Appl. Set-Valued Anal. Optim. 4 (2022), No. 3, pp. 277-292
Available online at http://asvao.biemdas.com
https://doi.org/10.23952/asvao.4.2022.3.03

NONLINEAR NONHOMOGENEOUS LOGISTIC EQUATIONS OF SUPERDIFFUSIVE TYPE

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Abstract. We consider a nonlinear logistic equation of superdiffusive type driven by a nonhomogeneous differential operator and a Robin boundary condition. We prove a multiplicity result for positive solutions which is global with respect to the parameter $\lambda > 0$ (bifurcation-type theorem). We also demonstrate the existence of a minimal positive solution u_λ^* and determine the monotonicity and continuity properties of the minimal solution map $\lambda \to u_\lambda^*$.
Keywords. Minimal positive solution; Nonhomogeneous differential operator; Nonlinear regularity theory; Nonlinear maximum principle; Superdiffusive reaction.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2- boundary $\partial\Omega$. In this paper, we study the following parametric nonlinear nonhomogeneous Robin problem

$$\begin{cases} -\operatorname{div} a(Du(z)) + \xi(z)u(z)^{p-1} = \lambda g(z,u(z)) - f(z,u(z)) \text{ in } \Omega, \\ \frac{\partial u}{\partial n_a} + \beta(z)u^{p-1} = 0 \text{ on } \partial\Omega,\ \lambda > 0,\ u \geq 0,\ 1 < p < \infty. \end{cases} \tag{P_λ}$$

The map $a : \mathbb{R}^N \to \mathbb{R}^N$ involved in the definition of the differential operator is continuous, monotone (thus maximal monotone too), and satisfies certain other regularity and growth conditions, listed in hypotheses $(\mathbf{H}_0)$ (see Section 2). These hypotheses come from the global (that is, up to the boundary) regularity theory of Lieberman [1] and are broad enough to incorporate in our framework many differential operators of interest, such as the $p-$Laplacian and the (p,q)-Laplacian (that is, the sum of a $p-$Laplacian and of a q-Laplacian, $1 < q < p$). There is also a potential term $\xi(z)u(z)^{p-1}$ with $\xi \in L^\infty(\Omega)$, $\xi(z) \geq 0$ for a.a. $z \in \Omega$. In the reaction (right hand side of (P_λ)), $\lambda > 0$ is a parameter and $g(z,x)$, $f(z,x)$ are both Carathéodory functions (that is, for all $x \in \mathbb{R}$, $z \to f(z,x)$ and $z \to g(z,x)$ are measurable, and for a.a. $z \in \Omega$, $x \to f(z,x)$ and $x \to g(z,x)$ are continuous). Both $g(z,\cdot)$ and $f(z,\cdot)$ exhibit a $(p-1)$-superlinear growth as $x \to \infty$ with $f(z,\cdot)$ growing faster. So, the reaction of the problem has a structure of a logistic reaction of superdiffussive type. Logistic equations arise in many physical applications, such as

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Received September 9, 2021; Accepted December 13, 2021.

mathematical biology where they describe the dynamics of biological populations whose mobility is density dependent. For such applications, we refer to the work of Gurtin-MacCamy [2].

In the boundary condition, $\frac{\partial u}{\partial n_a}$ denotes the conormal derivative of u corresponding to the map $a(.)$. If $u \in C^1(\overline{\Omega})$, then

$$\frac{\partial u}{\partial n_a} = (a(Du), n)_{\mathbb{R}^N}$$

where $n(.)$ is the outward unit normal on $\partial\Omega$. In general, the boundary condition is interpreted using the nonlinear Green theorem (see [3, p.34]). The boundary coefficient β satisfies $\beta \in C^{0,\alpha}(\partial\Omega)$ with $0 < \alpha < 1$, and $\beta(z) \geq 0$ for all $z \in \partial\Omega$.

Using variational tools based on the critical point theory together with truncation and comparison techniques, we prove a multiplicity theorem for positive solutions, which is global in the parameter $\lambda > 0$ (a bifurcation type result for large vales of the parameter). We also show that for every admissible parameter $\lambda > 0$, problem (P_λ) has a smallest positive solution u^*_λ and determine the monotonicity and continuity properties of the map $\lambda \to u^*_\lambda$.

To the best of our knowledge, this is the first global multiplicity result for positive solutions of nonlinear, nonhomogeneous equations with general logistic reaction and Robin boundary condition.

In the past, such results were proved primarily in the context of Dirichlet semilinear (driven by the Laplacian) and nonlinear problems (driven by the $p-$Laplacian), with a reaction of a particular (power) form.

We refer to the semilinear works of Afrouzi-Brown [4], Papageorgiou-Radulescu-Repovs [5], Radulescu-Repovs [6], and the nonlinear works of Dong [7], Gasinski-Papageorgiou [8], Papageorgiou-Radulescu-Repovs [9] (anisotropic problems), Takeuchi [10], [11]. We also mention the work of Gasinski-O'Regan-Papageorgiou [12], where the differential operator is similar to the one used here, but the aim there is to produce nodal solutions.

2. Mathematical Background-Hypotheses

The main spaces in the analysis of problem (P_λ) are the Sobolev space $W^{1,p}(\Omega)$, the Banach space $C^1(\overline{\Omega})$, and the "boundary" Lebesgue spaces $L^s(\partial\Omega)$ $(1 \leq s < \infty)$.

By $\|.\|$, we denote the norm of the Sobolev space $W^{1,p}(\Omega)$ defined by

$$\|u\| = \left[\|u\|_p^p + \|Du\|_p^p\right]^{\frac{1}{p}}, \text{ for all } u \in W^{1,p}(\Omega),$$

where $\|.\|_p$ stands for the L^p-norm.

The Banach space $C^1(\overline{\Omega})$ is ordered with positive (order) cone

$$C_+ = \left\{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior, given by

$$int\, C_+ = \left\{u \in C^1(\overline{\Omega}) : u(z) > 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

We also use another open cone in $C^1(\overline{\Omega})$, namely

$$D_+ = \left\{u \in C^1(\overline{\Omega}) : u(z) > 0 \text{ for all } z \in \Omega, \frac{\partial u}{\partial n}\,|_{\partial\Omega \cap u^{-1}(0)} < 0\right\}.$$

On $\partial\Omega$, we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(.)$. Using this measure, we can define in the usual way the "boundary" Lebesgue spaces $L^s(\partial\Omega)$ $(1 \leq s < \infty)$.

The theory of Sobolev spaces specifies that there exists a unique continuous linear map $\widehat{\gamma}_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, known as the "trace operator", such that

$$\widehat{\gamma}_0(u) = u\,|_{\partial\Omega} \text{ for all } u \in W^{1,p}(\Omega) \cap C\left(\overline{\Omega}\right).$$

So, the trace operator extends the notion of boundary values to all Sobolev functions.

In the sequel, for the sake of notational economy, we drop the use of the trace map $\widehat{\gamma}_0$. All restrictions of Sobolev functions to the boundary $\partial\Omega$ are understood in the sense of traces. Via the trace map, we show that $W^{1,p}(\Omega) \hookrightarrow L^s(\partial\Omega)$ compactly for all $s \in \left[1, \frac{(N-1)p}{N-p}\right)$ when $p < N$ and for all $1 \leq s < \infty$ when $N \leq p$.

Let $u : \Omega \to \mathbb{R}$ be measurable. For every $z \in \Omega$, we define $u^{\pm}(z) = \max\{\pm u(z), 0\}$. If $u \in W^{1,p}(\Omega)$, then $u^{\pm} \in W^{1,p}(\Omega)$, $|u| = u^+ + u^-$, and $u = u^+ - u^-$.

We set

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if} \quad p < N, \\ +\infty & \text{if} \quad p \geq N \end{cases}$$

(the critical Sobolev exponent for p).

We now introduce the conditions on the map $a(\cdot)$ that appears in the differential operator in (P_λ). Let $d \in C^1(0,\infty)$ with $d(t) > 0$ for all $t > 0$, and assume that

$$0 < C \leq \frac{d'(t)\,t}{d(t)} \leq \widehat{C} \text{ and } C_0 t^{p-1} \leq d(t) \leq C_1\left[t^{\mu-1} + t^{p-1}\right]$$
$$\text{for all } t > 0, \text{ some } C_0,\ C_1 > 0,\ 1 < \mu < p.$$

The hypotheses on the map $a(.)$ are the following:

$(\mathbf{H}_0)$: $a(y) = a_0(|y|)\,y$, for all $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$ and:

(i) $a_0 \in C^1(0,\infty)$, $t \to a_0(t)\,t$ is strictly increasing on $(0,\infty)$,

$$a_0(t)\,t \to 0^+ \text{ as } t \to 0^+ \text{and } \lim_{t\to 0^+} \frac{a_0'(t)\,t}{a_0(t)} > -1;$$

(ii) there exists $C_2 > 0$ such that

$$|\nabla a(y)| \leq C_2 \frac{d(|y|)}{|y|} \text{ for all } y \in \mathbb{R}^N \setminus \{0\};$$

(iii) for all $y \in \mathbb{R}^N \setminus \{0\}$ and all $\xi \in \mathbb{R}^N$, one has

$$(\nabla a(y)\,\xi, \xi)_{\mathbb{R}^N} \geq \frac{d(|y|)}{|y|}\,|\xi|^2.$$

Here and in what follows, $|y|$ denotes the $\mathbb{R}^N$ norm of $y \in \mathbb{R}^N$, and $(\cdot,\cdot)_{\mathbb{R}^N}$ denotes the $\mathbb{R}^N$ inner product.

Remark 2.1. Hypotheses $(\mathbf{H}_0)$ (i), (ii), (iii) are motivated by the nonlinear regularity theory of Lieberman [1] and the nonlinear maximum principle of Pucci-Serrin [13] (pp. 111, 120). Similar conditions were also used by Aizicovici-Papageorgiou-Staicu [14] and Papageorgiou-Radulescu [15].

Let

$$\widehat{G}_0(t) = \int_0^t a_0(s)\, s\, ds, \text{ for } t \geq 0.$$

Evidently the primitive $\widehat{G}_0(\cdot)$ introduced above is strictly increasing and strictly convex. We set

$$\widehat{G}(y) = \widehat{G}_0(|y|), \text{ for all } y \in \mathbb{R}^N.$$

It follows that $\widehat{G}(\cdot)$ is differentiable on $\mathbb{R}^N \setminus \{0\}$, convex, $\widehat{G}(0) = 0$, and (by the chain rule)

$$\nabla \widehat{G}(y) = \widehat{G}_0'(|y|) \frac{y}{|y|} = a_0(|y|)\, y = a(y) \text{ for all } y \in \mathbb{R}^N \setminus \{0\}.$$

Thus $\widehat{G}(.)$ is the primitive of the map $a(.)$. The convexity of $\widehat{G}(.)$ in conjunction with $\widehat{G}(0) = 0$ implies that

$$\widehat{G}(y) \leq (a(y), y)_{\mathbb{R}^N} \text{ for all } y \in \mathbb{R}^N. \tag{2.1}$$

The next lemma states the main properties of the map $a(.)$, which follow from hypotheses $(\mathbf{H}_0)$ (see Papageorgiou-Radulescu [15]).

Lemma 2.1. *If hypotheses* $(\mathbf{H}_0)$ (i), (ii), *and* (iii) *hold, then*

(a) *the map* $y \to a(y)$ *is continuous and strictly monotone, hence maximal monotone too;*

(b) $|a(y)| \leq C_3\left(|y|^{\mu-1} + |y|^{p-1}\right)$ *for some* $C_3 > 0$, *all* $y \in \mathbb{R}^N$;

(c) $(a(y), y)_{\mathbb{R}^N} \geq \frac{C_0}{p-1}|y|^p$ *for all* $y \in \mathbb{R}^N$.

This lemma and (2.1) lead to the following bilateral growth restrictions for the primitive $G(.)$:

Corollary 2.1. *If hypotheses* $(\mathbf{H}_0)$ (i), (ii), *and* (iii) *hold, then*

$$\frac{C_0}{p(p-1)}|y|^p \leq \widehat{G}(y) \leq C_4(1 + |y|^p) \text{ for some } C_4 > 0, \text{ all } y \in \mathbb{R}^N.$$

Remark 2.2. Note that $\widehat{G}(\cdot)$ has balanced growth. As a consequence of this fact, we have a global regularity theory (see Lieberman [1]).

The examples that follow show that these hypotheses provide a broad framework, which incorporates many nonlinear operators of interest (see, e.g., [14, 15]).

Example 2.1. The following maps $a(.)$ satisfy hypotheses $(\mathbf{H}_0)$:

(a) $a(y) = |y|^{p-2} y$ with $1 < p < \infty$.

This map corresponds to the $p-$Laplacian defined by

$$\triangle_p u = div\left(|Du|^{p-2} Du\right), \text{ for all } u \in W^{1,p}(\Omega).$$

(b) $a(y) = |y|^{p-2} y + |y|^{q-2} y$ with $1 < q < p$.

This map corresponds to the $(p,q)-$Laplacian defined by $\triangle_p u + \triangle_q u$ for all $u \in W^{1,p}(\Omega)$ Such operators arise in the mathematical modeling of many physical processes (see, Cherfils-Ilyasov [16]).

(c) $a(y) = \left(1+|y|^2\right)^{\frac{p-2}{2}} y$ with $1 < p < \infty$.

This map corresponds to the generalized $p-$ mean curvature differential operator defined by

$$div\left(\left(1+|Du|^2\right)^{\frac{p-2}{2}} Du\right), \text{ for all } u \in W^{1,p}(\Omega).$$

(d) $a(y) = |y|^{p-2} y\left(1+\frac{1}{1+|y|^p}\right)$ with $1 < p < \infty$.

This map corresponds to the following differential operator

$$\triangle_p u + div\left(\frac{|Du|^{p-2} Du}{1+|Du|^p}\right), \text{ for all } u \in W^{1,p}(\Omega),$$

which arises in plasticity theory (see Roubiceck [17]).

Let $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear operator defined by

$$\langle A(u), h\rangle = \int_\Omega (a(Du), Dh)_{\mathbb{R}^N}\, dz \text{ for all } u, h \in W^{1,p}(\Omega). \tag{2.2}$$

The properties of $A(\cdot)$ are summarized below (see [12]).

Proposition 2.1. *If hypotheses* $(\mathbf{H}_0)$ (i), (ii), *and* (iii) *hold, then the operator* $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ *defined by* (2.2) *is bounded (that is, it maps bounded sets to bounded sets), continuous, monotone (thus maximal monotone too) and of type* $(S)_+$, *that is, if* $\{u_n\}_{n\ge1} \subseteq W^{1,p}(\Omega)$ *is such that* $u_n \xrightarrow{w} u$ *in* $W^{1,p}(\Omega)$ *and* $\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \le 0$, *then* $u_n \to u$ *in* $W^{1,p}(\Omega)$ *as* $n \to \infty$.

Here and in what follows, $\xrightarrow{w}$ denotes the weak convergence, and $\langle .,.\rangle$ designates the duality pairing between $W^{1,p}(\Omega)^*$ and $W^{1,p}(\Omega)$.

Now, we introduce our conditions on the potential function $\xi(.)$ and on the boundary coefficient $\beta(.)$.

$(\mathbf{H}_1)$: $\xi \in L^\infty(\Omega)$, $\xi(z) \ge 0$ for a.a. $z \in \Omega$, $\beta \in C^{0,\alpha}(\partial\Omega)$ with $0 < \alpha < 1$, $\beta(z) \ge 0$ for all $z \in \partial\Omega$ and $\xi \not\equiv 0$ or $\beta \not\equiv 0$.

Remark 2.3. The case $\beta \equiv 0$ corresponds to the Neumann problem.

In what follows, by $\gamma : W^{1,p}(\Omega) \to \mathbb{R}$, we denote the C^1- functional defined by

$$\gamma(u) = \int_\Omega G(Du(z))\, dz + \frac{1}{p}\int_\Omega \xi(z)|u(z)|^p\, dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u(z)|^p\, d\sigma \quad \text{for all } u \in W^{1,p}(\Omega).$$

Also by $\gamma_p : W^{1,p}(\Omega) \to \mathbb{R}$, we denote the C^1- functional defined by

$$\gamma_p(u) = \frac{C_0}{p-1}\|Du\|_p^p + \int_\Omega \xi(z)|u(z)|^p\, dz + \int_{\partial\Omega} \beta(z)|u(z)|^p\, d\sigma \quad \text{for all } u \in W^{1,p}(\Omega).$$

Corollary 2.1 implies that

$$\frac{1}{p}\gamma_p(u) \leq \gamma(u) \text{ for all } u \in W^{1,p}(\Omega). \tag{2.3}$$

Also hypotheses $(\mathbf{H}_1)$ together with Mugnai-Papageorgiou [18, Lemma 4.11] and Gasinski-Papageorgiou [19, Proposition 2.4] imply that

$$C_5 \|u\|^p \leq \gamma_p(u) \text{ for some } C_5 > 0, \text{ all } u \in W^{1,p}(\Omega). \tag{2.4}$$

Finally, we introduce our hypotheses on the two nonlinearities g and f involved in the reaction of (P_λ).

$(\mathbf{H}_2)$: $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $g(z,0) = 0$ for a.a. $z \in \Omega$ and:

(i) for every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)$ such that

$$g(z,x) \leq a_\rho(z) \text{ for a.a. } z \in \Omega, \text{all } 0 \leq x \leq \rho;$$

(ii) there exists $\tau \in (p, p^*)$ such that

$$0 < \eta_\infty \leq \liminf_{x\to+\infty} \frac{g(z,x)}{x^{\tau-1}} \leq \limsup_{x\to+\infty} \frac{g(z,x)}{x^{\tau-1}} \leq \widehat{\eta}_\infty$$

uniformly for a.a. $z \in \Omega$;

(iii) with $s \in (p, \tau]$, we have

$$0 < \eta_0 \leq \liminf_{x\to 0^+} \frac{g(z,x)}{x^{s-1}} \leq \limsup_{x\to 0^+} \frac{g(z,x)}{x^{s-1}} \leq \widehat{\eta}_0$$

uniformly for a.a. $z \in \Omega$,

and for every $\theta > 0$, there exists $\delta_\theta > 0$ such that

$$\delta_\theta \leq g(z,x) \text{ for a.a. } z \in \Omega, \text{ all } x \geq \theta.$$

Remark 2.4. Since we are looking for positive solutions and all of the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, without any loss of generality, we may assume that

$$g(z,x) = 0 \text{ for a.a. } z \in \Omega, \text{ all } x \leq 0. \tag{2.5}$$

$(\mathbf{H}_3)$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and:

(i) $|f(z,x)| \leq \widehat{a}(z)\left(1 + x^{r-1}\right)$ for a.a. $z \in \Omega$, all $x \geq 0$ with $\widehat{a} \in L^\infty(\Omega)$, $\tau < r < p^*$ (see hypothesis $(\mathbf{H}_2)(ii)$);

(ii) $\lim_{x\to+\infty} \frac{f(z,x)}{x^{\tau-1}} = +\infty$ uniformly for a.a. $z \in \Omega$ and with $\tau \in (p, p^*)$ as in hypothesis $(\mathbf{H}_2)(ii)$;

(iii) with $s \in (p, \tau]$ as in hypothesis $(\mathbf{H}_2)(iii)$, we have

$$\liminf_{x\to 0^+} \frac{f(z,x)}{x^{s-1}} = 0 \text{ uniformly for a.a. } z \in \Omega.$$

Remark 2.5. Without any loss of generality, we may assume that

$$f(z,x) = 0 \text{ for a.a. } z \in \Omega, \text{ all } x \leq 0. \tag{2.6}$$

Also, we point out that no sign condition is imposed on $f(z,.)$, which may change sign.

$(\mathbf{H}_4)$: For any $\rho > 0$ and for every $\lambda > 0$, there exists $\widehat{\xi}_\rho^\lambda > 0$ such that for a.a. $z \in \Omega$, the function

$$z \to \lambda g(z,x) - f(z,x) + \widehat{\xi}_\rho^\lambda x^{p-1}$$

is nondecreasing on $[0,\rho]$.

Example 2.2. The following pairs of functions satisfy hypotheses $(\mathbf{H}_2)$ and $(\mathbf{H}_3)$. For the sake of simplicity, we drop the $z-$ dependence.

(a) $g_1(x) = (x^+)^{\tau-1}$ and $f_1(x) = (x^+)^{r-1}$with $\tau < r < p^*$.
This pair corresponds to the classical superdiffussive reaction which we encounter in most works in the literature; see, e.g., Takeuchi [10, 11];

(b)

$$g_2(x) = \begin{cases} (x^+)^{s-1} & \text{if } x \le 1 \\ x^{\tau-1} & \text{if } 1 < x \end{cases} \text{ with } p < s < \tau;$$

$$f_2(x) = \begin{cases} (x^+)^{\theta-1} & \text{if } x \le 1 \\ x^{\tau+1}[\ln x + 1] & \text{if } 1 < x \end{cases} \text{ with } \theta > \tau;$$

(c)

$$g_3(x) = \begin{cases} 2(x^+)^{s-1} - (x^+)^{\theta-1} & \text{if } x \le 1 \\ x^{\tau-1} & \text{if } 1 < x \end{cases}$$

with $p < s \le \tau < \theta$;

$$f_3(x) = \begin{cases} (x^+)^{\mu-1} - 2(x^+)^{\theta-1} & \text{if } x \le 1 \\ x^{r-1} - 2x^{\tau-1} & \text{if } 1 < x \end{cases}$$

with $\tau < \mu < \theta; \tau < r < p^*$.

We point out that $f_3(.)$ is sign-changing.

We next introduce the following two sets

$$\mathscr{L} = \{\lambda > 0: \text{ problem } (P_\lambda) \text{ admits a positive solution}\},$$

and

$$\mathscr{S}(\lambda) = \text{the set of positive solutions for problem } (P_\lambda).$$

We set

$$\lambda^* = \inf \mathscr{L}.$$

3. Positive Solutions

We start by showing that $\mathscr{L} \neq \varnothing$ and by determining the regularity properties of the positive solutions of (P_λ).

Proposition 3.1. *If hypotheses $(\mathbf{H}_0)$, $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, $(\mathbf{H}_3)$, and $(\mathbf{H}_4)$ hold, then $\mathscr{L} \neq \varnothing$ and for all $\lambda > 0$, $\mathscr{S}(\lambda) \subseteq int\ C_+$.*

Proof. Let $G(z,x)=\int\limits_0^x g(z,s)\,ds$ and $F(z,x)=\int\limits_0^x f(z,s)\,ds$. On account of hypotheses $(\mathbf{H}_2)\,(i)$ and (ii), we have

$$0\le G(z,x)\le C_6\left[1+x^{\tau}\right] \text{ for a.a. } z\in\Omega,\ \text{all } x\ge 0,\ \text{some } C_6>0. \tag{3.1}$$

Similarly, from hypotheses $(\mathbf{H}_3)\,(i)$ and (ii), we see that, for given $\eta>0$, we can find $C_\eta>0$ such that

$$\eta x^{\tau}-C_\eta\le F(z,x) \text{ for a.a. } z\in\Omega,\ \text{all } x\ge 0. \tag{3.2}$$

Let $\varphi_\lambda : W^{1,p}(\Omega)\to\mathbb{R}$ be the energy functional for problem (P_λ) defined by

$$\varphi_\lambda(u)=\gamma(u)+\int\limits_\Omega F(z,u)\,dz-\lambda\int\limits_\Omega G(z,u)\,dz \text{ for all } u\in W^{1,p}(\Omega).$$

We have $\varphi_\lambda\in C^1\left(W^{1,p}(\Omega)\right)$ and

$$\varphi_\lambda(u)\ge\frac{1}{p}\gamma_p(u)+\left[\eta-\lambda C_6\right]\left\|u^+\right\|_\tau^\tau-C_7$$

for some $C_7=C_7(\eta,\lambda)>0$ (see (2.3), (3.1), and (3.2)). Choosing $\eta>\lambda C_6$, we have

$$\varphi_\lambda(u)\ge\frac{1}{p}\gamma_p(u)-C_7\ge C_5\|u\|^p-C_7 \text{ (see (2.4))}.$$

Thus φ_λ is coercive. Also, using the Sobolev embedding theorem and the compactness of the trace map, we see that $\varphi_\lambda(.)$ is sequentially weakly lower semicontinuous. Invoking the Weierstrass-Tonelli theorem, we can find $u_\lambda\in W^{1,p}(\Omega)$ such that

$$\varphi_\lambda(u_\lambda)=\inf\left\{\varphi_\lambda(u): u\in W^{1,p}(\Omega)\right\}. \tag{3.3}$$

Let $u\equiv\theta\in(0,\infty)$. Then

$$\begin{aligned}\varphi_\lambda(\theta)&\le\frac{\theta^p}{p}\left[\|\xi\|_1+\|\beta\|_{L^1(\partial\Omega)}\right]+\int\limits_\Omega F(z,\theta)\,dz-\lambda\int\limits_\Omega G(z,\theta)\,dz\\&\le C_8-\lambda C_9 \text{ for some } C_8=C_8(\theta)>0,\ C_9=C_9(\theta)>0.\end{aligned}$$

Then, for $\lambda>0$ big, we have $\varphi_\lambda(\theta)<0$. Hence $\varphi_\lambda(u_\lambda)<0=\varphi_\lambda(0)$ (see (3.3)). Thus $u_\lambda\ne 0$ (when $\lambda>0$ is big). From (3.3), we have $\varphi_\lambda'(u_\lambda)=0$, and hence

$$\begin{aligned}&\langle A(u_\lambda),h\rangle+\int\limits_\Omega\xi(z)\,|u_\lambda|^{p-2}u_\lambda h\,dz+\int\limits_{\partial\Omega}\beta(z)\,|u_\lambda|^{p-2}u_\lambda h\,d\sigma\\&=\int\limits_\Omega\left[\lambda g(z,u_\lambda)-f(z,u_\lambda)\right]h\,dz \text{ for all } h\in W^{1,p}(\Omega).\end{aligned} \tag{3.4}$$

In (3.4), we choose $h=-u_\lambda^-\in W^{1,p}(\Omega)$. Then, using Lemma 2.1, (2.5) and (2.6), we obtain $\gamma_p\left(u_\lambda^-\right)\le 0$, which implies that $u_\lambda\ge 0,\ u_\lambda\ne 0$ (see (2.4)). From (3.4), we have

$$\begin{cases}-\operatorname{div}a\left(Du_\lambda(z)\right)+\xi(z)\,u_\lambda(z)^{p-1}\\ \qquad\qquad=\lambda g\left(z,u_\lambda(z)\right)-f\left(z,u_\lambda(z)\right) \text{ in } \Omega,\\ \frac{\partial u_\lambda}{\partial n_a}+\beta(z)\,u_\lambda^{p-1}=0 \text{ on } \partial\Omega.\end{cases} \tag{3.5}$$

From (3.5) and Papageorgiou-Radulescu [20, Proposition 2.10], we have $u_\lambda \in L^\infty(\Omega)$. Then the nonlinear regularity theory of Lieberman ([1, p.320]) implies that $u_\lambda \in C_+ \setminus \{0\}$. Let $\rho = \|u_\lambda\|_\infty$ and let $\widehat{\xi}_\rho^\lambda$ as postulated by hypothesis $(\mathbf{H}_4)$. We have

$$-\operatorname{div} a(Du_\lambda(z)) + \left(\xi(z) + \widehat{\xi}_\rho^\lambda\right) u_\lambda(z)^{p-1} \geq 0 \text{ in } \Omega,$$

and hence

$$\operatorname{div} a(Du_\lambda(z)) \leq \left(\|\xi\|_\infty + \widehat{\xi}_\rho^\lambda\right) u_\lambda(z)^{p-1} \quad \text{in } \Omega$$

(see hypotheses $(\mathbf{H}_0)$). Thus $u_\lambda \in int\ C_+$ (see Pucci-Serrin [13], pp. 111, 120). So we have proved that when $\lambda > 0$ is big, $\lambda \in \mathscr{L} \neq \varnothing$. Moreover, for any $\lambda > 0$, we have $\mathscr{S}(\lambda) \subseteq int\ C_+$. □

Proposition 3.2. *If hypotheses* $(\mathbf{H}_0), (\mathbf{H}_1), (\mathbf{H}_2), (\mathbf{H}_3)$, *and* $(\mathbf{H}_4)$ *hold, then* $\lambda_* > 0$.

Proof. Hypotheses $(\mathbf{H}_2)$ imply that, for $p^* > \mu > \tau$ and given $\varepsilon > 0$, we can find $C_{10} = C_{10}(\varepsilon) > 0$ such that

$$0 \leq g(z,x) \leq \varepsilon x^{\mu-1} + C_{10} x^{\tau-1} \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{3.6}$$

Similarly, hypotheses $(\mathbf{H}_3)$ imply that, given $\varepsilon > 0$, we can find $\theta = \theta(\varepsilon) > 0$ such that

$$f(z,x) \geq \theta x^{\tau-1} - \varepsilon x^{\mu-1} \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{3.7}$$

Using (3.6) and (3.7), we obtain

$$\begin{aligned} \lambda g(z,x) - f(z,x) \leq (\lambda+1)\varepsilon x^{\mu-1} - (\theta - \lambda C_{10}) x^{\tau-1} \\ \text{for a.a. } z \in \Omega, \text{ all } x \geq 0. \end{aligned} \tag{3.8}$$

Let $\lambda \in \left(0, \frac{\theta}{C_{10}}\right)$, and suppose that $\lambda \in \mathscr{L}$. Then we can find $u_\lambda \in \mathscr{S}(\lambda) \subseteq int\ C_+$ (see Proposition 3.1). We have

$$\begin{aligned} &\langle A(u_\lambda), h\rangle + \int_\Omega \xi(z) u_\lambda^{p-1} h dz + \int_{\partial\Omega} \beta(z) u_\lambda^{p-1} h d\sigma \\ &= \int_\Omega [\lambda g(z,u_\lambda) - f(z,u_\lambda)] h dz \\ &\leq (\lambda+1)\varepsilon \int_\Omega u_\lambda^{\mu-1} h dz \text{ for all } h \in W^{1,p}(\Omega),\ h \geq 0 \end{aligned} \tag{3.9}$$

(see (3.8) and recall that $0 < \lambda < \frac{\theta}{C_{10}}$). In (3.9), we choose $h = u_\lambda \in int\ C_+$ to derive

$$\gamma_p(u_\lambda) \leq (\lambda+1)\varepsilon \|u_\lambda\|_\mu^\mu,$$

and hence $C_5 \|u_\lambda\|^p \leq (\lambda+1)\varepsilon C_{11} \|u_\lambda\|^\mu$ for some $C_{11} > 0$ (see (2.4) and recall that $W^{1,p}(\Omega) \hookrightarrow L^\mu(\Omega)$ continuously). Thus $1 \leq (\lambda+1)\varepsilon C_{12} \|u_\lambda\|^{\mu-p}$ for some $C_{12} > 0$. Since $\varepsilon > 0$ is arbitrary, we let $\varepsilon \to 0^+$ to arrive at a contradiction. So, $\lambda \notin \mathscr{L}$ and we have$\lambda_* \geq \frac{\theta}{C_{10}} > 0$. □

The next proposition shows that $\mathscr{L}$ is an upper half line.

Proposition 3.3. *If hypotheses* $(\mathbf{H}_0), (\mathbf{H}_1), (\mathbf{H}_2), (\mathbf{H}_3)$, *and* $(\mathbf{H}_4)$ *hold,* $\lambda \in \mathscr{L}$, $u_\lambda \in \mathscr{S}(\lambda) \subseteq$ *int* C_+ *and* $\eta > \lambda$, *then* $\eta \in \mathscr{L}$ *and we can find* $u_\eta \in \mathscr{S}(\eta) \subseteq int\ C_+$ *such that* $u_\eta - u_\lambda \in D_+$.

Proof. We consider the following truncation of the reaction of problem (P_η) :

$$k_\eta(z,x) = \begin{cases} \eta g(z,u_\lambda(z)) - f(z,u_\lambda(z)) & \text{if} \quad x \le u_\lambda(z), \\ \eta g(z,x) - f(z,x) & \text{if} \quad u_\lambda(z) < x. \end{cases} \tag{3.10}$$

This is a Carathéodory function. We set $K_\eta(z,x) = \int_0^x k_\eta(z,s)\,ds$ and consider the C^1- functional $\psi_\eta : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\eta(u) = \gamma(u) - \int_\Omega K_\eta(z,u)\,dz \text{ for all } u \in W^{1,p}(\Omega).$$

Using (3.1), (3.2) and (3.10), we infer that $\psi_\eta(\cdot)$ is coercive. Also, the Sobolev embedding theorem and the compactness of the trace map imply that $\psi_\lambda(.)$ is sequentially weakly lower semicontinuous. Thuswe can find $u_\eta \in W^{1,p}(\Omega)$ such that

$$\psi_\eta(u_\eta) = \inf\left\{\psi_\eta(u) : u \in W^{1,p}(\Omega)\right\}.$$

Then$\psi_\eta'(u_\eta) = 0$, and

$$\begin{aligned} &\langle A(u_\eta), h\rangle + \int_\Omega \xi(z)\left|u_\eta\right|^{p-2} u_\eta h dz + \int_{\partial\Omega} \beta(z)\left|u_\eta\right|^{p-2} u_\eta h d\sigma \\ &= \int_\Omega k_\eta(z,u_\eta)\, h dz \text{ for all } h \in W^{1,p}(\Omega) \end{aligned} \tag{3.11}$$

In (3.11), we choose $h = [u_\lambda - u_\eta]^+ \in W^{1,p}(\Omega)$. It follows that

$$\begin{aligned} &\langle A(u_\eta), [u_\lambda - u_\eta]^+\rangle + \int_\Omega \xi(z)\left|u_\eta\right|^{p-2} u_\eta [u_\lambda - u_\eta]^+ dz \\ &+ \int_{\partial\Omega} \beta(z)\left|u_\eta\right|^{p-2} u_\eta [u_\lambda - u_\eta]^+ d\sigma \\ &= \int_\Omega [\eta g(z,u_\lambda) - f(z,u_\lambda)][u_\lambda - u_\eta]^+ dz \text{ (see (3.10))} \\ &\ge \int_\Omega [\lambda g(z,u_\lambda) - f(z,u_\lambda)][u_\lambda - u_\eta]^+ dz \text{ (since } \lambda < \eta,\ g \ge 0) \\ &= \langle A(u_\lambda), [u_\lambda - u_\eta]^+\rangle + \int_\Omega \xi(z) u_\eta^{p-1}[u_\lambda - u_\eta]^+ dz \\ &+ \int_{\partial\Omega} \beta(z) u_\lambda^{p-1}[u_\lambda - u_\eta]^+ d\sigma \text{ (since } u_\lambda \in \mathscr{S}(\lambda)), \end{aligned}$$

and hence

$$u_\lambda \le u_\eta \tag{3.12}$$

(see Proposition 2.1 and hypotheses $(\mathbf{H}_1)$). It follows from (3.10), (3.11), and (3.12) that

$$u_\eta \in \mathscr{S}(\eta) \subseteq \mathit{int}\ C_+, \text{ and so } \eta \in \mathscr{L}.$$

Let $\rho = \|u_\eta\|_\infty$ and set $\widehat{\xi}_\rho = \max\left\{\widehat{\xi}_\rho^\lambda, \widehat{\xi}_\rho^\eta\right\}$ (see hypotheses $(\mathbf{H}_4)$). We have

$$\begin{aligned}
&-\operatorname{div} a(Du_\lambda) + \left(\xi(z) + \widehat{\xi}_\rho\right) u_\lambda^{p-1} \\
&= \lambda g(z,u_\lambda) - f(z,u_\lambda) + \widehat{\xi}_\rho u_\lambda^{p-1} \\
&= \eta g(z,u_\lambda) - f(z,u_\lambda) + \widehat{\xi}_\rho u_\lambda^{p-1} + (\eta-\lambda) g(z,u_\lambda) \\
&\leq \eta g(z,u_\eta) - f(z,u_\eta) + \widehat{\xi}_\rho u_\eta^{p-1} \\
&\text{(see (3.12), hypotheses } (\mathbf{H}_4) \text{ and recall that } g \geq 0) \\
&= -\operatorname{div} a(Du_\eta) + \left(\xi(z) + \widehat{\xi}_\rho\right) u_\eta^{p-1}.
\end{aligned} \tag{3.13}$$

Since $u_\lambda \in int\, C_+$, we have $0 < \theta_\lambda = \min_{\overline{\Omega}} u_\lambda$, and hence

$$(\eta-\lambda) g(z,u_\lambda) \geq (\eta-\lambda)\delta_{\theta_\lambda} > 0$$

(see hypothesis $(\mathbf{H}_2)(iii)$). Then, from (3.13) and Aizicovici-Papageorgiou-Staicu [14, Proposition 3], we conclude that $u_\eta - u_\lambda \in D_+$. This completes the proof. □

For $\lambda > \lambda_*$, we have multiple positive solutions.

Proposition 3.4. *If hypotheses* $(\mathbf{H}_0)$, $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, $(\mathbf{H}_3)$, *and* $(\mathbf{H}_4)$ *hold and* $\lambda > \lambda_*$, *then* $\lambda \in \mathscr{L}$, *and problem* (P_λ) *has at least two positive solutions* $u_0, \widehat{u} \in int\, C_+$.

Proof. On account of Proposition 3.3, we have $\lambda \in \mathscr{L}$. Let $\mu \in (\lambda_*, \lambda)$. Then $\mu \in \mathscr{L}$ and we can find $u_\mu \in \mathscr{S}(\mu) \subset int\, C_+$. Truncating the reaction of problem (P_λ) at $u_\mu(z)$ and reasoning as in the proof of Proposition 3.3, via the direct method of the Calculus of Variations on the resulting C^1-functional ψ_λ, we produce $u_0 \in W^{1,p}(\Omega)$ such that

$$u_0 \in \mathscr{S}(\lambda) \subset int\, C_+ \text{ and } u_0 - u_\mu \in D_+. \tag{3.14}$$

Let

$$\left[u_\mu\right) = \left\{u \in W^{1,p}(\Omega) : u_\mu(z) \leq u(z) \text{ for a.a. } z \in \Omega\right\}.$$

Then, from (3.10), we see that

$$\varphi_\lambda \mid_{[u_\mu)} = \psi_\lambda \mid_{[u_\mu)} + \xi^* \text{ with } \xi^* \in \mathbb{R}. \tag{3.15}$$

From (3.14) and (3.15), it follows that u_0 is a local $C^1\left(\overline{\Omega}\right) -$ minimizer of $\varphi_\lambda(\cdot)$, and hence

$$u_0 \text{ is a local } W^{1,p}(\Omega) \text{ minimizer of } \varphi_\lambda(\cdot) \tag{3.16}$$

(see Papageorgiou-Radulescu [20]). On account of hypothesis $(\mathbf{H}_2)(iii)$ and $(\mathbf{H}_3)(iii)$, given $\varepsilon > 0$, we can find $\widehat{\delta} = \widehat{\delta}(\varepsilon) \in (0,1)$ such that

$$G(z,x) \leq \widetilde{\eta}_0 x^{s-1} \text{ for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \widehat{\delta} \text{ with } \widetilde{\eta}_0 > \widehat{\eta}_0,$$

$$F(z,x) \geq -\varepsilon x^{s-1} \text{ for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \widehat{\delta}.$$

So, if $u \in C^1\left(\overline{\Omega}\right)$ satisfies $\|u\|_{C^1(\overline{\Omega})} \leq \widehat{\delta}$, then

$$\begin{aligned}
\varphi_\lambda(u) &\geq \frac{1}{p}\gamma_p(u) - (\lambda\widetilde{\eta}_0 + \varepsilon)\|u\|_s^s \text{ (see (2.3))} \\
&\geq C_5\|u\|^p - (\lambda\widetilde{\eta}_0 + \varepsilon) C_{13}\|u\|^s \text{(for some } C_{13} > 0) \\
&\geq \left[C_5 - (\lambda\widetilde{\eta}_0 + \varepsilon) C_{14}\widehat{\delta}^{s-p}\right]\|u\|^p \text{(for some } C_{14} > 0)
\end{aligned}$$

(recall that $s > p$). Choosing $\widehat{\delta} \in (0,1)$ even smaller if necessary, we see that

$$\varphi_\lambda(u) \geq 0 = \varphi_\lambda(0) \text{ for all } u \in C^1(\overline{\Omega}),\ \|u\|_{C^1(\overline{\Omega})} \leq \widehat{\delta}.$$

So, we infer that $u = 0$ is a local $C^1(\overline{\Omega})-$ minimizer of $\varphi_\lambda(\cdot)$, and hence

$$u = 0 \text{ is a local } W^{1,p}(\Omega) \text{ minimizer of } \varphi_\lambda(\cdot) \tag{3.17}$$

If $K_{\varphi_\lambda} = \{u \in W^{1,p}(\Omega) : \varphi_\lambda'(u) = 0\}$ is the critical set of φ_λ, then $K_{\varphi_\lambda} \subset int\ C_+ \cup \{0\}$ (see (3.5) and (3.13)), and we may assume that

$$K_{\varphi_\lambda} \text{ is finite.} \tag{3.18}$$

Otherwise we already have an infinity of positive smooth solutions for problem (P_λ) and so, we are done. Also, we may assume that

$$0 = \varphi_\lambda(0) \leq \varphi_\lambda(u_0). \tag{3.19}$$

The reasoning is similar if the opposite inequality holds using this time (3.17) instead of (3.16). By (3.16), (3.18) and using Papageorgiou-Radulescu-Repovs [3, Theorem 5.7.6], we can find $\rho \in (0,1)$ small such that

$$\varphi_\lambda(u_0) < \inf\{\varphi_\lambda(u) : \|u - u_0\| = \rho\} = m_\lambda,\ \rho < \|u_0\|. \tag{3.20}$$

Recall that $\varphi_\lambda(\cdot)$ is coercive. So, from Papageorgiou-Radulesci-Repovs [3, Proposition 5.1.15], it follows that

$$\varphi_\lambda(\cdot) \text{ satisfies the PS-condition.} \tag{3.21}$$

Then, (3.20) and (3.21) permit the use of the mountain pass theorem. So, we can find $\widehat{u} \in W^{1,p}(\Omega)$ such that $\varphi_\lambda'(\widehat{u}) = 0$ and $0 < m_\lambda \leq \varphi_\lambda(\widehat{u})$ (see (3.19) and (3.20)). Hence

$$\widehat{u} \in \mathscr{S}(\lambda) \subset int\ C_+ \text{ and } \widehat{u} \notin \{0, u_0\}.$$

Therefore $\widehat{u} \in int\ C_+$ is a second positive solution of (P_λ), distinct from u_0. □

Next we prove the admissibility of the critical parameter $\lambda_* > 0$.

Proposition 3.5. *If hypotheses* $(\mathbf{H}_0), (\mathbf{H}_1), (\mathbf{H}_2), (\mathbf{H}_3)$*, and* $(\mathbf{H}_4)$ *hold, then* $\lambda_* \in \mathscr{L}$.

Proof. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be such that $\lambda_n \downarrow \lambda_*$. We can find $u_n \in \mathscr{S}(\lambda_n) \subset int\ C_+$ such that

$$\begin{aligned} &\langle A(u_n), h\rangle + \int_\Omega \xi(z) u_n^{p-1} h dz + \int_{\partial\Omega} \beta(z) u_n^{p-1} h d\sigma \\ &= \int_\Omega [\lambda_n g(z, u_n) - f(z, u_n)] h dz \text{ for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{3.22}$$

Hypotheses $(\mathbf{H}_2)(i), (ii)$ imply that there exists $C_{15} > 0$ such that

$$0 \leq g(z,x) \leq C_{15}\left[1 + x^{\tau-1}\right] \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{3.23}$$

Similarly, hypotheses $(\mathbf{H}_3)(i)$ and (ii) imply that, for given $\eta > 0$, we can find $\widehat{C}_\eta > 0$ such that

$$f(z,x) \geq \eta x^{\tau-1} - \widehat{C}_\eta \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{3.24}$$

In (3.22) ,we choose $h = u_n \in W^{1,p}(\Omega)$. Using (3.23), (3.24), and Lemma 2.1, we obtain

$$\gamma_p(u_n) \leq [\lambda_n C_{15} - \eta] \|u_n\|_\tau^\tau + C_{16} \text{ for some } C_{16} > 0, \text{ all } n \in \mathbb{N}. \tag{3.25}$$

Since $\eta > 0$ is arbitrary, we choose $\eta > \lambda_1 C_{15} \geq \lambda_n C_{15}$ for all $n \in \mathbb{N}$. From (3.25) and (2.4), we infer that $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega)$ is bounded. We conclude that $\{u_n\}_{n\in\mathbb{N}} \subseteq L^\infty(\Omega)$ is bounded (see Papageorgiou-Radulescu [20, Proposition 2.10]). Then, from Lieberman [1], it follows that there exists $\alpha \in (0,1)$ such that

$$u_n \in C^{1,\alpha}\left(\overline{\Omega}\right), \ \|u_n\|_{C^{1,\alpha}\left(\overline{\Omega}\right)} \leq C_{17} \text{ for some } C_{17} > 0, \text{all } n \in \mathbb{N}.$$

Exploiting the compact embedding of $C^{1,\alpha}\left(\overline{\Omega}\right)$ into $C^1\left(\overline{\Omega}\right)$, we say (at least for a subsequence) that

$$u_n \to u_* \text{ in } C^1\left(\overline{\Omega}\right). \tag{3.26}$$

If, in (3.22), we pass to the limit as $n \to \infty$ and use (3.26), then

$$\begin{aligned} &\langle A(u_*), h\rangle + \int_\Omega \xi(z) u_*^{p-1} h dz + \int_{\partial\Omega} \beta(z) u_*^{p-1} h d\sigma \\ &= \int_\Omega \left[\lambda_* g(z,u_*) - f(z,u_*)\right] h dz \text{ for all } h \in W^{1,p}(\Omega), \end{aligned}$$

and $u_* \in \mathscr{S}(\lambda_*)$. So, we need to show that $u_* \neq 0$. Arguing by contradiction, we suppose that $u_* = 0$. Then

$$u_n \to 0 \text{ in } C^1\left(\overline{\Omega}\right). \tag{3.27}$$

Let

$$y_n = \frac{u_n}{\|u_n\|}, \ n \in \mathbb{N}.$$

Then $\|y_n\| = 1$, $y_n \geq 0$ for all $n \in \mathbb{N}$. In (3.22), we choose $h = u_n \in W^{1,p}(\Omega)$ and multiply with $\|u_n\|^{-p}$. Using Lemma 2.1, we obtain

$$\gamma_p(y_n) \leq \int_\Omega \left[\lambda_n \frac{g(z,u_n)}{\|u_n\|^{p-1}} - \frac{f(z,u_n)}{\|u_n\|^{p-1}}\right] y_n dz \text{ for all } n \in \mathbb{N},$$

and hence

$$C_5 \leq \int_\Omega \left[\lambda_n \frac{g(z,u_n)}{\|u_n\|^{p-1}} - \frac{f(z,u_n)}{\|u_n\|^{p-1}}\right] y_n dz \text{ for all } n \in \mathbb{N} \tag{3.28}$$

(see (2.4) and recall that $\|y_n\| = 1$). On account of hypotheses $(\mathbf{H}_2)(iii)$ *and*, $(\mathbf{H}_3)(iii)$ (recall $s > p$) and of (3.27), we have

$$\int_\Omega \left[\lambda_n \frac{g(z,u_n)}{\|u_n\|^{p-1}} - \frac{f(z,u_n)}{\|u_n\|^{p-1}}\right] y_n dz \to 0 \text{ as } n \to \infty,$$

a contradiction (see (3.28)). Therefore $u_* \neq 0$ and so $u_* \in \mathscr{S}(\lambda_*) \subset int\, C_+$, $\lambda_* \in \mathscr{L}$. ☐

So, summarizing our findings for the positive solutions of problem (P_λ), we can state the following multiplicity theorem, which is global with respect to the parameter (bifurcation-type theorem).

Theorem 3.1. *If hypotheses* $(\mathbf{H}_0)$, $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, $(\mathbf{H}_3)$, *and* $(\mathbf{H}_4)$ *hold, then there exists* $\lambda_* > 0$ *such that*

(a) *for all* $\lambda > \lambda_*$, *problem* (P_λ) *has at least two positive solutions* $u_0, \widehat{u} \in int\, C_+$, $u_0 \neq \widehat{u}$;
(b) *for* $\lambda = \lambda_*$, *problem* (P_λ) *has at least one positive solutions* $u_* \in int\, C_+$;
(c) *for all* $\lambda \in (0,\lambda_*)$, *problem* (P_λ) *has no positive solutions.*

4. Minimal Positive Solution

In this section, we show that for every $\lambda \in \mathscr{L} = [\lambda_*, \infty)$ problem (P_λ) has a smallest positive solution u_λ^*, and we study the monotonicity and the continuity properties of the map $\lambda \to u_\lambda^*$.

Proposition 4.1. *If hypotheses* $(\mathbf{H}_0), (\mathbf{H}_1), (\mathbf{H}_2), (\mathbf{H}_3)$, *and* $(\mathbf{H}_4)$ *hold and* $\lambda \in \mathscr{L} = [\lambda_*, \infty)$, *then problem* (P_λ) *has a smallest positive solution* $u_\lambda^* \in \mathscr{S}(\lambda) \subset \operatorname{int} C_+$, *that is,* $u_\lambda^* \leq u$ *for all* $u \in \mathscr{S}(\lambda)$.

Proof. From the proof of the Proposition 7 in Papageorgiou-Radulescu-Repovs [21], we know that $\mathscr{S}(\lambda)$ is downward directed (that is, if $u_1, u_2 \in \mathscr{S}(\lambda)$, then there is $u \in \mathscr{S}(\lambda)$ such that $u \leq u_1$ and $u \leq u_2$). Then, invoking Hu-Papageorgiou [22, Lemma 3.10], we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathscr{S}(\lambda) \subset \operatorname{int} C_+$ decreasing such that

$$\inf \mathscr{S}(\lambda) = \inf \{u_n : n \in \mathbb{N}\}.$$

We have

$$\begin{aligned} &\langle A(u_n), h\rangle + \int_\Omega \xi(z) u_n^{p-1} h dz + \int_{\partial\Omega} \beta(z) u_n^{p-1} h d\sigma \\ &= \int_\Omega [\lambda g(z, u_n) - f(z, u_n)] h dz \text{ for all } h \in W^{1,p}(\Omega), \text{ all } n \in \mathbb{N}, \end{aligned} \tag{4.1}$$

$$0 \leq u_n \leq u_1 \text{ for all } n \in \mathbb{N}. \tag{4.2}$$

In (4.1), we choose $h = u_n \in W^{1,p}(\Omega)$ and use (4.2) and hypotheses $(\mathbf{H}_2)(i), (\mathbf{H}_3)(i)$ to conclude that $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(\Omega)$ is bounded. From this and the nonlinear regularity theory, it follows that at least for a subsequence $u_n \to u_\lambda^*$ in $C^1(\overline{\Omega})$. As in the proof of Proposition 3.5 (see the part of the proof after (3.26)), via a contradiction argument, we show that $u_\lambda^* \neq 0$, and hence $u_\lambda^* \in \mathscr{S}(\lambda) \subset \operatorname{int} C_+$ and $u_\lambda^* = \inf \mathscr{S}(\lambda)$. ☐

Consider the minimal solution map $\chi : (\lambda_*, \infty) \to C^1(\overline{\Omega})$ defined by $\chi(\lambda) = u_\lambda^*$. We say that $\chi(\cdot)$ is strictly increasing on (λ_*, ∞) if

$$\lambda_* < \lambda < \mu \Longrightarrow u_\eta^* - u_\lambda^* \in D_+. \tag{4.3}$$

Proposition 4.2. *If hypotheses* $(\mathbf{H}_0), (\mathbf{H}_1), (\mathbf{H}_2), (\mathbf{H}_3)$, *and* $(\mathbf{H}_4)$ *hold, then the minimal solution map* $\chi : (\lambda_*, \infty) \to C^1(\overline{\Omega})$ *is*

(a) *strictly increasing (see* (4.3)*);*

(b) *left continuous.*

Proof. (a) Let $\lambda_* < \mu < \lambda < \eta$. We have

$$-\operatorname{div} a\left(Du_\eta^*(z)\right) + \xi(z) u_\eta^*(z)^{p-1} \geq \lambda g\left(z, u_\eta^*(z)\right) - f\left(z, u_\eta^*(z)\right) \text{ in } \Omega, \tag{4.4}$$

$$-\operatorname{div} a\left(Du_\mu^*(z)\right) + \xi(z) u_\mu^*(z)^{p-1} \leq \lambda g\left(z, u_\mu^*(z)\right) - f\left(z, u_\mu^*(z)\right) \text{ in } \Omega, \tag{4.5}$$

We introduce the following truncation of the reaction of problem (P_λ)

$$\theta_\lambda(z,x) = \begin{cases} \lambda g\left(z, u_\mu^*(z)\right) - f\left(z, u_\mu^*(z)\right) & \text{if } \ x < u_\mu^*(z) \\ \lambda g(z,x) - f(z,x) & \text{if } \ u_\mu^*(z) \leq x \leq u_\eta^*(z) \\ \lambda g\left(z, u_\eta^*(z)\right) - f\left(z, u_\eta^*(z)\right) & \text{if } \ u_\eta^*(z) < x. \end{cases} \tag{4.6}$$

This is a Carathéodory function. We set

$$\Theta_\lambda(z,x) = \int_0^x \theta_\lambda(z,s)\, ds$$

and consider the C^1-functional $\widehat{\tau}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\tau}_\lambda(u) = \gamma(u) - \int_\Omega \Theta_\lambda(z,x)\,dz \text{ for all } u \in W^{1,p}(\Omega).$$

From (4.6), it is clear that $\widehat{\tau}_\lambda(\cdot)$ is coercive. Also it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}_\lambda \in W^{1,p}(\Omega)$ such that

$$\widehat{\tau}_\lambda(\widetilde{u}_\lambda) = \inf\left\{\widehat{\tau}_\lambda(u) : u \in W^{1,p}(\Omega)\right\}.$$

Then

$$\widehat{\tau}'_\lambda(\widetilde{u}_\lambda) = 0,$$

hence

$$\left\langle \widehat{\tau}'_\lambda(\widetilde{u}_\lambda), h\right\rangle = 0 \text{ for all } h \in W^{1,p}(\Omega). \tag{4.7}$$

Choosing in (4.7) $h = \left(\widetilde{u}_\lambda - u^*_\eta\right)^+ \in W^{1,p}(\Omega)$ and $h = \left(u^*_\mu - \widetilde{u}_\lambda\right)^+ \in W^{1,p}(\Omega)$ and using (4.4) and (4.5), we show that

$$u^*_\mu(z) \le \widetilde{u}_\lambda(z) \le u^*_\eta(z) \text{ for a.a. } z \in \Omega. \tag{4.8}$$

From (4.6), (4.7), and (4.8), it follows that $\widetilde{u}_\lambda \in \mathscr{S}(\lambda) \subset int\, C_+$. Moreover, as in the proof of Proposition 3.3, using [14, v], we show that $u^*_\eta - \widetilde{u}_\lambda \in D_+$, hence $u^*_\eta - u^*_\lambda \in D_+$. Thus $\chi : (\lambda_*, \infty) \to C^1(\overline{\Omega})$ is strictly increasing.

(b) Suppose that $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathscr{L}$ and assume that $\lambda_n \uparrow \lambda$. We have$u^*_{\lambda_1} \le u^*_{\lambda_n} = u^*_n \le u^*_\lambda$ for all $n \in \mathbb{N}$ (see part (a)). Hence the nonlinear regularity theory (see Lieberman [1]) implies that there exist $\alpha \in (0,1)$ and $C_{18} > 0$ such that

$$u^*_n \in C^{1,\alpha}(\overline{\Omega}) \text{ and } \|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \le C_{18} \text{ for all } n \in \mathbb{N}.$$

On account of the compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ and of the monotonicity of $\{u^*_n\}$ (see (a)), we conclude that

$$u^*_n \to \widehat{u}^*_\lambda \text{ in } C^1(\overline{\Omega}). \tag{4.9}$$

Suppose that $\widehat{u}^*_\lambda \neq u^*_\lambda$. Thus there exists $z_0 \in \Omega$ such that$u^*_\lambda(z_0) < \widehat{u}^*_\lambda(z_0)$, hence $u^*_\lambda(z_0) < u^*_n(z_0)$ for all large n (see (4.9)), which contradicts (a). Therefore $\widehat{u}^*_\lambda = u^*_\lambda$ and so, $\chi(\cdot)$ is left continuous. □

Acknowledgements

The third author acknowledges the partial support by the Portuguese Foundation for Science and Technology (FCT), through CIDMA - Center for Research and Development in Mathematics and Applications, within project UID/MAT/04106/2019.

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