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Maximal Operator in Variable Stummel Spaces

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Abstract
We prove that variable exponent Morrey spaces are closely embedded between variable exponent Stummel spaces. We also study the boundedness of the maximal operator in variable exponent Stummel spaces as well as in vanishing variable exponent Stummel spaces.

Keywords Variable Morrey spaces · Variable Stummel spaces · Maximal operator · Muckenhoupt weights

Mathematics Subject Classification 46E30 · 42B35

1 Introduction

The classical Stummel spaces $\mathfrak{S}^{p,\lambda}(\mathbb{R}^n)$, defined by the norm

$$\|f\|_{\mathfrak{S}^{p,\lambda}} := \sup_{x\in\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{|f(y)|^p}{|x-y|^\lambda} \mathrm{d}y \right)^{1/p}, \quad 1 \le p < \infty, \quad 0 < \lambda < n,$$

Dedicated to the 80th anniversary of Professor Stefan Samko.

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appeared for the first time in [31] in the case $p = 2$. These spaces are known to be used in applications to PDEs, see for instance [22–24]. The class of functions in Stummel space ($p = 1$) was also studied in [11, 27]. For $p = 1$ and $\lambda = n - 2$ it is also called Stummel–Kato class. Generalized Stummel spaces $\mathfrak{S}^{p,w}(\mathbb{R}^n)$, with the function $|x|^\lambda$ replaced by a more general weight function $w(x)$, were used in [2, 26, 30] in embedding results for global Morrey spaces.

The goal of the paper is twofold: on the one hand, to study the two-sided embeddings

$$\mathfrak{S}^{p(\cdot),\varphi(\cdot,\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p(\cdot),\varphi(\cdot,\cdot)}(\mathbb{R}^n) \hookrightarrow \mathfrak{S}^{p(\cdot),\frac{\varphi(\cdot,\cdot)}{w(\cdot,\cdot)}}(\mathbb{R}^n), \tag{1.1}$$

between Stummel and Morrey spaces as defined in (3.1) and (3.3); and on the other hand, to prove the boundedness of the Hardy–Littlewood maximal operator in Stummel type spaces; viz

$$M : \mathfrak{S}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n) \hookrightarrow \mathfrak{S}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n).$$

It is worth noting that such boundedness results were never studied, even in the constant exponent Stummel spaces, to the best of the authors' knowledge. This goal requires, in particular, to obtain refined quantitative estimates related to weighted norm inequalities for the maximal operator. Those estimates provide uniform bounds for a family of Muckenhoupt weights.

The two-sided embeddings (1.1) show that variable exponent generalized Morrey spaces are closely embedded between variable exponent generalized Stummel spaces. The latter spaces are introduced in Definition 3.1, while the former have been considered by several authors in diverse forms, e.g., [1, 14, 15, 25].

The paper is organized as follows. After some notations and preliminaries on variable exponent Lebesgue spaces, in Sect. 3 we introduce generalized Stummel spaces and prove the aforementioned two-sided embeddings. In Sect. 4, we prove several crucial quantitative results dealing with variable exponent Muckenhoupt weights, which play a key part in the proof of the main boundedness result. Finally, in Sect. 5 we study the boundedness of the maximal operator in variable exponent Stummel space and in the vanishing variable exponent Stummel spaces.

Notation

- $\mathbf{1}_E$ denotes the characteristic function of E;
- $\tau_h f(x) := f(h - x)$;
- $B(x, r)$ stands for the open ball centered at $x \in \mathbb{R}^n$ and radius $r > 0$;
- $B_0 := B(0, 1)$;
- $T : X \hookrightarrow Y$ means that T is a continuous mapping from X into Y.

2 Preliminaries

We refer to the books [5, 8] for the basics on the theory of variable exponent Lebesgue spaces. For applications of variable exponent type spaces to integral operators, see [16, 17].

Let $\Omega \subset \mathbb{R}^n$ be a measurable set. We define $\mathcal{P}(\Omega)$ as the class of all bounded measurable functions (usually called variable exponents) $p : \Omega \to [1, \infty)$. For $p \in \mathcal{P}(\Omega)$ we denote

$$p_\Omega^- := \operatorname{ess\,inf}_{x\in\Omega} p(x) \quad \text{and} \quad p_\Omega^+ := \operatorname{ess\,sup}_{x\in\Omega} p(x).$$

If $\Omega = \mathbb{R}^n$ we simply write p^- and p^+ instead of $p_{\mathbb{R}^n}^-$ and $p_{\mathbb{R}^n}^+$, respectively. By p' we denote the conjugate exponent function of p given by $p'(x) = \frac{p(x)}{p(x)-1}$, $x \in \Omega$. The following relations hold

$$(p_\Omega^+)' = (p')_\Omega^-, \quad (p_\Omega^-)' = (p')_\Omega^+. \tag{2.1}$$

We say that a function $g : \Omega \to \mathbb{R}$ is *locally* log-*Hölder continuous* if there exists $c_{\log}(g) > 0$ such that

$$|g(x) - g(y)| \le \frac{c_{\log}(g)}{\log(\mathrm{e} + 1/|x-y|)}, \quad \text{for all } x, y \in \Omega. \tag{2.2}$$

The function g is said to satisfy the log-*Hölder continuity condition at infinity*, also known as the *decay condition*, if there exist $g_\infty \in \mathbb{R}$ and $c_{\log}^*(g) > 0$ such that

$$|g(x) - g_\infty| \le \frac{c_{\log}^*(g)}{\log(\mathrm{e} + |x|)}, \quad \text{for all } x \in \Omega, \tag{2.3}$$

which is equivalent to

$$|g(x) - g(y)| \le \frac{C_{\log}^*(g)}{\log(\mathrm{e} + |x|)}, \quad \text{for all } x, y \in \Omega \text{ with } |y| > |x|. \tag{2.4}$$

Usually, we say that g is log-*Hölder continuous* when it satisfies conditions (2.2) and (2.3) simultaneously. Condition (2.3) is of interest only when Ω is unbounded. We denote by $\mathcal{P}^{\log}(\Omega)$ the class of all exponents $p \in \mathcal{P}(\Omega)$ which are log-Hölder continuous.

The next two lemmas emphasize the important role played by the log-Hölder continuity conditions, which are crucial tools when dealing with variable exponents. The first one is due to Diening [7], while the second one to Capone, Cruz-Uribe and Fiorenza [3].

Lemma 2.1 *If $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ satisfies (2.2), then there exists $C > 0$ such that*

$$|B|^{p_B^- - p_B^+} \le C,$$

for every ball B in $\mathbb{R}^n$.

Lemma 2.2 *Let G be a given set, μ a non-negative measure, and r and s two variable exponents such that*

$$|r(y)-s(y)| \le \frac{c^*}{\log(\mathrm{e}+|y|)}$$

for some $c^*>0$ and all $y\in\mathbb{R}^n$. Then for every $t\ge 1$ there exists $C=C(t,c^*)$ such that for all functions g with $|g|\le 1$,

$$\int_G |g(y)|^{s(y)}\mathrm{d}\mu(y) \le C\int_G |g(y)|^{r(y)}\mathrm{d}\mu(y) + \int_G \frac{\mathrm{d}\mu(y)}{(\mathrm{e}+|y|)^{tnr_G^-}}. \tag{2.5}$$

Remark 2.3 An analysis of the proof of [3,Lemma 2.7] shows that the constant in (2.5) is $C=\mathrm{e}^{tnc^*}$.

For our further purposes, we need a yet another result.

Lemma 2.4 *Let $p\in\mathcal{P}^{\log}(\mathbb{R}^n)$ and $\alpha|y|\le|z|\le\beta|y|$, where $0<\alpha<1<\beta$ are fixed real numbers. Then*

$$|z|^{p(y)-p(z)} \le C, \quad |y|^{p(y)-p(z)} \le C, \tag{2.6}$$

where C only depends on α, β, $c_{\log}(p)$, $c^*_{\log}(p)$, p^+, and p^-.

Proof We need to consider the following two cases.
Case 1: $\alpha|y|\ge\mathrm{e}$. It suffices to prove the estimates in (2.6) when $p(y)-p(z)\ge 0$. From the hypothesis of the lemma, we have

$$a|y|^{|p(y)-p(z)|} \le |z|^{|p(y)-p(z)|} \le b|y|^{|p(y)-p(z)|},$$

for appropriate $a,b\in\mathbb{R}_+$, depending only on α,β,p^+, and p^-. The result now follows using the decay condition (2.4).
Case 2: $\beta|y|<1/2$. We only need to address the case $p(y)-p(z)<0$. Using the elementary estimates

$$|z-y|\le|z|(1+1/a), \quad |z-y|\le|y|(1+b),$$

we obtain

$$|y|^{-|p(y)-p(z)|} \le |z-y|^{-|p(y)-p(z)|}, \quad |z|^{-|p(y)-p(z)|} \le |z-y|^{-|p(y)-p(z)|}. \tag{2.7}$$

From the local logarithmic condition (2.2) and (2.7), we get (2.6). □

The *variable exponent Lebesgue space*, denoted by $L^{p(\cdot)}(\Omega)$, with $p\in\mathcal{P}(\Omega)$, is the space of all measurable functions f on Ω such that

$$\varrho_{p(\cdot),\Omega}(f) := \int_\Omega |f(x)|^{p(x)}\mathrm{d}x < \infty.$$

Equipped with the norm

$$\|f\|_{p(\cdot),\Omega} := \inf\left\{\eta > 0 : \varrho_{p(\cdot)}\left(\frac{f}{\eta}\right) \leq 1\right\}, \tag{2.8}$$

$L^{p(\cdot)}(\Omega)$ is a Banach function space. We shall omit the set Ω in the notation when we deal with the whole space $\Omega = \mathbb{R}^n$.

From the definition of the norm, we have the equality

$$\|f\|_{p(\cdot)} = \|\tau_h f\|_{(\tau_h p)(\cdot)}, \tag{2.9}$$

as well as the following inequalities for $\Omega \subset \mathbb{R}^n$ and $p \in \mathcal{P}(\mathbb{R}^n)$:

$$\|f\,\mathbf{1}_\Omega\|_{p(\cdot)}^{p_\Omega^+} \leq \varrho_{p(\cdot)}(f\,\mathbf{1}_\Omega) \leq \|f\,\mathbf{1}_\Omega\|_{p(\cdot)}^{p_\Omega^-}, \quad \text{for all } f \text{ with } \|f\,\mathbf{1}_\Omega\|_{p(\cdot)} \leq 1, \tag{2.10}$$

and

$$\|f\,\mathbf{1}_\Omega\|_{p(\cdot)}^{p_\Omega^-} \leq \varrho_{p(\cdot)}(f\,\mathbf{1}_\Omega) \leq \|f\,\mathbf{1}_\Omega\|_{p(\cdot)}^{p_\Omega^+}, \quad \text{for all } f \text{ with } \|f\,\mathbf{1}_\Omega\|_{p(\cdot)} \geq 1. \tag{2.11}$$

Hölder's inequality holds in the form

$$\|fg\|_1 \leq 2\,\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}.$$

By $L_w^{p(\cdot)}(\mathbb{R}^n)$ we mean the weighted $L^{p(\cdot)}$ space normed by

$$\|f\|_{L_w^{p(\cdot)}} := \|fw\|_{p(\cdot)}.$$

As usual, by a *weight* we mean a measurable function w on $\mathbb{R}^n$ such that $0 < w(x) < \infty$ almost everywhere.

3 Embedding Results

In this section we compare variable exponent generalized Morrey spaces with variable exponent generalized Stummel spaces.

Recall that, for $p \in \mathcal{P}(\mathbb{R}^n)$, the Morrey space $L^{p(\cdot),\varphi(\cdot,\cdot)}(\mathbb{R}^n)$ is defined as the set of measurable functions f such that

$$\|f\|_{L^{p(\cdot),\varphi(\cdot,\cdot)}} := \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi(x,r)}\|f\mathbf{1}_{B(x,r)}\|_{p(\cdot)} < \infty, \tag{3.1}$$

where φ is assumed to be a measurable function satisfying

$$\inf_{x\in\mathbb{R}^n} \varphi(x,t) > 0 \text{ for every } t > 0. \tag{3.2}$$

We introduce variable exponent generalized Stummel spaces as follows:

Definition 3.1 For $p \in \mathcal{P}(\mathbb{R}^n)$ and a function φ satisfying (3.2), we define the *variable exponent generalized Stummel spaces* $\mathfrak{S}^{p(\cdot),\varphi(\cdot,\cdot)}(\mathbb{R}^n)$ as the set of measurable functions f such that

$$\|f\|_{\mathfrak{S}^{p(\cdot),\varphi(\cdot,\cdot)}} := \sup_{x\in\mathbb{R}^n} \left\| \frac{f}{\varphi(x,|x-\cdot|)} \right\|_{p(\cdot)} < \infty. \tag{3.3}$$

Remark 3.2 Stummel classes appeared for the first time in the literature in [31] in connection to PDEs with $\varphi(x,t) = \varphi(t) = t^{\lambda}$, see also [22–24]. As spaces of functions, Stummel spaces $\mathfrak{S}^{p,\varphi}(\mathbb{R}^n)$ were used in [2, 26, 30] in the study of embedding results involving Morrey spaces, with constant p and φ only depending on t.

We start with the observation that generalized Morrey spaces are not, in general, embedded into Lebesgue spaces. For example, taking

$$f_0(x) = |x|^{-\frac{n}{p}} \varphi(0,|x|)$$

we have that $f_0 \in L^{p,\varphi(\cdot,\cdot)}(\mathbb{R}^n)$ but $f_0 \notin L^p(\mathbb{R}^n)$ if φ satisfies the following assumptions:

$$t \mapsto \varphi^p(0,t) \text{ is almost increasing,} \tag{3.4}$$

$$t \mapsto \frac{\varphi^p(0,t)}{t^{\varepsilon}} \text{ is almost increasing,} \tag{3.5}$$

for some $\varepsilon > 0$,

$$t \mapsto \frac{\varphi^p(0,t)}{t^{n}} \text{ is almost decreasing,} \tag{3.6}$$

and

$$\frac{\varphi(0,t)}{\varphi(x,t)} \le C, \tag{3.7}$$

where C does not depend on $x \in \mathbb{R}^n$ or $t > 0$. Detailed calculations can be found in [2] (see the proof of Lemma 2.1).

Remark 3.3 The function $\varphi^p(x,t) = t^{\lambda(x,t)}$, with $\lambda(x,t)$ given by

$$\lambda(x,t) = \begin{cases} (n-\delta)s(x), & |x| > t > 1, \\ (n-\delta)s(0), & \text{otherwise,} \end{cases}$$

satisfies the conditions listed above, where $\delta > 0$ is a small real number and s is a function satisfying $|s(x) - s(0)| \log(\mathrm{e} + |x|) \le C$, with C independent of x, and $0 < (n - \delta)s(0) < n$.

For the embedding results between Morrey and Stummel spaces we need to introduce the following monotonicity conditions.

We say that $t \mapsto \varphi(x,t)$ is *x-almost uniformly increasing* in the interval (a,b) if, for all $a < t < s < b$, we have

$$\varphi(x,t) \le C\varphi(x,s),$$

where $C > 0$ does not depend on x, t or s. The notion of *x-almost uniformly decreasing* is introduced similarly with natural modifications. The function $t \mapsto \varphi(x,t)$ satisfies the *x-uniform doubling condition* if

$$\varphi(x,2s) \le C\varphi(x,s),$$

where $C > 0$ does not depend on x or s.

Theorem 3.4 *Let $t \mapsto \varphi(x,t)$ be x-almost uniformly increasing, $t \mapsto w(x,t)$ be x-almost uniformly increasing in $(0,r_x)$ and x-almost uniformly decreasing in (r_x,∞), for some $r_x > 0$. Assume, moreover, that $\sup_{x\in\mathbb{R}^n}\int_0^\infty \frac{w(x,t)}{t}\mathrm{d}t < \infty$, $t \mapsto w(x,t)/\varphi(x,t)$ is x-almost uniformly decreasing, and $t \mapsto \varphi(x,t)$ satisfies the x-uniform doubling condition. Then*

$$\mathfrak{S}^{p(\cdot),\varphi(\cdot,\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p(\cdot),\varphi(\cdot,\cdot)}(\mathbb{R}^n) \hookrightarrow \mathfrak{S}^{p(\cdot),\frac{\varphi(\cdot,\cdot)}{w(\cdot,\cdot)}}(\mathbb{R}^n). \tag{3.8}$$

Proof Fix $x \in \mathbb{R}^n$ and $r > 0$. Since $t \mapsto \varphi(x,t)$ is x-almost uniformly increasing, we have

$$\left\|\frac{1}{\varphi(x,r)} f\mathbf{1}_{B(x,r)}\right\|_{p(\cdot)} \le C\left\|\frac{1}{\varphi(x,|x-\cdot|)} f\mathbf{1}_{B(x,r)}\right\|_{p(\cdot)} \le C\left\|\frac{1}{\varphi(x,|x-\cdot|)} f\right\|_{p(\cdot)}$$

which yields, after taking supremum, the left-hand side embeddding in (3.8).

On the other hand, for fixed $x \in \mathbb{R}^n$, take r_x the monotony inflection point of the function $w(x,t)$ and $A_k(x){:=}B(x,2^{k+1}r_x) \setminus B(x,2^k r_x)$. Since $t \mapsto w(x,t)/\varphi(x,t)$ is x-almost uniformly decreasing and $t \mapsto \varphi(x,t)$ satisfies the x-uniform doubling condition, we have

$$
\begin{aligned}
\left\| f \frac{w(x,|x-\cdot|)}{\varphi(x,|x-\cdot|)} \right\|_{p(\cdot)} &= \left\| \sum_{k\in\mathbb{Z}} f \frac{w(x,|x-\cdot|)}{\varphi(x,|x-\cdot|)} \mathbf{1}_{A_k(x)} \right\|_{p(\cdot)} \\
&\le \sum_{k\in\mathbb{Z}} \left\| f \frac{w(x,|x-\cdot|)}{\varphi(x,|x-\cdot|)} \mathbf{1}_{A_k(x)} \right\|_{p(\cdot)} \\
&\le C \sum_{k\in\mathbb{Z}} \frac{w(x,2^k r_x)}{\varphi(x,2^k r_x)} \left\| f \mathbf{1}_{B(x,2^{k+1}r_x)} \right\|_{p(\cdot)} \\
&\le C \sum_{k\in\mathbb{Z}} w(x,2^k r_x) \|f\|_{L^{p(\cdot),\varphi(\cdot,\cdot)}}.
\end{aligned} \tag{3.9}
$$

From the sum-to-integral Lemma 5.5 and the condition on $w(x,t)$, we have

$$
\begin{aligned}
\sum_{k\in\mathbb{Z}} w(x,2^k r_x) &= \sum_{k=-\infty}^{0} w(x,2^k r_x) + \sum_{k=1}^{\infty} w(x,2^k r_x) \\
&\le C\left(\int_0^{\frac{r_x}{2}} \frac{w(x,t)}{t} \mathrm{d}t + \int_{\frac{r_x}{2}}^{\infty} \frac{w(x,t)}{t} \mathrm{d}t \right) \le C,
\end{aligned} \tag{3.10}
$$

since $\sup_{x\in\mathbb{R}^n} \int_0^\infty \frac{w(x,t)}{t}\mathrm{d}t < \infty$. The right-hand side embedding in (3.8) follows now from (3.9) and (3.10). □

4 Variable Exponent Muckenhoupt Weights

4.1 Weighted Norm Inequalities for the Maximal Operator

It is well-known that the classical Muckenhoupt class $\mathcal{A}_p$, $1 < p < \infty$, governs the boundedness of the (Hardy–Littlewood) maximal operator M,

$$
Mf(x) = \sup_{r>0} \frac{1}{r^n} \int_{B(x,r)} |f(y)|\mathrm{d}y, \tag{4.1}
$$

on weighted L^p-spaces. More precisely, M is bounded on $L^p_w(\mathbb{R}^n)$, $1 < p < \infty$, if and only if $w \in \mathcal{A}_p$ (cf. [10, 13]). Recall that $w \in \mathcal{A}_p$ if

$$
\|w\mathbf{1}_B\|_p \, \|w^{-1}\mathbf{1}_B\|_{p'} \le c\,|B| \tag{4.2}
$$

for every ball B (this includes the case $p = 1$). The $\mathcal{A}_p$ constant of w, denoted by $[w]_{\mathcal{A}_p}$, is the smallest constant $c \ge 1$ for which (4.2) holds. This definition considers the weight as a multiplier. If one treats the weight as a measure then we use the class A_p instead, where $w \in A_p$ if $[w]_{A_p} := [w^{1/p}]^p_{\mathcal{A}_p} < \infty$.

For later use, observe that $w \in A_1$ if

$$[w]_{A_1} := \sup_B |B|^{-1} \operatorname{ess\,sup}_{x\in B} w(x)^{-1} \int_B w(x)\mathrm{d}x < \infty,$$

or, equivalently, if $Mw(y) \le c\, w(y)$ almost everywhere in $\mathbb{R}^n$. The smallest $c \ge 1$ in the previous inequality gives $[w]_{A_1}$, the A_1 constant of w. Recall that

$$A_1 \subset A_p \subset A_q, \qquad \text{if} \quad 1 < p < q.$$

For an account on the theory of Muckenhoupt weights, we refer the reader to [13].

Weighted norm inequalities for the maximal operator on variable Lebesgue spaces have been investigated in various papers, e.g., [4, 6, 9, 18–21, 28] and in the monograph [8,Section 5.8]. One can find different formulations in the literature (not always equivalent) for corresponding Muckenhoupt classes for variable exponents. We refer to [4] for a detailed comparison among such classes.

Treating the weight as a measure, in [9] the authors introduced the class $A_{p(\cdot)}$ consisting of all weights w such that

$$\|w\mathbf{1}_B\|_1 \, \|w^{-1}\mathbf{1}_B\|_{\frac{p'(\cdot)}{p(\cdot)}} \le c\,|B|^{p_B}$$

for some constant $c > 0$ and all balls B in $\mathbb{R}^n$, where p_B is the harmonic mean of p on B, i.e., $p_B^{-1} := |B|^{-1} \int_B p(x)^{-1}\mathrm{d}x$ (in the case $p'(\cdot)/p(\cdot) \in (0,1)$, the quasi-norm $\|\cdot\|_{p'(\cdot)/p(\cdot)}$ is defined as in (2.8)). Note that $A_{p(\cdot)}$ coincides with the classical Muckenhoupt class A_p when $p(x) \equiv p$ is constant. As observed in [4,p. 364], for $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \le p^+ < \infty$,

$$w \in A_{p(\cdot)} \quad \text{if and only if} \quad \|w^{1/p(\cdot)}\mathbf{1}_B\|_{p(\cdot)} \, \|w^{-1/p(\cdot)}\mathbf{1}_B\|_{p'(\cdot)} \le c\,|B| \tag{4.3}$$

for some $c > 0$ and all balls B in $\mathbb{R}^n$.

We have

$$A_1 \subset A_{p^-} \subset A_{p(\cdot)} \subset A_{p^+}$$

for $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \le p^+ < \infty$. Moreover, under the same assumptions on p, the maximal operator is bounded on $L^{p(\cdot)}_{w^{1/p(\cdot)}}(\mathbb{R}^n)$ if and only if $w \in A_{p(\cdot)}$. These properties were proved in [9].

For our purposes, we prefer to treat weights as multipliers and follow the notation from [4]. Given $p \in \mathcal{P}(\mathbb{R}^n)$, we say that a weight w belongs to the class $\mathcal{A}_{p(\cdot)}$ if

$$[w]_{\mathcal{A}_{p(\cdot)}} := \sup_B |B|^{-1} \, \|w\mathbf{1}_B\|_{p(\cdot)} \, \|w^{-1}\mathbf{1}_B\|_{p'(\cdot)} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$. Note that $w^{-1} \in \mathcal{A}_{p'(\cdot)}$ if and only if $w \in \mathcal{A}_{p(\cdot)}$ with $[w^{-1}]_{\mathcal{A}_{p'(\cdot)}} = [w]_{\mathcal{A}_{p(\cdot)}}$, $1 < p^- \le p^+ < \infty$.

In view of (4.3), we have

$$w \in \mathcal{A}_{p(\cdot)} \Longleftrightarrow w(\cdot)^{p(\cdot)} \in A_{p(\cdot)}$$

for all exponents $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$.

The following result is taken from [4,Theorem 1.3].

Proposition 4.1 *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$. If $w \in \mathcal{A}_{p(\cdot)}$, then there exists $C_M > 0$ such that*

$$\|(Mf)\, w\|_{p(\cdot)} \leq C_M \, \|f\, w\|_{p(\cdot)}, \tag{4.4}$$

for all $f \in L^{p(\cdot)}_w(\mathbb{R}^n)$.

Remark 4.2 The embedding constant in (4.4) depends on n, p and w. A careful inspection of the long proof given in [4] allows us to highlight the dependence of C_M on the weight w. In fact, C_M may be taken as follows:

$$C_M = c(n, p, s, u, v) \times [1]_{\mathcal{A}_{u(\cdot)}} \times [1]_{\mathcal{A}_{v(\cdot)}} \times [1]_{\mathcal{A}_{p(\cdot)}} \times \left[w^{1/s}\right]^s_{\mathcal{A}_{sp(\cdot)}} \times \left[w^{-1/s}\right]^s_{\mathcal{A}_{sp'(\cdot)}}$$

with the intermediate constant $c(n, p, s, u, v) > 0$ depending only on the dimension n, the exponent p, and the auxiliary parameters s, u and v. Here s is chosen in the interval $(\min\{1/p^-, 1/(p')^-\}, 1)$, thus depending only on p, and $u, v \in \mathcal{P}^{\log}(\mathbb{R}^n)$ are given in terms of p and s:

$$\frac{1}{u'(x)} = s - \frac{1}{p(x)} \quad \text{and} \quad \frac{1}{v(x)} = s - \frac{1}{p'(x)}, \quad x \in \mathbb{R}^n.$$

Therefore, we can rewrite the constant as

$$\begin{aligned} C_M &= c(n, p) \times \left[w^{1/s}\right]^s_{\mathcal{A}_{sp(\cdot)}} \times \left[w^{-1/s}\right]^s_{\mathcal{A}_{sp'(\cdot)}} \\ &= c(n, p) \times \left[w^{1/s}\right]^s_{\mathcal{A}_{sp(\cdot)}} \times \left[w^{1/s}\right]^s_{\mathcal{A}_{(sp')'(\cdot)}} \end{aligned} \tag{4.5}$$

for some $c(n, p) > 0$ depending on n and p only.

4.2 Quantitative Results and Weight Dependence

In this section, we present some weighted estimates involving special A_1 weights. The dependence of the constants appearing in such estimates on the weight is crucial for our goals.

Lemma 4.3 *Let $p \in \mathcal{P}(\mathbb{R}^n)$ and w be a weight such that $w(\cdot)^{p(\cdot)} \in A_1$. For every ball B the following inequality holds*

$$\|w^{-1}\mathbf{1}_B\|_{p'(\cdot)} \leq [w(\cdot)^{p(\cdot)}]_{A_1}\,|B|\,\max\left\{\varrho_{p(\cdot)}(w\mathbf{1}_B)^{-\frac{1}{p_B^+}},\,\varrho_{p(\cdot)}(w\mathbf{1}_B)^{-\frac{1}{p_B^-}}\right\}. \tag{4.6}$$

Proof By the A_1 condition,

$$\|w^{-1}\mathbf{1}_B\|_{p'(\cdot)} \leq [w(\cdot)^{p(\cdot)}]_{A_1}\,|B|\,\varrho_{p(\cdot)}(w\mathbf{1}_B)^{-1}\|w^{p(\cdot)-1}\mathbf{1}_B\|_{p'(\cdot)}. \tag{4.7}$$

On the other hand, by (2.10), (2.11) and (2.1), we have

$$\|w^{p(\cdot)-1}\mathbf{1}_B\|_{p'(\cdot)} \leq \max\left\{\varrho_{p(\cdot)}(w\mathbf{1}_B)^{\frac{1}{(p_B^+)'}},\,\varrho_{p(\cdot)}(w\mathbf{1}_B)^{\frac{1}{(p_B^-)'}}\right\}.$$

Using this estimate in (4.7), we get (4.6) □

Lemma 4.4 *Let $p \in \mathcal{P}(\mathbb{R}^n)$ and w be a weight such that $w(\cdot)^{p(\cdot)} \in A_1$. Then*

$$\frac{|E|}{|B|} \leq 2\,[w(\cdot)^{p(\cdot)}]_{A_1}\,\frac{\|w\,\mathbf{1}_E\|_{p(\cdot)}}{\min\left\{\varrho_{p(\cdot)}(w\,\mathbf{1}_B)^{1/p_B^+},\,\varrho_{p(\cdot)}(w\,\mathbf{1}_B)^{1/p_B^-}\right\}} \tag{4.8}$$

for any ball B and any measurable set $E \subset B$.

Proof For a fixed ball B and a measurable set $E \subset B$, Hölder's inequality yields

$$|E| = \int_{\mathbb{R}^n} w(x)\mathbf{1}_E(x)w(x)^{-1}\mathbf{1}_B(x)\mathrm{d}x \leq 2\,\|w\mathbf{1}_E\|_{p(\cdot)}\|w^{-1}\mathbf{1}_B\|_{p'(\cdot)}.$$

Combining this with (4.6) we get (4.8). □

The following lemma gives a weighted version of Diening's result (cf. Lemma 2.1) for some special weights. Estimate (4.9) is given in [6,Lemma 3.3] for weights $w \in \mathcal{A}_{p(\cdot)}$. The novelty in our result is the explicit form of the constant given in (4.10), which will be useful later on. For reader's convenience, we give details in order to show how the successive constants depend on the weight w.

Lemma 4.5 *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$ and let w be a weight such that $w(\cdot)^{p(\cdot)} \in A_1$. Then there exists $C_0 \geq 1$ such that*

$$\|w\,\mathbf{1}_B\|_{p(\cdot)}^{p_B^- - p_B^+} \leq C_0, \tag{4.9}$$

for all balls B. The involved constant has the form

$$C_0 = c_{n,p}\,[w(\cdot)^{p(\cdot)}]_{A_1}^{p^+-p^-}\left(1+\varrho_{p(\cdot)}(w\mathbf{1}_{B_0})^{-1}\right)^{p^+-p^-}, \tag{4.10}$$

with $c_{n,p} > 0$ *depending only on n and p.*

Proof For short we use the notation $W := w(\cdot)^{p(\cdot)}$ and $B_0 = B(0, 1)$ as before.

Let $B = B(x_0, r_0)$ be an arbitrary (but fixed) ball. It suffices to consider $\|w\,\mathbf{1}_B\|_{p(\cdot)} \le 1$ (otherwise the inequality is obvious with $C_0 = 1$). We consider three different cases. In all of them we use Lemma 4.4 properly.

Case 1: $r_0 \le 1$ and $|x_0| \le 2$. Since $B \subset B(0, 3) = 3B_0$, Lemma 4.4 yields

$$|B| \le 2\,|3B_0|\,[W]_{A_1}\,\|w\,\mathbf{1}_B\|_{p(\cdot)}\big(1 + \varrho_{p(\cdot)}(w\,\mathbf{1}_{B_0})^{-1}\big).$$

Putting the factors in the right position, raising to the power $p_B^+ - p_B^-$ and using Lemma 2.1, we get

$$\begin{aligned}\|w\,\mathbf{1}_B\|_{p(\cdot)}^{p_B^- - p_B^+} &\le |B|^{p_B^- - p_B^+}\left(2\,|3B_0|\,[W]_{A_1}\left(1 + \varrho_{p(\cdot)}(w\,\mathbf{1}_{B_0})^{-1}\right)\right)^{p_B^+ - p_B^-}\\ &\le c_{n,p}\left([W]_{A_1}\left(1 + \varrho_{p(\cdot)}(w\,\mathbf{1}_{B_0})^{-1}\right)\right)^{p^+ - p^-},\end{aligned}$$

since $[W]_{A_1} \ge 1$.

Case 2: $r_0 \ge 1$ and $|x_0| \le 2r_0$. In this case we have $B, B_0 \subset B(x_0, 1+2r_0) =: B'$. As in the previous case (now with B' playing the role of $3B_0$), and observing that $\varrho_{p(\cdot)}(w\,\mathbf{1}_{B_0}) \le \varrho_{p(\cdot)}(w\,\mathbf{1}_{B'})$, we get

$$\|w\,\mathbf{1}_B\|_{p(\cdot)}^{-1} \le 2\,[W]_{A_1}\,|B'|\,|B|^{-1}\big(1 + \varrho_{p(\cdot)}(w\,\mathbf{1}_{B_0})^{-1}\big). \tag{4.11}$$

This gives the desired estimate since $|B'|\,|B|^{-1} \le 3^n$.

Case 3: $|x_0| \ge 2\max\{1, r_0\}$. Now we have $B, B_0 \subset B(0, 2|x_0|) =: B''$. Proceeding as in Case 2, we obtain an estimate like (4.11) with $|B''|$ in place of $|B'|$. Hence, it remains to show that both powers

$$|B|^{p_B^- - p_B^+} \quad \text{and} \quad |B''|^{p_B^+ - p_B^-}$$

are bounded by a constant independent of r_0 and x_0. The first follows from Lemma 2.1. The bound for the power involving B'' can be obtained directly from the decay condition (2.3).

Since p is continuous there are $y_1, y_2 \in \overline{B}$ for which $p(y_1) = p_B^-$ and $p(y_2) = p_B^+$. On the other hand, $|y_1|, |y_2| \ge \frac{|x_0|}{2}$. Hence, by (2.3) we get

$$p_B^+ - p_B^- \le |p(y_2) - p_\infty| + |p(y_1) - p_\infty| \le \frac{2\,c_{\log}^*(p)}{\log(\mathrm{e} + |x_0|/2)}.$$

Thus

$$\log\left(|B''|^{p_B^+ - p_B^-}\right) \le \frac{2\,c_{\log}^*(p)}{\log(\mathrm{e} + |x_0|/2)}\,\log|B''| \le c_{n,p},$$

with $c_{n,p}$ depending only on n and p. □

Remark 4.6 The estimate (4.9) corresponds to the result given in [6,Lemma 3.3] for weights $w \in \mathcal{A}_{p(\cdot)}$. Our statement here refines such a result from [6], since now we make explicit the dependence of the constant on the weight w, under the assumption $w(\cdot)^{p(\cdot)} \in A_1$.

The following result will be useful in the proof of the proposition below. It is taken from [6,Lemma 3.6], but the formulation in [6] is rough for our purposes. Again, we need to know how the involved constant depends on the weight and since this information is not available in [6] we give details for reader's convenience.

Lemma 4.7 *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \le p^+ < \infty$. If w is a weight such that $w(\cdot)^{p(\cdot)} \in A_1$, then there exists $C_1 \ge 1$ such that for every ball B with $\|w\,\mathbf{1}_B\|_{p(\cdot)} \ge 1$, the following inequality holds*

$$\|w\,\mathbf{1}_B\|_{p(\cdot)} \le C_1\,\varrho_{p(\cdot)}(w\,\mathbf{1}_B)^{\frac{1}{p_\infty}}. \tag{4.12}$$

The involved constant has the form

$$C_1 = c_{n,p}\,[w(\cdot)^{p(\cdot)}]_{A_1}^{\frac{p^+}{p_\infty}}\left(1 + \varrho_{p(\cdot)}(w\,\mathbf{1}_{B_0})\right)^{\frac{p^+}{p_\infty}}. \tag{4.13}$$

Proof Let us denote $W := w(\cdot)^{p(\cdot)}$. Since $\|w\,\mathbf{1}_B\|_{p(\cdot)} \ge 1$, an application of Lemma 2.2 (complemented with Remark 2.3) with $\mathrm{d}\mu(y) = W(y)\mathrm{d}y$, $g \equiv \|w\,\mathbf{1}_B\|_{p(\cdot)}^{-1}$, $s(y) = p(y)$ and $r(y) = p_\infty$, yields

$$\begin{aligned} 1 = \varrho_{p(\cdot)}\left(\|w\,\mathbf{1}_B\|_{p(\cdot)}^{-1} w\,\mathbf{1}_B\right) &= \int_B \|w\,\mathbf{1}_B\|_{p(\cdot)}^{-p(y)}\,W(y)\mathrm{d}y \\ &\le \mathrm{e}^{tnc^*_{\log}(p)} \int_B \|w\,\mathbf{1}_B\|_{p(\cdot)}^{-p_\infty}\,W(y)\mathrm{d}y + \int_B \frac{W(y)}{(\mathrm{e}+|y|)^{tnp_-}}\,\mathrm{d}y \end{aligned}$$

for every $t \ge 1$. As claimed in the proof of [6,Lemma 3.4], the second integral above is at most $1/2$ if t is taken large enough. This claim gives the desired result since then we have

$$\|w\,\mathbf{1}_B\|_{p(\cdot)}^{p_\infty} \le 2\,\mathrm{e}^{tnc^*_{\log}(p)}\,\varrho_{p(\cdot)}(w\,\mathbf{1}_B) \tag{4.14}$$

(for large t). However, the choice of t depends on the weight w, so we shall provide some details here in order to see how that dependence works.

We have

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{W(y)}{(\mathrm{e}+|y|)^{tnp_-}}\,\mathrm{d}y &\le \mathrm{e}^{-tnp_-}\int_{B_0} W(y)\,\mathrm{d}y + \int_{\mathbb{R}^n\setminus B_0} (2\sqrt{\mathrm{e}|y|})^{-tnp_-} W(y)\,\mathrm{d}y \\ &= \mathrm{e}^{-tnp_-}\varrho_{p(\cdot)}(w\,\mathbf{1}_{B_0}) + (2\sqrt{\mathrm{e}})^{-tnp^-}\sum_{k=1}^{\infty}\int_{2^{k-1}\le|y|<2^k} |y|^{-\frac{tnp^-}{2}} W(y)\,\mathrm{d}y \\ &\le \mathrm{e}^{-tnp_-}\varrho_{p(\cdot)}(w\,\mathbf{1}_{B_0}) + (2\mathrm{e})^{-\frac{tnp^-}{2}}\sum_{k=1}^{\infty} 2^{-k\frac{tnp^-}{2}}\varrho_{p(\cdot)}(w\,\mathbf{1}_{B_k}),\end{aligned}$$

where $B_k := B(0, 2^k)$. By Lemma 4.4 and the fact that $\|\cdot\|_{p(\cdot)} \le 1 + \varrho_{p(\cdot)}(\cdot)$, we have

$$\varrho_{p(\cdot)}(w\,\mathbf{1}_{B_k}) \le \left(2^{(k+1)n}[W]_{A_1}(1+\varrho_{p(\cdot)}(w\,\mathbf{1}_{B_0}))\right)^{p^+} =: 2^{knp^+}\, C'(n, p, w).$$

So, the series above can be estimated as

$$C'(n, p, w)\sum_{k=1}^{\infty} 2^{-k(\frac{tnp^-}{2}-np^+)} \le C'(n, p, w)\times 1$$

if we choose $t > 2p^+/p^-$ large enough, say $t \ge t_0$ for some fixed t_0 depending only on n and p. Therefore, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{W(y)}{(\mathrm{e}+|y|)^{tnp_-}}\,\mathrm{d}y &\le \mathrm{e}^{-\frac{tnp_-}{2}}\left(\varrho_{p(\cdot)}(w\,\mathbf{1}_{B_0}) + C'(n, p, w)\right) \\ &\le 2\,\mathrm{e}^{-\frac{tnp_-}{2}}\, C'(n, p, w).\end{aligned}$$

The right-hand side in the above estimate is at most $1/2$ for t satisfying

$$\mathrm{e}^{\frac{tnp^-}{2}} \ge 4\, C'(n, p, w).$$

Finally, choosing such a t (also bigger than t_0) we see that (4.14) can be written as

$$\|w\,\mathbf{1}_B\|_{p(\cdot)}^{p_\infty} \le c_{n,p}\,[W]_{A_1}^{p^+}\left(1+\varrho_{p(\cdot)}(w\,\mathbf{1}_{B_0})\right)^{p^+}\varrho_{p(\cdot)}(w\,\mathbf{1}_B)$$

with $c_{n,p}$ depending only on n and p. This gives (4.13). □

In contrast to $A_{p(\cdot)}$, the $\mathcal{A}_{p(\cdot)}$ classes are not monotone. Below we give a monotonicity type result for A_1 weights of the form $w(\cdot)^{p(\cdot)}$. Again, we control the dependence of the embedding constant with respect to the weight.

Proposition 4.8 *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \le p^+ < \infty$. If $w(\cdot)^{p(\cdot)} \in A_1$, then $w \in \mathcal{A}_{p(\cdot)}$ and*

$$[w]_{\mathcal{A}_{p(\cdot)}} \le c_{n,p(\cdot)}\,[w(\cdot)^{p(\cdot)}]_{A_1}^{1+2(p^+/p^-)}\left[1+\varrho_{p(\cdot)}(w\mathbf{1}_{B_0})+\varrho_{p(\cdot)}(w\mathbf{1}_{B_0})^{-1}\right]^{2(p^+/p^-)}. \tag{4.15}$$

Proof It is easy to see that $w \in \mathcal{A}_{p(\cdot)}$, since $A_1 \subset A_{p(\cdot)}$. For any ball B, using (4.6) we obtain

$$\begin{aligned}&|B|^{-1}\,\|w\mathbf{1}_B\|_{p(\cdot)}\,\|w^{-1}\mathbf{1}_B\|_{p'(\cdot)}\\&\quad\le [w(\cdot)^{p(\cdot)}]_{A_1}\,\|w\mathbf{1}_B\|_{p(\cdot)}\,\max\left\{\varrho_{p(\cdot)}(w\mathbf{1}_B)^{-\frac{1}{p_B^+}},\ \varrho_{p(\cdot)}(w\mathbf{1}_B)^{-\frac{1}{p_B^-}}\right\}.\end{aligned}$$

We consider two separate cases.

The case $\|w\mathbf{1}_B\|_{p(\cdot)} \le 1$: By (2.10), (2.11) we have $1 \ge \varrho_{p(\cdot)}(w\mathbf{1}_B) \ge \|w\mathbf{1}_B\|_{p(\cdot)}^{p_B^+}$. From (4.6) we derive

$$\begin{aligned}|B|^{-1}\,\|w\mathbf{1}_B\|_{p(\cdot)}\,\|w^{-1}\mathbf{1}_B\|_{p'(\cdot)} &\le [w(\cdot)^{p(\cdot)}]_{A_1}\,\|w\mathbf{1}_B\|_{p(\cdot)}\,\varrho_{p(\cdot)}(w\mathbf{1}_B)^{-\frac{1}{p_B^-}}\\&\le [w(\cdot)^{p(\cdot)}]_{A_1}\,\|w\mathbf{1}_B\|_{p(\cdot)}^{\frac{p_B^- - p_B^+}{p_B^-}}\\&\le [w(\cdot)^{p(\cdot)}]_{A_1}\,C_0^{\frac{1}{p_B^-}}\end{aligned}$$

for any ball B with $\|w\mathbf{1}_B\|_{p(\cdot)} \le 1$, where $C_0 \ge 1$ is the constant given in (4.10). Since $C_0 \ge 1$, we get the estimate

$$|B|^{-1}\,\|w\mathbf{1}_B\|_{p(\cdot)}\,\|w^{-1}\mathbf{1}_B\|_{p'(\cdot)} \le [w(\cdot)^{p(\cdot)}]_{A_1}\,C_0^{\frac{1}{p^-}}. \tag{4.16}$$

The case $\|w\mathbf{1}_B\|_{p(\cdot)} \ge 1$: By (4.7), have

$$|B|^{-1}\,\|w\mathbf{1}_B\|_{p(\cdot)}\,\|w^{-1}\mathbf{1}_B\|_{p'(\cdot)} \le [w(\cdot)^{p(\cdot)}]_{A_1}\,\|w\mathbf{1}_B\|_{p(\cdot)}\,\varrho_{p(\cdot)}(w\,\mathbf{1}_B)^{-1}\,\|w^{p(\cdot)-1}\mathbf{1}_B\|_{p'(\cdot)}.$$

Noticing that $p' \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $(p')_\infty = (p_\infty)'$ and $\varrho_{p'(\cdot)}\big(w^{p(\cdot)-1}\mathbf{1}_B\big) = \varrho_{p(\cdot)}(w\,\mathbf{1}_B) \ge 1$ (cf. (2.11)), we apply Lemma 4.7 to both norms on the right-hand side of the estimate above and get

$$\begin{aligned}|B|^{-1}\,\|w\mathbf{1}_B\|_{p(\cdot)}\,\|w^{-1}\mathbf{1}_B\|_{p'(\cdot)} &\le C_1^2\,[w(\cdot)^{p(\cdot)}]_{A_1}\,\varrho_{p(\cdot)}(w\,\mathbf{1}_B)^{\frac{1}{p_\infty}-1+\frac{1}{(p')_\infty}}\\&= C_1^2\,[w(\cdot)^{p(\cdot)}]_{A_1}\end{aligned} \tag{4.17}$$

with $C_1 \ge 1$ independent of B.

The inequality (4.15) follows now from (4.16) and (4.17) combined with the expressions in (4.10) and (4.13). □

For our goals, we need to obtain a uniform estimate, in $\xi \in \mathbb{R}^n$, for the A_1 condition of the family of weights $x \mapsto |x|^{-\gamma(\xi-x)}$.

Lemma 4.9 *Let γ be a function satisfying (2.2), (2.3), and $0 \le \gamma^- \le \gamma^+ < n$. For $w_\gamma(x) := |x|^{-\gamma(x)}$, we have that $w_\gamma \in A_1$ with*

$$[w_\gamma]_{A_1} \le \frac{C}{n-\gamma^+}, \tag{4.18}$$

where $C > 0$ only depends on the characteristics of the function γ (i.e., $c_{\log}(\gamma)$, $c^*_{\log}(\gamma)$, γ^+, and γ^-).

Proof Let $0 \ne y \in \mathbb{R}^n$ and $r > 0$ be fixed. We split the proof in three cases.
Case 1: $r < |y|/2$. When $z \in B(y,r)$ we have $\frac{|y|}{2} \le |z| \le \frac{3|y|}{2}$. Taking Lemma 2.4 into account, since the proof of this Lemma only used (2.2) and (2.3), we obtain

$$\int_{B(y,r)} |z|^{-\gamma(z)} \mathrm{d}z \le C\,|y|^{-\gamma(y)} \int_{B(y,r)} |y|^{\gamma(y)-\gamma(z)} \mathrm{d}z \le C r^n |y|^{-\gamma(y)},$$

with C independent of r and y.
Case 2: $|y|/2 < r < 2|y|$. We decompose the integral $\int_{B(y,r)} |z|^{-\gamma(z)}\mathrm{d}z$ as

$$\int_{\substack{B(y,r)\\ |y|<|z|<3|y|}} |z|^{-\gamma(z)}\mathrm{d}z + \int_{\substack{B(y,r)\\ 1<|z|<|y|}} |z|^{-\gamma(z)}\mathrm{d}z + \int_{\substack{B(y,r)\\ |z|<|y|\wedge|z|<1}} |z|^{-\gamma(z)}\mathrm{d}z =: I_1 + I_2 + I_3.$$

The estimate for I_1 follows the same lines as Case 1.

We now estimate I_2. Since $|z| < |y|$, we have $|z|^{|\gamma(y)-\gamma(z)|} \le C$ by the condition (2.4). Taking also into consideration that $B(y,r) \subset B(0, r+|y|)$ and using polar coordinates, we obtain

$$\begin{aligned} I_2 &\le \int_{\substack{B(y,r)\\ 1<|z|<|y|}} |z|^{-\gamma(y)}|z|^{|\gamma(y)-\gamma(z)|}\mathrm{d}z \\ &\le C\int_{\substack{B(y,r)\\ 1<|z|<|y|}} |z|^{-\gamma(y)}\mathrm{d}z \\ &\le C\int_{B(0,r+|y|)} |z|^{-\gamma(y)}\mathrm{d}z \\ &\le \frac{C}{(n-\gamma^+)} r^n |y|^{-\gamma(y)}, \end{aligned}$$

where the constant C does not depend on r and y.

For I_3, taking into account that $|B(y,r)| \approx |y|^n$ we have $|y|^{\gamma^-(B(y,r))-\gamma^+(B(y,r))} \le C$ by (2.2). Therefore, using again polar coordinates and studying the cases $|y| < 1$

and $|y| \geq 1$ separately,

$$I_3 \leq \int_{\substack{B(y,r)\\|z|\leq|y|}} |z|^{-\gamma^+(B(y,r))}\mathrm{d}z \leq \frac{C}{n-\gamma^+} r^n |y|^{-\gamma^+(B(y,r))} \leq \frac{C}{n-\gamma^+} r^n |y|^{-\gamma(y)},$$

with C independent of r and y.
Case 3: $r > 2|y|$. Splitting the ball $B(y,r)$, we have

$$\begin{aligned}\int_{B(y,r)} |z|^{-\gamma(z)}\mathrm{d}z &\leq \int_{B(y,2|y|)} |z|^{-\gamma(z)}\mathrm{d}z + \int_{B(y,r)\setminus B(y,2|y|)} |z|^{-\gamma(z)}\mathrm{d}z \\ &=: J_1 + J_2.\end{aligned}$$

The integral J_1 is analogous to the Case 2, but now studying the cases $r < 1$ and $r \geq 1$ separately. To estimate J_2, note that, for $2|y| < |z-y| < r$, we have $|z| \geq |y|$. Defining $\Xi := B(y,r) \setminus B(y,2|y|)$, we have

$$\begin{aligned}J_2 &\leq \int_{\substack{\Xi\\|y|\geq 1}} |z|^{-\gamma(z)}\mathrm{d}z + \int_{\substack{\Xi\\|z|\leq 1}} |z|^{-\gamma(z)}\mathrm{d}z + \int_{\substack{\Xi\\|y|\leq 1\wedge|z|\geq 1}} |z|^{-\gamma(z)}\mathrm{d}z \\ &=: A_1 + A_2 + A_3.\end{aligned}$$

Using the decay condition (2.4) and the inequality $|y| \leq |z|$, we estimate A_1 by

$$A_1 \leq |y|^{-\gamma(y)} \int_{B(y,r)} |y|^{|\gamma(y)-\gamma(z)|}\mathrm{d}z \leq C r^n |y|^{-\gamma(y)},$$

with C not depending on γ, r, or y.

For A_2, we have that $|z-y| < 2$, then

$$A_2 \leq \int_{B(y,r)} |y|^{-\gamma^+(B(y,2))}\mathrm{d}z \leq C|y|^{-\gamma(y)} r^n,$$

where we use the fact that $|y|^{\gamma(y)-\gamma^+(B(y,2))} \leq C$, which follows from (2.4).

The case of A_3 is estimated, taking into account that $|z| \geq 1$ and $|y| < 1$, as

$$A_3 \leq \int_{B(y,r)} \mathrm{d}z \leq C r^n |y|^{-\gamma(y)},$$

where C does not depend on γ, r, or y. The lemma now follows. $\square$

5 Boundedness of the Maximal Operator

We introduce yet another Stummel type space. By $\mathfrak{S}^{p(\cdot),\phi(\cdot,\cdot,\cdot)}(\mathbb{R}^n)$ we denote the set of measurable functions for which

$$\|f\|_{\mathfrak{S}^{p(\cdot),\phi(\cdot,\cdot,\cdot)}} := \sup_{x\in\mathbb{R}^n} \left\| \frac{f}{\phi(x,\cdot,|x-\cdot|)} \right\|_{p(\cdot)} < \infty, \tag{5.1}$$

where ϕ is an appropriate function. We are interested in just the following cases:

$$\mathfrak{S}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n) := \mathfrak{S}^{p(\cdot),\phi(\cdot,\cdot,\cdot)}(\mathbb{R}^n)\Big|_{\phi(x,y,t):=t^{\lambda(y)}}. \tag{5.2}$$

5.1 Stummel Spaces

Our main result in this subsection is Theorem 5.2, which deals with the boundedness of the maximal operator in variable exponent Stummel spaces. Since the proof of this result in non-variable Stummel spaces can be simplified, we provide a streamlined version of it in Theorem 5.1.

Theorem 5.1 *Let* $1 < p < \infty$ *and* $0 \le \lambda < n$. *Then*

$$M : \mathfrak{S}^{p,\lambda}(\mathbb{R}^n) \hookrightarrow \mathfrak{S}^{p,\lambda}(\mathbb{R}^n).$$

Proof Under the condition $0 \le \lambda < n$, we have $|\cdot|^{-\lambda} \in A_1$. Observing that

$$\frac{1}{r^n}\int_{B(y,r)} (\tau_h f)(\xi)\mathrm{d}\xi = \frac{1}{r^n}\int_{B(h-y,r)} f(z)\mathrm{d}z$$

it follows that

$$M(\tau_h f) = \tau_h(Mf). \tag{5.3}$$

By (5.3) and the Fefferman-Stein inequality (see [12]), we have

$$\begin{aligned}\int_{\mathbb{R}^n} Mf(y)^p|x-y|^{-\lambda}\mathrm{d}y &\le C\int_{\mathbb{R}^n} |f(y)|^p M\big(|x-\cdot|^{-\lambda}\big)(y)\mathrm{d}y\\ &= C\int_{\mathbb{R}^n} |f(y)|^p M\big(|\cdot|^{-\lambda}\big)(x-y)\mathrm{d}y\\ &\le C\int_{\mathbb{R}^n} |f(y)|^p|x-y|^{-\lambda}\mathrm{d}y,\end{aligned}$$

where the constants $C > 0$, in the previous chain of inequalities, do not depend on x. The result now follows. □

Theorem 5.2 *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 < p^- \le p^+ < \infty$, λ satisfies (2.2), (2.3) with $\lambda^- \ge 0$, and $(\lambda p)^+ < n$. Then*

$$M : \mathfrak{S}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n) \hookrightarrow \mathfrak{S}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n).$$

Proof By (2.9), (5.3), and Proposition 4.1, we have

$$\begin{aligned}\left\| \frac{Mf}{|x-\cdot|^{\lambda(\cdot)}} \right\|_{p(\cdot)} &= \left\| \frac{\tau_x(Mf)}{|\cdot|^{\lambda(x-\cdot)}} \right\|_{(\tau_x p)(\cdot)} \\ &= \left\| \frac{M(\tau_x f)}{|\cdot|^{\lambda(x-\cdot)}} \right\|_{(\tau_x p)(\cdot)} \\ &\le C_M \left\| \frac{\tau_x f}{|\cdot|^{\lambda(x-\cdot)}} \right\|_{(\tau_x p)(\cdot)} \\ &= C_M \left\| \frac{f}{|x-\cdot|^{\lambda(\cdot)}} \right\|_{p(\cdot)},\end{aligned}$$

the applicability of Proposition 4.1 being justified by the fact that $|\cdot|^{-\tau_x\lambda(\cdot)} \in \mathcal{A}_{(\tau_x p)(\cdot)}$ since $|\cdot|^{-\tau_x\lambda(\cdot)\tau_x p(\cdot)} \in A_1$, where the membership into the A_1 class follows from Lemma 4.9 and the hypotheses on the function λ.

The proof is completed by showing that C_M is uniformly bounded for $x \in \mathbb{R}^n$. Taking $w(y) = |y|^{-\tau_x\lambda(y)}$, from (4.4) and (4.5), we have

$$C_M \le c(n,p) \times \left[w^{1/s}\right]^s_{\mathcal{A}_{s(\tau_x p)(\cdot)}} \times \left[w^{1/s}\right]^s_{\mathcal{A}_{(s(\tau_x p)')'(\cdot)}},$$

with s independent of x, since $(\tau_x p)^+ = p^+$ and $(\tau_x p)^- = p^-$.

Using (4.15) it follows that

$$\begin{aligned}&\left[w^{1/s}\right]_{\mathcal{A}_{s(\tau_x p)(\cdot)}} \\ &\quad\le c_{n,p(\cdot)}\, [|\cdot|^{-\tau_x\lambda(\cdot)\tau_x p(\cdot)}]_{A_1}^{1+2(p^+/p^-)} \left(1 + \varrho_{s(\tau_x p)(\cdot)}(w\mathbf{1}_{B_0}) + \varrho_{s(\tau_x p)(\cdot)}(w\mathbf{1}_{B_0})^{-1}\right)^{2(p^+/p^-)}.\end{aligned} \tag{5.4}$$

The A_1 condition in (5.4) is bounded, using Lemma 4.9, by $C/(n-(\lambda p)^+)$ with the constant C not depending on x. From the estimates

$$\varrho_{s(\tau_x p)(\cdot)}(w\mathbf{1}_{B_0}) \le \int_{|y|<1} |y|^{-(\lambda p)^+}\,\mathrm{d}y$$

and

$$\varrho_{s(\tau_x p)(\cdot)}(w\mathbf{1}_{B_0}) \ge \int_{|y|<1} \mathrm{d}y,$$

we obtain that $\left[w^{1/s}\right]^s_{\mathcal{A}_{s(\tau_x p)(\cdot)}}$ is uniformly bounded in x.

To estimate the constant $[w^{1/s}]_{\mathcal{A}_{(s(\tau_x p)')'(\cdot)}}$ as in (5.4), note that

$$(s(\tau_x p)')'(y) = \frac{s(\tau_x p)(y)}{s(\tau_x p)(y) - (\tau_x p)(y) + 1}.$$

Since we can choose s as close to 1, from the left, as needed, then we can find s such that

$$\varrho_{\frac{s(\tau_x p)(\cdot)}{s(\tau_x p)(\cdot) - \tau_x p(\cdot) + 1}}(w\mathbf{1}_{B_0}) \le \int_{|y|<1} |y|^{-\frac{n+(\lambda p)^+}{2}} \mathrm{d}y,$$

$$\varrho_{\frac{s(\tau_x p)(\cdot)}{s(\tau_x p)(\cdot) - \tau_x p(\cdot) + 1}}(w\mathbf{1}_{B_0}) \ge \int_{|y|<1} \mathrm{d}y,$$

and

$$\left[|\cdot|^{-\frac{\tau_x \lambda(\cdot) \tau_x p(\cdot)}{s(\tau_x p)(\cdot) - (\tau_x p)(\cdot) + 1}} \right]_{A_1} \le \frac{C}{n - \frac{n+(\lambda p)^+}{2}} = \frac{C}{n - (\lambda p)^+},$$

where C does not depend on x, using Lemma 4.9. Indeed, due to the assumptions on λ and p, the choice $1 > s > \frac{2(\lambda p)^+}{p^+(n+(\lambda p)^+)} + 1 - \frac{1}{p^+}$ is enough. This completes the proof. $\square$

5.2 Vanishing Stummel Spaces

We introduce the variable exponent generalized vanishing Stummel space, denoted by $V_0\mathfrak{S}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$, as the collection of all $f \in \mathfrak{S}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$ such that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \left\| \frac{f}{|x-\cdot|^{\lambda(\cdot)}} \mathbf{1}_{B(x,r)} \right\|_{p(\cdot)} = 0. \tag{5.5}$$

Theorem 5.3 *Under the condition of Theorem 5.2, we have*

$$M : V_0\mathfrak{S}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n) \hookrightarrow V_0\mathfrak{S}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n).$$

Proof The boundedness of M acting from $V_0\mathfrak{S}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$ into $\mathfrak{S}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$ follows from Theorem 5.2. It remains to show that M preserves the vanishing property defining (5.5).

For fixed x and r, we take $f_1(y) := f(y)\mathbf{1}_{B(x,2r)}(y)$ and $f_2 := f(y) - f_1(y)$. Thus,

$$Mf(y) \le Mf_1(y) + Mf_2(y).$$

Estimation for Mf_1. Using the boundedness of M in the weighted Lebesgue space and the fact that $t \mapsto \|f\|_{L^{p(\cdot)}(B(x,t))}$ is non-decreasing, we have for fixed $\alpha > 0$

$$\begin{aligned}\left\|\frac{Mf_1}{|x-\cdot|^{\lambda(\cdot)}}\right\|_{p(\cdot)} &\le C\left\|\frac{f}{|x-\cdot|^{\lambda(\cdot)}}\mathbf{1}_{B(x,2r)}\right\|_{p(\cdot)}\\ &\le Cr^{\alpha}\int_r^{\infty} t^{-\alpha-1}\left\|\frac{f}{|x-\cdot|^{\lambda(\cdot)}}\mathbf{1}_{B(x,t)}\right\|_{p(\cdot)}\mathrm{d}t\\ &\le C\int_1^{\infty} t^{-\alpha-1}\left\|\frac{f}{|x-\cdot|^{\lambda(\cdot)}}\mathbf{1}_{B(x,rt)}\right\|_{p(\cdot)}\mathrm{d}t,\end{aligned}$$

thus

$$\sup_{x\in\mathbb{R}^n}\left\|\frac{Mf_1}{|x-\cdot|^{\lambda(\cdot)}}\right\|_{p(\cdot)} \le C\int_1^{\infty} t^{-\alpha-1}\sup_{x\in\mathbb{R}^n}\left\|\frac{f}{|x-\cdot|^{\lambda(\cdot)}}\mathbf{1}_{B(x,rt)}\right\|_{p(\cdot)}\mathrm{d}t. \tag{5.6}$$

Estimation for Mf_2. For $z \in B(x,r)$, we have

$$Mf_2(z) \le C\int_{\mathbb{R}^n\setminus B(x,2r)}\frac{|f(y)|}{|y-z|^n}\mathrm{d}y \le C\int_{\mathbb{R}^n\setminus B(x,r)}\frac{|f(y)|}{|x-y|^n}\mathrm{d}y. \tag{5.7}$$

From (5.7), we have for fixed $\beta > 0$

$$\begin{aligned}\left\|\frac{Mf_2}{|x-\cdot|^{\lambda(\cdot)}}\mathbf{1}_{B(x,r)}\right\|_{p(\cdot)} &\le C\left\|\frac{\mathbf{1}_{B(x,r)}}{|x-\cdot|^{\lambda(\cdot)}}\right\|_{p(\cdot)}\int_{\mathbb{R}^n\setminus B(x,r)}\frac{|f(y)|}{|x-y|^n}\mathrm{d}y\\ &\le Cr^{\frac{n}{p(x)}-\lambda(x)}\int_{\mathbb{R}^n\setminus B(x,r)}\frac{|f(y)|}{|x-y|^n}\mathrm{d}y\\ &\le Cr^{\frac{n}{p(x)}-\lambda(x)}\int_{\mathbb{R}^n\setminus B(x,r)}\frac{|f(y)|}{|x-y|^{n-\beta}}\mathrm{d}y\int_{|x-y|}^{\infty}\frac{\mathrm{d}s}{s^{\beta+1}}\\ &\le Cr^{\frac{n}{p(x)}-\lambda(x)}\int_r^{\infty}\frac{\mathrm{d}s}{s^{\beta+1}}\int_{r\le|x-y|<s}\frac{|f(y)|}{|x-y|^{n-\beta}}\mathrm{d}y\\ &\le Cr^{\frac{n}{p(x)}-\lambda(x)}\int_r^{\infty}\\ &\quad \frac{1}{s^{\beta+1}}\left\|\frac{f}{|x-\cdot|^{\lambda(\cdot)}}\mathbf{1}_{B(x,s)}\right\|_{p(\cdot)}\left\|\frac{\mathbf{1}_{B(x,s)}}{|x-\cdot|^{n-\beta-\lambda(\cdot)}}\right\|_{p'(\cdot)}\mathrm{d}s\\ &\le Cr^{\frac{n}{p(x)}-\lambda(x)}\int_r^{\infty} s^{\lambda(x)-\frac{n}{p(x)}}\left\|\frac{f}{|x-\cdot|^{\lambda(\cdot)}}\mathbf{1}_{B(x,s)}\right\|_{p(\cdot)}\frac{\mathrm{d}s}{s}\\ &\le C\int_1^{\infty} s^{\lambda(x)-\frac{n}{p(x)}}\left\|\frac{f}{|x-\cdot|^{\lambda(\cdot)}}\mathbf{1}_{B(x,rs)}\right\|_{p(\cdot)}\frac{\mathrm{d}s}{s}\end{aligned}$$

from which it follows

$$\sup_{x\in\mathbb{R}^n} \Big\| \frac{Mf_2}{|x-\cdot|^{\lambda(\cdot)}} \mathbf{1}_{B(x,r)} \Big\|_{p(\cdot)} \le C \int_1^\infty s^{\frac{(\lambda p)^+ - n}{p^-}} \sup_{x\in\mathbb{R}^n} \left\| \frac{f}{|x-\cdot|^{\lambda(\cdot)}} \mathbf{1}_{B(x,rs)} \right\|_{p(\cdot)} \frac{\mathrm{d}s}{s}. \tag{5.8}$$

By the Lebesgue dominated convergence theorem, estimates (5.6) and (5.8), and the fact that $f \in V_0\mathfrak{S}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$, we have

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \left\| \frac{Mf}{|x-\cdot|^{\lambda(\cdot)}} \mathbf{1}_{B(x,r)} \right\|_{p(\cdot)} = 0,$$

which proves the theorem. □

The boundedness of the maximal operator on vanishing Stummel spaces, given in Theorem 5.2, seems to be new even in the constant exponent case.

Corollary 5.4 *Let* $1 < p < \infty$ *and* $0 \le \lambda < n$. *Then*

$$M : V_0\mathfrak{S}^{p,\lambda}(\mathbb{R}^n) \hookrightarrow V_0\mathfrak{S}^{p,\lambda}(\mathbb{R}^n).$$

Appendix

Variants of the sum-to-integral lemma are known and scattered in the literature, see e.g. [2, 29]. For a proof of Lemma 5.5 see [26,Lemma 3.1].

Lemma 5.5 *(sum-to-integral) Let* $\alpha, \beta : \mathbb{R}_+ \to \mathbb{R}_+$ *be functions such that* α *is almost decreasing and* β *is almost increasing. Then*

$$\sum_{k=0}^{\infty} \alpha(2^{k+1}r)\beta(2^k r) \le C \int_r^\infty \alpha(t)\beta(t)\frac{\mathrm{d}t}{t}, \quad 0 < r < \infty, \tag{5.9}$$

and

$$\sum_{k=-\infty}^{0} \alpha(2^{k+1}r)\beta(2^k r) \le C \int_0^{2r} \alpha(t)\beta(t)\frac{\mathrm{d}t}{t}, \quad 0 < r < \infty. \tag{5.10}$$

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