \{v : v \text{ is a negative solution of } (1.1) \text{ in } [-\theta,0]\}.$$

From Papageorgiou-Radulescu-Repovs [13] (see the proof of Proposition 7)

$$\mathcal{S}_+ \text{ is downward directed}$$

(that is, if $u_1, u_2 \in \mathcal{S}_+$, then we can find $u \in \mathcal{S}_+$ such that $u \leq u_1$ and $u \leq u_2$) and

$$\mathcal{S}_- \text{ is upward directed}$$

(that is, if $v_1, v_2 \in \mathcal{S}_-$, then we can find $v \in \mathcal{S}_-$ such that $v_1 \leq v$ and $v_2 \leq v$).

Reasoning as in the first part of proof of Proposition 3.1, with $k(z,x)$ replaced by $f_0(z,x)$, we obtain the following result.

**Proposition 3.2.** *If hypotheses* $(\mathbf{H_0})$, $(\mathbf{H_1})$ *hold, then*

$$\varnothing \neq \mathcal{S}_+ \subseteq [0,\theta] \cap intC_+ \text{ and } \varnothing \neq \mathcal{S}_- \subseteq [-\theta,0] \cap (-intC_+).$$

In the next proposition, we produce a lower bound for the elements of $\mathcal{S}_+$ and an upper bound for the elements of $\mathcal{S}_-$.

**Proposition 3.3.** *If hypotheses* $(\mathbf{H_0})$, $(\mathbf{H_1})$ *hold, then*

$$\overline{u} \leq u \text{ for all } u \in \mathcal{S}_+ \text{ and } v \leq \overline{v} \text{ for all } v \in \mathcal{S}_-.$$

*Proof.* Let $u \in \mathcal{S}_+ \subseteq intC_+$. We introduce the following Carathéodory function

$$\tau_+(z,x) = \begin{cases} \widehat{k}(z,x^+) & \text{if } x \leq u(z) \\ \widehat{k}(z,u(z)) & \text{if } u(z) < x. \end{cases} \tag{3.7}$$

We set $T_+(z,x) = \int_0^x \tau_+(z,s)\,ds$ and consider the the $C^1-$functional $\sigma_+ : W^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned}\sigma_+(u) &= \int_\Omega \frac{1}{p(z)} |Du|^{p(z)} dz + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)} dz \\ &\quad + \int_\Omega \frac{|\xi(z)|+1}{p(z)} |u(z)|^{p(z)} dz - \int_\Omega T_+(z,u)\,dz\end{aligned}$$

for all $u \in W^{1,p(z)}(\Omega)$. From (3.7) it is clear that $\sigma_+(\cdot)$ is coercive. Also it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u} \in W^{1,p(z)}(\Omega)$ such that

$$\sigma_+(\widetilde{u}) = \inf\left\{\sigma_+(u) : u \in W^{1,p(z)}(\Omega)\right\} < 0 = \sigma_+(0).$$

(see the proof of Proposition 3.1). Hence

$$\widetilde{u} \neq 0.$$

We have

$$\sigma_+'(\widetilde{u}) = 0,$$

hence

$$\begin{aligned}&\left\langle A_{p(z)}(\widetilde{u}), h\right\rangle + \left\langle A_{q(z)}(\widetilde{u}), h\right\rangle + \int_\Omega [|\xi(z)|+1] |\widetilde{u}|^{p(z)-2} \widetilde{u} h dz \\ &= \int_\Omega \sigma_+(z,\widetilde{u}) h dz \text{ for all } h \in W^{1,p(z)}(\Omega)\end{aligned} \tag{3.8}$$

In (3.8) we choose $h = -\widetilde{u}^- \in W^{1,p(z)}(\Omega)$ and obtain

$$\widetilde{u} \geq 0,\ \widetilde{u} \neq 0.$$

Next in (3.8) we choose $h = [\widetilde{u} - u]^{+} \in W^{1,p(z)}(\Omega)$. Then

$$\begin{aligned}
&\left\langle A_{p(z)}(\widetilde{u}), [\widetilde{u}-u]^{+}\right\rangle + \left\langle A_{q(z)}(\widetilde{u}), [\widetilde{u}-u]^{+}\right\rangle \\
&+\int_{\Omega} [|\xi(z)|+1] |\widetilde{u}|^{p(z)-1} (\widetilde{u}-u)^{+} dz \\
&= \int_{\Omega} \widehat{k}(z,u)(\widetilde{u}-u)^{+} dz \text{ (see (3.7))} \\
&= \int_{\Omega} \left(k(z,u) + u^{p(z)-1}\right)(\widetilde{u}-u)^{+} dz \text{ (see (3.2) and recall that } u \in \mathcal{S}_{+}) \\
&= \int_{\Omega} \left(\beta(u) f(z,u) + u^{p(z)-1}\right)(\widetilde{u}-u)^{+} dz \text{ (see (2.5), (2.1), (2.2))} \\
&\leq \int_{\Omega} \left[f(z,u) + u^{p(z)-1}\right](\widetilde{u}-u)^{+} dz \text{ (see hypothesis } (\mathbf{H_1})(i)) \\
&= \left\langle A_{p(z)}(u), (\widetilde{u}-u)^{+}\right\rangle + \left\langle A_{q(z)}(u), (\widetilde{u}-u)^{+}\right\rangle \\
&+\int_{\Omega} [\xi(z)+1] u^{p(z)-1} (\widetilde{u}-u)^{+} dz \text{ (since } u \in \mathcal{S}_{+}), \\
&\leq \left\langle A_{p(z)}(u), (\widetilde{u}-u)^{+}\right\rangle + \left\langle A_{q(z)}(u), (\widetilde{u}-u)^{+}\right\rangle \\
&+\int_{\Omega} [|\xi(z)|+1] u^{p(z)-1} (\widetilde{u}-u)^{+} dz
\end{aligned}$$

hence

$$\widetilde{u} \leq u.$$

So, we have proved that

$$\widetilde{u} \in [0,u], \ \widetilde{u} \neq 0.$$

hence

$$\widetilde{u} = \overline{u} \in intC_{+} \text{ (see (3.7), (3.2) and Proposition 3.1)}$$

therefore

$$\overline{u} \leq u \text{ for all } u \in \mathcal{S}_{+}.$$

Similarly we show that

$$v \leq \overline{v} \text{ for all } v \in \mathcal{S}_{-}.$$

□

Using these bounds we can show the existence of extremal constant sign solutions, namely that $\mathcal{S}_{+}$ has a smallest element $u_{*} \in intC_{+}$ (hence $u_{*} \leq u$ for all $u \in \mathcal{S}_{+}$) and $\mathcal{S}_{-}$ has a biggest element $v_{*} \in -intC_{+}$ (hence $v \leq v_{*}$ for all $v \in \mathcal{S}_{-}$). These extremal solutions will lead to nodal ones.

**Proposition 3.4.** *If hypotheses* $(\mathbf{H_0})$, $(\mathbf{H_1})$ *hold, then there exist extremal constant sign solutions of* (1.1)

$$u_{*} \in intC_{+} \text{ and } v_{*} \in -intC_{+}$$

*Proof.* Recall that the set $\mathcal{S}_{+}$ is downward directed. So, according to Lemma 3.10, p. 178, of Hu-Papageorgiou [7], we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{S}_{+}$ decreasing such that

$$\inf_{n\geq 1} u_n = \inf \mathcal{S}_{+}.$$

We have

$$\begin{aligned}&\left\langle A_{p(z)}\left(u_n\right),h\right\rangle+\left\langle A_{q(z)}\left(u_n\right),h\right\rangle+\int_\Omega \xi\left(z\right)u_n^{p(z)-1}hdz\\&=\int_\Omega f\left(z,u_n\right)hdz \text{ for all } h\in W^{1,p(z)}\left(\Omega\right),\ \text{all } n\in\mathbb{N},\end{aligned}\tag{3.9}$$

$$\overline{u}\leq u_n\leq u_1 \text{ for all } n\in\mathbb{N} \text{ (see Proposition 3.3).}\tag{3.10}$$

Choosing $h=u_n\in W^{1,p(z)}\left(\Omega\right)$ in (3.9) and using (3.10) and hypothesis $(\mathbf{H_1})\,(ii)$, we infer that

$$\left\{u_n\right\}_{n\in\mathbb{N}}\subseteq W^{1,p(z)}\left(\Omega\right) \text{ is bounded.}$$

So, we may assume that

$$u_n\xrightarrow{w}u_* \text{ in } W^{1,p(z)}\left(\Omega\right) \text{ and } u_n\to u_* \text{ in } L^{p(z)}\left(\Omega\right) \text{ as n}\to\infty.\tag{3.11}$$

In (3.9) we choose $h=u_n-u_*\in W^{1,p(z)}\left(\Omega\right)$, pass to the limit as n$\to\infty$ and use (3.11). We obtain

$$\lim_{n\to\infty}\left\langle A_{p(z)}\left(u_n\right),u_n-u_*\right\rangle+\left\langle A_{q(z)}\left(u_n\right),u_n-u_*\right\rangle=0,$$

hence

$$\limsup_{n\to\infty}\left\langle A_{p(z)}\left(u_n\right),u_n-u_*\right\rangle+\left\langle A_{q(z)}\left(u_*\right),u_n-u_*\right\rangle\leq 0,$$

from the monotonicity of $A_{q(z)}\left(\cdot\right)$, therefore

$$\limsup_{n\to\infty}\left\langle A_{p(z)}\left(u_n\right),u_n-u_*\right\rangle\leq 0.$$

We conclude that

$$u_n\to u_* \text{ in } W^{1,p(z)}\left(\Omega\right),\ \overline{u}\leq u_*\tag{3.12}$$

(see Proposition 2.2 and (3.10)). Passing to the limit as n$\to\infty$ in (3.9) and using (3.12) we conclude that

$$u_*\in\mathcal{S}_+ \text{ and } u_*=\inf\mathcal{S}_+.$$

Similarly we produce the biggest negative solution

$$v_*\in\mathcal{S}_- \text{ and } v_*=\sup\mathcal{S}_-.$$

□

Now let $\eta_0>\left\|\xi\right\|_\infty$ and consider the following Carathéodory function

$$\tau_0\left(z,x\right)=\begin{cases}f_0\left(z,v_*\left(z\right)\right)+\eta_0\left|v_*\left(z\right)\right|^{p(z)-2}v_*\left(z\right) & \text{if } x<v_*\left(z\right)\\ f_0\left(z,x\right)+\eta_0\left|x\right|^{p(z)-2}x & \text{if } v_*\left(z\right)\leq x\leq v_*\left(z\right)\\ f_0\left(z,u_*\left(z\right)\right)+\eta_0u_*\left(z\right)^{p(z)-1} & \text{if } u_*\left(z\right)<x\end{cases}\tag{3.13}$$

Let

$$T_0\left(z,x\right)=\int_0^x\tau_0\left(z,s\right)ds$$

and consider the the $C^1-$functional $\varphi_0 : W^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \int_\Omega \frac{1}{p(z)} |Du|^{p(z)} dz + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)} dz$$
$$+ \int_\Omega \frac{\xi(z)+\eta_0}{p(z)} |u(z)|^{p(z)} dz - \int_\Omega T_0(z,u)\, dz \text{ for all } u \in W^{1,p(z)}(\Omega).$$

The functional $\varphi_0(\cdot)$ is even (see hypothesis $(\mathbf{H_1})(i)$) and on account of $(3.13)$, we have that $\varphi_0(\cdot)$ is coercive (recall that $\eta_0 > \|\xi\|_\infty$), hence bounded below and satisfies the $PS-$condition (see Proposition 5.1.15, p. 369, of Papageorgiou-Radulescu-Repovs [14]).

Let $V \subseteq W^{1,p(z)}(\Omega)$ be a finite dimensional subspace

**Proposition 3.5.** *If hypotheses* $(\mathbf{H_0})$, $(\mathbf{H_1})$ *hold, then there exist* $\rho_V > 0$ *such that*

$$\sup\{\varphi_0(u) : u \in V, \|u\| = \rho_V\} < 0.$$

*Proof.* Let $m_* = \min\left\{\min_{\overline{\Omega}} u_*,\ \min_{\overline{\Omega}}(-v_*)\right\} > 0$ (recall $u_* \in intC_+$ and $v_* \in -intC_+$). On account of hypothesis $(\mathbf{H_1})(iii)$, given any $\eta > 0$, we can find $\delta = \delta(\eta) \in (0, m_*)$ such that

$$F(z,x) \geq \frac{\eta}{q_-}|x|^{q_-} \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \delta. \tag{3.14}$$

Since $V$ is finite dimensional, all norms are equivalent. Therefore we can find $\rho_V \in (0,1)$ such that

$$u \in V,\ \|u\| \leq \rho_V \Longrightarrow |u(z)| \leq \delta \text{ for a.a. } z \in \Omega. \tag{3.15}$$

Therefore if $u \in V$ with $\|u\| = \rho_V$, then in view of $(3.13)$, $(3.14)$, $(3.15)$, hypothesis $(\mathbf{H_1})(i)$ and Proposition 2.1, we have

$$\varphi_0(u) \leq \frac{1}{p_-}\|u\|^{p_-} + \frac{1}{q_-}\left[\|u\|^{q_-} - \eta\|u\|_{q_-}^{q_-}\right]$$
$$\leq \frac{1}{p_-}\|u\|^{p_-} + \frac{1}{q_-}\left[\|u\|^{q_-} - \eta C_5\|u\|^{q_-}\right] \text{ for some } C_5 > 0.$$

(recall that on $V$ all the norms are equivalent).

Choosing $\eta > \frac{1}{C_5} > 0$, it follows that

$$\varphi_0(u) \leq \frac{1}{p_+}\rho_V^{p_-} - C_6\rho_V^{q_-} \text{ for some } C_6 > 0.$$

Since $\rho_V \in (0,1)$ and $q_- < p_-$, choosing $\rho_V \in (0,1)$ even smaller if necessary we conclude that

$$\sup\{\varphi_0(u) : u \in V, \|u\| = \rho_V\} < 0.$$

□

Now we are ready for the main result of this work.

**Theorem 3.6.** *If hypotheses* $(\mathbf{H_0})$, $(\mathbf{H_1})$ *hold, then there exists a sequence* $\{u_n\}_{n\in\mathbb{N}} \subseteq C^1(\overline{\Omega})$ *of nodal solutions of* $(1.1)$ *such that*

$$u_n \to 0 \text{ in } C^1(\overline{\Omega}) \text{ as } n \to \infty.$$

*Proof.* We use Proposition 2.3 and have a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq K_{\varphi_0}$ such that

$$u_n \to 0 \text{ in } W^{1,p(z)}(\Omega). \tag{3.16}$$

Using (3.13) and the anisotropic regularity theory (see Fan [4]) we obtain that

$$K_{\varphi_0} \subseteq [v_*, u_*] \cap C^1\left(\overline{\Omega}\right),$$

therefore $\{u_n\}_{n\in\mathbb{N}}$ are solutions of (1.1) when $f$ is replaced by $f_0$.

Proposition 3.1 of Gasinski-Papageorgiou [6], implies that we can find $C_7 > 0$ such that

$$\|u_n\|_\infty < C_7 \text{ for all } n \in \mathbb{N}.$$

Then Theorem 1.3 of Fan [4] (see also Lieberman [10]) implies that we can find $\alpha \in (0,1)$ and $C_8 > 0$ such that

$$u_n \in C^{1,\alpha}\left(\overline{\Omega}\right), \ \ \|u_n\|_{C^{1,\alpha}\left(\overline{\Omega}\right)} < C_8 \text{ for all } n \in \mathbb{N}. \tag{3.17}$$

Then from (3.16), (3.17) and the compact embedding of $C^{1,\alpha}\left(\overline{\Omega}\right)$ into $C^1\left(\overline{\Omega}\right)$, we have

$$u_n \to 0 \text{ in } C^1\left(\overline{\Omega}\right) \text{ as } n \to \infty. \tag{3.18}$$

Let

$$\widehat{\mu} = \min\left\{\mu, \ \min_{\overline{\Omega}} u_*, \ \min_{\overline{\Omega}} (-v_*)\right\} > 0.$$

From (3.18) it follows that

$$\{u_n\}_{n\geq n_0} \subseteq [-\widehat{\mu}, \widehat{\mu}] \text{ for some } n_0 \in \mathbb{N}. \tag{3.19}$$

Then from (3.19), (2.2) and the extremality of the solutions $u_*$ and $v_*$ we conclude that $\{u_n\}_{n\geq n_0}$ are nodal solutions of (1.1) and

$$u_n \to 0 \text{ in } C^1\left(\overline{\Omega}\right) \text{ as } n \to \infty.$$

□

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