A Wave Diffraction Problem with Higher Order Impedance Boundary Conditions

A. M. Simões

Center of Mathematics - University of Beira Interior (CM-UBI),
Department of Mathematics of University of Beira Interior,
6200-001 Covilhã, Portugal,

Abstract—In this paper, we consider an impedance boundary transmission problem for the Helmholtz equation originated by a problem of wave diffraction by an infinite strip with higher order imperfect boundary conditions. Operator theoretical methods and relations between operators are built to deal with the problem and, as consequence, a transparent interpretation of the problem in an operator theory framework are associated to the problem. In particular, different types of operator relations are exhibited for different types of operators acting between Lebesgue and Sobolev spaces on a finite interval and the positive half-line. All this has consequences in the understanding of the structure of this type of problems. At the end, we describe when the operators associated with the problem enjoy the Fredholm property in terms of the initial space order parameters.

Keywords—Helmholtz equation, higher order impedance boundary condition, Bessel potential space, convolution type operator, Fredholm operator, Wiener-Hopf operator, wave diffraction.

I. Introduction

By using methods from operator theory, in this paper, inspired by the work [14], we will consider a boundary-transmission problem for the Helmholtz equation which arises within the context of wave diffraction theory [3]–[5], [7]–[19], [20], [21] and [24]–[28] on a finite strip [9], [10] and [15] with impedance boundary conditions [7] and [9].

Was A. Sommerfeld the first one to consider canonical boundary value problems for time-harmonic waves governed by the Helmholtz equation in the famous work entitled *Mathematische Theorie der Diffraction*, [29]. Since then, a great number of researchers have made such a study their priority and a great number of different approaches have been presented and developed in the applied mathematics literature for studying canonical problems of plane wave diffraction. The most known and efficient methods and procedures to study such kind of problems are based on the classical Wiener-Hopf technique and the Maliushinets method [21], [28].

In the present work we will consider a Sommerfeld type problem where the geometry comprises a strip facing higher order imperfect boundary conditions. We want to understand better what are the operators behind such a problem. Thus, one of the main goals of the present work is the use of an operator theoretical machinery that will translate the problem into the study of properties of certain known types of operators associated to the problem.

To be more concrete, we will consider Wiener-Hopf operators and convolution type operators on finite intervals with semi-almost periodic Fourier symbol matrices. Convolution type operators $\mathcal{W}$ on finite intervals $\mathcal{I}$,

$$\mathcal{W}\varphi(x) = c\varphi(x) + \int_{\mathcal{I}} K(x-y)\varphi(y)\,dy, \qquad x \in \mathcal{I}.$$

are one-dimensional linear integral operators where the integration kernels K depend on the difference of the arguments and the domain of integration as well as the range of the independent variable are given by the same interval. In a constructive way, we will obtain this type of operators in Sobolev and Lebesgue spaces. This is because we will consider the problem formulated between Bessel potential spaces and defined with a complex wave number k which also allows a certain freedom in the corresponding smoothness orders.

II. Preliminaries and Formulation of the Problem

In this section we establish the notation and some preliminary concepts in view of presenting the mathematical formulation of the problem.

We denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of all rapidly decreasing functions and by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of tempered distributions on $\mathbb{R}^n$. As mentioned in the previous section, we will develop our study in a framework of Bessel potential spaces $\mathcal{H}^s$ defined by the elements $\varphi \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|\varphi\|_{\mathcal{H}^s(\mathbb{R}^n)} := \left\| \mathcal{F}^{-1}(1+|\xi|^2)^{s/2} \cdot \mathcal{F}\varphi \right\|_{L^2(\mathbb{R}^n)} < +\infty,$$

with $s \in \mathbb{R}$ and where $\mathcal{F} = \mathcal{F}_{x\mapsto\xi}$ is the Fourier transformation in $\mathbb{R}^n$ defined by

$$(\mathcal{F}\phi)(\xi) = \int_{\mathbb{R}^n} e^{i\xi \cdot x}\phi(x)dx, \qquad \xi \in \mathbb{R}^n.$$

For a given Lipschitz domain $\mathcal{D}$, on $\mathbb{R}^n$, by $\widetilde{\mathcal{H}}^s(\mathcal{D})$ we mean the closed subspace of $\mathcal{H}^s(\mathbb{R}^n)$ whose elements have supports in $\overline{\mathcal{D}}$, and by $\mathcal{H}^s(\mathcal{D})$ the space of distributions on $\mathcal{D}$ which have extensions into $\mathbb{R}^n$ belonging to $\mathcal{H}^s(\mathbb{R}^n)$. The space $\widetilde{\mathcal{H}}^s(\mathcal{D})$ is endowed with the subspace topology, and on $\mathcal{H}^s(\mathcal{D})$ we introduce the norm of the quotient space $\mathcal{H}^s(\mathbb{R}^n)/\widetilde{\mathcal{H}}^s(\mathbb{R}^n\backslash\overline{\mathcal{D}})$. Throughout the paper we will use the notation

$$\mathbb{R}^n_\pm := \{x = (x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n : \pm x_n > 0\}.$$

Adopting cartesian axes $Oxyz$ with the y-axis vertically upwards, we will consider a perpendicular time-harmonic

electromagnetic plane wave incident on a strip Σ in $\mathbb{R}^3$ where the material is considered to be invariant under the z-axis direction. Thus, the geometry of the problem is two dimensional and the strip will be therefore represented by

$$\Sigma :=]0, a[\quad \text{for} \quad 0 < a < \infty.$$

We are now in position to formulate our impedance boundary conditions problem.

For $\Omega := \mathbb{R}^2 \backslash \overline{\Sigma}$ and given $n \in \mathbb{N}_0$, we are interested in studying the properties of an element $u \in \mathcal{H}^{1+\varepsilon}(\Omega)$, for some $\varepsilon \geq 0$, which satisfies the Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2 \right) u = 0 \qquad \text{in} \quad \Omega,$$

together with the impedance boundary condition

$$\begin{cases} p^+ u_{n+1}^+ + q^+ u_n^+ = h^+ \\ p^- u_{n+1}^- + q^- u_n^- = h^- \end{cases} \quad \text{on} \quad \Sigma, \qquad (1)$$

where the wave number $k \in \mathbb{C}$ is given, as well as the impedance parameters $p^{\pm}, q^{\pm} \in \mathbb{C}$,

$$u_n^{\pm} := \left(\frac{\partial^n u}{\partial y^n} \right)_{|y=\pm 0}$$

denote the traces of u on the upper and lower banks of Σ, respectively, and $h^{\pm} \in \mathcal{H}^{-\frac{1}{2}-n+\varepsilon}(\Sigma)$ are arbitrarily given elements. For instance, is well known that for $n = 0$ and $n = 1$ we have $u_n^{\pm}$ as the traditional Dirichlet and Neumann traces, respectively.

III. REDUCTION OF THE PROBLEM TO A SYSTEM OF CONVOLUTION TYPE OPERATORS

In this section we will use operator techniques in view of a characterization of the problem by means of finite interval convolution type operators. In the next section, such characterization of the problem, will be used to present certain extension methods in view to obtain corresponding operator relations, between the operator related to the problem and new Wiener-Hopf operators.

We will consider the densities ϑ and φ defined by

$$\begin{bmatrix} \vartheta \\ \varphi \end{bmatrix} = \begin{bmatrix} u_1^+ - u_1^- \\ u_0^+ - u_0^- \end{bmatrix} \in \widetilde{\mathcal{H}}^{-\frac{1}{2}+\varepsilon}(\Sigma) \times \widetilde{\mathcal{H}}^{\frac{1}{2}+\varepsilon}(\Sigma).$$

For an integer j, it follows

$$u_j^+ = (-1)^j \mathcal{F}^{-1} t^j \cdot \mathcal{F} u_0^+$$

and

$$u_j^- = \mathcal{F}^{-1} t^j \cdot \mathcal{F} u_0^-,$$

where

$$t(\xi) = (\xi^2 - k^2)^{\frac{1}{2}} = t_+(\xi) t_-(\xi)$$

with $t_{\pm}$ the squareroot functions

$$t_{\pm}(\xi) = (\xi \pm k)^{\frac{1}{2}} = |\xi \pm k|^{\frac{1}{2}} e^{\frac{1}{2} i \arg(\xi \pm k)},$$

$\xi \in \mathbb{R}$, with branch cuts $\Gamma_{\mp} = \{ \pm k \pm it, \, t \geq 0 \}$, respectively,

$$\arg(\xi - k) \in \left] -\frac{3\pi}{2}, \frac{\pi}{2} \right[$$

and

$$\arg(\xi + k) \in \left] -\frac{\pi}{2}, \frac{3\pi}{2} \right[.$$

Using these formulas, we can define an invertible convolution operator

$$B_{\Phi_B, \Sigma} := \mathcal{F}^{-1} \Phi_B \cdot \mathcal{F}$$

which maps $\widetilde{\mathcal{H}}^{-\frac{1}{2}+\varepsilon}(\Sigma) \times \widetilde{\mathcal{H}}^{\frac{1}{2}+\varepsilon}(\Sigma)$ into $\widetilde{\mathcal{H}}^{-\frac{1}{2}-n+\varepsilon}(\Sigma) \times \widetilde{\mathcal{H}}^{\frac{1}{2}-n+\varepsilon}(\Sigma)$ as

$$B_{\Phi_B, \Sigma} \begin{bmatrix} \vartheta \\ \varphi \end{bmatrix} = \begin{bmatrix} u_{n+1}^+ - u_{n+1}^- \\ u_n^+ - u_n^- \end{bmatrix}, \qquad (2)$$

with Fourier symbol

$$\Phi_B = \frac{1}{2} \begin{bmatrix} (1 + (-1)^n) t^n & (1 - (-1)^n) t^{n+1} \\ (1 - (-1)^n) t^{n-1} & (1 + (-1)^n) t^n \end{bmatrix}.$$

Now, by the use of (2), it is possible to rewrite the boundary condition (1) as

$$C_{\Phi_C, \Sigma} \begin{bmatrix} u_{n+1}^+ - u_{n+1}^- \\ u_n^+ - u_n^- \end{bmatrix} = \begin{bmatrix} h^+ \\ h^- \end{bmatrix} \qquad (3)$$

where we define a convolution type operator

$$C_{\Phi_C, \Sigma} := r_\Sigma \mathcal{F}^{-1} \Phi_C \cdot \mathcal{F}$$

which maps the spaces $\widetilde{\mathcal{H}}^{-\frac{1}{2}-n+\varepsilon}(\Sigma) \times \widetilde{\mathcal{H}}^{\frac{1}{2}-n+\varepsilon}(\Sigma)$ into the spaces $\mathcal{H}^{-\frac{1}{2}-n+\varepsilon}(\Sigma) \times \mathcal{H}^{-\frac{1}{2}-n+\varepsilon}(\Sigma)$ with Fourier symbol

$$\Phi_C = \frac{1}{2} \begin{bmatrix} p^+ - q^+ t^{-1} & -p^+ t + q^+ \\ -p^- - q^- t^{-1} & -p^- t - q^- \end{bmatrix}. \qquad (4)$$

Throughout the paper, we are using r_Σ to denote the restriction operator to $\Sigma \subset \mathbb{R}$ and in the particular case of $r_{\mathbb{R}_+}$ we will simply write r_+ for this restriction.

From (2) and (3), we obtain

$$C_{\Phi_C, \Sigma} B_{\Phi_B, \Sigma} \begin{bmatrix} \vartheta \\ \varphi \end{bmatrix} = \begin{bmatrix} h^+ \\ h^- \end{bmatrix}.$$

Our immediate goal will be to extend this last convolution type operator on a finite interval into a convolution type operator on the half-line. In view of this, we will need to consider some extension operator relations.

IV. EXTENSION METHODS AND RELATIONS BETWEEN OPERATORS

We will now perform some operator extension procedures in view of obtaining corresponding operator relations between the operators presented in the last section and new Wiener-Hopf operators. These operator relations will be used in the last section to study the Fredholm property of the operators associated with the problem.

Definition 4.1: [15] Let us consider two operators

$$A : X_1 \to Y_1$$

and

$$B : X_2 \to Y_2,$$

acting between Banach spaces.

(i) The operators A and B are said to be algebraically equivalent after extension if there exist additional Banach spaces Z_1 and Z_2 and invertible linear operators

$$E : Y_2 \times Z_2 \to Y_1 \times Z_1$$

and

$$F : X_1 \times Z_1 \to X_2 \times Z_2$$

such that

$$\begin{bmatrix} A & 0 \\ 0 & I_{Z_1} \end{bmatrix} = E \begin{bmatrix} B & 0 \\ 0 & I_{Z_2} \end{bmatrix} F. \qquad (5)$$

(ii) If, in addition to (i), the invertible and linear operators E and F in (5) are bounded, then we will say that A and B are topologically equivalent after extension operators, or simply say that A and B are equivalent after extension operators, [1].

(iii) A and B are said to be equivalent operators in the particular case when

$$A = E\,B\,F,$$

for some bounded invertible linear operators

$$E : Y_2 \to Y_1$$

and

$$F : X_1 \to X_2.$$

The above notion of topological equivalence after extension relation is equivalente to the concept of *matricial coupling* [1]. We refer to [4], [6] and [15] for a discussion on the differences between algebraic and topological equivalence after extension relations between convolution type operators.

We will now apply some results of [6] to our convolution type operator $C_{\Phi_C,\Sigma}$.

Theorem 4.1: The convolution type operator $C_{\Phi_C,\Sigma}$ with Fourier symbol (4) is algebraically equivalent after extension to the Wiener-Hopf operator $\mathcal{C}_{\Phi_C,\mathbb{R}_+}$ which maps $\widetilde{\mathcal{H}}^{-\frac{1}{2}-n+\varepsilon}(\mathbb{R}_+) \times \widetilde{\mathcal{H}}^{\frac{1}{2}-n+\varepsilon}(\mathbb{R}_+) \times \widetilde{\mathcal{H}}^{-\frac{1}{2}-n+\varepsilon}(\mathbb{R}_+) \times \widetilde{\mathcal{H}}^{-\frac{1}{2}-n+\varepsilon}(\mathbb{R}_+)$ into $\mathcal{H}^{-\frac{1}{2}-n+\varepsilon}(\mathbb{R}_+) \times \mathcal{H}^{\frac{1}{2}-n+\varepsilon}(\mathbb{R}_+) \times \mathcal{H}^{-\frac{1}{2}-n+\varepsilon}(\mathbb{R}_+) \times \mathcal{H}^{-\frac{1}{2}-n+\varepsilon}(\mathbb{R}_+)$ given by

$$\mathcal{C}_{\Phi_C,\mathbb{R}_+} := r_+\mathcal{F}^{-1}\Phi_C \cdot \mathcal{F},$$

and with Φ_C being the Fourier symbol defined by

$$\Phi_C(\xi) = \begin{bmatrix} e^{-i\xi a} & 0 & 0 & 0 \\ 0 & e^{-i\xi a} & 0 & 0 \\ \frac{1}{2}(p^+ - q^+ t^{-1}(\xi)) & \frac{1}{2}(-p^+ t(\xi) + q^+) & e^{i\xi a} & 0 \\ \frac{1}{2}(-p^- - q^- t^{-1}(\xi)) & \frac{1}{2}(-p^- t(\xi) - q^-) & 0 & e^{i\xi a} \end{bmatrix}.$$

So, there are Banach spaces X_1 and Y_1 and linear homeomorphisms E_1 and F_1 such that

$$\begin{bmatrix} C_{\Phi_C,\Sigma} & 0 \\ 0 & I_{Z_1} \end{bmatrix} = E_1 \begin{bmatrix} \mathcal{C}_{\Phi_C,\mathbb{R}_+} & 0 \\ 0 & I_{Z_2} \end{bmatrix} F_1.$$

The proof is omitted in here because is a well-known result addressed in [22]. For some generalizations see [6], [23].

Due to the use of the lifting procedure, and choosing convenient auxiliary bounded invertible operators, we now

obtain a new operator relation for an operator acting between Lebesgue spaces – which is presented in the next result.

We will use the notation $L_+^2(\mathbb{R}) := \widetilde{\mathcal{H}}^0(\mathbb{R}_+)$.

Theorem 4.2: The Wiener-Hopf operator $\mathcal{C}_{\Phi_C,\mathbb{R}_+}$ defined above between Bessel potential spaces is equivalent to the Wiener-Hopf operator

$$\widehat{\mathcal{C}}_{\Phi_{\widehat{C}},\mathbb{R}_+} := r_+\mathcal{F}^{-1}\Phi_{\widehat{C}} \cdot \mathcal{F} : \left[L_+^2(\mathbb{R})\right]^4 \to \left[L^2(\mathbb{R}_+)\right]^4,$$

where $\Phi_{\widehat{C}}$ has the block matricial representation

$$\Phi_{\widehat{C}}(\xi) = \begin{bmatrix} \mathfrak{A}(\xi) & 0_2 \\ \mathfrak{C}(\xi) & \mathfrak{B}(\xi) \end{bmatrix} \qquad (6)$$

where

$$\mathfrak{A}(\xi) = \begin{bmatrix} \zeta^{-\frac{1}{2}-n+\varepsilon}(\xi)e^{-i\xi a} & 0 \\ 0 & \zeta^{\frac{1}{2}-n+\varepsilon}(\xi)e^{-i\xi a} \end{bmatrix},$$

$$\mathfrak{B}(\xi) = \begin{bmatrix} \zeta^{-\frac{1}{2}-n+\varepsilon}(\xi)e^{i\xi a} & 0 \\ 0 & \zeta^{-\frac{1}{2}-n+\varepsilon}(\xi)e^{i\xi a} \end{bmatrix},$$

$$\mathfrak{C}(\xi) = \begin{bmatrix} \mathfrak{C}_{11}(\xi) & \mathfrak{C}_{12}(\xi) \\ \mathfrak{C}_{21}(\xi) & \mathfrak{C}_{22}(\xi) \end{bmatrix},$$

with

$$\mathfrak{C}_{11}(\xi) = \frac{1}{2}(p^+\zeta^{-\frac{1}{2}-n+\varepsilon}(\xi) - q^+\zeta^{-n+\varepsilon}(\xi)(\xi-k)^{-1}),$$

$$\mathfrak{C}_{12}(\xi) = \frac{1}{2}(-p^+\zeta^{-n+\varepsilon}(\xi) + q^+\zeta^{-\frac{1}{2}-n+\varepsilon}(\xi)(\xi+k)^{-1}),$$

$$\mathfrak{C}_{21}(\xi) = \frac{1}{2}(-p^-\zeta^{-\frac{1}{2}-n+\varepsilon}(\xi) - q^-\zeta^{-n+\varepsilon}(\xi)(\xi-k)^{-1}),$$

$$\mathfrak{C}_{22}(\xi) = \frac{1}{2}(-p^-\zeta^{-n+\varepsilon}(\xi) - q^-\zeta^{-\frac{1}{2}-n+\varepsilon}(\xi)(\xi+k)^{-1}),$$

$\zeta(\xi) = \frac{\xi-k}{\xi+k} = \frac{\lambda_-}{\lambda_+}$, $\xi \in \mathbb{R}$ and 0_2 denotes the 2×2 zero matrix.

Proof: The equivalence relation can be directly obtained by computing the following operator composition

$$\mathcal{C}_{\Phi_C,\mathbb{R}_+} = \mathrm{W}_{\Phi_E,\mathbb{R}_+}\, l_0\, \widehat{\mathcal{C}}_{\Phi_{\widehat{C}},\mathbb{R}_+}\, l_0 \mathrm{W}_{\Phi_F,\mathbb{R}_+},$$

where

$$l_0 : \left[L^2(\mathbb{R}_+)\right]^4 \to \left[L_+^2(\mathbb{R})\right]^4$$

denotes de zero extension operator and where $\mathrm{W}_{\Phi_E,\mathbb{R}_+} l_0$ is defined between the spaces $\left[L^2(\mathbb{R}_+)\right]^4$ and $\mathcal{H}^{-\frac{1}{2}-n+\varepsilon}(\mathbb{R}_+) \times \mathcal{H}^{\frac{1}{2}-n+\varepsilon}(\mathbb{R}_+) \times \mathcal{H}^{-\frac{1}{2}-n+\varepsilon}(\mathbb{R}_+) \times \mathcal{H}^{-\frac{1}{2}-n+\varepsilon}(\mathbb{R}_+)$ by

$$\mathrm{W}_{\Phi_E,\mathbb{R}_+} l_0 := r_+\mathcal{F}^{-1}\Phi_E \cdot \mathcal{F}l_0$$

with

$$\Phi_E(\xi) = \begin{bmatrix} \lambda_-^{\frac{1}{2}+n-\varepsilon} & 0 & 0 & 0 \\ 0 & \lambda_-^{-\frac{1}{2}+n-\varepsilon} & 0 & 0 \\ 0 & 0 & \lambda_-^{\frac{1}{2}+n-\varepsilon} & 0 \\ 0 & 0 & 0 & \lambda_-^{\frac{1}{2}+n-\varepsilon} \end{bmatrix}$$

and $l_0\mathrm{W}_{\Phi_F,\mathbb{R}_+}$ is defined between $\widetilde{\mathcal{H}}^{-\frac{1}{2}-n+\varepsilon}(\mathbb{R}_+) \times \widetilde{\mathcal{H}}^{\frac{1}{2}-n+\varepsilon}(\mathbb{R}_+) \times \widetilde{\mathcal{H}}^{-\frac{1}{2}-n+\varepsilon}(\mathbb{R}_+) \times \widetilde{\mathcal{H}}^{-\frac{1}{2}-n+\varepsilon}(\mathbb{R}_+)$ and $\left[L_+^2(\mathbb{R})\right]^4$ by

$$l_0\mathrm{W}_{\Phi_F,\mathbb{R}_+} := l_0 r_+\mathcal{F}^{-1}\Phi_F \cdot \mathcal{F}$$

with

$$\Phi_F(\xi)=\begin{bmatrix} \lambda_+^{-\frac{1}{2}-n+\varepsilon} & 0 & 0 & 0 \\ 0 & \lambda_+^{\frac{1}{2}-n+\varepsilon} & 0 & 0 \\ 0 & 0 & \lambda_+^{-\frac{1}{2}-n+\varepsilon} & 0 \\ 0 & 0 & 0 & \lambda_+^{-\frac{1}{2}-n+\varepsilon} \end{bmatrix}.$$

Notice that the bounded operators $W_{\Phi_E,\mathbb{R}_+} l_0$ and $l_0 W_{\Phi_F,\mathbb{R}_+}$ are invertible as pointed out in [30, §2.10.3]. ∎

V. FREDHOLM ANALYSIS

Our main goal is to study and characterize the Fredholm property of the finite interval convolution type operator $C_{\Phi_C,\Sigma}$ for general ε. We will use different factorization procedures applied to the operators introduced in the last section. We start by recalling the definition of Fredholm operator.

Definition 5.1: Let X, Y be two Banach spaces and $A : X \to Y$ a bounded linear operator with closed image. The operator A is called a Fredholm operator if

$$n(A) := \dim Ker\,A < \infty$$

and

$$d(A) := \dim Y/Im\,A < \infty.$$

If A is a Fredholm operator, then the Fredholm index of A is the integer defined by

$$Ind\,A = n(A) - d(A).$$

Theorem 5.1: Let $\Phi_{\widehat{C}}$ be defined by (6) and

$$\det \mathfrak{C}(\pm\infty) \neq 0.$$

The operator $\widehat{\mathcal{C}}_{\Phi_{\widehat{C}},\mathbb{R}_+}$ presented in the last theorem admits the factorization

$$\widehat{\mathcal{C}}_{\Phi_{\widehat{C}},\mathbb{R}_+} = \widehat{W}_{\widehat{\Phi}_-,\mathbb{R}_+} \widetilde{\mathcal{C}}_{\Phi_{\widetilde{C}},\mathbb{R}_+} \widehat{W}_{\widehat{\Phi}_+,\mathbb{R}_+}$$

where $\widehat{W}_{\widehat{\Phi}_-,\mathbb{R}_+}$ and $\widehat{W}_{\widehat{\Phi}_+,\mathbb{R}_+}$ are invertible operators having Fourier symbols

$$\widehat{\Phi}_-(\xi) = \begin{bmatrix} -1 & 0 & -e^{-ia\xi}\tau^-(\xi) \\ 0 & -1 & \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

and

$$\widehat{\Phi}_+(\xi) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & e^{ia\xi}\tau^+(\xi) \\ 0 & 1 & \end{bmatrix},$$

which admit bounded analytic extensions in $\Im m\,\xi < 0$ and $\Im m\,\xi > 0$, respectively, and with

$$\tau^-(\xi) = \frac{1-S(\xi)}{2}\mathfrak{C}^{-1}(-\infty) +$$
$$\frac{1+S(\xi)}{2}\begin{bmatrix} e^{i\pi(-1-2n+2\varepsilon)} & 0 \\ 0 & e^{i\pi(1-2n+2\varepsilon)} \end{bmatrix}\mathfrak{C}^{-1}(+\infty)$$

and

$$\tau^+(\xi) = \frac{1-S(\xi)}{2}\mathfrak{C}^{-1}(-\infty) +$$
$$\frac{1+S(\xi)}{2}\begin{bmatrix} e^{i\pi(-1-2n+2\varepsilon)} & 0 \\ 0 & e^{i\pi(-1-2n+2\varepsilon)} \end{bmatrix}\mathfrak{C}^{-1}(+\infty)$$

where $S : \mathbb{C} \to \mathbb{C}$ is the normalized sine-integral function given by

$$S(\xi) = \frac{2}{\pi} \int_0^\xi \frac{\sin x}{x} dx$$

and where $\mathfrak{C}^{-1}(-\infty)$ and $\mathfrak{C}^{-1}(+\infty)$ are defined by

$$\mathfrak{C}^{-1}(-\infty) = \begin{bmatrix} \frac{1}{p^+} & -\frac{1}{p^-} \\ -\frac{1}{p^+} & -\frac{1}{p^-} \end{bmatrix}$$

and

$$\mathfrak{C}^{-1}(+\infty) = \begin{bmatrix} \frac{1}{p^+}e^{i\pi(1+2n-2\varepsilon)} & -\frac{1}{p^-}e^{i\pi(1+2n-2\varepsilon)} \\ -\frac{1}{p^+}e^{i\pi(2n-2\varepsilon)} & -\frac{1}{p^-}e^{i\pi(2n-2\varepsilon)} \end{bmatrix}$$

if

$$\det \mathfrak{C}(-\infty) = -\frac{p^+ p^-}{2} \neq 0$$

and

$$\det \mathfrak{C}(+\infty) = -\frac{p^+ p^-}{2}e^{i\pi(-1-4n+4\varepsilon)} \neq 0,$$

respectively.

The Fourier symbol $\Phi_{\widetilde{C}}$ belongs to $PC^{4\times4}(\overset{\bullet}{\mathbb{R}})$, the space of four by four matrix-valued functions with piecewise continuous entries on $\overset{\bullet}{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, and is given by

$$\Phi_{\widetilde{C}}(\xi) = \left[\begin{array}{c|c} \mathfrak{A}(\xi) & \mathfrak{B}(\xi) \\ \hline \mathfrak{D}(\xi) & -\mathfrak{C}(\xi) \end{array} \right] \tag{7}$$

where

$$\mathfrak{A}(\xi)=\left(\begin{bmatrix} \zeta^{-\frac{1}{2}-n+\varepsilon}(\xi) & 0 \\ 0 & \zeta^{\frac{1}{2}-n+\varepsilon}(\xi) \end{bmatrix}-\tau^-(\xi)\mathfrak{C}(\xi)\right)\tau^+(\xi)+$$
$$\tau^-(\xi)\begin{bmatrix} \zeta^{-\frac{1}{2}-n+\varepsilon}(\xi) & 0 \\ 0 & \zeta^{-\frac{1}{2}-n+\varepsilon}(\xi) \end{bmatrix},$$
$$\mathfrak{B}(\xi)=e^{-ia\xi}\left(\tau^-(\xi)\mathfrak{C}(\xi)-\begin{bmatrix} \zeta^{-\frac{1}{2}-n+\varepsilon}(\xi) & 0 \\ 0 & \zeta^{\frac{1}{2}-n+\varepsilon}(\xi) \end{bmatrix}\right),$$
$$\mathfrak{D}(\xi)=e^{ia\xi}\left(\mathfrak{C}(\xi)\tau^+(\xi)-\begin{bmatrix} \zeta^{-\frac{1}{2}-n+\varepsilon}(\xi) & 0 \\ 0 & \zeta^{-\frac{1}{2}-n+\varepsilon}(\xi) \end{bmatrix}\right).$$

The proof of the last result can be done by direct computation and therefore is here omitted. Anyway, we have,

$$\lim_{\xi\to\pm\infty}\left(\tau^-(\xi)\mathfrak{C}(\xi)-\begin{bmatrix} \zeta^{-\frac{1}{2}-n+\varepsilon}(\xi) & 0 \\ 0 & \zeta^{\frac{1}{2}-n+\varepsilon}(\xi) \end{bmatrix}\right)=0, \tag{8}$$
$$\lim_{\xi\to\pm\infty}\left(\mathfrak{C}(\xi)\tau^+(\xi)-\begin{bmatrix} \zeta^{-\frac{1}{2}-n+\varepsilon}(\xi) & 0 \\ 0 & \zeta^{-\frac{1}{2}-n+\varepsilon}(\xi) \end{bmatrix}\right)=0. \tag{9}$$

These last two results are a consequence of the fact that we agree that

$$\lim_{\xi\to-\infty}\zeta^\sigma(\xi) = 1$$

and

$$\lim_{\xi\to+\infty}\zeta^\sigma(\xi) = e^{i2\pi\sigma},$$

for $\sigma \in \mathbb{R}$.

In order to continue, let us consider, for $\Phi \in PC^{n \times n}(\dot{\mathbb{R}})$, the function

$$\overline{\Phi} : \dot{\mathbb{R}} \times [0, 1] \to \mathbb{C}^{n \times n}$$

defined by

$$\overline{\Phi}(\xi, \mu) := (1 - \mu)\Phi(\xi - 0) + \mu\Phi(\xi + 0),$$

$(\xi, \mu) \in \dot{\mathbb{R}} \times [0, 1]$, where

$$\Phi(\infty - 0) := \Phi(+\infty)$$

and

$$\Phi(\infty + 0) := \Phi(-\infty).$$

The following result [2, Theorem 5.9] helps us to study the Fredholm property for the operator $C_{\Phi_C, \Sigma}$.

Theorem 5.2: For $\Phi \in PC^{n \times n}(\dot{\mathbb{R}})$, it follows that

$$\det \overline{\Phi}(\xi, \mu) \neq 0$$

for all $(\xi, \mu) \in \dot{\mathbb{R}} \times [0, 1]$ if and only if

$$\mathcal{W}_{\Phi, \mathbb{R}_+} := r_+ \mathcal{F}^{-1}\Phi \cdot \mathcal{F} : \left[L_+^2(\mathbb{R})\right]^n \to \left[L^2(\mathbb{R}_+)\right]^n$$

is a Fredholm operator.

In case of having the Fredholm property, the Fredholm index of $\mathcal{W}_{\Phi, \mathbb{R}_+}$ is given by

$$Ind\, \mathcal{W}_{\Phi, \mathbb{R}_+} = -wind(\det \overline{\Phi}),$$

where $wind$ denotes the winding number.

Finally, we are able to present the Fredholm characterization to our operator $C_{\Phi_C, \Sigma}$ and, consequently, to our initial problem.

Theorem 5.3: The finite interval convolution type operator $C_{\Phi_C, \Sigma}$ is a Fredholm operator with zero Fredholm index if and only if

$$\varepsilon \neq \frac{q}{2} \quad \text{for} \quad q \in \mathbb{Z}. \tag{10}$$

Proof: First of all, we notice that from Theorems 4.1–5.1 we conclude that the operator $C_{\Phi_C, \Sigma}$ is algebraically equivalent after extension to the operator

$$\widetilde{\mathcal{C}}_{\Phi_{\widetilde{C}}, \mathbb{R}_+} := r_+ \mathcal{F}^{-1}\Phi_{\widetilde{C}} \cdot \mathcal{F} : \left[L_+^2(\mathbb{R})\right]^4 \to \left[L^2(\mathbb{R}_+)\right]^4$$

where $\Phi_{\widetilde{C}}$ is given by (7). Therefore, in view to obtain the desired conclusion, that $C_{\Phi_C, \Sigma}$ is a Fredholm operator, we start by deducing the conditions which characterize the Fredholm property of $\widetilde{\mathcal{C}}_{\Phi_{\widetilde{C}}, \mathbb{R}_+}$.

Letting

$$\overline{\Phi}_{\widetilde{C}}(\xi, \mu) = (1 - \mu)\Phi_{\widetilde{C}}(\xi - 0) + \mu\Phi_{\widetilde{C}}(\xi + 0)$$

and

$$\Phi_{\widetilde{C}}(\infty \pm 0) := \Phi_{\widetilde{C}}(\mp\infty),$$

by Theorem 5.2, we have that

$$\det \overline{\Phi}_{\widetilde{C}}(\xi, \mu) \neq 0$$

for $(\xi, \mu) \in \dot{\mathbb{R}} \times [0, 1]$ if and only if the operator $\widetilde{\mathcal{C}}_{\Phi_{\widetilde{C}}, \mathbb{R}_+}$ has the Fredholm property. Additionally, from Theorem 5.1, we already know that the Fourier symbol $\Phi_{\widetilde{C}}$ can be written as

$$\Phi_{\widetilde{C}}(\xi) = \widehat{\Phi}_-^{-1}(\xi)\Phi_{\widehat{C}}(\xi)\widehat{\Phi}_+^{-1}(\xi).$$

Thus, for any $\xi \in \mathbb{R}$ we have

$$\det \Phi_{\widetilde{C}}(\xi \pm 0) = \det \Phi_{\widehat{C}}(\xi)$$

because $\Phi_{\widehat{C}}(\xi)$ has no discontinuities on the real line, $\det \widehat{\Phi}_\pm^{-1}$ also have no discontinuities on the real line and, moreover, $\det \widehat{\Phi}_\pm^{-1} \equiv 1$. Therefore,

$$\begin{aligned}
\det \overline{\Phi}_{\widetilde{C}}(\xi, \mu) &= \det \left[(1 - \mu)\Phi_{\widehat{C}}(\xi) + \mu\Phi_{\widehat{C}}(\xi)\right] \\
&= \det \Phi_{\widehat{C}}(\xi) \\
&= \zeta^{-1-4n+4\varepsilon}(\xi) \\
&\neq 0,
\end{aligned}$$

in the case of $\xi \in \mathbb{R}$.

For $\xi = \infty$, we have,

$$\det \overline{\Phi}_{\widetilde{C}}(\infty, \mu) = \det \left[(1 - \mu)\Phi_{\widetilde{C}}(+\infty) + \mu\Phi_{\widetilde{C}}(-\infty)\right].$$

Appealing to the limits (8)–(9), we obtain

$$\Phi_{\widetilde{C}}(-\infty) = \left[\begin{array}{c|c} \mathfrak{C}^{-1}(-\infty) & 0_2 \\ \hline 0_2 & -\mathfrak{C}(-\infty) \end{array}\right]$$

and

$$\Phi_{\widetilde{C}}(+\infty) = \left[\begin{array}{c|c} \mathfrak{A}\mathfrak{C}^{-1}(+\infty)\mathfrak{B} & 0_2 \\ \hline 0_2 & -\mathfrak{C}(+\infty) \end{array}\right],$$

with

$$\mathfrak{A} = \left[\begin{array}{cc} e^{i\pi(-1-2n+2\varepsilon)} & 0 \\ 0 & e^{i\pi(1-2n+2\varepsilon)} \end{array}\right],$$

and

$$\mathfrak{B} = \left[\begin{array}{cc} e^{i\pi(-1-2n+2\varepsilon)} & 0 \\ 0 & e^{i\pi(-1-2n+2\varepsilon)} \end{array}\right].$$

Thus, by direct computation, we have

$$\Phi_{\widetilde{C}}(-\infty) = \left[\begin{array}{cccc} \frac{1}{p^+} & -\frac{1}{p^-} & 0 & 0 \\ -\frac{1}{p^+} & -\frac{1}{p^-} & 0 & 0 \\ 0 & 0 & -\frac{1}{2}p^+ & \frac{1}{2}p^+ \\ 0 & 0 & \frac{1}{2}p^- & \frac{1}{2}p^- \end{array}\right]$$

and

$$\Phi_{\widetilde{C}}(+\infty) = \left[\begin{array}{cccc} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{array}\right],$$

with

$$a_{11} = \frac{1}{p^+}e^{i\pi(-1-2n+2\varepsilon)},$$

$$a_{12} = -\frac{1}{p^-}e^{i\pi(-1-2n+2\varepsilon)},$$

$$a_{21} = -\frac{1}{p^+}e^{i\pi(-2n+2\varepsilon)},$$

$$a_{22} = -\frac{1}{p^-}e^{i\pi(-2n+2\varepsilon)},$$

$$a_{33} = -\frac{p^+}{2}e^{i\pi(-1-2n+2\varepsilon)},$$

$$a_{34} = \frac{p^+}{2}e^{i\pi(-2n+2\varepsilon)},$$

$$a_{43} = \frac{p^-}{2}e^{i\pi(-1-2n+2\varepsilon)},$$

$$a_{44} = \frac{p^-}{2}e^{i\pi(-2n+2\varepsilon)}.$$

Finally, the last results, tell us that

$$\det \overline{\Phi}_{\widetilde{C}}(\infty,\mu) = \begin{vmatrix} b_{11} & b_{12} & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ 0 & 0 & b_{33} & b_{34} \\ 0 & 0 & b_{43} & b_{44} \end{vmatrix},$$

where

$$b_{11} = \frac{1-\mu}{p^+}e^{i\pi(-1-2n+2\varepsilon)} + \frac{\mu}{p^+},$$

$$b_{12} = -\frac{1-\mu}{p^-}e^{i\pi(-1-2n+2\varepsilon)} - \frac{\mu}{p^-},$$

$$b_{21} = -\frac{1-\mu}{p^+}e^{i\pi(-2n+2\varepsilon)} - \frac{\mu}{p^+},$$

$$b_{22} = -\frac{1-\mu}{p^-}e^{i\pi(-2n+2\varepsilon)} - \frac{\mu}{p^-},$$

$$b_{33} = -\frac{(1-\mu)p^+}{2}e^{i\pi(-1-2n+2\varepsilon)} - \frac{\mu p^+}{2},$$

$$b_{34} = \frac{(1-\mu)p^+}{2}e^{i\pi(-2n+2\varepsilon)} + \frac{\mu p^+}{2},$$

$$b_{43} = \frac{(1-\mu)p^-}{2}e^{i\pi(-1-2n+2\varepsilon)} + \frac{\mu p^-}{2},$$

$$b_{44} = \frac{(1-\mu)p^-}{2}e^{i\pi(-2n+2\varepsilon)} + \frac{\mu p^-}{2}.$$

So,

$$\det \overline{\Phi}_{\widetilde{C}}(\infty,\mu) =$$
$$\left[(1-\mu)e^{i\pi(-1-2n+2\varepsilon)} + \mu\right]^2 \left[(1-\mu)e^{i\pi(-2n+2\varepsilon)} + \mu\right]^2.$$

As a consequence, $\widetilde{\mathcal{C}}_{\Phi_{\widetilde{C}},\mathbb{R}_+}$ is a Fredholm operator if and only if

$$(1-\mu)e^{i\pi(-1-2n+2\varepsilon)} + \mu \neq 0 \qquad (11)$$

and

$$(1-\mu)e^{i\pi(-2n+2\varepsilon)} + \mu \neq 0, \qquad (12)$$

$\mu \in [0,1].$

Since the sets

$$\mathcal{S}_1 = \left\{(1-\mu)e^{i\pi(-1-2n+2\varepsilon)} + \mu \; : \; \mu \in [0,1]\right\}$$

and

$$\mathcal{S}_2 = \left\{(1-\mu)e^{i\pi(-2n+2\varepsilon)} + \mu \; : \; \mu \in [0,1]\right\}$$

define the line segments joining 1 to $e^{i\pi(-1-2n+2\varepsilon)}$ and 1 to $e^{i\pi(-2n+2\varepsilon)}$, respectively, for holding the inequalities in (11) and (12), we need that

$$e^{i\pi(-1-2n+2\varepsilon)} \notin \mathbb{R}_-$$

and

$$e^{i\pi(-2n+2\varepsilon)} \notin \mathbb{R}_-.$$

Thus

$$\pi(-1-2n+2\varepsilon) \neq \pi + 2\pi q$$

and

$$\pi(-2n+2\varepsilon) \neq \pi + 2\pi q,$$

$q \in \mathbb{Z}$, i.e.,

$$\varepsilon \neq 1 + n + q \quad \text{and} \quad \varepsilon \neq \frac{1}{2} + n + q, \qquad q \in \mathbb{Z}.$$

So, we have $\varepsilon \neq \frac{q}{2}$, $q \in \mathbb{Z}$.

Therefore, from the operator identities provided by both the above mentioned algebraic and topological equivalence relations, given in Theorems 4.1–5.1, we conclude that $\widetilde{\mathcal{C}}_{\Phi_{\widetilde{C}},\mathbb{R}_+}$ and $C_{\Phi_C,\Sigma}$ are Fredholm operators if and only if condition (10) holds, and that the corresponding defect spaces of these operators have the same dimensions. From this, and since by [1, Theorem 3] Fredholm operators in Banach spaces are equivalent after extension if and only if their corresponding defect spaces have equal dimensions, we even arrive at the conclusion that $\widetilde{\mathcal{C}}_{\Phi_{\widetilde{C}},\mathbb{R}_+}$ and $C_{\Phi_C,\Sigma}$ are not only algebraically equivalent after extension but also topologically equivalent after extension.

Finally, jointing the last conclusion with Theorem 5.2, we obtain the following result for the Fredholm index of $C_{\Phi_C,\Sigma}$,

$$Ind\, C_{\Phi_C,\Sigma} = Ind\, \widetilde{\mathcal{C}}_{\Phi_{\widetilde{C}},\mathbb{R}_+}$$
$$= -wind\left(\det \overline{\Phi}_{\widetilde{C}}(\xi,\mu)\right)$$
$$= -\frac{1}{2\pi}\left(\left[\arg \det \overline{\Phi}_{\widetilde{C}}(\xi,\mu)\right]_{\mathbb{R}} + \left[\arg \det \overline{\Phi}_{\widetilde{C}}(\infty,\mu)\right]_{[0,1]}\right)$$
$$= -\frac{1}{2\pi}\left(\left[\arg \det \Phi_{\widetilde{C}}(\xi)\right]_{\mathbb{R}} + \left[\arg \det \overline{\Phi}_{\widetilde{C}}(\infty,\mu)\right]_{[0,1]}\right),$$

where $[f(\xi)]_{\mathbb{R}}$ denotes the increment of $f(\xi)$ when ξ varies through $\mathbb{R}$ from $-\infty$ to $+\infty$ and $[f(\infty,\mu)]_{[0,1]}$ is the increment of $f(\infty,\mu)$ when μ varies through $\mathbb{R}$ from 0 to 1. Directly, we obtain

$$\left[\arg \det \Phi_{\widetilde{C}}(\xi)\right]_{\mathbb{R}} = \pi(-2-8n+8\varepsilon)$$

and

$$\left[\arg \det \overline{\Phi}_{\widetilde{C}}(\infty,\mu)\right]_{[0,1]} = \pi(2+8n-8\varepsilon).$$

So, we have the desired result $Ind\, C_{\Phi_C,\Sigma} = 0.$ ∎

VI. Conclusion

In the present paper we were able to characterize the Fredholm property of particular operators associated with an impedance boundary problem which are a generalization of the results presented in [14]. For practical and theoretical reasons, with the Fredholm property we are able to answer further questions about this kind of diffraction problems in particular the invertibility and the image normalization of the operators related with the problem. We plan to do this in future works.

Acknowledgment

References

[1]   H. Bart and V. E. Tsekanovskii, Matricial coupling and equivalence after extension, *Oper. Theory Adv. Appl.* **59**, 143–160, 1992.

[2]   A. Böttcher, Yu. I. Karlovich, and I. M. Spitkovsky, *Convolution Operators and Factorization of Almost Periodic Matrix Functions*, Birkhäuser Verlag, Basel, 2002.

[3]   A. Büyükaksoy and G. Çýnar, Solution of a matrix Wiener-Hopf equation connected with the plane wave diffraction by an impedance loaded parallel plate waveguide, *Math. Methods Appl. Sci.* **28**, 1633–1645, 2005.

[4]   L. P. Castro, Regularity of convolution type operators with PC symbols in Bessel potential spaces over two finite intervals, *Mathematische Nachrichten* **261-262**(1), 23–36, 2003.

[5]   L. P. Castro, Solution of a Sommerfeld diffraction problem with a real wave number, in: C. Constanda, M. Ahues, A. Largillier (eds.), *Integral Methods in Science and Engineering*, Birkhäuser, Boston, MA, 25–30, 2004.

[6]   L. P. Castro, Wiener-Hopf operators on unions of finite intervals: relations and generalized inversion, in: F. J. Cobos (ed.) et al., *Proceedings of the Meeting on Matrix Analysis and Applications*, University of Sevilha, Sevilha, 148–155, 1997.

[7]   L. P. Castro and D. Kapanadze, The impedance boundary-value problem of diffraction by a strip, *Journal of Mathematical Analysis and Applications* **337**(2), 1031–1040, 2008.

[8]   L. P. Castro and D. Kapanadze, Wave diffraction by a half-plane with an obstacle perpendicular to the boundary, *J. Differential Equations*, **254**, 493–510, 2013.

[9]   L. P. Castro and D. Kapanadze, Wave diffraction by a strip with first and second kind boundary conditions: the real wave number case, *Mathematische Nachrichten*, **281**, 1400–1411, 2008.

[10]   L. P. Castro and A. Moura Santos, An operator approach for an oblique derivative boundary-transmission problem, *Math. Meth. Appl. Sci.*, **27**, 1469–1491, 2004.

[11]   L. P. Castro and D. Natroshvili, The potential method for the reactance wave diffraction problem in a scale of spaces, *Georgian Math. J.* **13**, 251–260, 2006.

[12]   L. P. Castro and A. M. Simões, Fredholm analysis for a wave diffraction problem with higher order boundary conditions on a strip, in: *Mathematical Problems in Engineering and Aerospace Sciences*, Cambridge Scientific Publishers, Cambridge, 535–542, 2009.

[13]   L. P. Castro and A. M. Simões, Mathematical treatment of a wave diffraction problem with higher order boundary conditions, in: *Trends and Challenges in Applied Mathematics*, Matrix Rom Publishers, Bucharest, 44–49, 2007.

[14]   L. P. Castro and A. M. Simões, The impedance problem of wave diffraction by a strip with higher order boundary conditions, *AIP Conference Proceedings*, **1561**(1), 184–193, 2013.

[15]   L. P. Castro and F.-O. Speck, Relations between convolution type operators on intervals and on the half-line, *Integral Equations Operator Theory* **37**, 169–207, 2000.

[16]   L. P. Castro, F.-O. Speck, and F.S. Teixeira, Explicit solution of a Dirichlet-Neumann wedge diffraction problem with a strip, *J. Integral Equations Appl.* **5**, 359–383, 2003.

[17]   L. P. Castro, F.-O. Speck, and F.S. Teixeira, On a class of wedge diffraction problems posted by Erhard Meister, *Oper. Theory Adv. Appl.* **147**, 213–240, 2004.

[18]   S. N. Chandler-Wilde, The impedance boundary value problem for the Helmholtz equation in a half-plane, *Math. Methods Appl. Sci.* **20**, 813–840, 1997.

[19]   D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer-Verlag, Berlin, 1998.

[20]   A. K. Gautesen, Diffraction of plane waves by a wedge with impedance boundary conditions, *Wave Motion* **41**, 239–246, 2005.

[21]   D. S. Jones, *Methods in Electromagnetic Wave Propagation. Volume 1: Theory and Guided Waves. Volume 2: Radiating Waves*, Clarendon Press, Oxford, 1987.

[22]   A. B. Kuijper, A note on first kind convolution equations on a finite interval, *Integral Equations Operator Theory* **14**, 146–152, 1991.

[23]   A. B. Kuijper and I. M. Spitkovskij, On convolution equations with semi-almost periodic symbols on a finite interval, *Integral Equations Operator Theory* **16**, 530–538, 1993.

[24]   W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge Univ. Press, Cambridge, UK, 2000.

[25]   E. Meister, Some multiple-part Wiener-Hopf problems in mathematical physics, in: *Mathematical Models and Methods in Mechanics*, Banach Center Publ., vol. 15, PWN, Polish Scientific Publishers, Warsaw, 359–407, 1985.

[26]   E. Meister and F.-O. Speck, Modern Wiener-Hopf methods in diffraction theory, in: *Ordinary and Partial Differential Equations, vol. II*, Pitman Res. Notes Math. Ser., vol. 216, Longman Sci. Tech., Harlow, 130–171, 1989.

[27]   A. Moura Santos, F.-O. Speck, and F. S. Teixeira, Compatibility conditions in some diffraction problems, in: *Direct and Inverse Electromagnetic Scattering*, Pitman Res. Notes Math. Ser., vol. 361, Longman, Harlow, 25–38, 1996.

[28]   B. Noble, *Methods based on the Wiener-Hopf Technique for the Solution of Partial Differential Equations*, Pergamon Press, London, 1958; Chelsea Publishing Company, New York, 1988.

[29]   A. Sommerfeld, Mathematische Theorie der Diffraction, *Math. Ann.* **47**, 317–374, 1896.

[30]   H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, second ed., Barth, Leipzig, 1995.